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FREE VIBRATION OF RECTANGULAR PLATES WITH A SMALL INITIAL CURVATURE

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16. ABSTRACT

The method of matched asymptotic expansions is used to solve the transverse free vibration of a slightly curved, thin rectangular plate. Analytical results for natural frequencies and mode shapes are presented in the limit when the dimensionless bending rigidity, ε, is small compared with in-plane forces. Results for different boundary conditions are obtained when the initial deflection is (1) a polynomial in both directions, (2) the product of a polynomial and a trigonometric function, and arbitrary.

For the arbitrary initial deflection case, the Fourier series technique is used to define the initial deflection.

The results obtained show that the natural frequencies of vibration of slightly curved plates are coincident with those of perfectly flat, prestressed rectangular plates. However, the eigenmodes are very different from those of initially flat prestressed rectangular plates. The total deflection is found to be the sum of the initial deflection, the deflection resulting from the solution of the flat plate problem, and the deflection resulting from the static problem.

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SUMMARY

The method of matched asymptotic expansions is used to solve the transverse free vibration of a slightly curved, thin rectangular plate. Analytical results for natural frequencies and mode shapes are presented in the limit when the dimensionless bending rigidity, ε, is small compared with in-plane forces. Results for different boundary conditions are obtained when the initial deflection is (1) a polynomial in both directions, (2) the product of a polynomial and a trigonometric function, and arbitrary.

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I. INTRODUCTION

In many practical applications of plates, it is almost impossible to start out with a perfectly flat rectangular plate. It is usually observed that some initial curvature is present in one or two orthogonal directions. The effect of various mathematical forms of the initial curvature on the vibration of rectangular plates under different boundary conditions is the subject of the present study. This report therefore complements the earlier work of Adeniji-Fashola and Oyediran [1] in which the effect of a small arbitrary initial curvature on the free vibration of clamped rectangular plates was discussed.

The effect of initial deflections on the deformation of elastic plates is treated here using small deflection theory for which superposition is applicable. The forms of the initial deflection considered include (1) second order polymonial in two orthogonal directions, (2) a second order polynomial in one direction and a trigonometric function in the orthogonal direction, and (3) an arbitrary shape which can be decomposed into Fourier components in two orthogonal directions. Results are presented for different boundary conditions. In the literature [2], explicit analytical results for the free vibration of clamped, perfectly flat rectangular plates are not known. Even for the static problems of curved plates, solutions for only a few special cases exist [3].

It is easily seen that the normalized equations of linear vibration of curved plates are readily amenable to the technique of singular perturbations since a small parameter, ε, multiplies the highest derivative terms. This method has been used successfully by various researchers [4,5,6] to solve the problem of vibration of prestressed, flat rectangular plates. These authors neglected the corner regions present in the rectangular plates. Indeed, few solutions exist in the literature which include the corner problem for a rectangular plate [7]. Neyfeh and coworkers [8,9] have, more recently, extended the method to solve the vibration problems of prestressed circular plates and cylindrical shells.

The method of singular perturbations is therefore used in the present study to analyze the free vibration problem of a slightly curved rectangular plate under various boundary conditions and for various mathematical forms of the initial curvature. The effects of corner regions are neglected as are the effects of rotatory and shear deformations.

In the next section, the mathematical problem is formulated and the dimensionless form of the governing differential equation is obtained. The method of solution is presented in Section 3 while the method is applied to a slightly curved plate with various mathematical forms for the initial curvature in Section 4. The report ends in Section 5 with a discussion of the results obtained.

II. PROBLEM FORMULATION

We consider the linear, transverse, free vibration of a slightly curved, thin, rectangular plate. In the absence of shearing prestress, the equilibrium equation of curved plates given by Timoshenko and Woinowski-Krieger [3] in rectangular Cartesian coordinates is

$$D\nabla^4 w_1' - N_x \frac{\partial^2 w_1'}{\partial x'^2} - N_y \frac{\partial^2 w_1'}{\partial y'^2} + m \frac{\partial^2 w_1'}{\partial \tau^2} = N_x \frac{\partial^2 w_0'}{\partial x'^2} + N_y \frac{\partial^2 w_0'}{\partial y'^2} \tag{2.1}$$

where

$$w'(x',y',\tau) = w_0'(x',y') + w_1'(x',y',\tau) \quad . \tag{2.2}$$

The deflection $w_0'(x',y')$ is termed the initial deflection and $w_1'(x',y',\tau)$ is the additional deflection due to vibration, while $w'(x',y',\tau)$ is the total deflection.

In non-dimensional form, equation (2.1) becomes

$$\varepsilon^2\nabla^4 w_1 - \beta_1^2 \frac{\partial^2 w_1}{\partial x^2} - \beta_2^2 \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial t^2} = \beta_1^2 \frac{\partial^2 w_0}{\partial x^2} + \beta_2 \frac{\partial^2 w_0}{\partial y^2} \tag{2.3}$$

with

$$\varepsilon^2 = D/N_o L^2 << 1 \quad , \quad \beta_1^2 = N_x/N_o \quad , \quad \beta_2^2 = N_y/N_o \tag{2.4}$$

L is the characteristic length used in rendering w', x', and y' in equation (2.1) dimensionless. Time is rendered dimensionless by using the characteristic frequency ω so that

$$t = \omega\tau \quad \text{and} \quad N_o/\mu\omega^2 L^2 = 1 \tag{2.5}$$

and μ is the mass per unit area of the plate.

In this work, we shall examine three different cases of initial curvature, viz:

(1) When w_o is a second order polynomial in both the x and y directions.

(2) When w_o is a second order polynomial in one direction and sinusoidal in the other direction.

(3) When w_o is arbitrary.

In this last case, the arbitrary w_o can then be expressed by a double Fourier series.

If we make the substitution

$$w_1(x,y,t) = \bar{u}(x,y,t) + \Omega(x,y) \tag{2.6}$$

in equation (2.3), we obtain two different equations as follows:

$$\epsilon^2 \nabla^4 \bar{u} - \beta_1^2 \frac{\partial^2 \bar{u}}{\partial x^2} - \beta_2^2 \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial t^2} = 0 \tag{2.7}$$

and

$$\epsilon^2 \nabla^4 \Omega - \beta_1^2 \frac{\partial^2 \Omega}{\partial x^2} - \beta_2^2 \frac{\partial^2 \Omega}{\partial y^2} = \beta_1^2 \frac{\partial^2 w_o}{\partial x^2} + \beta_2^2 \frac{\partial^2 w_o}{\partial y^2} \tag{2.8}$$

Equation (2.7) is the governing differential equation describing the dynamic response of an initially-flat, normally-prestressed rectangular plate, while equation (2.8) corresponds to the static problem of a similar plate but having an initial curvature, with the initial curvature terms on the right-hand side, rendering the differential equation inhomogeneous.

To complete the problem formulation, we specify the boundary conditions. For definiteness, we require that

$$w(0,y) = \frac{\partial w}{\partial x}(0,y) = 0 \quad ;$$

$$w(1,y) = \frac{\partial w}{\partial x}(1,y) = 0 \quad ;$$

$$w(x,0) = \frac{\partial w}{\partial y}(x,0) = 0 \quad ; \qquad (2.9)$$

$$w(x,b) = \frac{\partial w}{\partial y}(x,b) = 0 \quad , \qquad 0 < b \leq 1$$

for a clamped plate. It is to be noted that the characteristic length, L, has been taken equal to the plate length in the x direction.

Alternatively, the conditions of equation (2.9) can be written as $w = \partial w/\partial n = 0$ on all boundaries while the boundary conditions for a fully-hinged rectangular plate takes the form

$$w = \partial^2 w/\partial n^2 = 0 \qquad (2.10)$$

on all boundaries. Here $\partial/\partial n$ denotes the derivative normal to the boundary.

Equations (2.7) and (2.8) are solved in conjunction with appropriate boundary conditions using the method of singular perturbations. The natural frequencies and the flat plate component of total deflection are obtained from equation (2.7) while the component of the total deflection resulting from the static conditions are obtained from equation (2.8).

III. METHOD OF SOLUTION

The asymptotic method of singular perturbations is used here to analyze the linear, transverse free vibration of a highly-prestressed, initially-curved plate with rectangular edges. For infinitesimal transverse vibrations, the effect of mid-plane stretching on the in-plane loads can be neglected. With this technique, explicit analytical results are obtained in the limit when the applied loads are very large compared with D/L^2.

In the limit as $\varepsilon \to 0^+$, a state of membrane prevails over the plate, except in a thin layer adjacent to the boundaries where higher derivatives become important.

Under time-independent boundary conditions, we seek for a separable solution of equation (2.7) of the form

$$\bar{u}(x,y,t;\varepsilon) = u(x,y;\varepsilon)\ \chi(t;\varepsilon) \qquad (3.1)$$

describing standing waves. Equation (3.1) inserted into equation (2.7) leads to

$$\varepsilon^2\nabla^4 u - \beta_1^2 \frac{\partial^2 u}{\partial x^2} - \beta_2^2 \frac{\partial^2 u}{\partial y^2} - \lambda^2 u = 0 \tag{3.2}$$

as the dynamic problem governing the transverse vibration of the initially-curved plate.

The governing differential equations (g.d.e) for the vibration of curved plates can thus be written as

$$\begin{aligned} &\varepsilon^2\nabla^4 u - \beta_1^2 \frac{\partial^2 u}{\partial x^2} - \beta_2^2 \frac{\partial^2 u}{\partial y^2} - \lambda^2 u = 0 \quad ; \\ &\varepsilon^2\nabla^4 \Omega - \beta_1^2 \frac{\partial^2 \Omega}{\partial x^2} - \beta_2^2 \frac{\partial^2 \Omega}{\partial y^2} = \beta_1^2 \frac{\partial^2 w_0}{\partial x^2} + \beta_2^2 \frac{\partial^2 w_0}{\partial y^2} \end{aligned} \tag{3.3}$$

subject to appropriate boundary conditions.

<u>Outer Solution</u>

We seek an approximate solution, otherwise known as the outer solution, in the form

$$\begin{aligned} u^0 &= u_0^{\,0} + \varepsilon u_1^{\,0} + \varepsilon^2 u_2^{\,0} + 0(\varepsilon^3) \\ \lambda^2 &= \lambda_0^{\,2} + \varepsilon\lambda_1^{\,2} + \varepsilon^2\lambda_2^{\,2} + 0(\varepsilon^3) \\ \Omega^0 &= \Omega_0^{\,0} + \varepsilon\Omega_1^{\,0} + \varepsilon^2\Omega_2^{\,0} + 0(\varepsilon^3) \end{aligned} \tag{3.4}$$

The g.d.e. can then be written as

$$\begin{aligned} &\beta_1^2 u^0_{\nu_{xx}} + \beta_2^2 u^0_{\nu_{yy}} + \lambda_0^2 u_\nu^0 = \nabla^4 u^0_{\nu-2} - \sum_{\sigma=1}^{\nu} \lambda_\nu^2 u^0_{\nu-\sigma} \\ &\beta_1^2 \Omega^0_{\nu_{xx}} + \beta_2^2 \Omega^0_{\nu_{yy}} = \nabla^4 \Omega^0_{\nu-2} - \beta_1^2 \frac{\partial^2 w_0}{\partial x^2} - \beta_2^2 \frac{\partial^2 w_0}{\partial y^2} \quad . \end{aligned} \tag{3.5}$$

The subscripts denote the order in ε, while superscript 0 denotes the outer solutions. Variables with negative indices (subscripts) are discarded. Leading and higher order equations are given by the recurrence relations equations (3.5) when ν takes the values 0, 1, 2, and so on. It is pertinent to point out that the order of the g.d.e. has been reduced by two from fourth-order equations (3.3) to second-order equations (3.5).

The solutions of equations (3.5) will not, in general, satisfy the prescribed boundary conditions. Thus, the solutions are not valid near the plate boundaries. Therefore, a thin layer where plate displacements change very rapidly from a membrane type [see equations (3.5)] problem to a bending type problem must exist near the boundaries. These solutions, valid in the thin layer adjacent to the boundaries, are called inner solutions and are matched with the outer solutions using the Van Dyke [11] matching principle.

Inner Solution

Near the boundaries, where the fourth-order and second-order derivatives are of identical orders of magnitude in ε, new coordinates (stretched regular coordinates) are defined. For example, near x = 0, the new coordinate takes the form

$$X = x/\varepsilon \tag{3.6}$$

while we seek an expansion of the form

$$\Omega^i \equiv \psi^i(X,y)$$

$$\psi^i(X,y) = \psi_0{}^i + \varepsilon\,\psi_1{}^i + \varepsilon^2\,\psi_2{}^i + 0(\varepsilon^3) \tag{3.7}$$

where, as before, subscripts denote the order in ε and superscript i denotes inner solution.

The g.d.e. then take the forms

$$u^i_{\nu_{XXXX}} - \beta_1{}^2\,u^i_{\nu_{XX}} = -2\,u^i_{\nu-2_{XXyy}} + \beta_2{}^2\,u^i_{\nu-2_{yy}} - u^i_{\nu-4_{yyyy}} - \sum_{\sigma=0}^{\nu-2}(\lambda_\mu{}^2\,u^i_{\nu-\sigma-2})$$

and

$$\psi^i_{\nu_{XXXX}} - \beta_1{}^2\,\psi^i_{\nu_{XX}} = -2\,\psi^i_{\nu-2_{XXyy}} + \beta_2{}^2\,\psi^i_{\nu-2_{yy}} - \psi^i_{\nu-4_{yyyy}} - \left(\beta_1{}^2\frac{\partial^2}{\partial(\varepsilon X)^2} + \beta_2{}^2\frac{\partial^2}{\partial y^2}\right)w_o(\varepsilon X,y) \tag{3.8}$$

Expansions similar to equations (3.7) and (3.8) are also written near $x = 1$, $y = 0$, and $y = b$, respectively.

The procedure for obtaining the inner and outer solutions to various orders is as follows: If we consider the static problem of equation (3.3), the leading order outer solution, Ω_0^0 is first obtained. The leading order inner solution, Ω_0^i is then obtained from equation (3.8) and to satisfy all the boundary conditions. These two solutions are subsequently matched. The procedure for matching is simple: we write the outer solution in inner variables and expand to 1 term while the inner solution is written in outer variables and expanded also to 1 term. These two expansions are then equated (matched) to determine the unknowns. Next, we solve for Ω_1^i and carry out a 2-1 matching. The solution of Ω_1^0 is then sought, with a 2-2 matching providing all the unknowns of Ω_1^0. Next, the inner solution, Ω_2^i, is determined from a 3-2 matching and, finally, the problem is completed to $0(\varepsilon^2)$ by the determination of Ω_2^0 from 3-3 matching.

In the next section, solutions to equations (3.5) and (3.8) are obtained and these are demonstrated for various mathematical forms of the initial curvature.

IV. APPLICATION TO VARIOUS CASES

The method of singular perturbations (otherwise known as the method of matched asymptotic expansions, MMAE) is used in this section to solve the problem of the linear vibration of curved rectangular plates. Essentially, the problem has been reduced to that of solving two auxiliary problems, equations (3.3), subject to appropriate boundary conditions. As stated earlier, equation (3.3a) is identical to the equation governing the linear vibration of a perfectly flat, prestressed rectangular membrane. We shall henceforth refer to this as Problem I or the dynamic problem. Equation (3.3b) is Problem II or the static problem. The initial curvature effect is included in Problem II.

In this section, various mathematical forms of the initial curvature will also be examined.

4.1 The Dynamic Problem

The governing differential equation for a full-clamped rectangular plate undergoing sinusoidal vibration is given by

$$\varepsilon^2 \nabla^4 u - \beta_1^2 \frac{\partial^2 u}{\partial x^2} - \beta_2^2 \frac{\partial^2 u}{\partial y^2} - \lambda^2 u = 0 \quad ,$$

subject to

$$u(0,y) = \frac{\partial u}{\partial x}(0,y) = 0 \quad ;$$

$$u(1,y) = \frac{\partial u}{\partial x}(1,y) = 0 \quad ;$$

$$0 < b \leq 1 \tag{4.1}$$

$$y(x,0) = \frac{\partial u}{\partial y}(x,0) = 0 \quad ;$$

$$u(x,b) = \frac{\partial u}{\partial y}(x,b) = 0 \quad .$$

Schneider [4] was the first to provide an asymptotic solution to the problem defined by equation (4.1) for $\beta_1^{\,2} = \beta_2^{\,2} = 1$. The solution he provided is good to $0(\varepsilon)$. This solution was later generalized for variable $\beta_1^{\,2}$ and $\beta_2^{\,2}$ and to $0(\varepsilon^2)$ by Hutten and Olunloyo [5]. More recently, Oyediran and Gbadeyan [6] used the same asymptotic method to investigate the vibration of a prestressed, rectangular thin plate exhibiting natural material orthotropy. Various boundary conditions were considered.

It is pertinent to point out that a similar problem has been solved by Nayfeh et al. [8] for circular as well as near-circular and annular prestressed plates.

We shall merely infer the result to Problem I from the literature. The uniformly valid eigenfunction is given [4,5,6] as

$$\begin{aligned}
u(x,y) = {} & A_0 \sin (n\pi x) \sin (m\pi y/b) \\
& + \varepsilon A_0 \{(n\pi/\beta_1)(2x-1) \cos (n\pi x) \sin (m\pi y/b) \\
& \qquad + (m\pi/\beta_2 b^2)(2y-b) \sin (n\pi x) \cos (m\pi y/b)\} \\
& + \varepsilon^2 A_0 \{(2n^2\pi^2/\beta_1^{\,2})\, x\, (1-x) \cos (n\pi x) \cos (m\pi y/b) \\
& \qquad + (n\pi/\beta_1^{\,2}\beta_2)(2\beta_2 b - \beta_1)(2x-1) \cos (n\pi x) \sin (m\pi y/b) \\
& \qquad + (m\pi/\beta_1\beta_2^{\,2} b^3)(2\beta_1 - \beta_2 b)(2y-b) \sin (n\pi x) \cos (m\pi y/b) \\
& \qquad + (2m^2\pi^2/\beta_2^{\,2} b^4)\, y\, (b-y) \sin (n\pi x) \sin (m\pi y/b)\}
\end{aligned} \tag{4.2}$$

(continued)

$$+ \varepsilon \{(\tilde{b}_1/\beta_1) \exp(-\beta_1 x/\varepsilon) + (\tilde{\tilde{b}}_1/\beta_1) \exp(-\beta_1(1-x)/\varepsilon)$$

$$+ (\tilde{f}_1/\beta_2) \exp(-\beta_2 y/\varepsilon) + (\tilde{\tilde{f}}_1/\beta_2) \exp(-\beta_2(b-y)/\varepsilon)\}$$

$$+ \varepsilon^2 \{(\tilde{b}_2/\beta_1) \exp(-\beta_1 x/\varepsilon) + (\tilde{\tilde{b}}_2/\beta_1) \exp(-\beta_1(1-x)/\varepsilon)$$

$$+ (\tilde{f}_2/\beta_2) \exp(-\beta_2 y/\varepsilon) + (\tilde{\tilde{f}}_2/\beta_2) \exp(-\beta_2 (b-y)/\varepsilon)\}$$

$$+ 0(\varepsilon^3) \tag{4.2}$$

while

$$\lambda^2(\varepsilon) = \beta_1^2 n^2 \pi^2 + \beta_2^2 m^2 \pi^2/b^2 + \varepsilon \{4 \beta_1 n^2 \pi^2 + 4 \beta_2 m^2 \pi^2/b^2\}$$

$$+ \varepsilon^2 \{(\pi^4/b^4) (n^2b^2 + m^2)^2 + (12 \pi^2/b^4) (n^2 b^4 + m^2)\} + O(\varepsilon^3) \tag{4.3}$$

where

$$\left.\begin{aligned} \tilde{b}_1 &= A_0 n \pi \sin (m\pi y/b) \\ \tilde{\tilde{b}}_1 &= (-1)^{n+1} \tilde{b}_1 \\ \tilde{f}_1 &= (A_0 m\pi/b) \sin (n\pi x) \\ \tilde{\tilde{f}}_1 &= (-1)^{m+1} \tilde{f}_1 \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{aligned} \tilde{b}_2 &= (A_0 n\pi/b^2\beta_1\beta_2) \{2b^2 \beta_2 \sin (m\pi y/b) + (m\pi \beta_1) (2y-b) \cos (m\pi y/b)\} \\ \tilde{\tilde{b}}_2 &= (-1)^{n+1} \tilde{b}_2 \\ \tilde{f}_2 &= (A_0 m\pi/b^2\beta_1\beta_2) \{n\pi b\beta_2 (2x-1) \cos (n\pi x) + 2 \beta_1 \sin (n\pi x)\} \\ \tilde{\tilde{f}}_2 &= (-1)^{m+1} \tilde{f}_2 \end{aligned}\right\} \tag{4.5}$$

We shall now consider the static problem (Problem II) for various mathematical forms of the initial curvature.

4.2 The Static Problem

For a fully clamped plate, the governing equations for the static problem take the form

$$\varepsilon^2 \nabla^4 \Omega - \beta_1^2 \frac{\partial^2 \Omega}{\partial x^2} - \beta_2^2 \frac{\partial^2 \Omega}{\partial y^2} = \beta_1^2 \frac{\partial^2 w_0}{\partial x^2} + \beta_2^2 \frac{\partial^2 w_0}{\partial y^2} \tag{4.6}$$

Subject to

$$\begin{aligned} &\Omega(0,y) = \frac{\partial \Omega}{\partial x}(0,y) = 0 \quad ; \\ &\Omega(1,y) = \frac{\partial \Omega}{\partial x}(1,y) = 0 \quad ; \qquad\qquad 0 < b \leq 1 \\ &\Omega(x,0) = \frac{\partial \Omega}{\partial y}(x,0) = 0 \quad ; \\ &\Omega(x,b) = \frac{\partial \Omega}{\partial y}(x,b) = 0 \end{aligned} \tag{4.7}$$

Here, we shall consider various mathematical forms for $w_0(x,y)$ and systematically obtain $\Omega(x,y;\varepsilon)$ to $0(\varepsilon^2)$.

4.3 Initial Deflection Form – Case 1

– When the initial curvature is second-order polynomial in both x and y directions

For this case, we let the initial curvature take the form

$$w_0(x,y) = d_{11} xy\,(1-x)\,(b-y) \tag{4.8}$$

where d_{11} is the initial amplitude.

4.3-1 Fully Clamped Plate [C-C-C-C]

It can be seen from equation (4.8) that the initial curvature, w_0, vanishes on all the boundaries as it must. However, there are discontinuities in the slopes of w_0 along the boundaries; that is equation (4.8) does not satisfy the relation

$$\frac{\partial w_0}{\partial x}(0,y) = \frac{\partial w_0}{\partial x}(1,y) = \frac{\partial w_0}{\partial y}(x,0) = \frac{\partial w_0}{\partial y}(x,b) = 0 \quad . \tag{4.9}$$

It is easy to see that equations (4.6) and (4.8) can be combined to take the form

$$\varepsilon^2 \nabla^4 \Omega - \beta_1^2 \frac{\partial^2 \Omega}{\partial x^2} - \beta_2^2 \frac{\partial^2 \Omega}{\partial y^2} = -2\, d_{11}\, \beta_1^2\, y(b-y) - 2\, b_{11}\, \beta_2^2\, x\,(1-x) \tag{4.10}$$

As outlined previously in Section 3, we write

$$\Omega^0 = \sum_{\nu=0}^{\infty} \varepsilon^{\nu}\, \Omega_{\nu} \quad . \tag{4.11}$$

If we now use equation (4.11) in equation (4.10), we obtain

$$\beta_1^2 \frac{\partial^2 \Omega_\nu^0}{\partial x^2} + \beta_2^2 \frac{\partial^2 \Omega_\nu^0}{\partial y^2} = \begin{cases} 2d_{11}[\beta_1^2\, y(b-y) + \beta_2^2\, x(1-x)] & \text{for } \nu = 0 \\ 0 & \text{for } \nu = 1 \\ \nabla^4\, \Omega_{\nu-2} & \text{for } \nu \geq 2 \end{cases} \quad . \tag{4.12}$$

Solving equations (4.12) for Ω_ν^0 provides the outer solution to $O(\varepsilon^\nu)$.

<u>Leading Order Outer Problem</u>:

If we set $\nu = 0$ in equations (4.12), the resulting differential equation is

$$\beta_1^2 \frac{\partial^2 \Omega_0^0}{\partial x^2} + \beta_1^2 \frac{\partial^2 \Omega_0^0}{\partial y^2} = 2d_{11}\, \beta_1^2\, y(b-y) + 2d_{11}\, \beta_2^2\, x(1-x) \quad . \tag{4.13}$$

It can easily be seen that the solution of equation (4.13) takes the form

$$\Omega_0{}^0 = a_0 + a_1 x + a_2 y + a_3\, xy + \alpha_1\, x^2 y^2 + \alpha_2\, x^2 y + \alpha_3\, xy^2 + \alpha_4\, x^3 y + \ldots \quad . \tag{4.14}$$

Clearly, the solution equation (4.14) does not satisfy the boundary conditions of equations (4.7) and so is not uniformly valid.

It becomes apparent, after some manipulation, that

$$a_0 = a_1 = a_3 = \alpha_4 = \ldots = 0 \tag{4.15}$$

while compatibility requires that

$$\alpha_1 = d_{11} \quad , \quad \alpha_2 = -bd_{11} \quad , \quad \alpha_3 = -d_{11} \tag{4.16}$$

and a_3 is yet to be determined. We anticipate that the remaining unknown, a_3, will be determined when the problem is solved near the boundaries, that is, when the inner problem is considered.

The Inner Problem

Near the boundaries, we need new coordinates to define the rapidly varying deflections accurately. We thus set

$$\begin{aligned} X &= x/\varepsilon \quad , & &\text{near } x = 0 \\ X &= (1-x)/\varepsilon \quad , & &\text{near } x = 1 \\ Y &= y/\varepsilon \quad , & &\text{near } y = 0 \\ Y &= (b-y)/\varepsilon \quad , & &\text{near } y = b \quad . \end{aligned} \tag{4.17}$$

Near $x = 0$ and $x = 1$, the differential equation for the inner problem takes the form

$$\psi^i_{\nu_{XXXX}} - \beta_1^2\, \psi^i_{\nu_{XX}} = \begin{cases} 0 & \text{for } \nu = 0,1 \\ -2\, \psi^i_{\nu-2_{XXyy}} + \beta_2^2\, \psi^i_{\nu-2_{yy}} - 2\, d_{11}\, \beta_1^2\, y(b-y) & \text{for } \nu = 2 \\ -2\, \psi^i_{\nu-2_{XXyy}} + \beta_2^2\, \psi^i_{\nu-2_{yy}} - 2\, d_{11}\, \beta_2^2\, X & \text{for } \nu = 3 \end{cases} \tag{4.18}$$

Near y = 0 and y = b, the inner problem is obtained by using equation (4.17) and following the same procedure, as

$$\psi^i_{\nu_{YYYY}} - \beta_2^2\, \psi^i_{\nu_{YY}} = \begin{cases} 0 & \text{for } \nu = 0,1 \\ -2\, \psi^i_{\nu-2_{xxYY}} + \beta_1^2\, \psi^i_{\nu-2_{xx}} - 2\, d_{11}\, \beta_2^2\, x(1-x) & \text{for } \nu = 2 \\ -2\, \psi^i_{\nu-2_{xxYY}} + \beta_1^2\, \psi^i_{\nu-2_{xx}} - 2\, d_{11}\, \beta_1^2\, bY & \text{for } \nu = 3 \end{cases} \quad . \tag{4.19}$$

<u>Leading Order Inner Problem</u>

If we set $\nu = 0$ in equations (4.18) and (4.19), the resulting differential equations are

$$\psi^i_{0_{XXXX}} - \beta_1^2\, \psi^i_{0_{XX}} = 0 \quad , \qquad \text{near } x = 0,1 \quad ; \tag{4.20}$$

and

$$\psi^i_{0_{YYYY}} - \beta_2^2\, \psi^i_{0_{YY}} = 0 \quad , \qquad \text{near } y = 0,b \quad . \tag{4.21}$$

The solutions to equations (4.20) and (4.21) that satisfy the boundary condition equations (4.7) can be written as

$$\psi^i_0 = \begin{cases} K_1(y)\ [X + (1/\beta_1)\ (\exp(-\beta_1 X)-1)] \quad , & \text{near } x = 0 \quad , \\ K_2(y)\ [X + (1/\beta_1)\ (\exp(-\beta_1 X)-1)] \quad , & \text{near } x = 1 \quad , \\ K_3(x)\ [Y + (1/\beta_2)\ (\exp(-\beta_2 Y)-1)] \quad , & \text{near } y = 0 \quad , \\ K_4(x)\ [Y + (1/\beta_2)\ (\exp(-\beta_2 Y)-1)] \quad , & \text{near } y = b \quad . \end{cases} \tag{4.22}$$

where functions $K_1(y)$, $K_2(y)$, $K_3(x)$, and $K_4(x)$ are to be determined from matching. Exponentially growing terms have been discarded as unmatchable. Equations (4.14) and (4.22), being the solutions to the leading order outer and inner problems, respectively, are now to be matched.

From 1-1 matching, it becomes immediately obvious that these two solutions are unmatchable. We thus set

$$K_1(y) = K_2(y) = K_3(x) = K_4(x) = 0 \tag{4.23}$$

and

$$\psi_0^i = 0 \quad . \tag{4.24}$$

Equation (4.24) is intuitively obvious since a zero value for the leading order inner solution is in agreement with the boundary condition of zero deflection on all the boundaries.

To the leading order, the unknown constant, a_3, in equation (4.14) remains as yet undetermined. It may be determined, however, by considering the first order inner problem and, consequently, a 2-1 matching.

First Order Inner Problem

The differential equations for ψ_1^i are similar to those of ψ_0^i. Hence, it is easy to see that

$$\psi_1^i = \begin{cases} K_5(y)\ [X + (1/\beta_1)\ (\exp(-\beta_1 X)-1)] \quad , & \text{near } x = 0 \ ; \\ K_6(y)\ [X + (1/\beta_1)\ (\exp(-\beta_1 X)-1)] \quad , & \text{near } x = 1 \ ; \\ K_7(x)\ [Y + (1/\beta_2)\ (\exp(-\beta_2 Y)-1)] \quad , & \text{near } y = 0 \ ; \\ K_8(x)\ [Y + (1/\beta_2)\ (\exp(-\beta_2 Y)-1)] \quad , & \text{near } y = b \ . \end{cases} \tag{4.25}$$

The unknowns K_5, ... K_8 and a_3 will now be determined from 2-1 matching as illustrated below.

2-1 Matching

Near $x = 0$, the 2-term inner solution is

$$\psi^i(X,y) = \varepsilon\, \psi_1^i\, (X,y) + 0\,(\varepsilon^2)$$

$$= K_5(y)\, [\varepsilon X + (\varepsilon/\beta_1)\,(\exp(-\beta_1 X)-1)] + 0\,(\varepsilon^2) \quad (4.26)$$

which can be rewritten in outer variables, $x = \varepsilon X$, and expanded to 1 term for x fixed and $\varepsilon \to 0^+$ to give

$$\psi^i(x,y) = K_5(y)\, x + 0\,(\varepsilon) \quad . \quad (4.27)$$

Similarly, 1-term outer solution, rewritten in inner variables $X = x/\varepsilon$, and expanded to 2 terms for X fixed and $\varepsilon \to 0^+$ yields

$$\Omega^0(X,y) = \varepsilon\, [a_3 y + \alpha_3\, y^2]\, X + 0\,(\varepsilon^2)$$

or, rewritten in outer variables,

$$\Omega^0(x,y) = (a_3 y + \alpha_3\, y^2)\, x + 0\,(\varepsilon^2) \quad (4.28)$$

Van Dyke's [11] matching principle states that these two expansions, equations (4.27) and (4.28), are equal. Hence, we have that

$$K_5(y) = a_3 y + \alpha_3\, y^2 \quad . \quad (4.29)$$

Similarly, near $x = 1$, we set $1-x = \varepsilon X$. Using Van Dyke's principle, it is easily verified that

$$a_3 = d_{11} b \quad (4.30)$$

and

$$K_6(y) = -d_{11} y\,(b-y) \quad , \quad K_5(y) = K_6(y) \quad . \quad (4.31)$$

The same procedure can be carried out near $y = 0$ and $y = b$. It can be readily shown that $K_7(x)$ and $K_8(x)$ are determined as

$$K_7 = K_8 = -d_{11} b\, x(1-x) \quad . \tag{4.32}$$

To complete the first order problem, the outer solution is also required to $0(\varepsilon)$.

First Order Outer Problem

From equations (4.12), the first order problem takes the form

$$\beta_1^{\,2}\, \Omega^0_{1_{xx}} + \beta_2^{\,2}\, \Omega^0_{1_{yy}} = 0 \quad . \tag{4.33}$$

The resulting solution, Ω^0_1, is to be matched with

$$\begin{aligned} &- K_5(y)/\beta_1 \quad , \quad \text{near } x = 0 \\ &- K_6(y)/\beta_1 \quad , \quad \text{near } x = 1 \\ &- K_7(x)/\beta_2 \quad , \quad \text{near } y = 0 \\ &- K_8(x)/\beta_2 \quad , \quad \text{near } y = b \quad . \end{aligned} \tag{4.34}$$

The requirement that Ω^0_1 match with equations (4.34) is equivalent to a 2-2 matching. Alternatively, the problem defined by equations (4.33) and (4.34) can now be rewritten as

$$\beta_1^{\,2}\, \Omega^0_{1_{xx}} + \beta_2^{\,2}\, \Omega^0_{1_{yy}} = 0 \quad . \tag{4.35a}$$

Subject to

$$\begin{aligned} &\Omega^0_1 = (d_{11}/\beta_1)\, y(b-y) \quad ; \quad \text{near } x = 0 \text{ and } x = 1 \\ &\Omega^0_1 = (d_{11}/\beta_2)\, x(1-x) \quad ; \quad \text{near } y = 0 \text{ and } y = b \quad . \end{aligned} \tag{4.35b}$$

We seek a solution to equations (4.35) of the form

$$\Omega_1^0 = (d_{11}/\beta_1)\ y(b-y) + (d_{11}b/\beta_2)\ x(1-x) + \Phi(x,y) \tag{4.36}$$

such that $\Phi(x,y)$ is determined by inserting Ω_1^0 defied by equation (4.36) into equation (4.35a). Thus, the equation for $\Phi(x,y)$ takes the form

$$\beta_1^{\ 2}\ \Phi_{xx} + \beta_2^{\ 2}\ \Phi_{yy} + D = 0 \tag{4.37}$$

where

$$D = -(2d_{11}/\beta_1\beta_2)\ (b\beta_1^{\ 3} + \beta_2^{\ 3}) \quad . \tag{4.38}$$

The form of the solution suggested in equation (4.36) ensures that $\Omega_1^0(x,y)$ is matched along the boundaries provided $\Phi(x,y)$ vanishes on the boundaries.

It is interesting to note from the symmetric end conditions of equations (4.35b), that

$$\left.\frac{\partial \Omega_1^0}{\partial x}\right|_{x=1/2} = \left.\frac{\partial \Omega_1^0}{\partial y}\right|_{y=b/2} = 0 \quad . \tag{4.39}$$

It can easily be shown that

$$\Phi(x,y) = \sum_n \left[A_{1n} \sinh\left(\frac{n\pi\beta_1}{\beta_2} y\right) + B_n \cosh\left(\frac{n\pi\beta_1}{\beta_2}\ y\right) - \frac{D_n}{n^2\pi^2\beta_1^{\ 2}}\right] \sin\ (n\pi x)$$

$$+ \sum_m \left[A_{2m} \sinh\left(\frac{m\pi\beta_2}{b\beta_1}\ x\right) + A_{3m} \cosh\left(\frac{m\pi\beta_2}{\beta_1}\ x\right) - \frac{E_m b^2}{m^2\pi^2\beta_2^{\ 2}}\right] \sin\left(\frac{m\pi y}{b}\right) \tag{4.40}$$

where the constants A_{1n}, B_n, A_{2m}, and A_{3m} are determined from a 2-2 matching and are given below as

$$B_n = \frac{D_n}{n^2\pi^2\beta_1^{\ 2}} \quad , \quad A_{1n} = \frac{B_n\left[1 - \cosh\left(\frac{n\pi\beta_1}{\beta_2}\right)b\right]}{\sinh\left(\frac{n\pi\beta_1}{\beta_2}\right)b} \tag{4.41}$$

(Continued)

$$A_{3m} = \frac{E_m b^2}{m^2\pi^2\beta_2^2} \quad , \quad A_{2m} = \frac{A_{3m}\left[1 - \cosh\left(\frac{m\pi\beta_2}{b\beta_1}\right)\right]}{\sinh\left(\frac{m\pi\beta_2}{b\beta_1}\right)}$$

(4.41)
(Concluded)

$$D_n = \frac{D}{n\pi}\{(-1)^n - 1\} \quad \text{and} \quad E_m = \frac{Db}{m\pi}\{(-1)^m - 1\} \quad .$$

This completes the problem to $0(\varepsilon)$.

Second Order Correction

Since we seek a solution correct to $O(\varepsilon^2)$, we have to solve the second-order problem. This is accomplished by first solving the inner problem and then the outer problem.

Inner Problem

Near $x = 0$, the solution to the inner problem of equation (4.18), to second order, is

$$\psi^i = K_5 x + d_{11} y(b-y)\, x^2 + \varepsilon\,(K_9 x - K_5/\beta_1) - \varepsilon^2 K_9/\beta_1 \tag{4.42}$$

where $K_5(y)$ is given by equation (4.31) and $K_9(y)$ is to be determined from a 2-3 matching, given below.

2-3 Matching

The 2-term outer solution

$$\Omega^0(x,y) = \Omega_0^{\,0}(x,y) + \varepsilon\, \Omega_1^{\,0}(x,y) + 0(\varepsilon^2) \quad ,$$

rewritten in inner variables near $x = 0$, say, and expanded to 3 terms for X fixed and $\varepsilon \to 0^+$ is to be matched with the appropriate terms of the inner solution given by equation (4.42). It is easily verified that, equating the εx terms, yields

$$K_9(y) = d_{11} b/\beta_2 + n\pi[A_n \sinh \tilde{\alpha}_1 y + B_n (\cosh \tilde{\alpha}_1 y - 1)] + A_{2m}\, \tilde{\alpha}_2 \sin(m\pi y/b) \tag{4.43}$$

where

$$\tilde{\alpha}_1 = (n\pi\beta_1/\beta_2) \quad , \quad \tilde{\alpha}_2 = (m\pi\beta_2/b\beta_1) \quad . \tag{4.44}$$

The same procedure can be carried out near x = 1, y = 0, and y = b, respectively. It is easy to see that the second order outer solution, Ω_2^0, will have to match with $-K_9/\beta_1$ near x = 0. This is equivalent to a 3-3 matching.

Outer Problem

For a 3-3 matching, it is obvious that Ω_2^0 must be obtained from the differential equation

$$\beta_1^2 \, \Omega_{2\,xx}^{0} + \beta_2^2 \, \Omega_{2\,yy}^{0} = \nabla^4 \, \Omega_0^0 \tag{4.45}$$

and must then be matched with the inner solution as

$$\Omega_2^0(0,y) = \psi_2^i(0,y) \doteq -(d_{11}b/\beta_1\beta_2) - (n\pi/\beta_1)\,[A_{1n} \sinh \tilde{\alpha}_1 y + B_{1n} (\cosh \tilde{\alpha}_1 y - 1)] - (A_2\tilde{\alpha}_2/\beta_2) \sin (m\pi y/b)$$

$$\Omega_2^0(1,y) = \psi_2^i(1,y) = -(d_{11}b/\beta_1\beta_2) + (-1)^n (n\pi/\beta)\,[A_{1n} \sinh \tilde{\alpha}_1 y + B_{1n} (\cosh \tilde{\alpha}_1 y - 1)] + (1/\beta_1)\,[A_{2m}\tilde{\alpha}_2 \cosh \tilde{\alpha}_2 + A_{3m}\tilde{\alpha}_2 \sinh \tilde{\alpha}_2] \sin (m\pi y/b)$$

$$\Omega_2^0(x,0) = \psi_2^i(x,0) = -(d_{11}b/\beta_1\beta_2) - (m\pi/b\beta_2)\,[A_{2m} \sinh \tilde{\alpha}_2 x + A_{3m} (\cosh \tilde{\alpha}_2 x - 1)] - (A_{1n}\tilde{\alpha}_1/\beta_2) \sin (n\pi x)$$

(4.46)
(Continued)

$$\Omega_2^0(x,b) = \psi_2^i(x,b) = -(d_{11}b/\beta_1\beta_2) + (-1)^m (m\pi/b\beta_2) [A_{2m} \sinh \tilde{\alpha}_2 x + A_{3m} (\cosh \tilde{\alpha}_2 x - 1)] + (1/\beta_2) [A_{1n}\tilde{\alpha}_1 \cosh \tilde{\alpha}_1 b + B_{1n} \tilde{\alpha}_1 \sinh \tilde{\alpha}_1 b] \sin (n\pi x) \quad . \tag{4.46}$$

(Concluded)

To solve this boundary value problem, it is convenient to introduce the decomposition

$$\Omega_2^0 = \phi_2^{(1)} + \phi_2^{(2)} + \phi_2^{(3)} \tag{4.47}$$

where $\phi_2^{(1)}$ is a solution of equation (4.45) but does not satisfy equation (4.46). We insist here that $\phi_2^{(1)}$ satisfy instead the following boundary condition extracted from equations (4.46)

$$\begin{aligned}
\phi_2^{(1)}(0,y) &= -(d_{11}b/\beta_1\beta_2) + (n\pi\, B_{1n}/\beta_1) \\
\phi_2^{(1)}(1,y) &= -(d_{11}b/\beta_1\beta_2) + (-1)^{n+1} (n\pi\, B_{1n}/\beta_1) \\
\phi_2^{(1)}(x,0) &= -(d_{11}b/\beta_1\beta_2) + (m\pi\, A_{3m}/b\beta_2) \\
\phi_2^{(1)}(x,b) &= -(d_{11}b/\beta_1\beta_2) + (-1)^{m+1} (m\pi\, A_{3m}/b\beta_2) \quad .
\end{aligned} \tag{4.48}$$

Similarly, the problem for $\phi_2^{(\alpha)}$ is described by

$$\beta_1^2 \frac{\partial^2\phi_2^{(\alpha)}}{\partial x^2} + \beta_2^2 \frac{\partial^2\phi_2^{(\alpha)}}{\partial y^2} = 0 \quad , \qquad \alpha = 2,3 \tag{4.49}$$

while the boundary values of $\phi_2^{(\alpha)}$ ($\alpha = 2,3$) are derived by insisting that $\phi_2^{(1)} + \phi_2^{(2)} + \phi_2^{(3)}$ must satisfy all the boundary conditions on Ω_2^0. Here, we require that $\phi_2^{(2)}$ must satisfy the following boundary conditions:

$$\phi_2^{(2)}(0,y) = -(A_{2m}\tilde{\alpha}_2/\beta_1) \sin (m\pi y/b)$$

$$\phi_2^{(2)}(1,y) = (\tilde{\alpha}_2/\beta_1) [A_{2m} \cosh \tilde{\alpha}_2 + A_{3m} \sinh \tilde{\alpha}_2] \sin (m\pi y/b)$$

$$\phi_2^{(2)}(x,0) = -(A_{1n}\tilde{\alpha}_1/\beta_2) \sin (n\pi x) \tag{4.50}$$

$$\phi_2^{(2)}(x,b) = (\tilde{\alpha}_1/\beta_2) [A_{1n} \cosh \tilde{\alpha}_1 b + B_{1n} \sinh \tilde{\alpha}_1 b] \sin (n\pi x) \quad .$$

With little effort, the solution to $\phi_2^{(1)}$ is obtained as

$$\phi_2^{(1)} = \bar{\phi}_2^{(1)}(x,m) \sin (m\pi y/b) + \bar{\phi}_2^{(1)}(n,y) \sin (n\pi x) \tag{4.51}$$

where

$$\bar{\phi}_2^{(1)}(x,m) = \tilde{F}_1 \sinh \tilde{\alpha}_2 x + \tilde{F}_2 \cosh \tilde{\alpha}_2 x - K_1/\tilde{\alpha}_2^2$$

$$\bar{\phi}_2^{(1)}(n,y) = \tilde{F}_3 \sinh \tilde{\alpha}_1 y + \tilde{F}_4 \cosh \tilde{\alpha}_1 y - \tilde{G}_1 \tag{4.52}$$

while

$$K_1 = \frac{2m^2 \pi^2 \beta_2 A_{3m}}{b^2} + \{(-1)^m - 1\} \left\{ \frac{8d_{11}b}{m\pi} + \frac{m\pi\beta_2 d_{11}}{\beta_1} \right\}$$

$$\tilde{F}_1 = \frac{b\tilde{\alpha}_2}{2m\pi\beta_1} \{1-(-1)^m\} \left\{ A_{2m} \cosh \tilde{\alpha}_2 + A_{3m} \sinh \tilde{\alpha}_2 \right\} + \frac{K_1}{\tilde{\alpha}_2^2} - \tilde{F}_2 \cosh \tilde{\alpha}_2$$

$$F_2 = \frac{K_1}{\tilde{\alpha}_2^2} - \frac{b}{2m\pi} \{1-(-1)^m\} \frac{A_{2m}\tilde{\alpha}_2}{\beta_1}$$

$$\tilde{F}_3 = \frac{\tilde{G}_1 + \tilde{G}_3 - \tilde{F}_4 \cosh \tilde{\alpha}_1 b}{\sinh \tilde{\alpha}_1 b} \tag{4.53}$$

(Continued)

$$\tilde{F}_4 = \tilde{G}_1 + \tilde{G}_2$$

where

$$\tilde{G}_1 = \frac{2\, n^2 \pi^2 B_{1n}\, \beta_1 + \{(-1)^n - 1\} \left(\frac{8 d_{11} \beta_1}{n\pi} + \frac{d_{11} b}{\beta_2} \right)}{n^2 \pi^2 \beta_1^2 \beta_2^2}$$

$$G_2 = \frac{\tilde{\alpha}_1}{2n\pi\beta_2} A_{1n} [1 - (-1)^n]$$

$$\tilde{G}_3 = \frac{\tilde{\alpha}_1}{2n\pi\beta_2} [A_{1n} \cosh \tilde{\alpha}_1 b + B_{1n} \sinh \tilde{\alpha}_1 b \qquad (4.53) \text{ (Concluded)}$$

The solution to equation (4.49) for $\alpha = 2$ can be easily shown to take the form

$$\phi_2^{(2)} = H_1(y) \sin (n\pi x) + H_2(x) \sin (n\pi y/b) \quad . \qquad (4.54)$$

It clearly follows from equations (4.54) and (4.49) that

$$\phi_2^{(2)} = [\tilde{L}_1 \cosh \tilde{\alpha}_1 y + \tilde{L}_2 \sinh \tilde{\alpha}_1 y] \sin (n\pi x)$$

$$+ [\tilde{L}_3 \cosh \tilde{\alpha}_2 x + \tilde{L}_4 \sinh \tilde{\alpha}_2 x] \sin (m\pi y/b) \qquad (4.55)$$

where

$$\tilde{L}_1 = -\frac{A_{1n}}{\beta_2} \tilde{\alpha}_1 \quad ; \quad \tilde{L}_3 = -\frac{A_{2m} \tilde{\alpha}_2}{\beta_1}$$

$$\tilde{L}_2 = \frac{2\, \tilde{\alpha}_1 A_{1n}}{\beta_2 \tanh \tilde{\alpha}_1 b} + \frac{\tilde{\alpha}_2 B_{1n}}{\beta_2} \qquad (4.56)$$

$$\tilde{L}_4 = \frac{2\, \tilde{\alpha}_2 A_{2m}}{\beta_1 \tanh \tilde{\alpha}_2} + \frac{\tilde{\alpha}_2 A_{3m}}{\beta_1}$$

In preparation for solving equation (4.49) for $\alpha = 3$, we first make the following definitions:

$$\theta_1 = \int_0^1 \sin(n\pi x) \cosh \tilde{\alpha}_2 x \, dx$$

$$\theta_2 = \int_0^1 \sin(n\pi x) \sinh \tilde{\alpha}_2 x \, dx$$

$$\theta_3 = \int_0^b \sin(m\pi y/b) \cosh \tilde{\alpha}_1 y \, dy \qquad (4.57)$$

$$\phi_4 = \int_0^b \sin(m\pi y/b) \sinh \tilde{\alpha}_1 y \, dy$$

Also,

$$\tilde{Z}_1 = -\frac{m\pi}{b\beta_2} [A_{2m} \theta_2 + A_{3m} \theta_1]$$

$$\tilde{Z}_2 = \frac{(-1)^m m\pi}{b\beta_2} [A_{2m} \theta_2 + A_{3m} \theta_1]$$

$$\tilde{Z}_3 = \frac{-n\pi}{\beta_1} [A_{1n} \theta_4 + B_{1n} \theta_3] \qquad (4.58)$$

$$\tilde{Z}_4 = \frac{(-1)^n n\pi}{\beta_1} [A_{1n} \theta_4 + B_{1n} \theta_3]$$

Then $\phi_2^{(3)}$ takes the form

$$\phi_2^{(3)}(x,y) = \bar{\phi}_2^{(3)}(n,y) \sin(n\pi x) + \bar{\phi}_2^{(3)}(x,m) \sin(m\pi y/b) \qquad (4.59)$$

where

$$\bar{\phi}_2^{(3)}(n,y) = \tilde{L}_5 \sinh \tilde{\alpha}_1 y + \tilde{L}_6 \cosh \tilde{\alpha}_1 y + \tilde{L}_7 y \sinh \tilde{\alpha}_1 y + \tilde{L}_8 y \cosh \tilde{\alpha}_1 y \quad (4.60a)$$

$$\bar{\phi}_2^{(3)}(x,m) = \tilde{E}_5 \sinh \tilde{\alpha}_2 x + \tilde{E}_6 \cosh \tilde{\alpha}_2 x + \tilde{E}_7 x \sinh \tilde{\alpha}_2 x + \tilde{E}_8 \cosh \tilde{\alpha}_2 x \quad (4.60b)$$

respectively.

From a 3-3 matching, it can easily be established that

$$\tilde{L}_5 = \left[\frac{\tilde{Z}_2}{\sinh \tilde{\alpha}_1 b} - \tilde{L}_6 \coth \tilde{\alpha}_1 b - \tilde{L}_7 b - \tilde{L}_8 b \coth \tilde{\alpha}_1 b\right]$$

$$\tilde{L}_6 = \tilde{Z}_1 \quad , \quad \tilde{L}_7 = \frac{n^2 \pi^2 \beta_1}{\tilde{\alpha}_1 \beta_2^2} B_{1n} \quad , \quad \tilde{L}_8 = \frac{\tilde{L}_7 A_{1n}}{B_{1n}}$$

$$(4.61)$$

$$E_5 = \left[\frac{\tilde{Z}_4}{\sinh \tilde{\alpha}_2} - (\tilde{E}_6 + \tilde{E}_8) \coth \tilde{\alpha}_2 - \tilde{E}_7\right]$$

$$\tilde{E}_6 = \tilde{Z}_3 \quad , \quad \tilde{E}_7 = \frac{m^2 \pi^2 \beta_2}{\tilde{\alpha}_2 \beta_1^2} A_{3m} \quad , \quad \tilde{E}_8 = \frac{\tilde{E}_7 A_{2m}}{A_{3m}}$$

This completes the static problem to $0(\varepsilon^2)$ for a clamped, rectangular plate whose initial curvature is second-order polynomial in x and y.

The result for a fully hinged, rectangular plate is now presented and is found to check with the exact solution presented by Timoshenko and Woinowski-Krieger [3].

4.3-2 Fully Hinged Plate [S-S-S-S]

For this case, the dynamic problem has been solved [5] and it is found that the eigenvalues can be written down exactly as

$$\lambda^2 = n^2\pi^2\beta_1^2 + m^2\pi^2\beta_2^2 + \varepsilon(n^4\pi^4b^4 + n^2m^2\pi^4b^2 + m^4\pi^4)/b^4 \quad (4.62)$$

while the dynamic eigensolution takes the form

$$u = A_0 \sin(n\pi x) \sin(m\pi y/b) \quad , \quad m,n = 1,2,\ldots \tag{4.63}$$

and the static deflection is given by

$$\Omega = \Omega_0^{\ 0} + \varepsilon^2 \Omega_2^{\ 0} \tag{4.64}$$

Here,

$$\Omega_0^{\ 0} = -d_{11}\, xy\, (1-x)\, (b-y) \tag{4.65}$$

while $\Omega_2^{\ 0}$ can be determined from the equation

$$\beta_1^{\ 2} \frac{\partial^2 \Omega_2^{\ 0}}{\partial x^2} + \beta_2^{\ 2} \frac{\partial^2 \Omega_2^{\ 0}}{\partial y^2} = \nabla^4 \Omega_0^{\ 0} = -8d_{11} \tag{4.66}$$

subject to $\Omega_2^{\ 0} = 0$ on all the boundaries. Therefore, we have that

$$\Omega_2^{\ 0}(x,y) = 2\sum_n [F_1 \cosh(\tilde{\alpha}_1 y) + F_2 \sinh(\tilde{\alpha}_1 y) - L_2/\tilde{\alpha}_1^{\ 2}] \sin(n\pi x)$$

$$+ 2\sum_m [\tilde{F}_1 \cosh(\tilde{\alpha}_2 x) + \tilde{F}_2 \sinh(\tilde{\alpha}_2 x) - L_2/\tilde{\alpha}_2^{\ 2}] \sin(m\pi y/b) \tag{4.67}$$

where

$$\left.\begin{aligned}
L_2 &= -\frac{8\, d_{11}\, \xi}{n\, \pi\, \beta_2^{\ 2}} \quad , \quad \xi = [1 + (-1)^{n+1}] \\
\tilde{L}_2 &= -\frac{8\, d_{11}\, b\, \tilde{\xi}}{m\, \pi\, \beta_1^{\ 2}} \quad , \quad \tilde{\xi} = [1 + (-1)^{m+1}] \\
F_1 &= L_2/\tilde{\alpha}_1^{\ 2} \quad , \quad F_2 = [F_1(1 - \cosh(\alpha_1 b))]/\sinh(\tilde{\alpha}_1 b) \quad , \\
\tilde{F}_1 &= \tilde{L}_2/\tilde{\alpha}_2^{\ 2} \quad , \quad \tilde{F}_2 = [\tilde{F}_1(1 - \cosh(\tilde{\alpha}_2))]/\sinh(\alpha_2) \quad .
\end{aligned}\right\} \tag{4.68}$$

It can be readily verified that

$$\Omega_\nu^0 = 0 \quad \text{for} \quad \nu = 3,4,\ldots \quad .$$

The static problem for a fully hinged, rectangular plate whose initial curvature is second-order polynomial in x and y is thus completed to all orders.

4.3-3 Partly Clamped and Partly Hinged Plate [C-C-S-S]

For this case, we shall infer the result to the dynamic problem from the literature. This is already given as equation (4.2) above and will not be repeated here.

The Static Problem

For a plate that is partly clamped and partly simply-supported on adjacent sides, the outer problem to solve takes the form

$$\beta_1^2 \frac{\partial^2\psi_\nu}{\partial x^2} + \beta_2^2 \frac{\partial^2\psi_\nu}{\partial y^2} = \begin{cases} 2\, d_{11}\, [\beta_1^2\, y(b-y) + \beta_2^2\, x(1-x)] & , \quad \nu = 0 \\ 0 & , \quad \nu = 1 \\ \nabla^4\, \psi_{\nu-2} & , \quad \nu \geq 2 \end{cases} \qquad (4.69)$$

in place of equation (4.12) and subject to

$$\left.\begin{aligned} &\psi_\nu(0,y) = \psi_\nu(x,0) = 0 \quad , \\ &\frac{\partial\psi_\nu}{\partial x}(0,y) = \frac{\partial\psi_\nu}{\partial y}(x,0) = 0 \quad , \\ &\psi_\nu(1,y) = \psi_\nu(x,b) = 0 \quad , \\ &\frac{\partial^2\psi_\nu}{\partial x^2}(1,y) = \frac{\partial^2\psi_\nu}{\partial y^2}(x,b) = 0 \quad . \end{aligned}\right\} \qquad (4.70)$$

Here

$$\Omega\,(x,y) \equiv \psi\,(x,y) \quad .$$

Following the procedure outlined previously in Section 4.3, the solutions to the outer and inner problems are summarized below:

$$\psi_0(x,y) = -d_{11}\, xy(1-x)(b-y) \quad , \tag{4.71}$$

$$\psi_1(x,y) = (d_{11}/\beta_1)\, y(1-x)(b-y) + (d_{11}/\beta_2)\, x(1-x)(b-y) + \phi_1(x,y) \tag{4.72}$$

where $\phi_1(x,y)$ satisfies the Poisson equation subject to homogeneous boundary conditions. With relative ease, it can be shown that

$$\begin{aligned}\phi_1(x,y) = {} & \{\bar{E}_1 \cosh(\tilde{\alpha}_1 y) + \bar{E}_2 \sinh(\alpha_1 y) - (G_1/\tilde{\alpha}_1^{\,2})(b-y) \\ & + (b^2\, \bar{G}_2/n^2\pi^2)\} \sin(n\pi x) + \{\bar{\bar{E}}_1 \cosh(\tilde{\alpha}_2 x) + \bar{\bar{E}}_2 \sinh(\tilde{\alpha}_2 x) \\ & - (\bar{\bar{G}}_1 + \bar{\bar{G}}_2)(1-x)\} \sin(m\pi y/b)/b \quad ,\end{aligned} \tag{4.73}$$

where

$$\begin{aligned}
&\bar{G}_1 = (2d_{11}\beta_1^{\,2}/n\pi\beta_2^{\,3})\, \xi_1 \quad , \quad \bar{\bar{G}}_1 = d_{11}b^2/\pi\beta_2 \\
&\bar{G}_2 = 2d_{11}/n\pi\beta_1 \quad , \quad \bar{\bar{G}}_2 = (2d_{11}b\beta_2^{\,2}/\beta_1^{\,3}\, m\pi)\, \xi_1 \\
&\bar{E}_1 = (\bar{G}_1 b/\tilde{\alpha}_1^{\,2}) - (b^2\bar{G}_2/m^2\pi^2) \quad , \\
&\bar{E}_2 = \bar{E}_1 \coth(\tilde{\alpha}_1 b) + b^2\bar{G}_2/m^2\pi^2 \sinh(\tilde{\alpha}_1 b) \quad , \\
&\bar{\bar{E}}_1 = (\bar{\bar{G}}_1 + \bar{\bar{G}}_2)/\tilde{\alpha}_2^{\,2} \quad , \\
&\bar{\bar{E}}_2 = \bar{\bar{E}}_1 [1 - \cosh(\tilde{\alpha}_2)]/\tilde{\alpha}_2^{\,2} \sinh(\tilde{\alpha}_2) - \bar{\bar{G}}_2 \coth(\tilde{\alpha}_2)/\tilde{\alpha}_2^{\,2}
\end{aligned} \tag{4.74}$$

Similarly, we have

$$\psi_2(x,y) = \psi_2^{(\alpha)}(x,y) + \psi_2^{(\beta)}(x,y) + \psi_2^{(\gamma)}(x,y) \tag{4.75}$$

where

$$\psi_2^{(\alpha)}(x,y) = \{F_1 \cosh(\tilde{\alpha}_1 y) + F_2 \sinh(\tilde{\alpha}_1 y)$$

$$- [(\tilde{\alpha}_1^2 P_0 - 2P_2) - \tilde{\alpha}_1^2 y(P_1+P_2 y)]/\tilde{\alpha}_1^4\} \sin(n\pi x)$$

$$+ \{\tilde{F}_1 \cosh(\tilde{\alpha}_2 x) + \tilde{F}_2 \sinh(\tilde{\alpha}_2 x)$$

$$- [(\tilde{\alpha}_2^2 \tilde{P}_0 - 2P_2) - \tilde{\alpha}_2^2 x(\tilde{P}_1+\tilde{P}_2 x)]/\tilde{\alpha}_2^4\} \sin(m\pi y/b) \quad (4.76)$$

with

$$\left.\begin{aligned}
P_0 &= (\tilde{\alpha}_1^2/\beta_1 n\pi)\,[S_1 + S_2 b - (8d_{11}\,\beta_1/\alpha_1^2\,\beta_2^2)\,\xi_1] \quad ,\\
P_1 &= (\tilde{\alpha}_1^2/\beta_1 n\pi)\,(S_2 - S_3 b) \quad ,\\
P_2 &= -\,(\tilde{\alpha}_1^2\,d_{11}/\beta_1^2\,n\pi) \quad ,\\
\tilde{P}_0 &= (\tilde{\alpha}_2^2\,b/\beta_2\,m\pi)\,[\tilde{S}_1 + \tilde{S}_2 - (8d_{11}\,\beta_2/\tilde{\alpha}_2^2)\,\bar{\xi}_1] \quad ,\\
\tilde{P}_1 &= (\tilde{\alpha}_2^2\,b/\beta_2\,m\pi)\,(\tilde{S}_2 - \tilde{S}_3) \quad ,\\
\tilde{P}_2 &= -\,(\alpha_2^2\,b/\beta_2^2\,m\pi) \quad .
\end{aligned}\right\} \quad (4.77)$$

F_1, F_2, $\tilde{F}_1$, and $\tilde{F}_2$ remain to be determined.

$$\psi_2^{(\beta)}(x,y) = (\tilde{S}_6/\beta_2)\,[\coth(\tilde{\alpha}_1 b)\sinh(\tilde{\alpha}_1 y) - \cosh(\tilde{\alpha}_1 y)]\sin(n\pi x)$$

$$+ (S_6/\beta_1)\,[\coth(\tilde{\alpha}_2)\sinh(\tilde{\alpha}_2 x) - \cosh(\tilde{\alpha}_2 x)]\sin(m\pi y/b) \quad (4.78)$$

with

$$\tilde{S}_6 = 2\,[\tilde{\alpha}_1\,\bar{E}_1 + \bar{G}_1/\tilde{\alpha}_1^{\,2}] \quad ,$$

$$S_6 = 2\,[\tilde{\alpha}_1\,\bar{E}_2 + \bar{G}_2/\tilde{\alpha}_1^{\,2}] \quad . \tag{4.79}$$

and

$$\psi_2^{(\gamma)}(x,y) = [(N_1 + N_3 y)\cosh(\tilde{\alpha}_1 y) + (N_2 + N_4 y)\sinh(\tilde{\alpha}_1 y)]\sin(n\pi x)$$
$$+ [(\tilde{N}_1 + \tilde{N}_3 x)\cosh(\tilde{\alpha}_2 x) + (\tilde{N}_2 + \tilde{N}_4 x)\sinh(\tilde{\alpha}_2 x)]\sin(m\pi y/b) \tag{4.80}$$

where

$$\left.\begin{aligned}
&N_3 = \tilde{\alpha}_1\,\bar{E}_1/\beta \quad , \quad N_4 = 2\,\tilde{\alpha}_1^{\,2}\,\bar{E}_1/\beta_1 \\
&N_1 = -(\tilde{S}_4\,\theta_1 + \tilde{S}_5\,\theta_2)/\beta_2 \\
&N_2 = -\,[N_4 b + (N_1 + N_3 b)\coth(\tilde{\alpha}_1 b)] \\
&\tilde{S}_4 = 2\,m\,\pi\,\bar{\bar{E}}_1/b^2 \quad , \quad \tilde{S}_5 = 2\,m\,\pi\,\bar{\bar{E}}_2/b^2 \quad , \\
&S_1 = 2\,n\,\pi\,b^2\,\bar{G}_2/m^2\,\pi^2 \quad , \\
&S_2 = d_{11}/\beta_2 - 2\,n\,\pi\,\bar{G}_1/\tilde{\alpha}_1^{\,2} \quad , \\
&S_3 = -d_{11}/\beta_1 \quad , \quad S_4 = 2\,n\,\pi\,\bar{E}_1 \quad , \quad S_5 = 2\,n\,\pi\,\bar{E}_2 \\
&\tilde{S}_1 = 2\,m\,\pi\,\bar{\bar{G}}_2/b^2\,n^2\,\pi^2 \quad , \\
&\tilde{S}_2 = d_{11} b/\beta_1 - 2\,m\,\pi\,\bar{\bar{G}}_1/b^2\,\tilde{\alpha}_2^{\,2}
\end{aligned}\right\}$$

$$S_3 = - d_{11} b/\beta_1 \tag{4.81 (Concluded)}$$

and

$$\tilde{N}_1 = - (S_1\theta_1 + S_2\theta_2)/\beta_1$$

$$\tilde{N}_2 = - [\tilde{N}_4 + (\tilde{N}_1 + \tilde{N}_3) \coth (\tilde{\alpha}_2)]$$

$$\tilde{N}_3 = \tilde{\alpha}_2 \tilde{S}_2 \, b/2 \, n \, \pi \, \beta_2$$

$$\tilde{N}_4 = \tilde{\alpha}_2 \tilde{S}_1 \, b/2 \, m \, \pi \, \beta_1 \quad .$$

We now determine the remaining constants, F_1, ..., $\tilde{F}_2$ as follows. We set

$$V_2 = \{-(\tilde{S}_1 + \tilde{S}_2)\, \xi_1 + (-1)^{n+1} (\tilde{S}_2 - \tilde{S}_3) - \tilde{S}_3 \, (2\xi_1 + (-1)^m n^2 \pi^2)/n^2\pi^2\}/\beta_2 \, m\pi$$

and

$$\tilde{V}_2 = \{-(S_1 + S_2 b)\, \bar{\xi}_1 + (-1)^{m+1} (S_2 - S_3 b) - S_3 \, (2\bar{\xi}_1 + (-1)^m m^2 \pi^2)\, b^2/m^2 \pi^2\}\, b/\beta_1 \, m\pi \tag{4.82}$$

and define

$$F_1 = V_2 + (\tilde{\alpha}_1^{\,2} P_0 - 2P_2)/\tilde{\alpha}_1^{\,4}$$

$$F_2 = -F_1 \coth (\tilde{\alpha}_1 b) + [(\alpha_1^{\,2} P_0 - 2P_2) - \tilde{\alpha}_1^{\,2}\, b(P_1 + P_2 b)]/\alpha_1^{\,4} \sinh (\tilde{\alpha}_1 b) \tag{4.83}$$

$$\tilde{F}_1 = \tilde{V}_2 + (\tilde{\alpha}_2^{\,2} \tilde{P}_0 - 2P_2)/\tilde{\alpha}_2^{\,4} \quad ,$$

$$\tilde{F}_2 = -\tilde{F}_1 \coth (\tilde{\alpha}_2) + [\alpha_2^{\,2} \tilde{P}_0 - 2\tilde{P}_2 - \tilde{\alpha}_2^{\,2}(\tilde{P}_1 + \tilde{P}_2)]/\alpha_2^{\,4} \sinh (\tilde{\alpha}_2) \quad .$$

Finally, the solution to the inner problem to $0(\varepsilon^2)$ takes the form

$$\Psi(x,y) = \begin{cases} T_1(y)\, x + d_{11}\, y(b-y)\, x^2 + \varepsilon[\bar{T}_1(y)\, x - T_1(y)/\beta_1] - \varepsilon^2\, \bar{T}_1(y)/\beta_1 \ , & \text{near } x = 0 \\ T_2(y)\, (1-x) + d_{11}\, y(b-y)\, (1-x)^2 + \varepsilon[\bar{T}_2(y)\, (1-x)] \ , & \text{near } x = 1 \\ T_3(x)\, y + d_{11}\, x(1-x)\, y^2 + \varepsilon[\bar{T}_3(x)\, y - T_3(x)/\beta_2] - \varepsilon^2\, \bar{T}_3(x)/\beta_2 \ , & \text{near } y = 0 \\ T_4(x)\, (b-y) + d_{11}\, x(1-x)\, (b-y)^2 + \varepsilon[T_4(x)\, (b-y)] \ , & \text{near } y = b \end{cases} \tag{4.84}$$

where

$$T_1(y) = T_2(y) = -d_{11}\, y(b-y) \ ,$$

$$\left.\begin{aligned} T_3(x) &= T_4(x) = -d_{11}\, x(1-x) \ , \\ \bar{T}_1(y) &= S_1 + S_2(b-y) + S_3 y(b-y) + S_4 \cosh(\alpha_1 y) + S_5 \sinh(\alpha_1 y) \\ &\quad + S_6 \sin(m\pi y/b) \ , \\ \bar{T}_2(y) &= (-1)^{n+1}\, S_1 + S_7(b-y) + S_3\, y(b-y) + (-1)^{n+1}\, [S_4 \cosh(\tilde{\alpha}_1 y) \\ &\quad + S_5 \sinh(\tilde{\alpha}_1 y)] + S_8 \sin(m\pi y/b) \ , \\ \bar{T}_3(x) &= \tilde{S}_1 + \tilde{S}_2(1-x) + \tilde{S}_3\, x(1-x) + \tilde{S}_4 \cosh(\tilde{\alpha}_2 x) + \tilde{S}_5 \sinh(\tilde{\alpha}_2 x) \\ &\quad + \tilde{S}_6 \sinh(n\pi x) \ , \\ \bar{T}_4(x) &= (-1)^{m+1}\, \tilde{S}_1 + S_7(1-x) - \tilde{S}_3\, x(1-x) + (-1)^{m+1}\, [\tilde{S}_4 \cosh(\tilde{\alpha}_2 x) \\ &\quad + \tilde{S}_5 \sinh(\tilde{\alpha}_2 x)] + \tilde{S}_8 \sin(n\pi x) \ . \end{aligned}\right\} \tag{4.85}$$

This completes the static problem to $0(\varepsilon^2)$ for a rectangular plate that is partly clamped and partly hinged on adjacent sides and whose initial curvature is second-order polynomial in x and y directions.

4.4 Initial Deflection Form – Case 2

When the initial curvature is polynomial in one direction and sinusoidal in the other:

Here, we set the initial curvature to take the form

$$W_0(x,y) = d_{11}\ x(1-x)\ \sin\ (m\pi y/b) \tag{4.86}$$

with d_{11} being the amplitude.

4.4-1 Fully Clamped Plate [C-C-C-C]

As for the previous case, W_0 satisfies the zero deflection condition on all the edges while the condition of zero slope is not met. For a fully clamped, curved plate, the problem can be decomposed into two, viz: the dynamic problem and the static problem. The *former* is the same as was presented in equations (4.2) through (4.5), while the latter problem takes the same form as equations (4.6) and (4.7) with equation (4.8) being replaced by equation (4.86).

It is easily verified that the leading order solution takes the form

$$\Omega_0^{\ 0} = -d_{11}\ x(1-x)\ \sin\ (m\pi y/b) \tag{4.87}$$

while the inner solution, to $0(\varepsilon)$, can be written as

$$\psi^i = \begin{cases} \varepsilon\ u_0(y)\ (X + (1/\beta_1)\ (\exp(-\beta_1 X) - 1)) & , \quad \text{near } x = 0 \\ \varepsilon\ u_1(y)\ (X + (1/\beta_1)\ (\exp(-\beta_1 X) - 1)) & , \quad \text{near } x = 1 \\ \varepsilon\ u_2(x)\ (Y + (1/\beta_2)\ (\exp(-\beta_2 Y) - 1)) & , \quad \text{near } y = 0 \\ \varepsilon\ u_3(x)\ (Y + (1/\beta_2)\ (\exp(-\beta_2 Y) - 1)) & , \quad \text{near } y = b \end{cases} \tag{4.88}$$

where

$$\left. \begin{aligned} u_0(y) &= u_1(y) = -\ d_{11}\ \sin\ (m\pi y/b) \\ u_2(x) &= u_3(x) = -\ (d_{11} m\pi/b)\ x(1-x) \quad . \end{aligned} \right\} \tag{4.88a}$$

The outer problem to $0(\varepsilon)$ is thus identical to that of Case 1, equation (4.35a), but with the following boundary conditions:

$$\left.\begin{aligned}
\Omega_1^0(0,y) &= (d_{11}/\beta_1)\ \sin\ (m\pi y/b)\\
\Omega_1^0(1,y) &= (d_{11}/\beta_1)\ \sin\ (m\pi y/b)\\
\Omega_1^0(x,0) &= (d_{11}/\beta_2 b)\ x\ (1-x)\\
\Omega_1^0(x,b) &= (d_{11}/\beta_2 b)\ x\ (1-x)
\end{aligned}\right\} \qquad (4.89)$$

The solution to equations (4.35a) and (4.89) is easily obtained as

$$\Omega_1^0(x,y) = (d_{11}/\beta_1)\ \sin\ (m\pi y/b) + (d_{11}\ m\pi/\beta_2 b)\ x\ (1-x) + \tilde{\Phi}_1(x,y) \qquad (4.90)$$

such that $\tilde{\Phi}_1$ is determined from

$$\beta_1^2\ \tilde{\Phi}_{1_{xx}} + \beta_2^2\ \tilde{\Phi}_{1_{yy}} - \left(\frac{2d_{11} m\pi\ \beta_1^2}{\beta_2 b} + \frac{\beta_2^2\ d_{11}\ m^2\pi^2}{\beta_1 b^2}\ \sin\ (\frac{m\pi y}{b})\right) = 0 \qquad (4.91)$$

subject to $\tilde{\Phi}_1 = 0$ on all the boundaries.

The solution to equation (4.91) is obtained as

$$\begin{aligned}
\tilde{\Phi}_1 = {} & [K_{11}\ \cosh\ (\tilde{\alpha}_2 x) + K_{12}\ \sinh\ (\tilde{\alpha}_2 x) + N_2]\ \sin\ (m\pi y/b)\\
& + [K_{13}\ \cosh\ (\tilde{\alpha}_1 y) + K_{14}\ \sin\ (\tilde{\alpha}_1 y) + I_3]\ \sin\ (n\pi x)\\
& + K_\nu\ \sin\ (m\pi y/b)\ \sin\ (n\pi x)
\end{aligned} \qquad (4.92)$$

where

$$N_2 = -\frac{2\ d_{11}\ [1-(-1)^m]\ b^2\ \beta_1^2}{m^3\ \pi^2\ \beta_2^3} \qquad (4.93)$$

(Continued)

$$I_3 = - \frac{2\, d_{11}\, I_2}{b\, \beta_2\, n^2\, \pi} \quad , \quad I_2 = [1 - (-1)^n]/n\pi \qquad (4.93)$$
(Concluded)

$$K_\nu = - \frac{d_{11}\, I_2\, \beta_2^2}{\beta_1^3\, b^2\, n^2 + \beta_1\, \beta_2^2\, m^2} \quad .$$

The unknowns K_{11}, ..., K_{14} are determined from a 2-2 matching and are given as

$$K_{11} = - N_2 \quad , \quad K_{12} = \frac{N_2 (\cosh \tilde{\alpha}_2 - 1)}{\sinh \tilde{\alpha}_2} \qquad (4.94)$$

$$K_{13} = - I_3 \quad , \quad K_{14} = \frac{I_3 (\cosh (\tilde{\alpha}_1 b) - 1)}{\sin (\tilde{\alpha}_1 b)} \quad .$$

The problem is thus completed to $0(\varepsilon)$.

Second Order Correction

It is easily shown, from equation (4.12) and 3-2 matching, that Ω_2^0 must satisfy the equation

$$\beta_1^2 \frac{\partial^2 \Omega_2^0}{\partial x^2} + \beta_2^2 \frac{\partial^2 \Omega_2^0}{\partial y^2} = \nabla^4 \Omega_0^0 \qquad (4.95)$$

subject to

$$\Omega_2^0(0,y) = \frac{1}{\beta_1} \left(\frac{d_{11}\pi}{\beta_2 b} + I_3\, n\pi \right) - \frac{1}{\beta_1} (K_{12}\, \tilde{\alpha}_2 + K_\nu\, n\pi) \sin (\frac{m\pi y}{b})$$
$$- \frac{n\pi}{\beta_1} [K_{13} \cosh (\tilde{\alpha}_1 y) + K_{14} \sinh (\tilde{\alpha}_1 y)]$$

$$\Omega_2^0(1,y) = - \frac{1}{\beta_1} \left(\frac{d_{11}\pi}{\beta_2 b} + (-1)^{n+1} I_3\, n\pi \right) + \frac{1}{\beta_1} [K_{11}\, \tilde{\alpha}_2 \sinh \tilde{\alpha}_2 + K_{12}\, \tilde{\alpha}_2 \cosh \tilde{\alpha}_2$$
$$+ (-1)^{n+1} K_\nu\, n\pi] \sin (\frac{m\pi y}{b}) + (-1)^n \frac{n\pi}{\beta_1} [K_{13} \cosh (\tilde{\alpha}_1 y) + K_{14} \sinh (\tilde{\alpha}_1 y)]$$

$$\Omega_2^0(x,0) = -\frac{1}{\beta_2}\left(\frac{d_{11}\pi}{\beta_1 b} + N_2\frac{m\pi}{b}\right) - \frac{1}{\beta_2}\left(K_{14}\,\tilde{\alpha}_1\frac{m\pi}{b} + K_\nu\frac{m\pi}{b}\right)\sin(n\pi x)$$

$$-\frac{1}{\beta_2}\frac{m\pi}{b}\left[K_{11}\cosh(\tilde{\alpha}_2 x) + K_{12}\sinh(\tilde{\alpha}_2 x)\right]$$

$$\Omega_2^0(x,b) = \frac{(-1)^m}{\beta_2}\left(\frac{d_{11}\pi}{\beta_1 b} + N_2\frac{m\pi}{b}\right) + (-1)^m\frac{m\pi}{\beta_2 b}\left[K_{11}\cosh(\tilde{\alpha}_2 x) + K_{12}\sinh(\tilde{\alpha}_2 x)\right]$$

$$+\frac{1}{\beta_2}\left[K_{13}\,\tilde{\alpha}_1\sinh(\tilde{\alpha}_1 b) + K_{14}\,\tilde{\alpha}_1\cosh(\tilde{\alpha}_1 b) + (-1)^m K_\nu\frac{m\pi}{b}\right]\sin(n\pi x)$$

(4.96)
(Concluded)

The solution to this problem is analogous to that of the problem defined by equations (4.45) and (4.46) and can thus be inferred directly from equations (4.47) through (4.61). Thus, this completes the static problem to $0(\varepsilon^2)$ for a clamped, rectangular plate whose initial curvature is second-order polynomial in one direction and sinusoidal in an orthogonal direction.

4.4-2 Fully Hinged Plate [S-S-S-S]

For a fully hinged plate having its initial curvature given by equation (4.86), the solution to the associated dynamic problem is the same as given in equations (4.62) and (4.63). The result for the static problem is now summarized below:

We have, for the outer solution given to $0(\varepsilon^2)$,

$$\psi_0(x,y) = -d_{11}\,x(1-x)\sin(m\pi y/b) \tag{4.97}$$

$$\psi_1(x,y) = 0 \tag{4.98}$$

$$\psi_2(x,y) = B\sin(m\pi y/b) + D\,x(1-x)\sin(m\pi y/b) \tag{4.99}$$

where

$$B = -A\,m^2\,\pi^2/b^2\,\beta_2^2 \quad \text{and} \quad D = (4/\beta_2^2 + 2\,\beta_1^2\,A/\beta_2^4) \tag{4.100}$$

with $A = -d_{11}$. The inner solutions all vanish.

4.4-3 Partly Clamped and Partly Hinged Plate [C-C-S-S]

For a plate that is partly clamped and partly simply-supported on adjacent sides, the outer problem to solve is given by the differential equation

$$\beta_1^2 \frac{\partial^2 \psi_\nu}{\partial x^2} + \beta_2^2 \frac{\partial^2 \psi_\nu}{\partial y^2} = \begin{cases} 2\, d_{11}\, \beta_1^2 \left[1 + \dfrac{\beta_2^2\, m^2\, \pi^2}{2\, \beta_1^2\, b^2} x(1-x)\right] \sin\left(\dfrac{m\pi y}{b}\right) , & \nu = 0 \\ 0 , & \nu = 1 \\ \nabla^4\, \psi_{\nu-2} , & \nu \geq 2 \end{cases} \tag{4.101}$$

subject to the boundary conditions of equations (4.70). The method of solution is the same as was enumerated before and the solution is summarized below:

$$\psi_0(x,y) = -\, d_{11}\, x(1-x)\, \sin\,(m\pi y/b) \quad , \tag{4.102}$$

$$\begin{aligned} \psi_1(x,y) = {} & (d_{11}/\beta_1)\,(1-x)\,\sin\,(m\pi y/b) + (d_{11}\, m\pi/\beta_2 b_2)\, x(1-x)\,(b-y) \\ & + [E_1 \cosh\,(\tilde{\alpha}_1 y) + E_2 \sinh\,(\tilde{\alpha}_1 y) - (G_1/\tilde{\alpha}_1^2)\,(b-y) \\ & + (b^2/m^2\,\pi^2)\, G_2 \sin\,(m\pi y/b)]\, \sin\,(n\pi x) \\ & + [\tilde{E}_1 \cosh\,(\tilde{\alpha}_2 x) + \tilde{E}_2 \sinh\,(\tilde{\alpha}_2 x) - \{(\tilde{G}_1/\tilde{\alpha}_2^2) + (G_2(1-x)/\tilde{\alpha}_2^2)\}] \\ & \sin\,(m\pi y/b) \quad . \end{aligned} \tag{4.103}$$

The unknown constants here are defined later on in the section. The $0(\varepsilon^2)$ term, $\psi_2(x,y)$, is decomposed into three parts as

$$\psi_2(x,y) = \psi_2^{(\rho)}(x,y) + \psi_2^{(\sigma)}(x,y) + \psi_2^{(\eta)}(x,y) \tag{4.104}$$

where

$$\begin{aligned} \psi_2^{(\rho)}(x,y) = {} & [W_1 \cosh\,(\tilde{\alpha}_1 y) + W_2 \sinh\,(\tilde{\alpha}_1 y)]\, \sin\,(n\pi x) + [W_3 \cosh\,(\tilde{\alpha}_2 x) \\ & + W_4 \sinh\,(\tilde{\alpha}_2 x) + (d_{11}\, \beta_1^2/\beta_2^4)\, \{(4\, \beta_2^2/\beta_1^2) - 2 + \tilde{\alpha}_2^2\, x(1-x)\}]\, \sin\,(m\pi y/b \end{aligned} \tag{4.105}$$

with

$$\left.\begin{aligned}
W_1 &= -(\tilde{S}_6/\beta_2) \\
W_2 &= -W_1 \coth(\tilde{\alpha}_1 b) \\
W_3 &= -[(S_4/\beta_1) + (2\, d_{11}/\beta_2^{\,4})(2\,\beta_2^{\,2} - \beta_1^{\,2})] \\
W_4 &= -[W_3 \cosh(\tilde{\alpha}_2) + (2\, d_{11}/\beta_2^{\,4})(2\,\beta_2^{\,2} - \beta_1^{\,2})]/\sinh(\tilde{\alpha}_2) \quad .
\end{aligned}\right\} \quad (4.106)$$

$$\begin{aligned}
\psi_2^{(\sigma)}(x,y) = {} & [N_1 \cosh(\tilde{\alpha}_1 y) + N_2 \sinh(\tilde{\alpha}_1 y) + N_3\, y \cosh(\tilde{\alpha}_1 y) \\
& + N_4\, y \sinh(\tilde{\alpha}_1 y)] \sin(n\pi x) \\
& + [\tilde{N}_1 \cosh(\tilde{\alpha}_2 x) + \tilde{N}_2 \sinh(\tilde{\alpha}_2 x) + \tilde{N}_3\, x \cosh(\tilde{\alpha}_2 x) \\
& + N_4\, x \sinh(\tilde{\alpha}_2 x)] \sin(m\pi y/b)
\end{aligned}$$

with

$$\left.\begin{aligned}
N_3 &= (\tilde{\alpha}_1\, S_2/2n\pi\beta_1) \\
N_4 &= (\tilde{\alpha}_1\, S_1/2n\pi\beta_1) \\
N_1 &= -(\tilde{S}_1\, \theta_1 + \tilde{S}_2\, \theta_2)/\beta_2 \\
N_2 &= -[N_4 b + (N_1 + N_3 b) \coth(\tilde{\alpha}_1 b)] \\
\tilde{N}_3 &= (\tilde{\alpha}_2\, \tilde{S}_2\, b/2m\pi\beta_2) \\
\tilde{N}_4 &= (\tilde{\alpha}_2\, \tilde{S}_1\, b/2m\pi\beta_2)
\end{aligned}\right\}$$

$$\tilde{N}_1 = -(S_1\ \theta_1 + S_2\ \theta_2)/\beta_1$$

$$\tilde{N}_2 = -[\tilde{N}_4 + (\tilde{N}_1 + \tilde{N}_3)\ \coth\ (\tilde{\alpha}_2)] \qquad \text{(4.108) (Concluded)}$$

and

$$\psi_2^{(n)}(x,y) = [F_1 \cosh(\tilde{\alpha}_1 y) + F_2 \sinh(\tilde{\alpha}_1 y) - (S_3/\beta_1 n\pi)(b-y)] \sin(n\pi x)$$
$$+ [\tilde{F}_1 \cosh(\tilde{\alpha}_2 x) + \tilde{F}_2 \sinh(\tilde{\alpha}_2 x) \qquad (4.109)$$
$$+ \{2 P_2 + \tilde{\alpha}_2^2 P_0 - \tilde{\alpha}_2^2 x(P_1 - P_2 x)\}/\tilde{\alpha}_2^4] \sin(m\pi y/b)$$

with

$$P_0 = (\tilde{\alpha}_2^2\ b/\beta_2\ m\pi)\ (\tilde{S}_4 - \tilde{S}_5) \quad ,$$
$$P_1 = (\tilde{\alpha}_2^2\ b/\beta_2\ m\pi)\ (\tilde{S}_4 + \tilde{S}_3) \quad ,$$
$$P_2 = (\tilde{\alpha}_2^2\ b/\beta_2\ m\pi)\ \tilde{S}_3 \quad ,$$
$$\tilde{F}_1 = (2\ P_2 + \tilde{\alpha}_2^2\ P_0)/\tilde{\alpha}_2^4 - (S_3\ b^2/\beta_1\ m\pi) \quad ,$$
$$\tilde{F}_2 = \{-\tilde{F}_1 \coth \tilde{\alpha}_2 + 2\ P_2 + \tilde{\alpha}_2^2 - \tilde{\alpha}_2^2\ (P_1 - P_2)\}/(\tilde{\alpha}_2^2 \sinh \tilde{\alpha}_2) \quad , \qquad (4.110)$$
$$F_1 = \{(\tilde{S}_5 - \tilde{S}_4)\ \xi_1 + (-1)^{n+1}\ (\tilde{S}_4 + \tilde{S}_3) + (\tilde{S}_3/n^2\pi^2)\ (2\ \xi_1$$
$$+ (-1)^n\ n^2\ \pi^2)\}\ /\ (\beta_2 n\pi) - (S_3\ b/\beta_1 n\pi)$$
$$F_2 = -\ F_1 \coth(\tilde{\alpha}_1 b) \quad .$$

Also, we have that

$$G_1 = (2\ \beta_1^2\ m\ d_{11}/\beta_2^3\ b^2 n)\ \xi_1 \quad ; \quad G_2 = (m^2\ \pi\ d_{11}/\beta_1\ b^2 n) \quad ,$$

$$E_1 = (G_1\ b/\tilde{\alpha}_1^2) \quad ; \quad E_2 = -(G_1\ b/\tilde{\alpha}_1^2) \coth(\tilde{\alpha}_1 b) \quad ,$$

$$
\left.
\begin{aligned}
&\tilde{G}_1 = (2\, d_{11}/\beta_2) \quad ; \quad \tilde{G}_2 = (\tilde{\alpha}_2^{\,2}\, d_{11}\, b/2\beta_1) \quad , \\
&\tilde{E}_1 = (\tilde{G}_1 + \tilde{G}_2)/\tilde{\alpha}_2^{\,2} \quad ; \quad \tilde{E}_2 = (\tilde{G}_1/\tilde{\alpha}_2^{\,2} \sinh(\tilde{\alpha}_2)) - \tilde{E}_1 \coth(\tilde{\alpha}_2^{\,2}) \\
&S_1 = 2\, n\, \pi\, E_1 \quad ; \quad S_2 = 2\, n\, \pi\, E_2 \quad , \\
&S_3 = (d_{11}\, n\, \pi/\beta_2 b^2) - (2\, n\, \pi\, G_1/\tilde{\alpha}_1^{\,2}) \\
&S_4 = (2nb^2\, G_2/m^2\pi) + (2\, \tilde{\alpha}_2\, E_2/b) - (d_{11}/\beta_1) + (2\, \tilde{G}_2/\tilde{\alpha}_2^{\,2} b) \quad , \\
&S_5 = (d_{11} m\, \pi/\beta_2 b^2) + (-1)^n\, (2\, n\, \pi\, G_1/\tilde{\alpha}_1^{\,2}) \\
&S_6 = (d_{11}/\beta_1) - (2\, \tilde{\alpha}_2^{\,2}\, \tilde{R}_2/b) - (2\, \tilde{G}_2/b\, \tilde{\alpha}_2^{\,2}) \quad ,
\end{aligned}
\right\} \quad (4.111)
$$

where

$$
\tilde{R}_2 = \tilde{E}_1 \sinh(\tilde{\alpha}_2) + \tilde{E}_2 \cosh(\tilde{\alpha}_2) \quad .
$$

Finally, the solution to the inner problem complement of equation (4.101), to $0(\varepsilon^2)$, takes the form

$$
\Psi(x,y) = \begin{cases}
b_1(y)\, x + d_{11}\, x^2 \sin(m\pi y/b) + \varepsilon\, [b_2(y)\, x - b_1(y)/\beta_1] \\
\qquad - \varepsilon^2\, b_2^{\,0}(y)/\beta_1 \quad , & \text{near } x = 0 \quad ; \\
b_1''(y)\, (1-x) + d_{11}(1-x)^2 \sin(m\pi y/b) + \varepsilon\, b_2''(y)\, (1-x) \quad , & \text{near } x = 1 \quad ; \\
f_1'(x)\, y + \varepsilon\, [f_2'(x)\, y - f_1'(x)/\beta_2] - \varepsilon^2\, f_2'(x)/\beta_2 \quad , & \text{near } y = 0 \quad ; \\
f_1''(x)\, (b-y) + \varepsilon\, f_2''(x)\, (b-y) \quad , & \text{near } y = b \quad .
\end{cases}
\quad (4.112)
$$

where

$$
\left.\begin{aligned}
b_1'(y) &= -\, d_{11} \sin (m\pi y/b) \quad , \\
b_1''(y) &= b_1'(y) \quad , \\
f_1'(x) &= (d_{11}\, m\pi/b)\, x\, (1-x) \quad , \\
f_1''(x) &= (-1)^m\, f_1'(x)
\end{aligned}\right\} \qquad (4.113)
$$

and

$$
\left.\begin{aligned}
b_2'(y) &= S_1 \cosh (\tilde{\alpha}_1 y) + S_2 \sinh (\tilde{\alpha}_1 y) + S_3\, (b-y) \\
&\quad + S_4 \sin (m\pi y/b) \quad , \\
b_2''(y) &= (-1)^{n+1}\, [S_1 \cosh (\tilde{\alpha}_1 y) + S_2 \sinh (\tilde{\alpha}_1 y)] + S_5\, (b-y) \\
&\quad + S_6 \sin (m\pi y/b) \quad , \\
f_2'(x) &= \tilde{S}_1 \cosh (\tilde{\alpha}_2 x) + \tilde{S}_2 \sinh (\tilde{\alpha}_2 x) - \tilde{S}_3\, x(1-x) \\
&\quad + \tilde{S}_4\, (1-x) - \tilde{S}_5 + \tilde{S}_6 \sin (n\pi x) \quad , \\
f_2''(x) &= (-1)^{m+1}\, [\tilde{S}_1 \cosh (\tilde{\alpha}_2 x) + \tilde{S}_2 \sinh (\tilde{\alpha}_2 x)] + \tilde{S}_3\, x(1-x) \\
&\quad + (-1)^{m+1}\, \tilde{S}_4 (1-x) + (-1)^m\, \tilde{S}_5 - \tilde{S}_6 \sin (n\pi x) \quad .
\end{aligned}\right\} \qquad (4.114)
$$

This completes the static problem to $0(\varepsilon^2)$. In the next section, we consider the same plate problem with an arbitrary initial deflection.

4.5 Initial Deflection Form – Case 3

– When the initial curvature is of arbitrary form.

Here, we decompose the initial curvature into its Fourier components, i.e.:

$$\bar{W}_0(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin (n\pi x) \sin (m\pi y/b) \quad . \tag{4.115}$$

The non-dimensional governing differential equation for the static problem takes the form

$$\varepsilon^2 \nabla^4 \psi - \beta_1^2 \frac{\partial^2 \psi}{\partial x^2} - \beta_2^2 \frac{\partial^2 \psi}{\partial y^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin (n\pi x) \sin (m\pi y/b) \tag{4.116}$$

where

$$q_{mn} = - \frac{\pi^2 P_{mn}}{b^2} (b^2 \beta_1^2 + \beta_2^2) \quad .$$

4.5-1 Fully Clamped Plate

The problem definition is completed by specifying the pertinent boundary conditions. As in the previous cases, w_0 satisfies the zero deflection and zero moment conditions on all edges. The dynamic problem is the same as that presented in Section 4.3 and will not be repeated here.

As outlined previously in Section 3, we write

$$\psi^0 = \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \psi_{\nu} \tag{4.117}$$

for the outer solution as was done by Adeniji-Fashola and Oyediran [1] when they considered the same problem but for the fully-clamped rectangular plate case. The procedure followed is to solve equation (4.117) for $\nu = 0,1,2,...$ subject to appropriate boundary conditions. The fully-clamped case, which is discussed in Reference 1 will not be repeated here.

4.5-2 Fully Simply-Supported Rectangular Plate

For a fully simply-supported plate, it is easily seen that the boundary value problem for ψ_0 takes the form

$$\beta_1^2 \frac{\partial^2 \psi_0}{\partial x^2} + \beta_2^2 \frac{\partial^2 \psi_0}{\partial y^2} = - q_{mn} \sin (n\pi x) \sin (m\pi y/b) \tag{4.118}$$

Subject to ψ_0 vanishing on all boundaries.

The solution takes the form

$$\psi_0 = \gamma_{mn} \sin (n\pi x) \sin (m\pi y/b) \tag{4.119}$$

where

$$\gamma_{mn} = q_{mn}/(\beta_1^2 n^2 \pi^2 + \beta_2^2 m^2 \pi^2/b^2) \quad .$$

With relative ease, it can be verified that ψ_1 must vanish while the solution to the ψ_2 problem is

$$\psi_2 = \delta_5 \sin (n\pi x) \sin (m\pi y/b) \tag{4.120}$$

with

$$\delta_5 = - \frac{\gamma_{mn} (n^2 \pi^2 + m^2 \pi^2/b^2)^2}{(\beta_1^2 n^2 \pi^2 + \beta_2^2 m^2 \pi^2/b^2)} \quad .$$

This solution for the simply-supported case agrees with the solution by Timoshenko and Woinowski-Krieger [3].

4.5-3 Partly Clampled and Partly Hinged Plate [C-C-S-S]

The governing differential equations are those presented in Reference 1 plus the pertinent boundary conditions. It is therefore clear that the solution takes the form

$$\psi_0 = \gamma_{mn} \sin (n\pi x) \sin (m\pi y/b) \tag{4.121}$$

$$\psi_1 = L_1' \sinh \tilde{\alpha}_1 \sin (n\pi x) + L_2' \sinh (\tilde{\alpha}_2 x) \sin (m\pi y/b) \tag{4.122}$$

with

$$\left. \begin{aligned} L_1' &= \frac{(-1)^m \gamma_{mn} (m\pi/b)}{\beta_2 \sinh \tilde{\alpha}_1} \\ L_2' &= \frac{(-1)^n \gamma_{mn} n\pi}{\beta_1 \sinh \tilde{\alpha}_2} \quad . \end{aligned} \right\} \tag{4.123}$$

The second order problem, ψ_2, can be shown to take the form

$$\beta_1^2 \frac{\partial^2 \psi_2}{\partial x^2} + \beta_2 \frac{\partial^2 \psi_2}{\partial y^2} = \nabla^4 \psi_0 \tag{4.124}$$

subject to

$$\left. \begin{aligned} \psi_2(0,y) &= -\frac{1}{\beta_1} [L_1' n\pi \sin (\tilde{\alpha}_1 y) + L_2' \tilde{\alpha}_2 \sin (m\pi y/b) \\ \psi_2(1,y) &= -\frac{1}{\beta_1} [(-1)^{n+1} n\pi L_1' \sinh(\tilde{\alpha}_1 y) - L_2' \tilde{\alpha}_2 \cosh \tilde{\alpha}_2 \sin (m\pi y/b)] \\ \psi_2(x,0) &= -\frac{1}{\beta_2} [L_1' \tilde{\alpha}_1 \sin (n\pi x) + (m\pi/b) L_2' \sinh (\tilde{\alpha}_2 x) \\ \psi_2(x,1) &= -\frac{1}{\beta_2} [(m\pi/b) (-1)^{m+1} L_2' \sinh (\tilde{\alpha}_2 x) - L_1' \tilde{\alpha}_1 \cosh (\tilde{\alpha}_1 b) \sin (n\pi x) \end{aligned} \right\} \tag{4.125}$$

Clearly, we have the solution to ψ_2 as (see Appendix)

$$\psi_2 = [K_1 \sinh (\tilde{\alpha}_1 y) + K_2 \cosh (\tilde{\alpha}_1 y) + E_1 \sinh (\tilde{\alpha}_1 y) + E_2 \cosh (\tilde{\alpha}_1 y)$$

$$+ E_3 y \cosh (\tilde{\alpha}_1 y)] \sin (n\pi x)$$

$$+ [K_3 \sinh (\tilde{\alpha}_2 x) + K_4 \cosh (\tilde{\alpha}_2 x) + E_4 \sinh (\tilde{\alpha}_2 x) + E_5 \cosh (\tilde{\alpha}_2 x)$$

$$+ E_6 x \cosh(\tilde{\alpha}_2 x)] \sin(m\pi y/b)$$

$$+ K_5 \sin(n\pi x) \sin(m\pi y/b) \quad . \tag{4.126}$$

The constants are determined in the Appendix and are listed in equation (A.13).

V. CONCLUDING REMARKS

In this report, analytical results that include the small initial curvature effect on the free vibration of rectangular plates are presented in the limit when in-plane forces are much larger than the bending rigidity. These results are presented for various boundary conditions such as fully clamped, fully simply-supported, and partly clamped partly simply-supported cases. Different mathematical forms of the initial curvature are also considered.

Taking advantage of the linear nature of the problem, a decomposition method is used to break the problem into two parts, viz: the dynamic and the static problems. The solution to the dynamic problem which is readily available in the literature is recovered independently by the present authors while the solutions to the static problem for various boundary conditions and initial curvature forms are presented. The method of matched asumptotic expansions is used.

It is found that a small initial curvature has no effect on the natural frequency of these plates. However, the eigenmodes are greatly affected when compared with initially perfectly flat plates. The results presented here complement the earlier work of the present authors [1].

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APPENDIX

We consider the fourth-order problem

$$\beta_1^{\,2} \frac{\partial^2 \psi_2}{\partial x^2} + \beta_2^{\,2} \frac{\partial^2 \psi_2}{\partial y^2} = \nabla^4 \psi_0 \tag{A.1}$$

subject to

$$\left.\begin{aligned}
\psi_2(0,y) &= -\frac{1}{\beta_1}\left[L_1\, n\pi \sinh(\tilde{\alpha}_1 y) + L_2\, \tilde{\alpha}_2 \sin\left(\frac{m\pi y}{b}\right)\right] \\
\psi_2(1,y) &= -\frac{1}{\beta_1}\left[(-1)^{n+1}\, n\pi\, L_1 \sinh(\tilde{\alpha}_1 y) - L_2\, \tilde{\alpha}_2 \cosh \tilde{\alpha}_2 \sin\left(\frac{m\pi y}{b}\right)\right] \\
\psi_2(x,0) &= -\frac{1}{\beta_2}\left[L_1\, \tilde{\alpha}_1 \sin(n\pi x) + \frac{m\pi}{b} L_2 \sinh(\tilde{\alpha}_2 x)\right] \\
\psi_2(x,b) &= -\frac{1}{\beta_2}\left[\frac{m\pi}{b} (-1)^{m+1} L_2 \sinh(\tilde{\alpha}_2 x) - L_1\, \tilde{\alpha}_1 \cosh(\tilde{\alpha}_1 b) \sin(n\pi x)\right]
\end{aligned}\right\} \tag{A.2}$$

where L_1 and L_2 are known quantities (see text).

To solve this boundary value problem, it is convenient to introduce the decomposition

$$\psi_2 = \psi_2^{(1)} + \psi_2^{(2)} \tag{A.3}$$

where $\psi_2^{(1)}$ satisfies the homogeneous part of equation (A.1) and the following boundary conditions:

$$\begin{aligned}
\psi_2^{(1)}(0,y) &= \frac{n\pi\, L_1}{\beta_1} \sinh(\tilde{\alpha}_1 y) \\
\psi_2^{(1)}(1,y) &= \frac{(-1)^n\, n\pi}{\beta_1} L_1 \sinh(\tilde{\alpha}_1 y)
\end{aligned}$$

$$\left.\begin{aligned}\psi_2^{(1)}(x,0) &= -\frac{m\pi}{\beta_2 b} L_2 \sinh(\tilde{\alpha}_2 x)\\ \psi_2^{(1)}(x,b) &= \frac{(-1)^m m\pi}{\beta_2 b} L_2 \sinh(\tilde{\alpha}_2 x)\end{aligned}\right\} \quad \text{(A.4) (Concluded)}$$

The problem for $\psi_2^{(2)}$ is described by

$$\beta_1^2 \frac{\partial^2 \psi_2^{(2)}}{\partial x^2} + \beta_2^2 \frac{\partial^2 \psi_2^{(2)}}{\partial y^2} = \nabla^4 \psi_0 \quad \text{(A.5)}$$

and the boundary conditions of equations (A.2) less the portions satisfied by the $\psi_2^{(1)}$ problem in equations (A.4). Thus, we have for the boundary conditions of the $\psi_2^{(2)}$ problem

$$\left.\begin{aligned}\psi_2^{(2)}(0,y) &= -\frac{L_2 \tilde{\alpha}_2}{\beta_1} \sin\left(\frac{m\pi y}{b}\right)\\ \psi_2^{(2)}(1,y) &= \frac{L_2 \tilde{\alpha}_2}{\beta_1} \cosh \tilde{\alpha}_2 \sin\left(\frac{m\pi y}{b}\right)\\ \psi_2^{(2)}(x,0) &= -\frac{L_1 \tilde{\alpha}_1}{\beta_2} \sin(n\pi x)\\ \psi_2^{(2)}(x,b) &= \frac{L_1 \tilde{\alpha}_1}{\beta_2} \cosh(\tilde{\alpha}_1 b) \sin(n\pi x)\end{aligned}\right\} \quad \text{(A.6)}$$

The boundary conditions dictate the solution of $\psi_2^{(2)}$ to take the form

$$\psi_2^{(2)}(x,y) = \sum_n [K_1 \sinh(\tilde{\alpha}_1 y) + K_2 \cosh(\tilde{\alpha}_1 y)] \sin(n\pi x) + \sum_m [K_3 \sin(\tilde{\alpha}_2 x) + K_4 \cosh(\tilde{\alpha}_2 x)] \sin\left(\frac{m\pi y}{b}\right) + K_5 \sin(n\pi x) \sin\left(\frac{m\pi y}{b}\right) \quad \text{(A.7)}$$

with

$$\left.\begin{aligned}
K_1 &= \frac{2L_1 \tilde{\alpha}_1}{\beta_2} \coth (\tilde{\alpha}_1 b) \\
K_2 &= - \frac{L_1 \tilde{\alpha}_1}{\beta_2} \\
K_3 &= \frac{2 L_2 \tilde{\alpha}_2}{\beta_1} \coth \tilde{\alpha}_2 \\
K_4 &= - \frac{L_2 \tilde{\alpha}_2}{\beta_1} \\
K_5 &= \frac{\gamma_{mn} \left(n^2 \pi^2 + \frac{m^2 \pi^2}{b^2} \right)^2}{\beta_1^2 n^2 \pi^2 + \frac{\beta_2^2 m^2 \pi^2}{b^2}}
\end{aligned}\right\} \quad \text{(A.8)}$$

where γ_{mn} is defined in equation (4.119).

We attempt to solve for $\psi_2^{(1)}(x,y)$ using the finite Fourier transform method. The transform with respect to x gives

$$\beta_2^2 \bar{\psi}_{2_{yy}} - \beta_1^2 n^2 \pi^2 \bar{\psi}_2 = \beta_1^2 n\pi \left[(-1)^n \psi_2(1,y) - \psi_2(0,y)\right] , \quad \text{(A.9)}$$

where $\bar{\psi}_2(n,y)$ is the transform of ψ_2 with respect to x. Using equation (A.4) in equation (A.9), we have

$$\bar{\psi}_{2_{yy}} - \tilde{\alpha}_1^2 \bar{\psi}_2 = 2 n^2 \pi^2 L_1 \beta_1 \sinh (\tilde{\alpha}_1 y) \quad . \quad \text{(A.10)}$$

Thus, the solution to $\bar{\psi}_2^{(1)}$ is of the form

$$\bar{\psi}_2^{(1)}(n,y) = E_1 \sinh (\tilde{\alpha}_1 y) + E_2 \cosh (\tilde{\alpha}_1 y) + E_3 \, y \cosh (\tilde{\alpha}_1 y) \quad \text{(A.11)}$$

where compatibility requires that

$$E_3 = \frac{n^2 \pi^2 L_1 \beta_1}{\beta_2^2 \alpha_1} .$$

On taking the transform with respect to y, adding the result to equation (A.11) and inverting, we obtain

$$\psi_2^{(1)}(x,y) = [E_1 \sinh(\tilde{\alpha}_1 y) + E_2 \cosh(\tilde{\alpha}_1 y) + E_3 y \cosh(\tilde{\alpha}_1 y)] \sin(n\pi x)$$
$$+ [E_4 \sinh(\tilde{\alpha}_2 x) + E_5 \cosh(\tilde{\alpha}_2 x) + E_6 x \cosh(\tilde{\alpha}_2 x)] \sin\left(\frac{m\pi y}{b}\right) \quad (A.12)$$

where

$$E_6 = \frac{m^2 \pi^2 \beta_2 L_2}{\alpha_2 \beta_1^2} .$$

The unknowns E_1, E_2, E_4, and E_5 are, by requiring that $\psi_2^{(1)}(x,y)$ satisfy the boundary conditions of equation (A.4),

$$\left.\begin{aligned} E_1 &= \frac{(-1)^m m\pi L_2 K_2'}{\beta_2 b \sinh(\tilde{\alpha}_1 b)} + \frac{L_1 \tilde{\alpha}_1}{2\beta_2} \coth(\tilde{\alpha}_1 b) - E_3 b \coth(\tilde{\alpha}_1 b) \\ E_2 &= E_5 = 0 \\ E_4 &= \frac{(-1)^n n\pi L_1 K_4'}{\beta_1 \sinh \tilde{\alpha}_2} + \frac{L_2 \tilde{\alpha}_2}{2\beta_1} \coth \tilde{\alpha}_2 - E_6 \coth \tilde{\alpha}_2 \end{aligned}\right\} \quad (A.13)$$

where

$$K_2' = \frac{(-1)^{n+1} n\pi \sinh \tilde{\alpha}_2}{\tilde{\alpha}_2^2 + n^2 \pi^2}$$

$$K_4' = \frac{(-1)^{m+1} \frac{m\pi}{b} \sinh \tilde{\alpha}_1}{\tilde{\alpha}_1^2 + \frac{m^2 \pi^2}{b^2}}$$

APPROVAL

FREE VIBRATION OF RECTANGULAR PLATES WITH A SMALL INITIAL CURVATURE

By A. A. Adeniji-Fashola and A. A. Oyediran

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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