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ON THE APPROPRIATENESS OF APPLYING CHI-SQUARE DISTRIBUTION BASED CONFIDENCE INTERVALS TO SPECTRAL ESTIMATES OF HELICOPTER FLYOVER DATA

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SUMMARY

The validity of applying chi-square based confidence intervals to farfield acoustic flyover spectral estimates was investigated. Simulated data using a Kendall series and experimental acoustic data from the NASA/McDonnell Douglas 500E acoustics test were analyzed. Statistical significance tests to determine the equality of distributions of the simulated and experimental data relative to theoretical chi-square distributions were performed. Bias and uncertainty errors associated with the spectral estimates were easily identified from the data sets. A model relating the uncertainty and bias errors to the estimates resulted, which aided in determining the appropriateness of the chi-square distribution based confidence intervals. Such confidence intervals were appropriate for the non-tonally associated frequencies of the experimental data but were inappropriate for the tonally associated estimate distributions. The inappropriateness at the tonally associated frequencies was indicated by the presence of bias error and non-conformity of the distributions to the theoretical chi-square distribution. A technique for determining appropriate confidence intervals at the tonally associated frequencies was suggested.

SYMBOLS AND ABBREVIATIONS

Δt	digitization sample time interval
α	type 1 error, the accepted error rate of rejecting a true null hypothesis
a_k	real components from DFT
\hat{A}_k	one-sided amplitude spectra
\hat{A}_k^2	one-sided power spectra via amplitude spectra
b_k	imaginary components from DFT
$B(\omega)$	frequency dependent bias error
D	Kolmogorov-Smirnov test statistic
df	degrees of freedom
ϵ	uncertainty error
$H(\omega)$	transfer function for filter
ω	frequency, radians/second
p	probability of truth of a relationship
σ^2	parametric variance
$S_x(\omega)$	parametric power spectral density
$\hat{S}_x(\omega)$	estimate of power spectral density
$\bar{S}_x(\omega)$	mean of ensemble of $\hat{S}_x(\omega)$
s_y	sample standard deviation of Y
$t_{\alpha[n-1]}$	critical value of Student's t-distribution
μ	parametric mean
W_s	window correction factor
χ^2	chi-square
X_j	digitized time series data
DFT	discrete Fourier transform

INTRODUCTION

By including confidence interval or variance determinations with any reported experimental measurement data, a researcher more clearly describes his quantitative results. By doing so, the researcher attempts to convey the level of precision and accuracy that reflects the quality of the data. In aeroacoustic spectral data Rao and Preisser¹ have shown that asymptotic techniques for estimating the variance of power spectral density estimates work quite well. For acoustic spectral data, determined by digital techniques, a chi-square (χ^2) distribution based confidence interval formula is sometimes applied to determine confidence intervals for experimental data. The formula theoretically models the variation of the individual or averaged spectral estimates ($\hat{S}_x(\omega)$ or $\bar{S}_x(\omega)$ respectively) about their parametric values $S_x(\omega)$ at the frequency ω . Specifically the confidence intervals are determined as shown by Otnes and Enochson² as

$$\text{Prob} \left[\frac{df \cdot \bar{S}_x(\omega)}{\chi_{df; \alpha/2}^2} < S_x(\omega) \leq \frac{df \cdot \bar{S}_x(\omega)}{\chi_{df; 1 - \alpha/2}^2} \right] = p, \quad (1)$$

where p , the probability of truth of the relation is selected by the researcher, $\alpha = 1 - p$, $\chi_{df; \alpha/2}^2$, and $\chi_{df; 1 - \alpha/2}^2$, represent the abscissas of the χ^2 probability density function with probability areas of $\alpha/2$ and $1 - \alpha/2$ respectively. The number of degrees of freedom, df , is equal to twice the number of individual estimates $\hat{S}_x(\omega)$ which were averaged to obtain the mean $\bar{S}_x(\omega)$. As such, this formula establishes a probabilistic relationship between the averaged spectral estimates and the parametric values which they estimate.

It should be recalled from statistical sample theory³ that experimental measurements are performed to estimate parametric values that are not exactly determined (unless the sample universe is exhaustively sampled). The best that can be achieved is to estimate these parametric values in a probabilistic sense. As an example, confidence intervals serve to probabilistically relate the sample data estimates to the parametric values.

A major assumption of the χ^2 distribution based confidence intervals for spectral estimates is that the time series input data are Gaussian distributed. When this is true, Otnes and Enochson² have shown that the real and imaginary components associated with the Fourier transforms of the time series data are independent Gaussian random variables and as such squaring and adding them creates a χ^2 variable. By example, for the Gaussian distributed real components, a_k , a_k^2 is a χ^2 random variable with 1 associated degree of freedom. When added to the similarly distributed squared imaginary components b_k^2 , this yields a new χ^2 distribution with two associated degrees of freedom. This relationship is known as the additive property of the χ^2 distribution.

The purpose of this work was to assess the appropriateness of using the χ^2 distribution based confidence interval formula for complex helicopter flyover data. This can be achieved by determining statistically if the individual spectral estimates which are averaged to determine $\bar{S}_x(\omega)$ are indeed χ^2 distributed. The experimental acoustic data is from a recent NASA/McDonnell Douglas Helicopter Company (MDHC) farfield acoustics test using the MDHC supplied model 500E helicopter. Since the acoustic data for these analyses was collected and processed using ensemble averaging techniques⁴, the results are pertinent to this experimental design and analysis scheme.

COMPUTATIONAL METHODS

Power Spectra from Amplitude Analysis

Before the fast Fourier transform algorithms for computing the discrete Fourier transform (DFT) were commonly used on digital computers, banks of analog bandpass filters were used to measure the frequency content of acoustics signals. For continuity with this earlier work, power spectra (PS) are now determined from amplitude spectra using digital techniques for most far-field acoustic measurements.

Two major assumptions of this spectral analysis technique are (a) that the time series data are stationary, and (b) that the length of the time series data analyzed contains an integer multiple of periods. The experimental flyover acoustic data is non-stationary. This is largely due to the modulation of the acoustic sources by spherical spreading. The ensemble averaging experimental design and analysis scheme⁴ is used with the intent to make the data more locally stationary. This is achieved by averaging spectra which are determined from sequential short segments of the time series. Because helicopter acoustic time series data have signals of different periods within the data, the assumption of analyzing integer multiples of periods of data is difficult. This assumption is not adhered to and the ramifications (spectral leakage) have been discussed by Burgess.⁵ For calculating PS, continuous time histories $x(t)$ of length $N\Delta t$ are discretized every Δt seconds so that

$$X_j = x(j\Delta t) \text{ for } j = 0, N - 1.$$

The Fourier series resulting from the DFT in relation to X_j can be expressed as

$$X_j = \frac{a_0}{2} + \sum_{k=1}^M \left(a_k \cos \frac{2\pi k j}{N} + b_k \sin \frac{2\pi k j}{N} \right) \quad (2)$$

for $j = 0, N - 1$ and $N = M$, where the Fourier coefficients are defined as

$$a_k = \frac{2}{N} \sum_{j=0}^{N-1} X_j \cos \frac{2\pi k j}{N} \quad (3)$$

and

$$b_k = \frac{2}{N} \sum_{j=0}^{N-1} X_j \sin \frac{2\pi k j}{N} \quad (4)$$

for $k = 0, M$.

The one-sided spectra at the independent Fourier frequencies is

$$\hat{A}_k = 2\sqrt{a_k^2 + b_k^2}, \text{ so that the power spectra via amplitude spectra is equal to } \hat{A}_k^2.$$

These quantities are interpreted as the finite power or the mean square of the acoustic time series pressure data as a function of the specific Fourier frequencies.

When M , the number of Fourier coefficients, is chosen to be less than the number of discrete points in the time series, the transform is approximate. For the DFT, where by definition $N = M$, the relationship between the X_j and the transform is exact.

The frequency domain presentation of the time domain data is exact if the data were filtered to exclude all frequencies above the nyquist frequency.

Power Spectral Density

Another spectral analysis technique used in acoustics and other engineering disciplines is the power spectral density (PSD). Again the assumption of stationarity of the time series data is required by this analysis technique. It may be defined, as shown by Hardin⁶ for a signal $X(t)$ whose total duration is $b\Delta t$ from discrete data $X(n\Delta t)$ for $n = 0, b - 1$ as

$$\hat{S}_x(\omega) = W_S |X_F(\omega)|^2 \quad (5)$$

where

$$X_F(\omega) = \frac{\Delta t}{2\pi} \sum_{n=0}^{b-1} d(n\Delta t) X(n\Delta t) e^{-i\omega n\Delta t} \quad (6)$$

The data window $d(n\Delta t)$ used in this analysis is the rectangular (boxcar) window, resulting in the window correction factor W_S being unity. The appropriate interpretation of the PSD is as the power (or mean square pressure) per unit frequency.

As both spectral estimate techniques (PS and PSD) involve using the DFT, an exact transform, the variability observed from the spectral estimates is largely a function of the variability of the data itself. Figures 1A to 1E present frequency domain data using the PS method from one of the NASA/MDHC 500E flyovers. Because the range of the mean square pressures is so large a series of graphs is used to show the data distributions of interest. The ensemble averaging experimental design technique employed allows for thirty ensembles at each selected directivity angle pair, consequently there are thirty two-degree-of-freedom spectral estimates at each frequency analyzed. As the acoustic analysis procedures require entire spectra to be presented, which involves averaging of the mean square acoustic pressures within frequencies and then converting to the dB scale, an understanding of the intra-frequency variability is not generally presented. The standard acoustic analysis generates a single estimate with sixty associated degrees of freedom. Because distributions of estimates are used in the forthcoming analyses, thirty estimates with two associated degrees of freedom are used at each Fourier frequency. Such a presentation is fundamental to understanding the sample variability and defining the distributions of the frequency dependent mean square acoustic pressures which will direct the appropriate confidence interval analysis.

Ideal Measurement Data - Kendall Series

As the parametric functions of far-field acoustic data are not known during flyover situations and the data's character changes as a function of time, it is necessary to have a more well behaved time series for example cases. A test data set using the Kendall series can serve this purpose⁶. The stationary Kendall series Y_n for N points is determined using the relationship

$$Y_n = a_1 Y_{n-1} + a_2 Y_{n-2} + X_n \quad (7)$$

for $n = 0, N$. Selected initial conditions for Y_{n-1} and Y_{n-2} are set at 0.0 with $a_1 = 1.0$ and $a_2 = -0.5$ yielding a stable second order recursive filter with characteristics of a lowpass and bandpass hybrid filter. The input series X_n is a uniform distribution with mean of 0.0 and variance (σ_x^2) of $0.\bar{3}$. To associate the output series with time, each successive Y_n was assigned a Δt of 1 second. Hardin⁶ showed the PSD for such a series could be analytically defined as

$$S_y(\omega) = \frac{A}{2\pi} \left(\frac{1}{1 - S_1 e^{-i\omega}} + \frac{1}{1 - S_1 e^{i\omega}} - 1 \right) + \frac{B}{2\pi} \left(\frac{1}{1 - S_2 e^{-i\omega}} + \frac{1}{1 - S_2 e^{i\omega}} - 1 \right) \quad (8)$$

where

$$A = \frac{\sigma_x^2 S_1}{(S_1 - S_2)(1 - S_1 S_2)(1 - S_1^2)} \quad (9)$$

$$B = \frac{-\sigma_x^2 S_2}{(S_1 - S_2)(1 - S_1 S_2)(1 - S_2^2)} \quad (10)$$

and

$$S_{1,2} = a_1 \pm \frac{\sqrt{a_1^2 + 4 a_2^2}}{2}. \quad (11)$$

The theoretical $S_y(\omega)$ can be considered to be the parametric function that all estimates of the series approximate.

Kolmogorov-Smirnov One Sample Test

The Kolmogorov-Smirnov test statistic⁷ D can be used to perform a significance test which will facilitate accepting or rejecting the hypothesis of equality of a theoretical distribution and a measured sample distribution. Sokal and Rohlf⁷ have demonstrated that this nonparametric goodness of fit test works well for small sample sizes. Specifically, D is determined as the largest of the values d_i^+ or d_i^- where $d_i^+ = |F_i - \hat{F}_i|$ and $d_i^- = |F_{i-1} - \hat{F}_i|$. \hat{F}_i is the i -th standardized cumulative expected frequency of the theoretical distribution, and F_i is the i -th standardized cumulative observed frequency from the measured sample distribution. The standardization used for both distributions is to divide each element of the distribution by the distribution's mean and then to multiply the elements by the expected degree of freedom. The statistical significance of the D statistic is determined using tables of critical values of the Kolmogorov-Smirnov one sample test⁸. Alpha levels of 0.05 were used for all significance tests, therefore the accepted error rate of rejecting a true null hypothesis is five percent.

APPLICATIONS USING KENDALL SERIES TEST DATA

Some of the computational methods just described were used to analyze the Kendall series test data. A particular advantage of the Kendall series is that long time series of random test data can be generated. This allows for spectral estimates of high numbers of degrees of freedom to be determined. Figure 2-E displays the PSD of a

single estimate having 200,000 degrees of freedom in relation to its parametric function, the Kendall series theoretical PSD. This estimate is observed to closely follow its parametric function. One hundred thousand time series blocks of the Kendall series were analyzed to produce this estimate.

The series of graphs in Figure 2 serve to illustrate the two types of errors that occur in spectral estimates. These include uncertainty error and bias error.

Uncertainty error, which is characterized by the scatter of the estimates about their expected values, is observed to vary as a function of the number of degrees of freedom associated with the estimate. The uncertainty error decreases as the number of degrees of freedom increases and is random.

The two degree of freedom case (Fig. 2A), a single unaveraged estimate, is observed to not closely resemble the parametric spectra it estimates. Generally, as the number of degrees of freedom increases, the correlation between the estimate and the parametric values increases (Fig. 2B to 2E).

Bias error, which is systematic, can be seen most clearly in Figure 2E. For this 200000 degree of freedom estimate virtually all uncertainty error has been eliminated through the spectral averaging process. The difference between the estimate and the parametric function is the systematic bias error introduced into the estimate by the PSD analysis. Hardin⁶ has shown bias error to be proportional to the second derivative of the PSD parametric function. All of the estimates of Figure 2 (A to E) have a similar bias error but because of its magnitude relative to the uncertainty error it is only obvious for the 200000 degree of freedom case.

The bias error is proportionally largest at maxima and minima of the second derivative of this function. The bias error's behavior can be most easily explained as mimicking the output of a lowpass smoothing filter operating on the parametric PSD function. For the estimates, the peaks are not as high and the troughs are not as low as they would be if the spectral estimation techniques were unbiased estimators.

Within a single spectrum there may exist positive, negative, and zero bias depending on the shape of the parametric function.

Confidence Intervals

For the confidence intervals about spectral estimates to be appropriate, they should account for the errors associated with the estimate. Consequently, the confidence interval formula should include a function that models the distribution of spectral estimate uncertainty and bias. To test the hypothesis that this confidence interval function should be based on the χ^2 family of distributions for spectral estimate variability, 10000 estimates each from a range of numbers of degrees of freedom were generated. Figures 3A to 3E display these results. For all of these estimates, the parametric function is the same theoretical PSD as displayed in the Figure 2 series. These results show how the estimate variability changes as a function of the number of degrees of freedom associated with each of the 10000 estimates which make up these data. Figure 4A displays the distribution of estimates from a similar analysis as above but only presenting the PSD at a single frequency (0.113 Hz). The shapes of the distributions are observed to qualitatively agree with the standard χ^2 distribution. Generally this involves the transition from an L-shaped asymmetric distribution to a symmetric normal distribution when the number of degrees of freedom range from low to high. The cumulative probability distributions of these data were compared to theoretical χ^2 distributions with the appropriate number of degrees of freedom (Fig. 4B to 4F). There are two cumulative probability functions on each of the graphs (Fig. 4B to 4F) but because they are so similar they overlap each other and cannot be separately distinguished. The Kolmogorov-Smirnov test statistics (Table 1) were determined to test the hypothesis of equality of the associated cumulative frequency distributions. The significance of the test statistics was determined by comparing them to the critical value of the Kolmogorov-Smirnov one-sample statistic with the appropriate sample

size and alpha level. The insignificant ($\alpha = 0.05$) D statistics indicate that there is insufficient evidence to reject the hypothesis of equality, so the distributions of the Kendall series data and the theoretical χ^2 data are equal.

Interpretation of Confidence Intervals

As the PSD spectral estimates for the Kendall Series were shown to be χ^2 distributed then equation (1) for confidence intervals seems initially appropriate for these data. To illustrate this, eighty percent confidence intervals were selected. The correct interpretation of a confidence interval from equation 1 when $p = 0.8$ is that the probability is 0.8 that the parametric value will lie within this confidence interval. An alternate interpretation is that if a large number of estimates and their associated confidence intervals are determined, then 80 percent of such intervals would bound the parametric value.

To verify this interpretation four-hundred such estimates and intervals were calculated from the Kendall Series data at two different degrees of freedom (2 and 1000) and at two different frequencies of the PSD spectra (0.016 and 0.113 Hz). These frequencies were selected in relation to the amount of bias error associated with these frequencies. Specifically, as shown in Figure 2-E, 0.016 Hz is representative of a frequency with very little bias error and 0.113 Hz has the highest bias error for the Kendall series used.

Figure 5 A-D shows graphically the four-hundred individual confidence intervals for the estimates in relation to the parametric value μ which the estimates approximate. In these cases μ are the values of the theoretical PSD for the Kendall series evaluated at either 0.016 or 0.113 Hz. From the data presented in Figures 5A to 5D, the percentage of the 400 trials in which the intervals bounded μ is presented in Table 2.

For the low bias associated frequency both degree of freedom cases exhibited close to 80% of the intervals bounding μ . For the high bias associated frequency, the

2 degree of freedom case resulted in 80.5% of the 400 confidence intervals bounding μ , with only 63.5% for the 1000 degree of freedom case.

APPLICATIONS USING ACOUSTIC FLYOVER DATA

To determine the appropriateness of using χ^2 distribution based confidence intervals for experimental acoustic flyover data, a similar analysis was done. For the acoustic data the power spectra from amplitude spectra were used and the sample size was much smaller (30 versus 10000 as for the Kendall Series).

The ensemble averaged ($n = 30$) power spectrum of the example 500E data (Figure 6) is shown to be much more complex than the unimodal Kendall Series data previously examined. Because of the presence of "spikes" in the acoustic power spectrum, the difference in estimated power distributions relative to frequency was investigated. The "spike" characteristics observed in the graph can be associated with the helicopter's main rotor and tail rotor blade passage frequencies and their harmonics.

The term tonal will be used to denote these characteristic "spikes" and non-tonal will be associated with the frequency dependent power measurements which are not pronouncedly higher or lower than their nearby surrounding estimates.

Results of the frequency dependent Kolmogorov-Smirnov one sample tests are displayed by symbols in Figure 6 and in Table 3. The statistical tests were performed to determine if the distribution of the estimated power within the associated frequencies followed a χ^2 distribution of two degrees of freedom. The first four tones within the spectrum had significant ($\alpha = 0.05$) Kolmogorov-Smirnov statistics indicating the distribution of estimates associated with these frequencies are not from the χ^2 family of distributions.

The spectral analysis for these data had a Nyquist frequency of 12500 Hertz but only the estimated frequencies up to 500 Hertz are presented. For this spectrum there

were 45 (out of 1019) other frequencies which yielded significant ($\alpha = 0.05$) Kolmogorov-Smirnov test statistics. These non- χ^2 distributions were associated with non-tonal frequencies and were always at least 15 dB lower in power than the low frequency tones observed. They appeared to be randomly spaced within the non-tonally associated frequencies of the data.

The cumulative probability distribution functions for selected frequencies (Figure 7(A-I)) from these data graphically show the distribution differences relative to the theoretical two degree of freedom χ^2 distribution. Although the functions for the non-tonal frequencies (Fig. 7(A,C,D)) do not vary with the theoretical χ^2 cumulative probability functions as closely as did the Kendall series distributions, the results from the Komorogov-Smirnov tests indicate the observed variation is within acceptable limits for a χ^2 distributed sample from such a small sample size (30).

The cumulative probability distribution functions for the tonally associated frequencies (Figure 7(B,E,F,G,H,I)) present a range of covariation relative to the theoretical χ^2 distribution. Tone power is observed to monotonically decrease as frequency increases. The first four tones of the spectrum being non- χ^2 distributed with the following higher frequency tones following the χ^2 distribution.

DISCUSSION

Kendall Series

For the Kendall series the two measurement errors associated with the spectral estimates may be modeled with respect to an individual estimate $\hat{S}_{x_{df}}$ with an associated number of degrees of freedom df as

$$\hat{S}_{x_{df}}(\omega) = B_{\omega} S_x(\omega) + \epsilon_{df} \quad (12)$$

where $S_x(\omega)$ is the theoretical PSD, B_ω is the frequency dependent bias error and ϵ_{df} represents the random uncertainty error which is χ_{df}^2 distributed. The bias error varies proportionally to the second-derivative of the theoretical PSD and is therefore frequency dependent. From the results of the analyses of the Kendall series data it is observed that the PSD estimate distributions follow χ^2 distributions.

The theoretical PSD is unimodal with a maximum at approximately 0.11 hertz. The theoretical PSD, not being flat or a ramp function, has a nonzero and changing second derivative so a spectral estimate by either of the techniques will produce a biased estimate. When the PSD has a higher frequency dependent second derivative the bias error will be proportionally higher at these frequencies.

For equation 1 confidence intervals to be appropriate it should account for both bias and uncertainty errors within the estimate model. Since only the uncertainty component is modeled it may be expected that the relationship between the parametric value (theoretical PSD(ω)) and its estimates do not strictly follow equation 1. The confidence interval data from Figure 5 and Table 2 support this expectation. When the bias error is low, both sets of 400 trials at the two different degrees of freedom resulted in close to 80% of the confidence intervals bounding the parametric values. For the high bias associated frequency with 2 associated degrees of freedom, again close to 80% of the 400 trial confidence intervals bounded μ . The relative width of the confidence intervals for the 2 degree of freedom estimates compared to the systematic bias error in the estimates causes this result. When the confidence intervals are very wide (low associated df) the percent of the confidence interval trials bounding μ follows equation 1. If this is true (as in the 2 degree of freedom case), the systematic bias error is small in relation to the range in estimates. For the 1000 associated degrees of freedom case, where the range of estimates is much smaller, the bias error is large relative to the range of estimates and consequently much less than 80% of the confidence intervals bound μ .

For equation 1 to be appropriate it also must account for the bias error in the estimates for experimental data. Because the bias error for experimental data is not known (μ not known), the inclusion of this term in the confidence interval equation is impossible. Equation 1 can therefore be appropriate when there is no bias error. This is achieved when the data is white noise so that the PSD is flat or linear in ω .

It has been shown that the technique of prewhitening and postdarkening can be used to convert a biased estimate into an unbiased estimate.^{6,9,10} The technique involves designing a digital filter that will reduce $\hat{S}_x(\omega)$ to a spectra that is flat or at least linear as a function of frequency. The transfer function for the digital filter, $H(\omega)$, is then related to the original estimated $S_x(\omega)$ and the new flattened estimate $\hat{S}_y(\omega)$ by

$$\hat{S}_y(\omega) = |H(\omega)|^2 \hat{S}_x(\omega) \quad (13)$$

By reversing the process the unbiased estimate of the parametric spectra may be obtained by

$$\hat{S}_x(\omega) = \frac{\hat{S}_y(\omega)}{|H(\omega)|^2}. \quad (14)$$

Hardin⁶ further suggests that the selection of the digital filter is a difficult process and would probably be best implemented through an iterative approach. It is expected that such unbiased estimates would still conform to the χ^2 distribution in order to account for their uncertainty if the time series data are Gaussian distributed.

Acoustic Data

The compound problems of bias and the omission of bias error in the confidence interval equation are increased for the experimental acoustic data. The

numerous tones at various frequencies of the spectrum cause this situation. Consequently, the complexity of the process to define a digital filter to flatten the spectra is much increased.

The distributions of spectral estimates for the acoustic data can be divided into two groups, the tonally and non-tonally associated estimates. The distribution of spectral estimates at the non-tonally associated frequencies, which, by our definition, are roughly flat or at least linear in (ω) , follow the χ^2 distribution.

Ninety-five percent of the distributions associated with estimates at these nontonal frequencies were statistically χ^2 distributed. Because of the linearity of the second derivative of the power spectra at these frequencies, they may be considered unbiased, making B_{ω} (for equation 12) for these ω 's very close to 1.0. Consequently, for a large percentage of the power spectrum of the examined acoustic data, the application of equation 1 for confidence intervals is appropriate.

The distribution of spectral estimates at the tonally associated frequencies of the acoustic data were shown not to follow the χ^2 family of distributions for the frequencies of the spectrum with the most power. Efforts to define a single family of distributions which modeled the distribution of spectral estimates at these tonal frequencies failed. The tones, by our definition, include areas of the spectrum where the second derivative would be relatively high or low, so the amount of bias at these frequencies would be proportionately high or low. As such, the confidence interval calculations as in equation 1 would seem inappropriate to account for the variation in these tonally associated spectral estimates.

Confidence Interval Determinations at Tonally Associated Frequencies

Appropriate confidence intervals for the tonally associated frequencies may be determined using other techniques. If the bias and uncertainty errors of the estimates are accounted for by the technique, it should perform well. One such technique could

involve using the prewhitening and postdarkening technique as suggested by Hardin⁶ to remove the bias error from the estimates. Next, standard descriptive statistical techniques to quantitatively describe the sample distribution such as skewness and kurtosis could be computed. If one opted to use Gaussian distribution based confidence intervals, statistical significance tests for the skewness and kurtosis statistics could be performed to determine if the data were Gaussian distributed. If the distribution was not Gaussian, transformations which reshape the distribution could be performed. Each attempted transformation could be tested to determine if it successfully produced a normal distribution by again determining the significance of the skewness and kurtosis statistics relative to their expected values for a Gaussian distribution. Once a statistically significant Gaussian distribution had been obtained, an equation for the confidence interval bounding the mean of the transformed data such as

$$\text{Prob}\left[\bar{Y} - t_{\alpha[n-1]} s_{\bar{Y}} \leq \mu \leq \bar{Y} + t_{\alpha[n-1]} s_{\bar{Y}}\right] = p \quad (15)$$

could be used. Here \bar{Y} would be the mean of the transformed distribution, α is equal to $1 - p$, n is the number of samples in the distribution, $t_{\alpha[n-1]}$ is the α -th percentile critical value of Student's t-distribution with $n-1$ degrees of freedom, $s_{\bar{Y}}$ is the sample standard deviation and μ is the unknown parametric mean. Once the confidence interval of the transformed data had been determined it should be appropriately inverse transformed and then converted to the dB scale.

This technique, while being theoretically sound, would demand considerable computational expense, especially in the determination of the appropriate digital filter to flatten the observed spectra for the prewhitening phase.

General

The analysis of the experimental acoustic data used the individual frequency dependent spectral estimates to draw conclusions. This approach is not usually used when determining confidence intervals for an estimate. The average of the estimates with its appropriate associated degrees of freedom as dictated by the additive property of the χ^2 distribution is generally used. The conformity of the distributions of spectral estimates to the theoretical χ^2 distributions which resulted with the Kendall series data (Fig. (3A to 3E) and 4A) illustrates the soundness of this practice.

The inappropriateness of the χ^2 distribution based confidence intervals may have been predicted based on the assumptions of the analysis techniques. As Otnes and Enochson² have shown, if the time series data are random Gaussian distributed variables, the real and imaginary coefficients associated with the DFT are independent random variables. The periodic nature of helicopter acoustic data associated with the main rotor and tail rotor blade passage frequencies make these data not random. Consequently, one of the assumptions of the analysis of the data is not valid.

CONCLUSIONS

Chi-square distribution based confidence intervals may be appropriately applied to non-tonally associated spectral estimate distributions as determined by the PSD or PS spectral methods. This seems logical since, by definition, the bias error is near zero at the non-tonal frequencies and the estimate distributions are observed to follow the χ^2 distribution. Consequently, χ^2 distribution based confidence intervals account for the major errors associated with the estimates.

Because of the bias error and non- χ^2 distribution conformity of spectral estimate distributions at tonally associated frequencies, the application of χ^2 distribution based confidence intervals is inappropriate at these frequencies. An approach to determine

confidence intervals at the tonally associated frequencies involving (a) prewhitening and postdarkening to eliminate bias, (b) using descriptive statistical techniques as an aid in determining the appropriate transform to force the sample distribution to conform to a normal distribution, and then (c) using Gaussian techniques is suggested.

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TABLE 1

Kolmogorov-Smirnov one sample test statistics (D) for Kendall Series data with varying associated degrees of freedom. Sample size for all distributions, 10000.

Degrees of Freedom	D	Exact Probability	Significance
2	0.006458	0.798	NS
4	0.0081510	0.519	NS
10	0.0047907	0.975	NS
20	0.0072172	0.674	NS
100	0.0067225	0.756	NS

Critical $D_{(0.05,10000)} = 0.00886$

NS = not significant

Table 2 - Percentages of Confidence Intervals Bounding the Parametric Values

Frequency Hz	Relative Bias	<u>Degrees of Freedom</u>	
		2	1000
0.13	Low	78.5	79.5
0.16	High	80.5	63.5

TABLE 3 - Kolmogorov-Smirnov one-sample test statistics (D) for 500E power spectra data at different frequencies. Sample size for all distributions, 30.

Frequency (Hz)	D	Exact Probability	Significance
12.2	0.1498	0.510	NS
24.4	0.1115	0.849	NS
36.6	0.4267	0.000	S
48.8	0.1629	0.403	NS
61.0	0.1071	0.881	NS
73.2	0.4046	0.000	S
85.4	0.1507	0.502	NS
97.7	0.0850	0.981	NS
109.9	0.2796	0.018	S
122.1	0.1437	0.564	NS
134.3	0.2734	0.022	S
146.5	0.2836	0.016	S
158.7	0.0930	0.957	NS
170.9	0.1459	0.545	NS
183.1	0.1626	0.405	NS
195.3	0.0955	0.947	NS
207.5	0.1179	0.798	NS
219.7	0.2242	0.097	NS
231.9	0.0959	0.945	NS
244.1	0.1344	0.649	NS
256.3	0.1508	0.501	NS
268.6	0.1112	0.851	NS

Table 3 - Continued

Frequency (Hz)	D	Exact Probability	Significance
280.8	0.1312	0.679	NS
293.0	0.1705	0.347	NS
305.2	0.0886	0.972	NS
317.4	0.1567	0.452	NS
329.6	0.1423	0.577	NS
341.8	0.1030	0.907	NS
354.0	0.1239	0.746	NS
366.2	0.0907	0.965	NS
378.4	0.1348	0.646	NS
390.6	0.0894	0.969	NS
402.8	0.1059	0.889	NS
415.0	0.1251	0.735	NS
427.2	0.1049	0.895	NS
439.5	0.2019	0.173	NS
451.7	0.2359	0.070	NS
463.9	0.1215	0.767	NS
476.1	0.1568	0.451	NS
488.3	0.1441	0.560	NS

Critical $D_{(0.05,30)} = 0.24170$

NS = not significant

S = significant

Figure 1-A Frequency domain presentation of example 500E data showing distributions of interest.

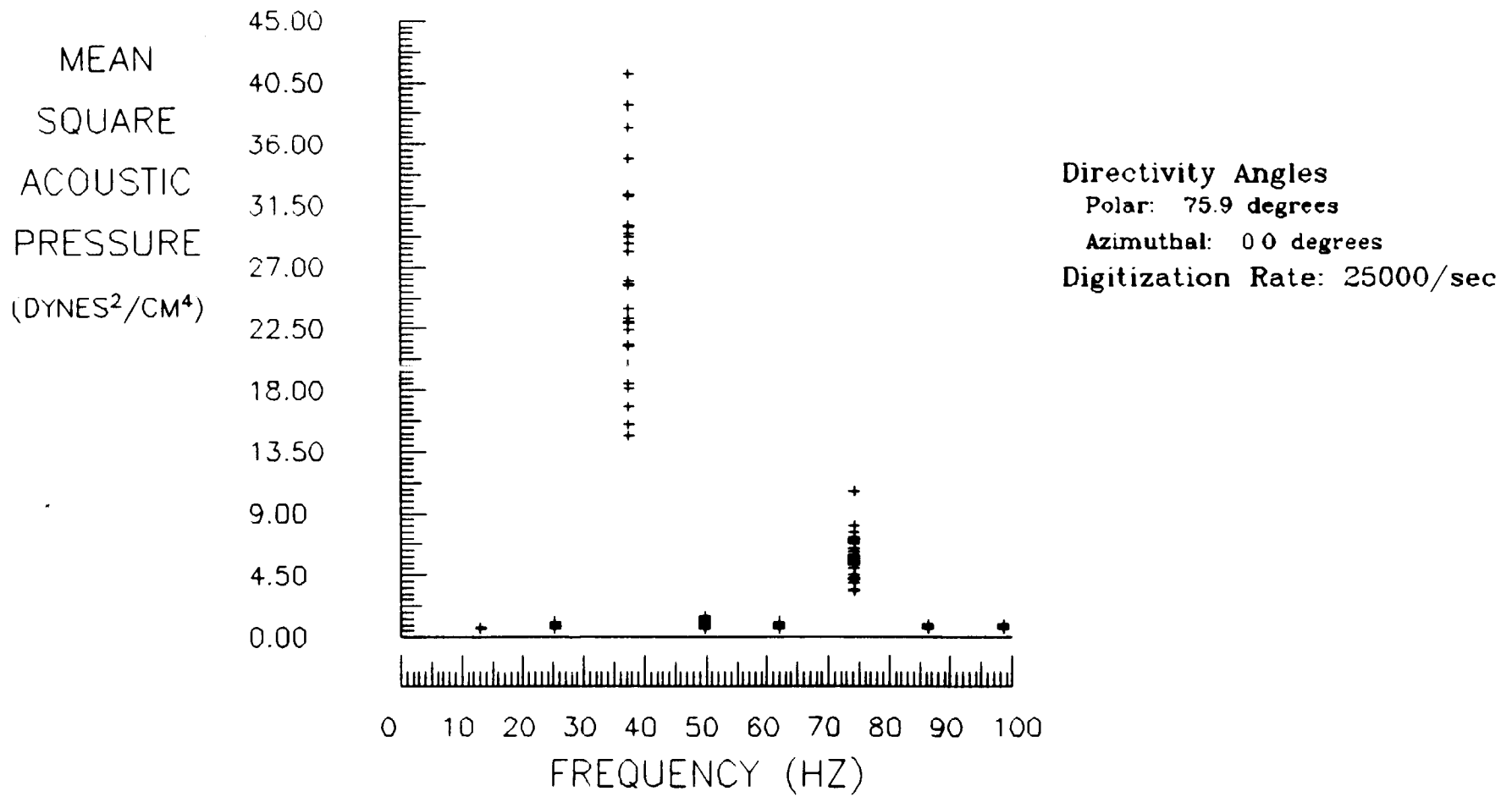


Figure 1-B Frequency domain presentation of example 500E data showing distributions of interest.

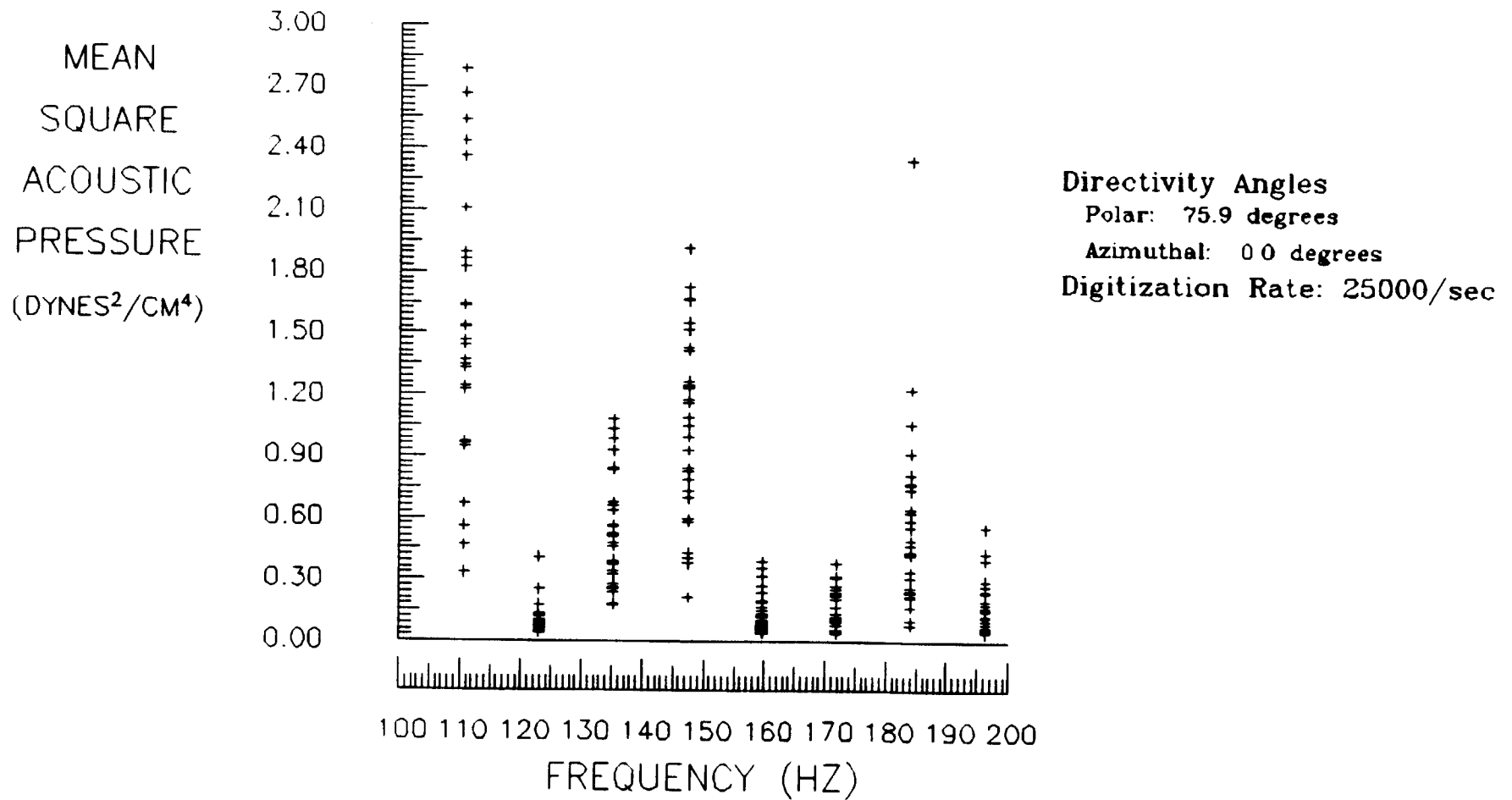


Figure 1-C Frequency domain presentation of example 500E data showing distributions of interest.

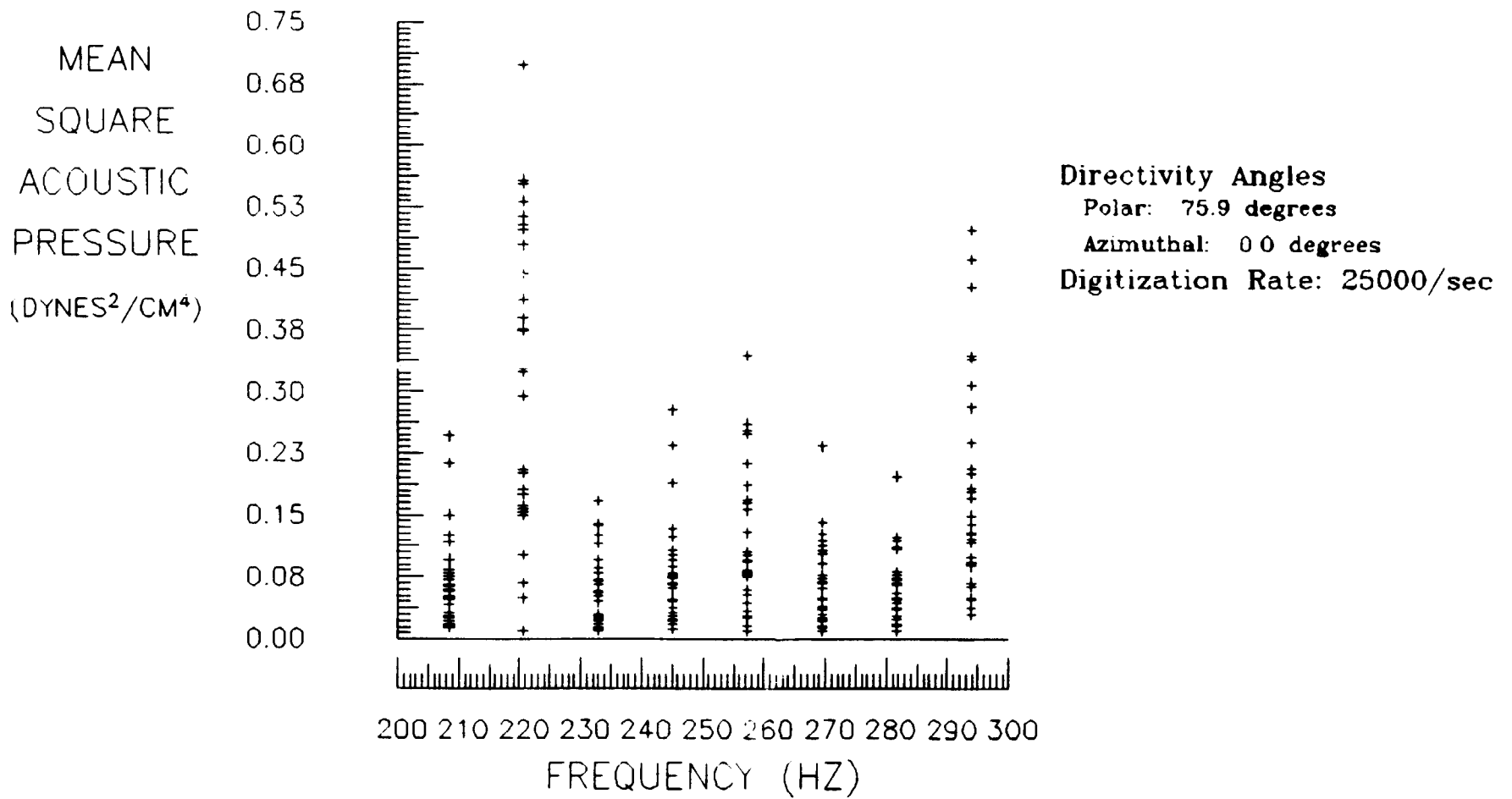


Figure 1-D Frequency domain presentation of example 500E data showing distributions of interest.

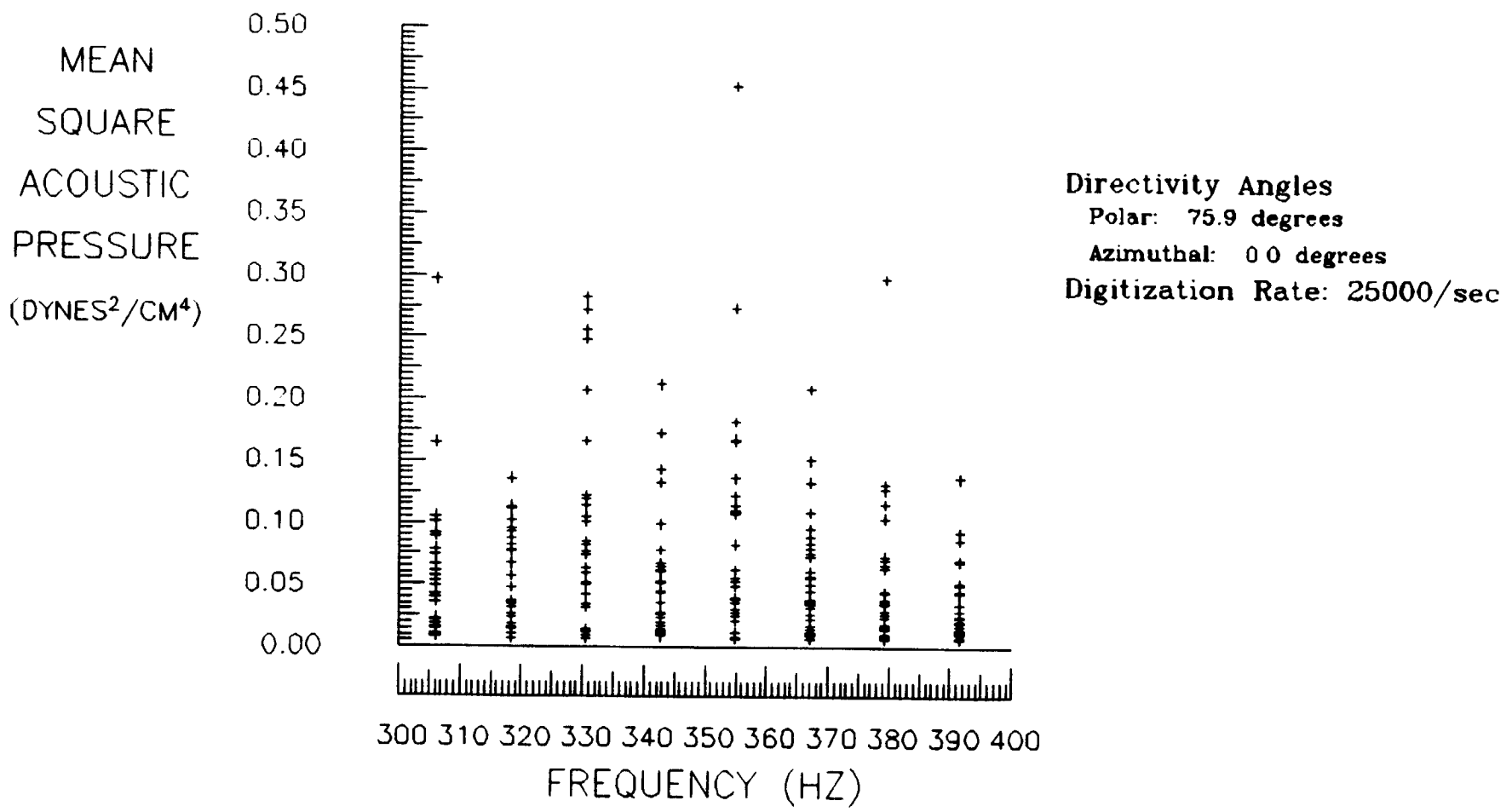


Figure 1-E Frequency domain presentation of example 500E data showing distributions of interest.

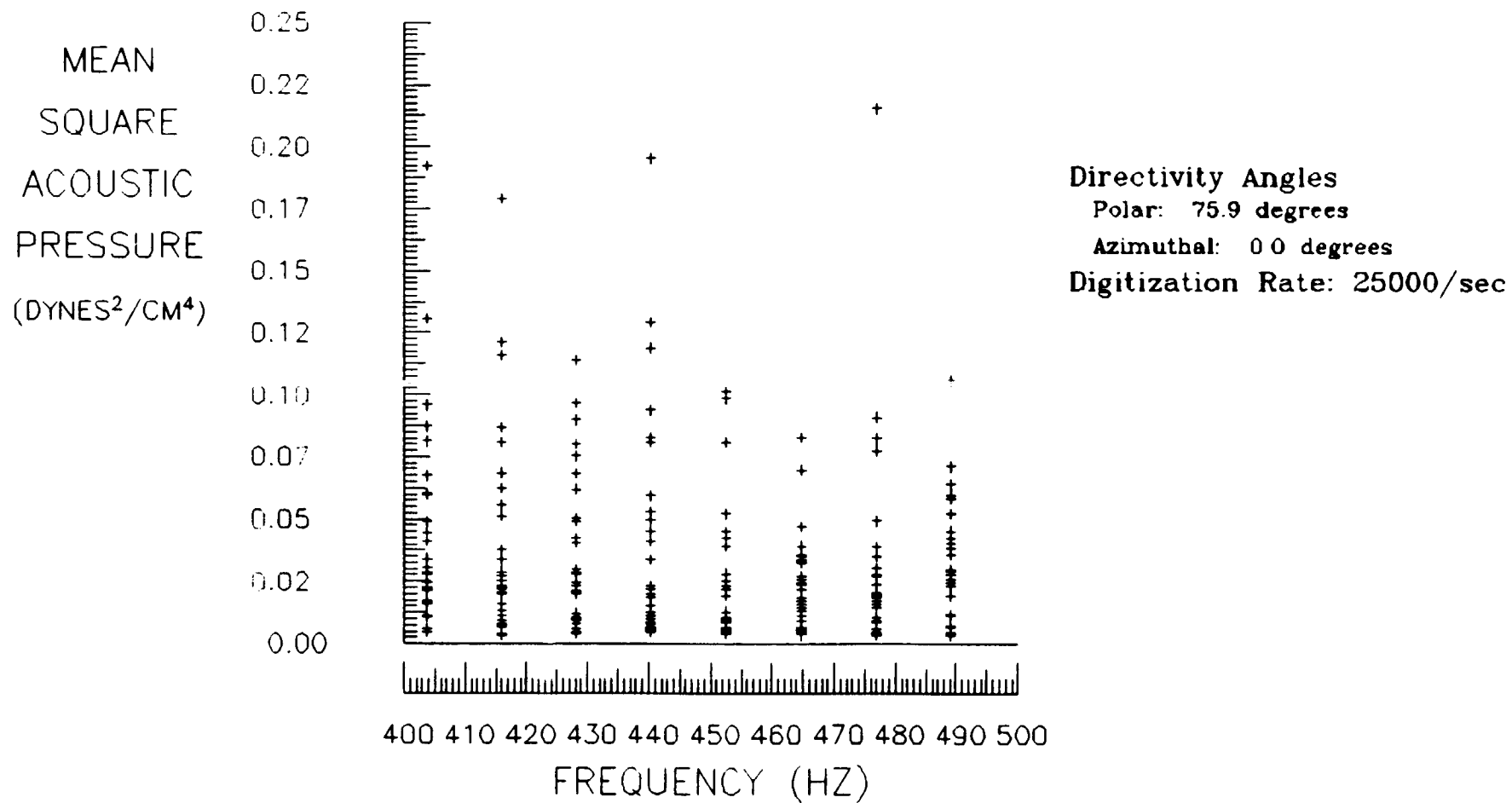


Figure 2-A Theoretical and simulated Kendall series power spectral density.
2 degrees of freedom estimate for simulated data.

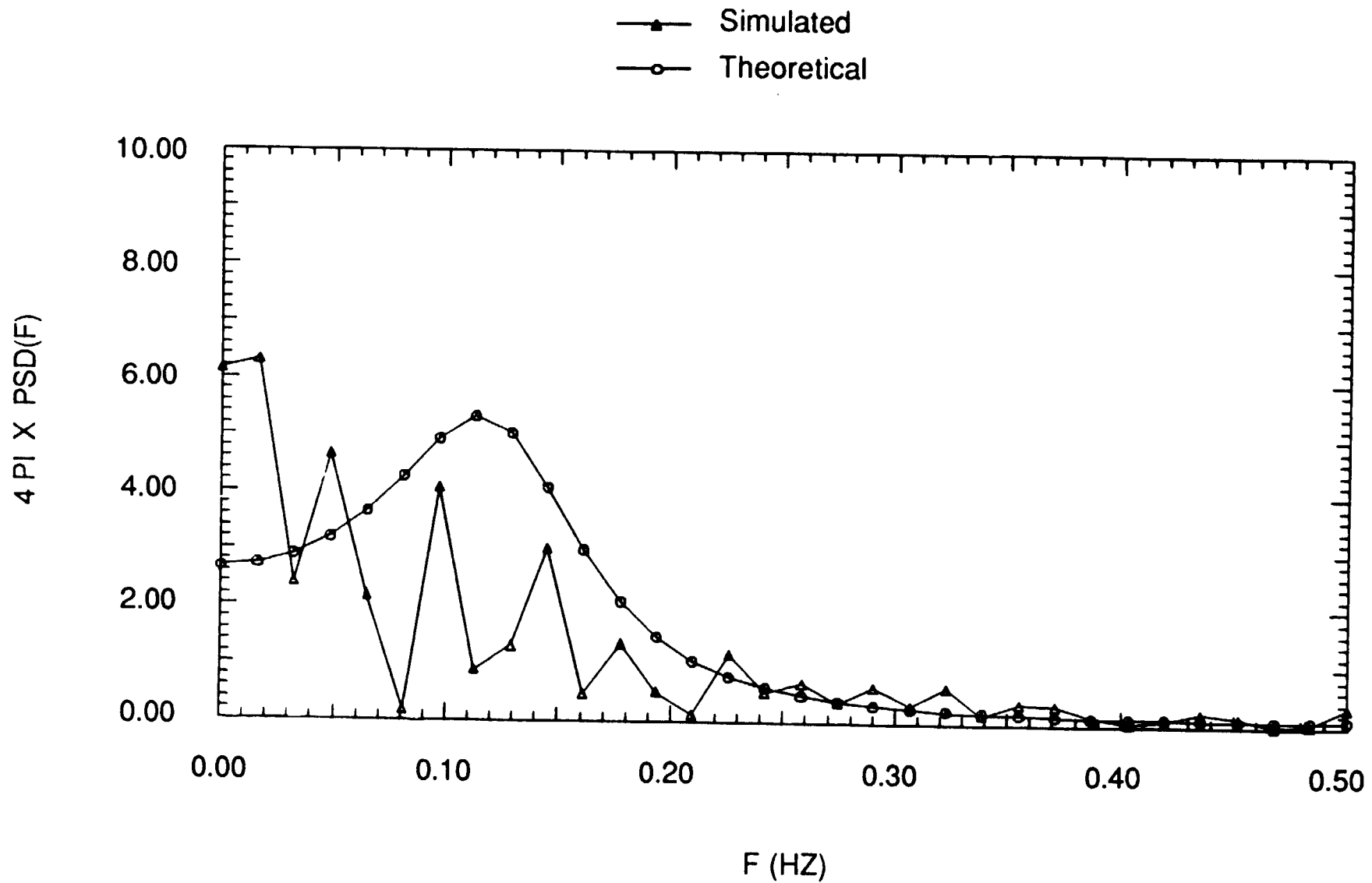


Figure 2-B Theoretical and simulated Kendall series power spectral density.
10 degrees of freedom estimate for simulated data.

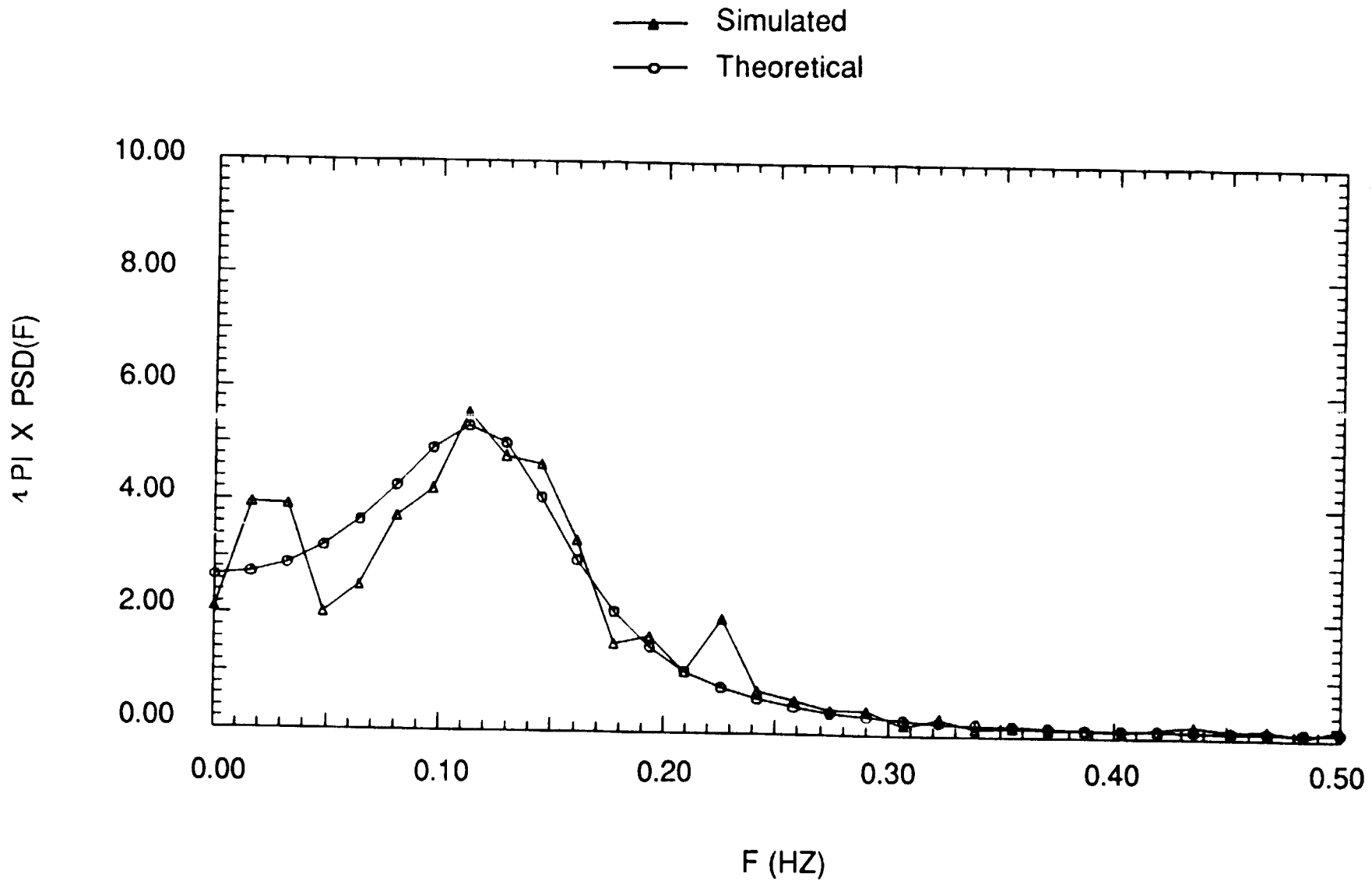


Figure 2-C Theoretical and simulated Kendall series power spectral density.
100 degrees of freedom estimate for simulated data.

—▲— Simulated
—○— Theoretical

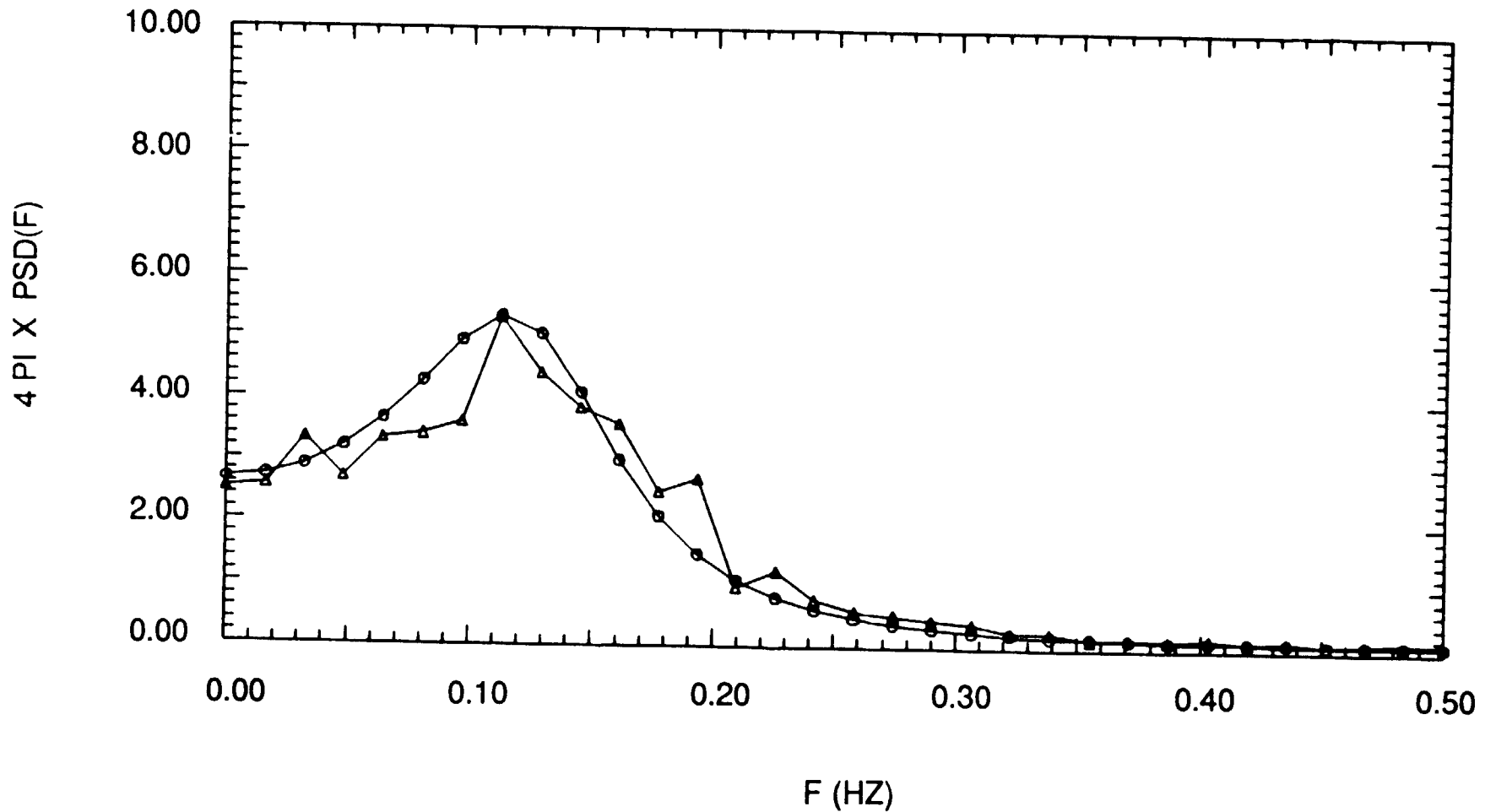


Figure 2-D Theoretical and simulated Kendall series power spectral density.
1000 degrees of freedom estimate for simulated data.

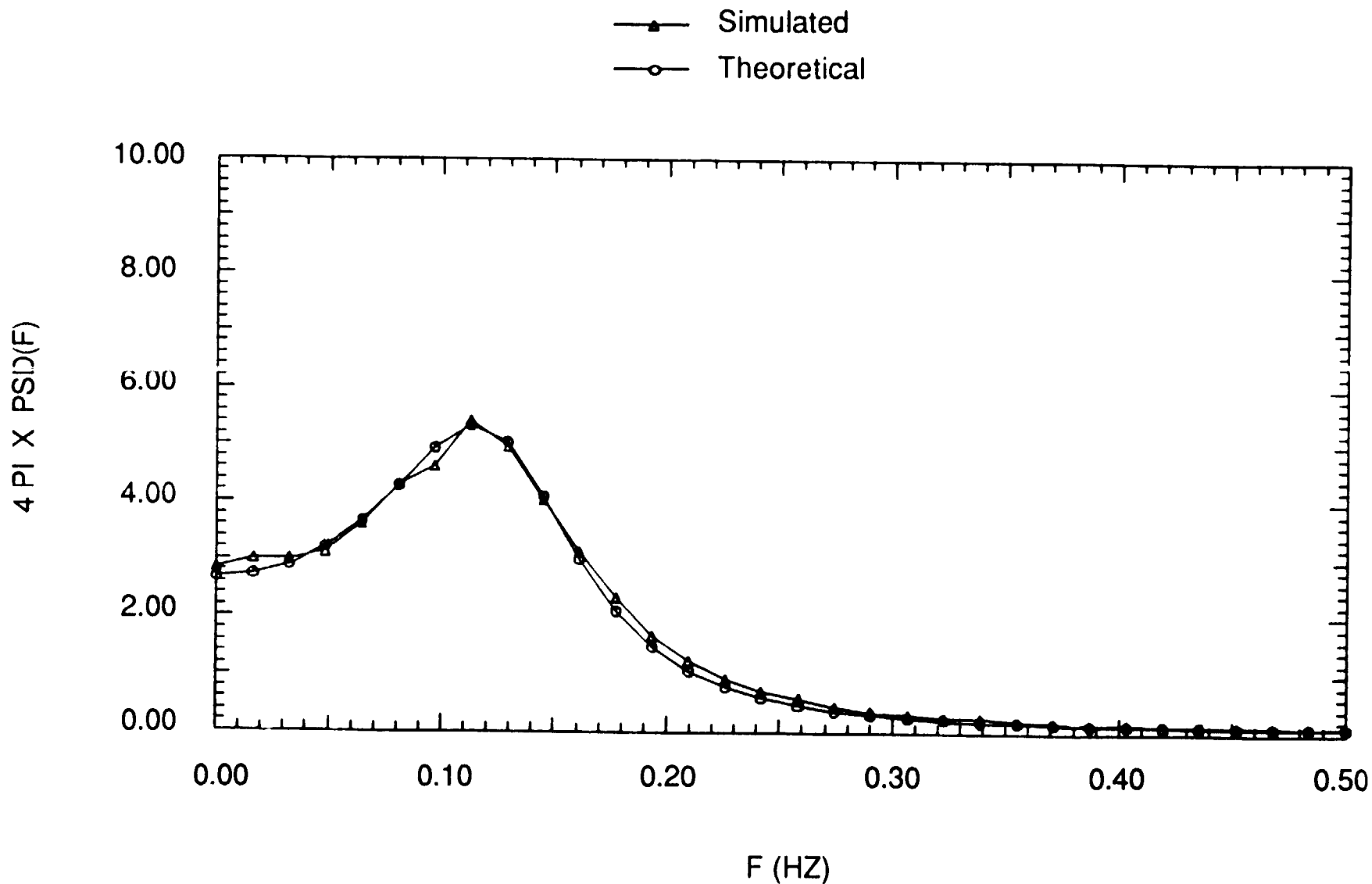
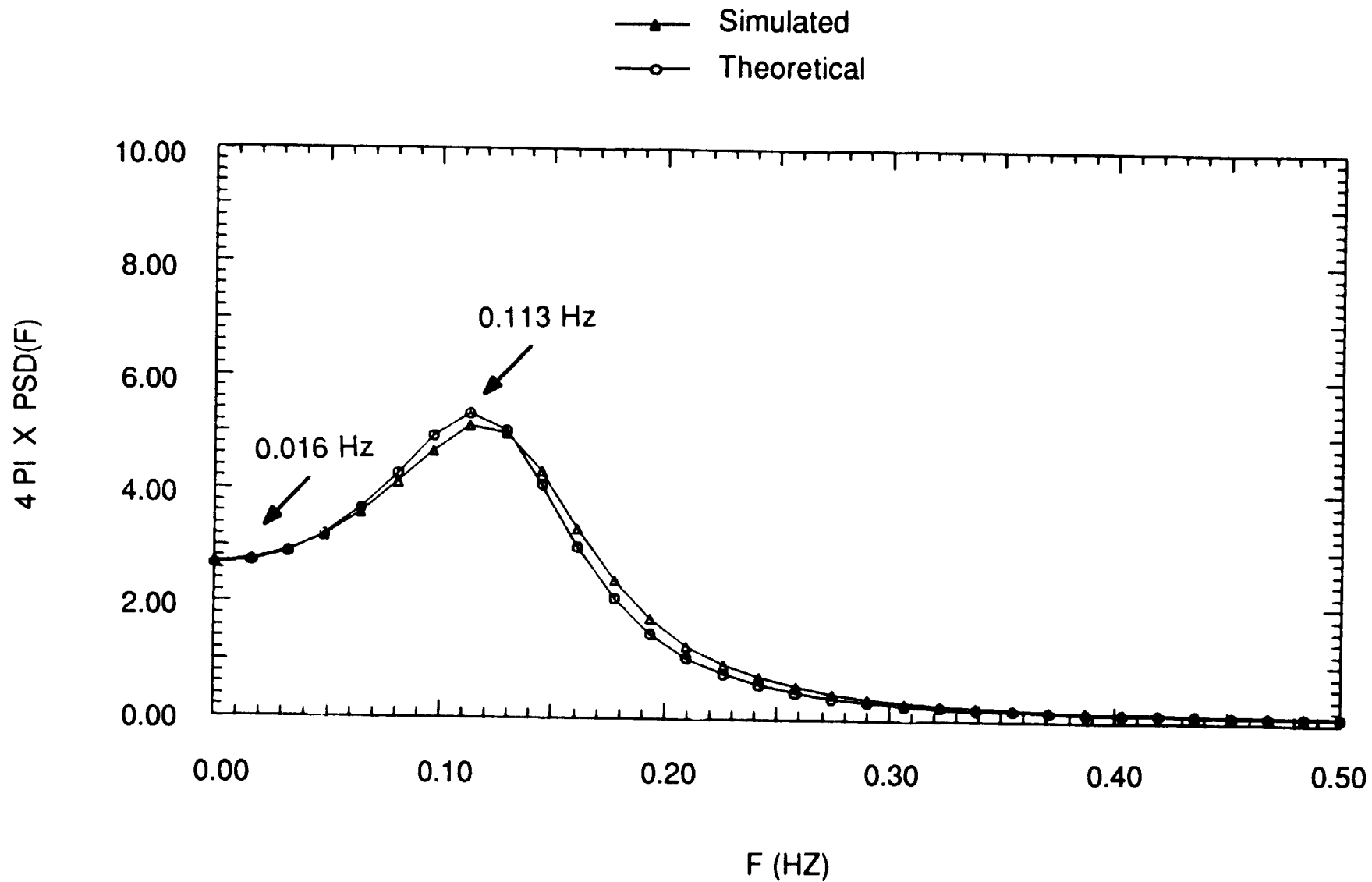
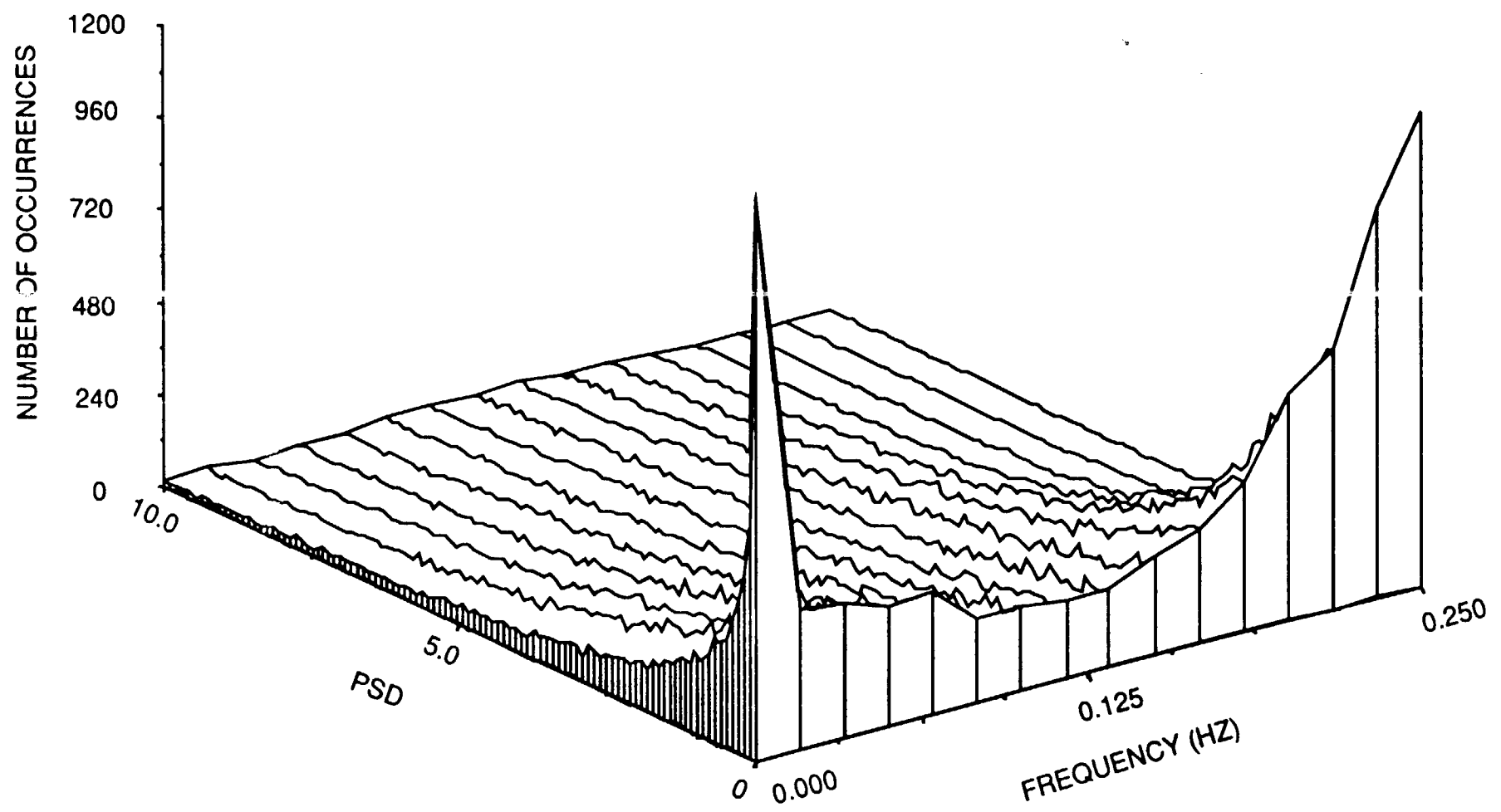


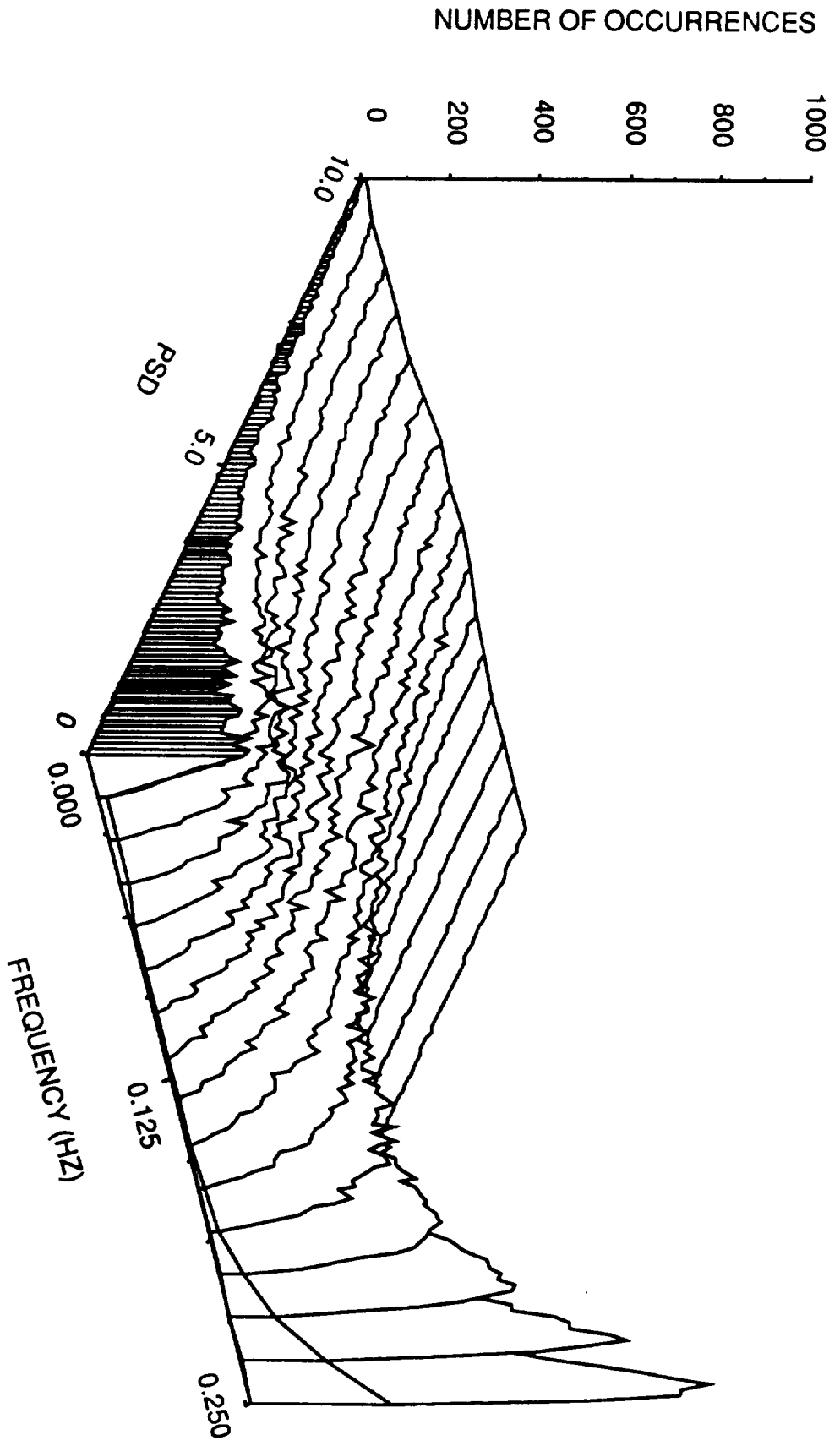
Figure 2-E Theoretical and simulated Kendall series power spectral density.
20000 degrees of freedom estimate for simulated data.



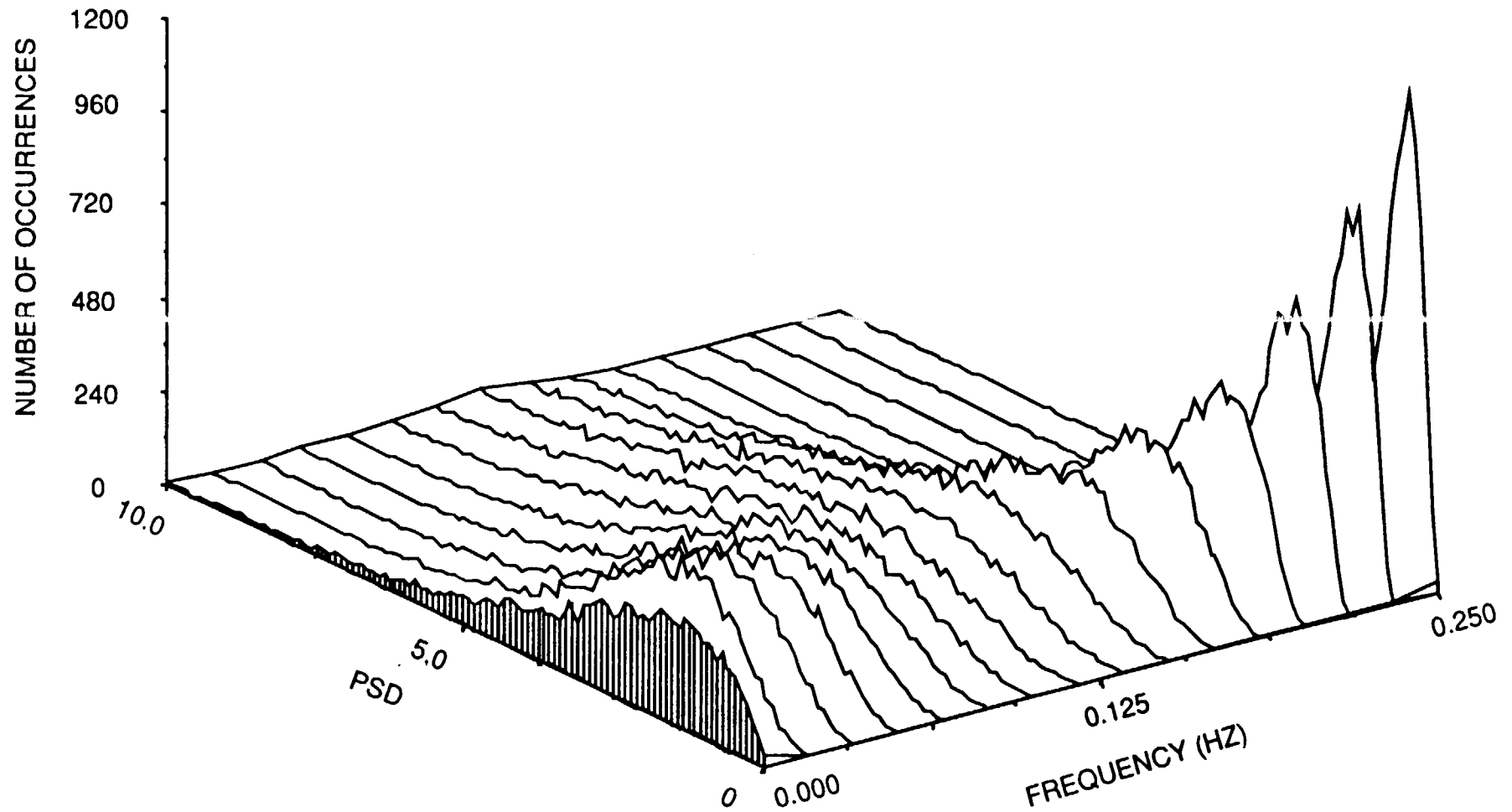
**FIGURE 3-A DISTRIBUTION OF KENDALL SERIES POWER SPECTRAL DENSITY
10000 REALIZATIONS WITH 2 DEGREES OF FREEDOM EACH**



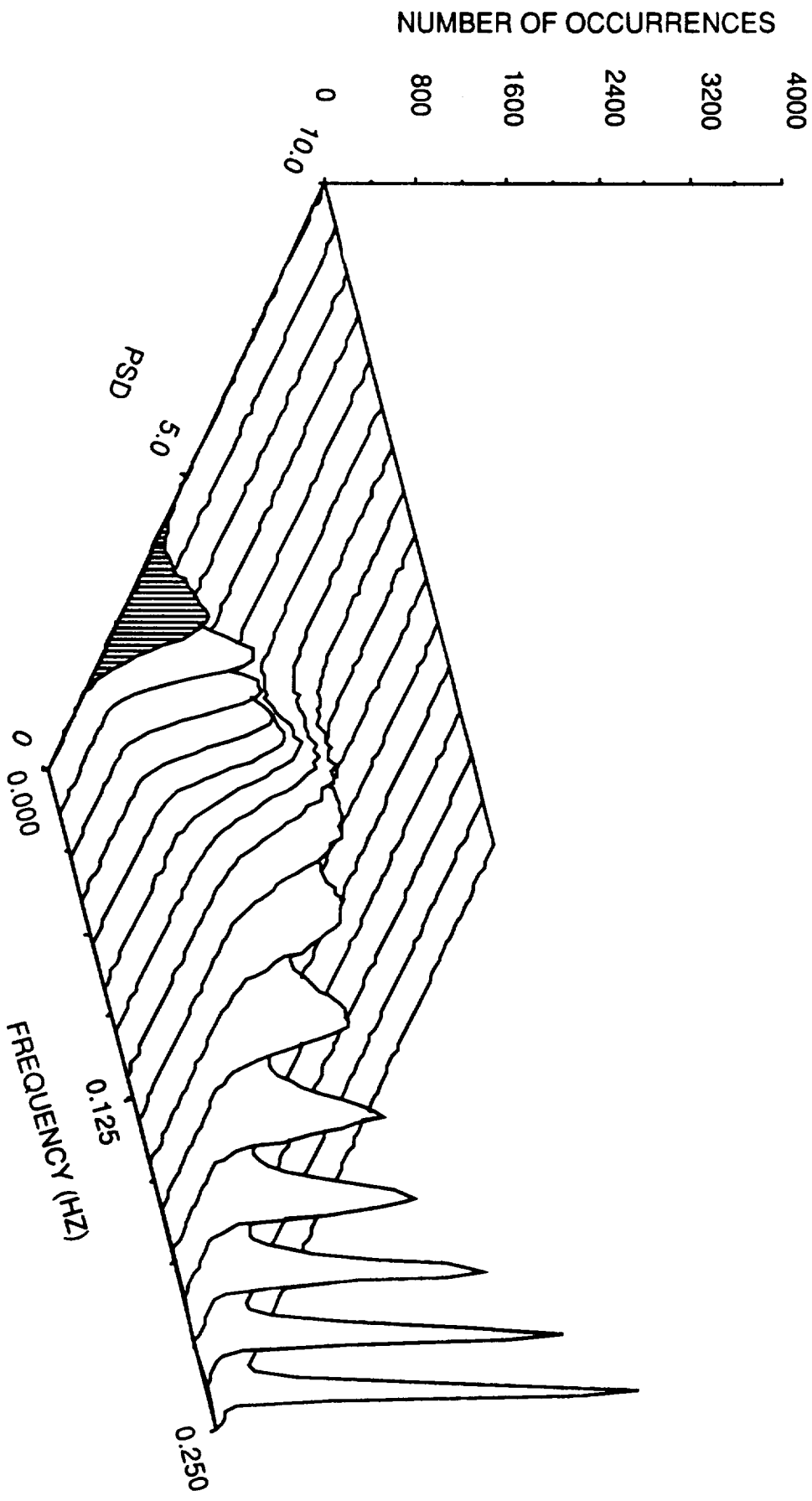
**Figure 3-B DISTRIBUTION OF KENDALL SERIES POWER SPECTRAL DENSITY
10000 REALIZATIONS WITH 4 DEGREES OF FREEDOM EACH**



**FIGURE 3-C DISTRIBUTION OF KENDALL SERIES POWER SPECTRAL DENSITY
10000 REALIZATIONS WITH 10 DEGREES OF FREEDOM EACH**



**FIGURE 3-D DISTRIBUTION OF KENDALL SERIES POWER SPECTRAL DENSITY
10000 REALIZATIONS WITH 100 DEGREES OF FREEDOM EACH**



**FIGURE 3-E DISTRIBUTION OF KENDALL SERIES POWER SPECTRAL DENSITY
10000 REALIZATIONS WITH 1000 DEGREES OF FREEDOM EACH**

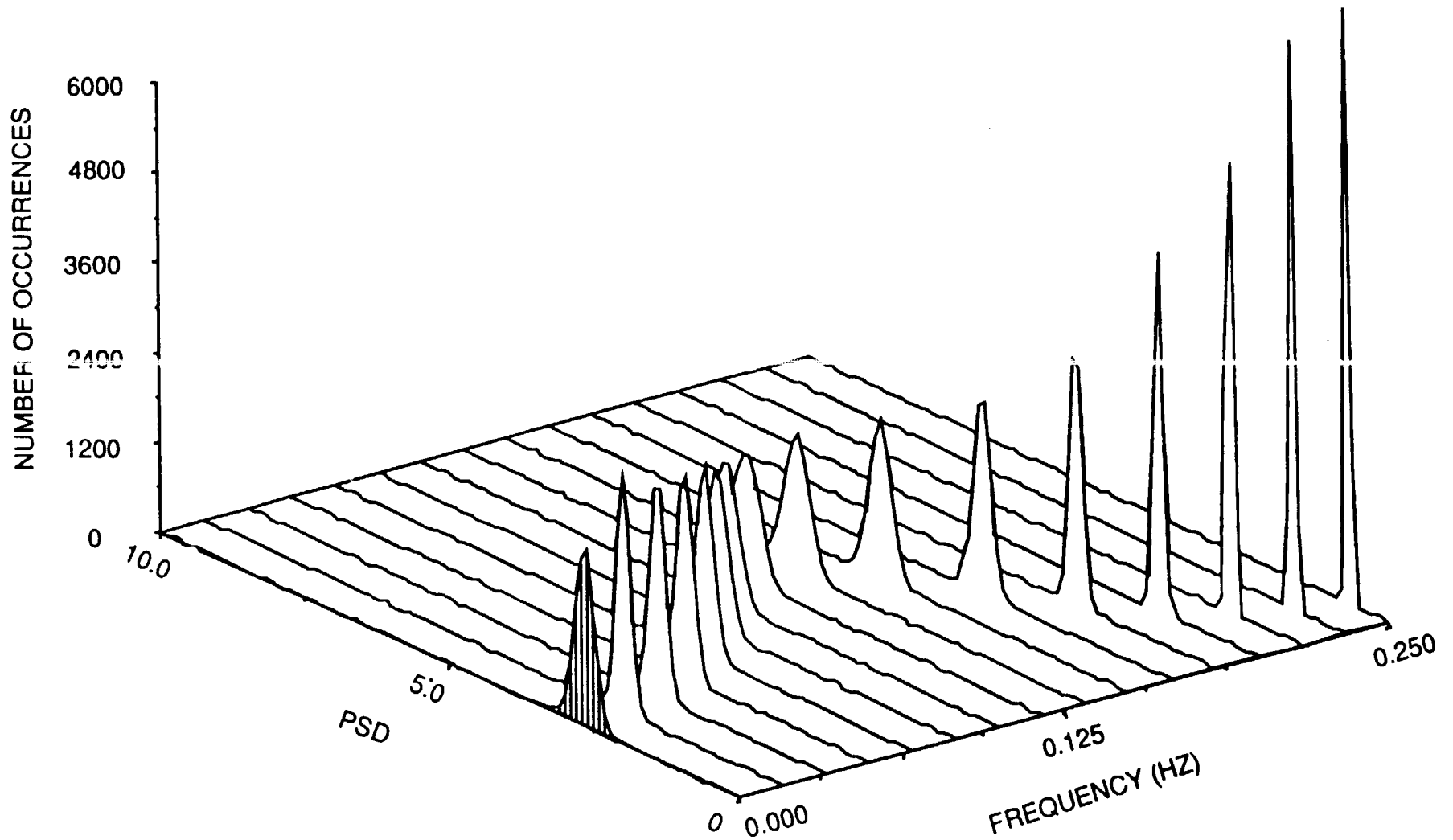


Figure 4A. Histograms of PSD Spectral Estimates of Kendall Series Data at Fixed Frequency (0.113 Hz)

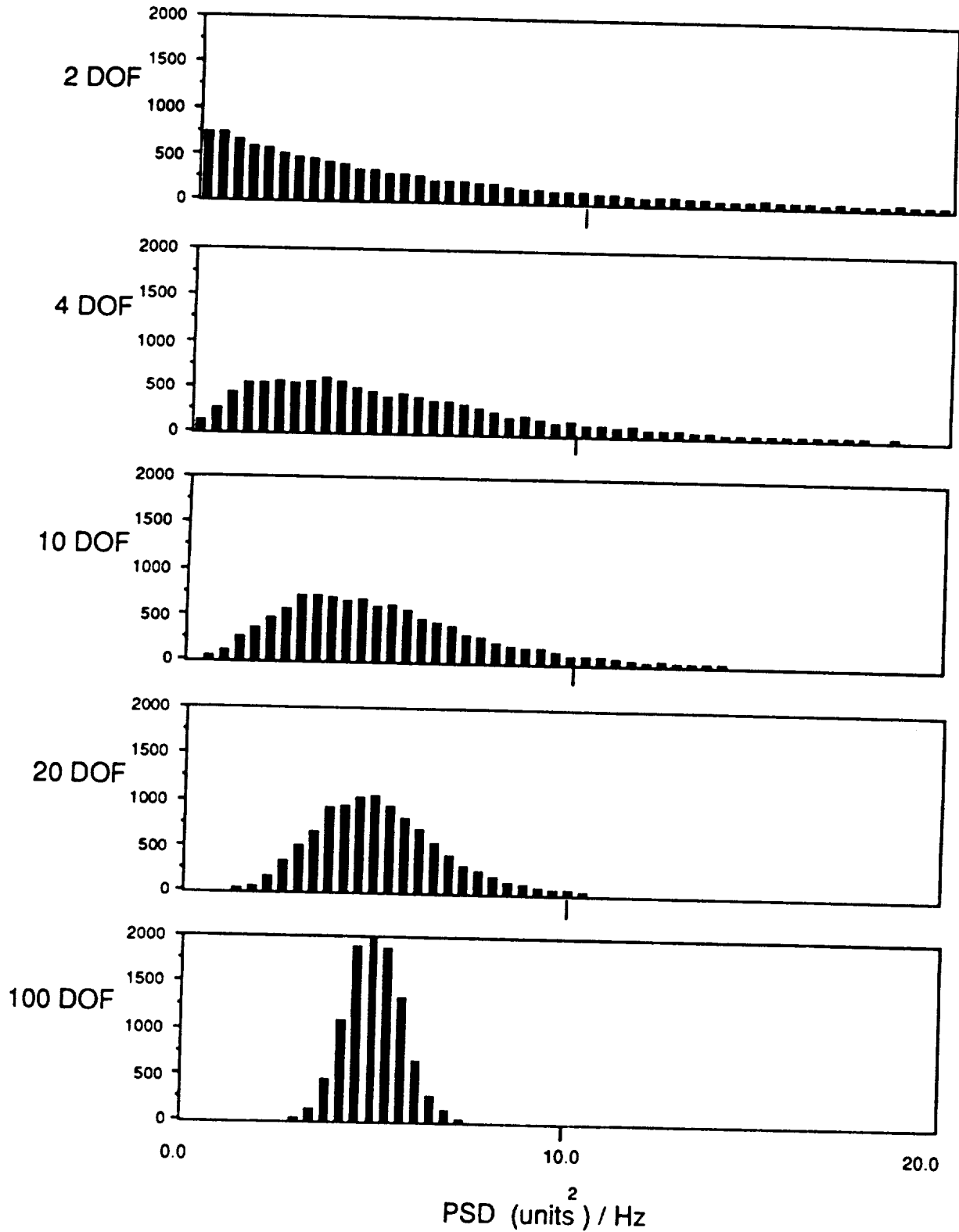


Figure 4-B Cumulative probability density functions for theoretical chi-square distribution with 2 degrees of freedom and a distribution of 10000 spectral estimates from the Kendall series data evaluated at 0.113 Hz.

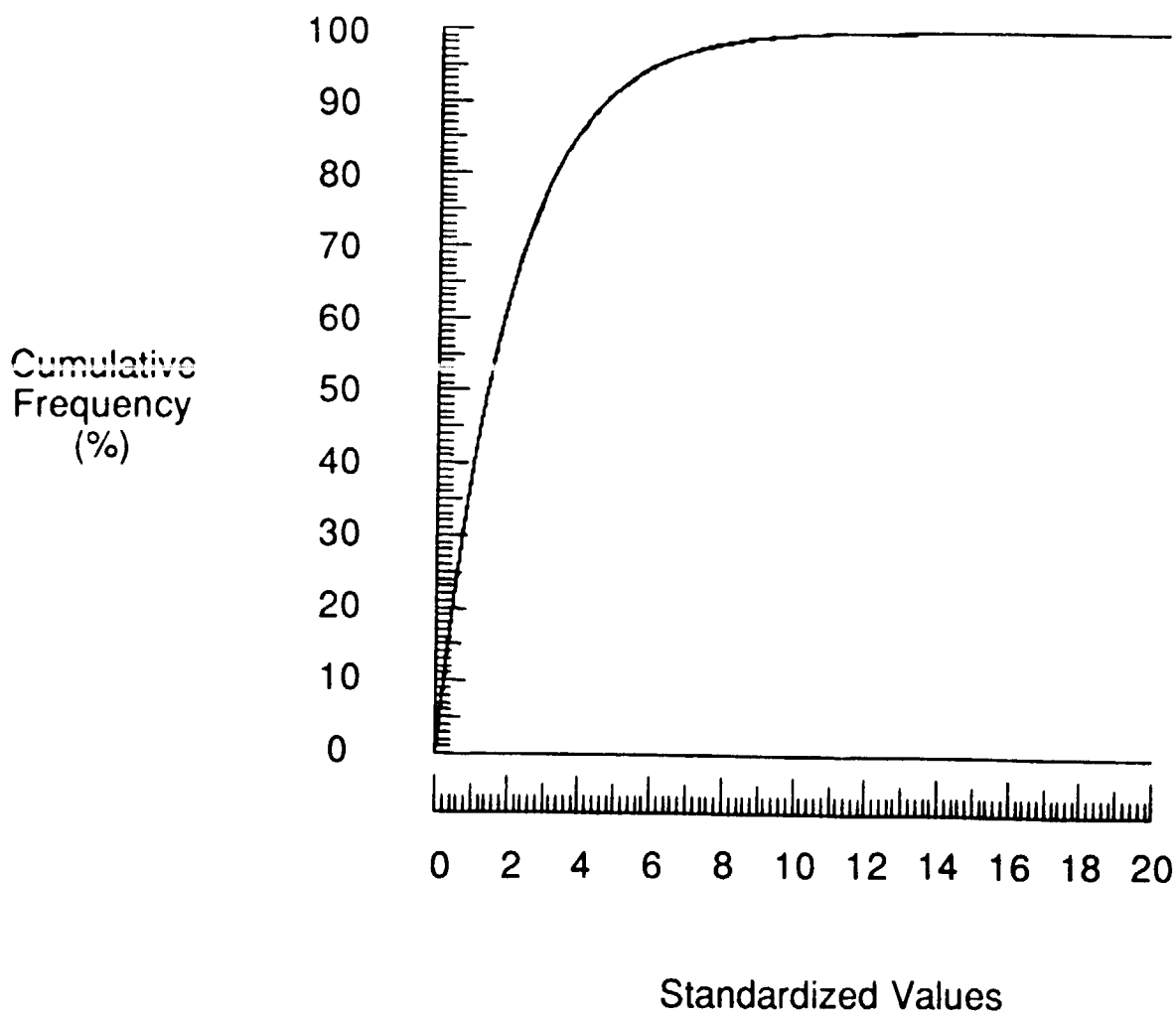


Figure 4-C Cumulative probability density functions for theoretical chi-square distribution with 4 degrees of freedom and a distribution of 10000 spectral estimates from the Kendall series data evaluated at 0.113 Hz.

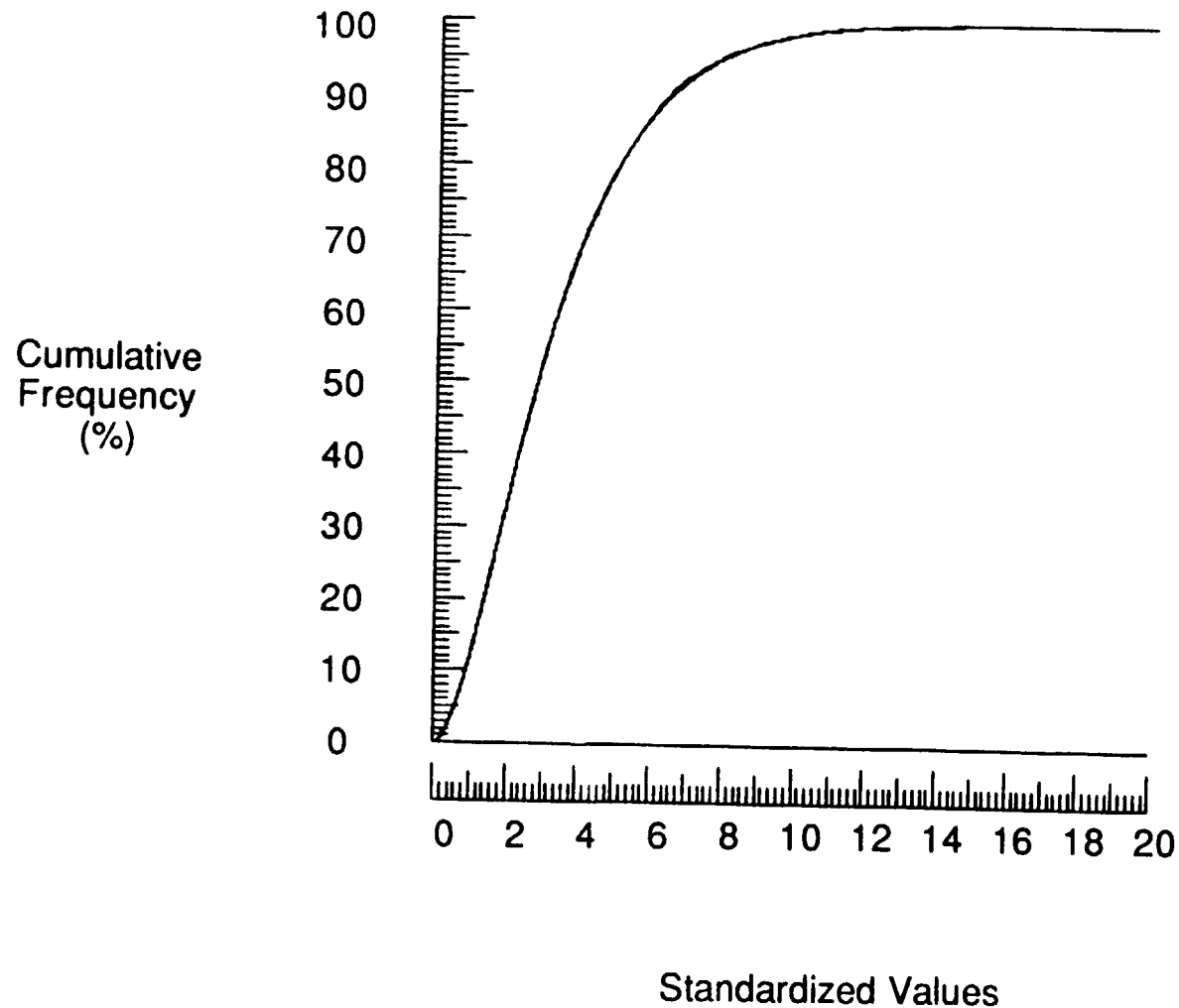


Figure 4-D Cumulative probability density functions for theoretical chi-square distribution with 10 degrees of freedom and a distribution of 10000 spectral estimates from the Kendall series data evaluated at 0.113 Hz.

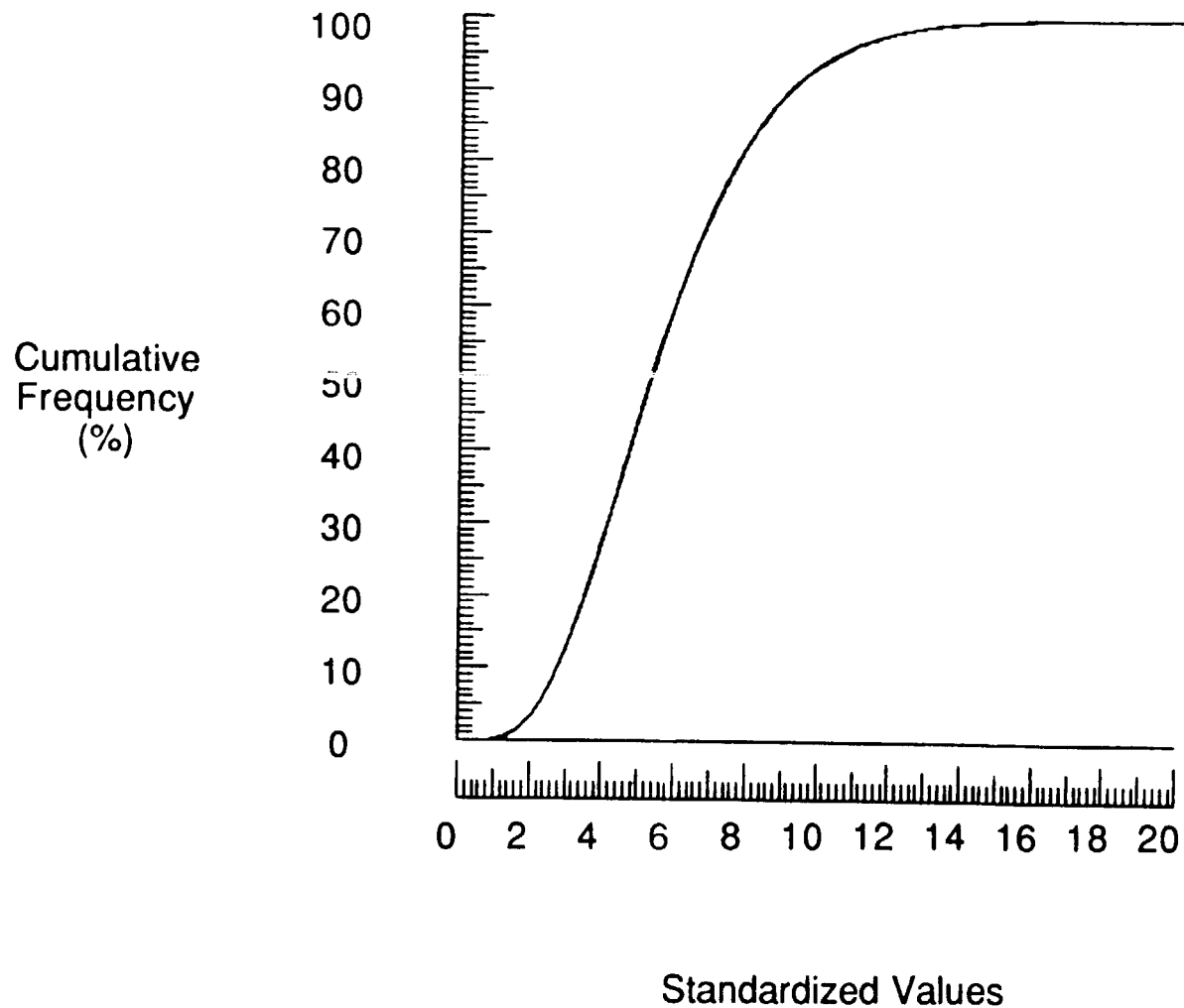


Figure 4-E Cumulative probability density functions for theoretical chi-square distribution with 20 degrees of freedom and a distribution of 10000 spectral estimates from the Kendall series data evaluated at 0.113 Hz.

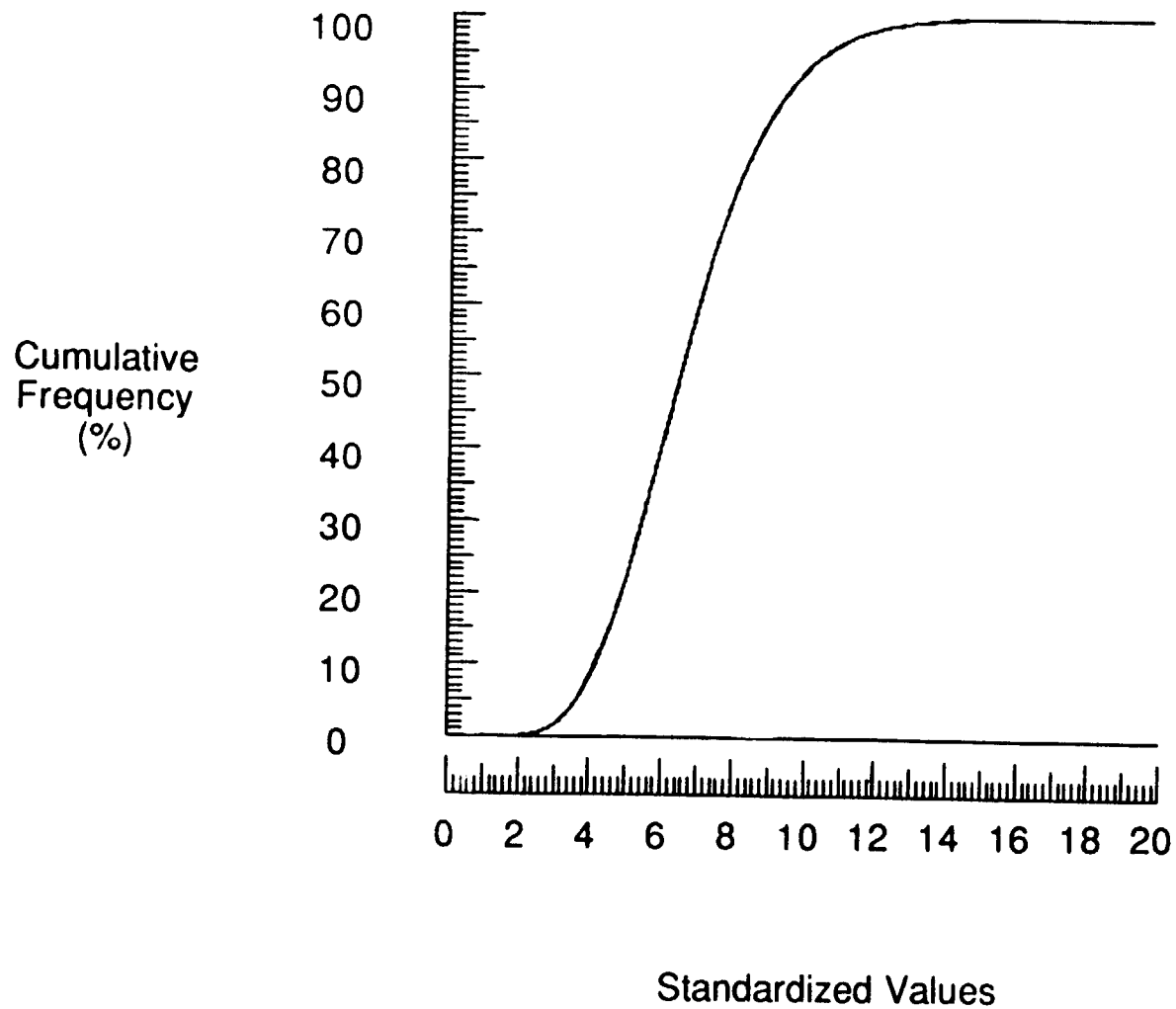


Figure 4-F Cumulative probability density functions for theoretical chi-square distribution with 100 degrees of freedom and a distribution of 10000 spectral estimates from the Kendall series data evaluated at 0.113 Hz.

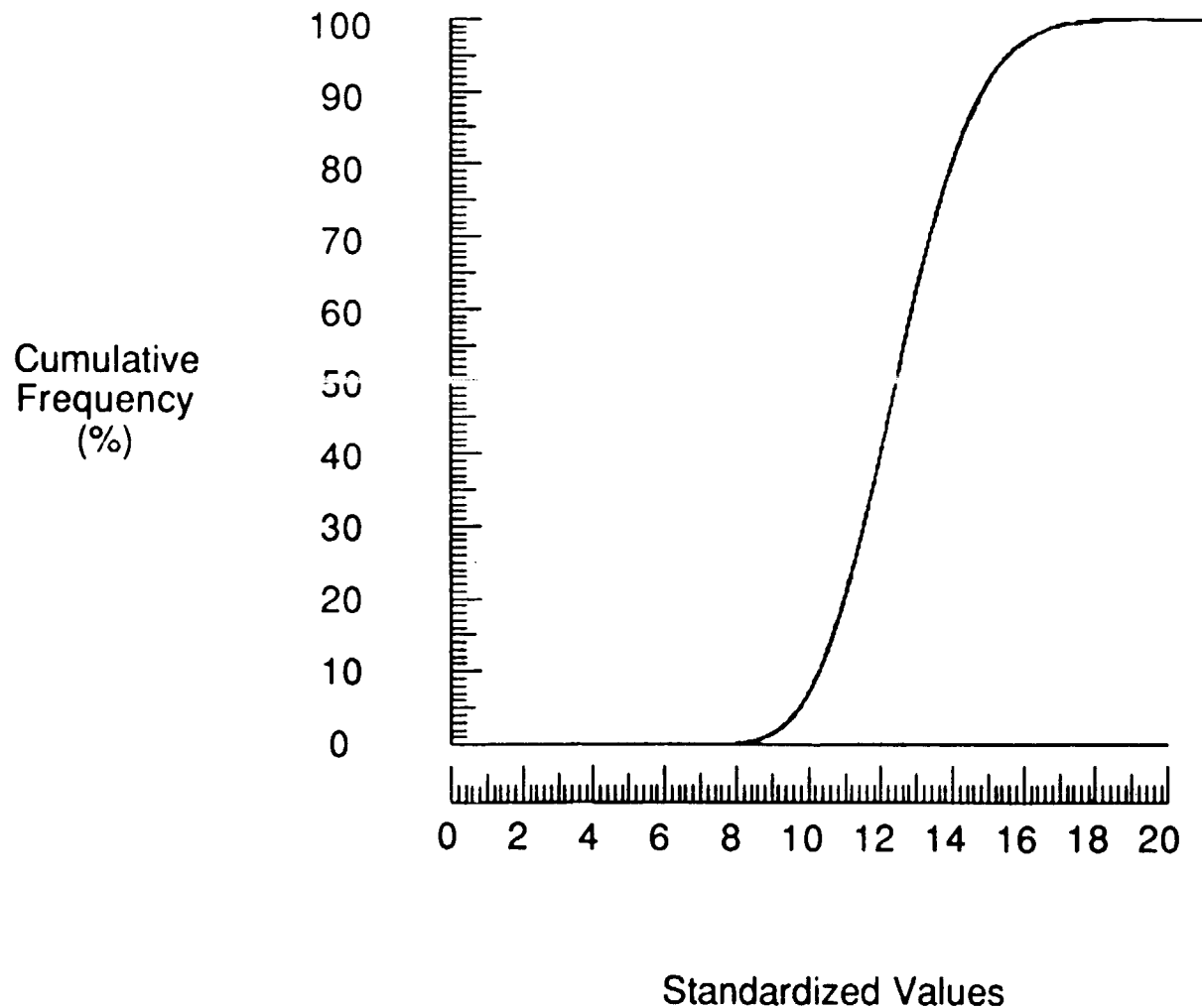


Figure 5-A Eighty percent confidence intervals of 2 DOF spectral estimates from 400 trials.

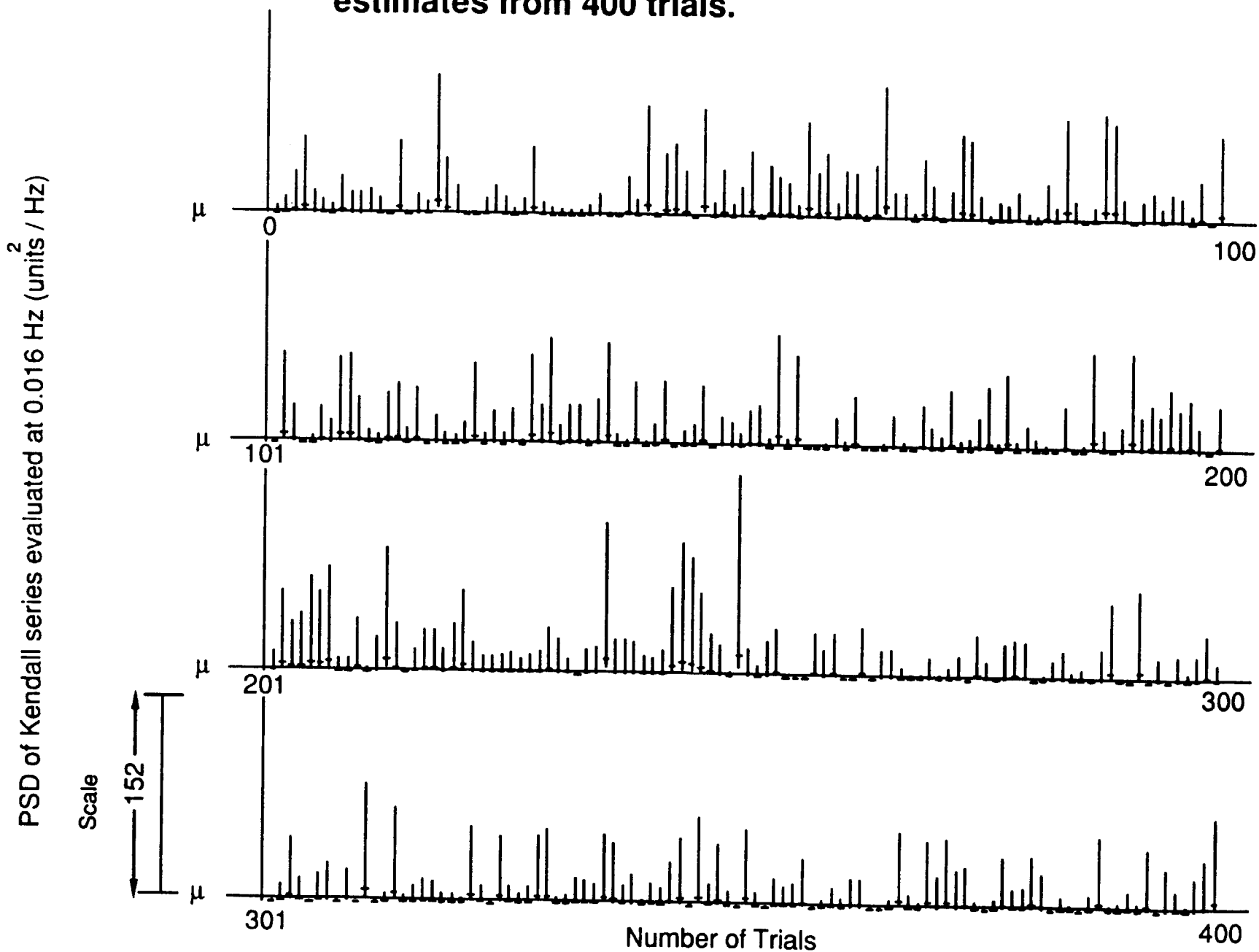


Figure 5-B Eighty percent confidence intervals of 2 DOF spectral estimates from 400 trials.

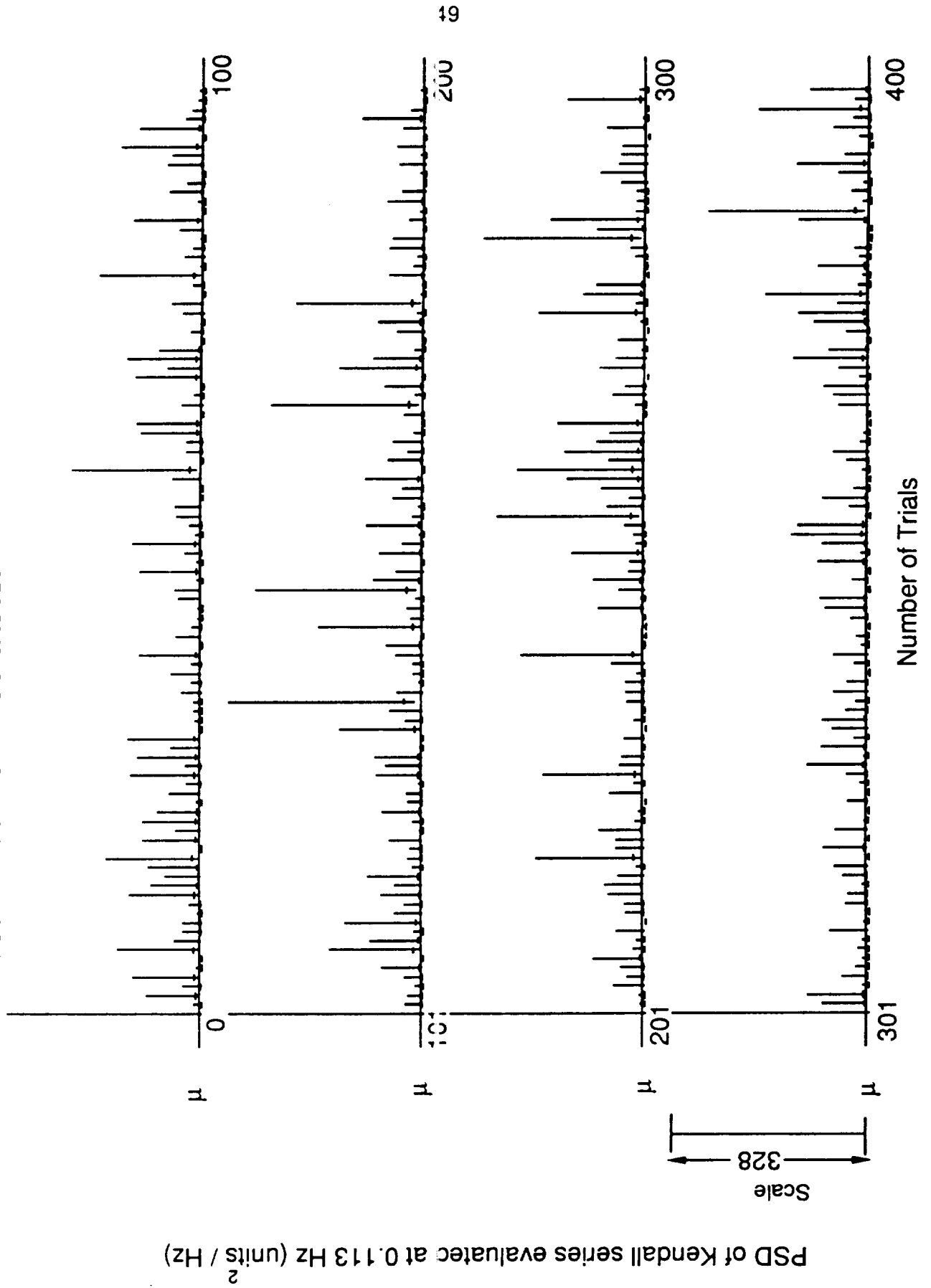


Figure 5-C Eighty percent confidence intervals of 1000 DOF spectral estimates from 400 trials.

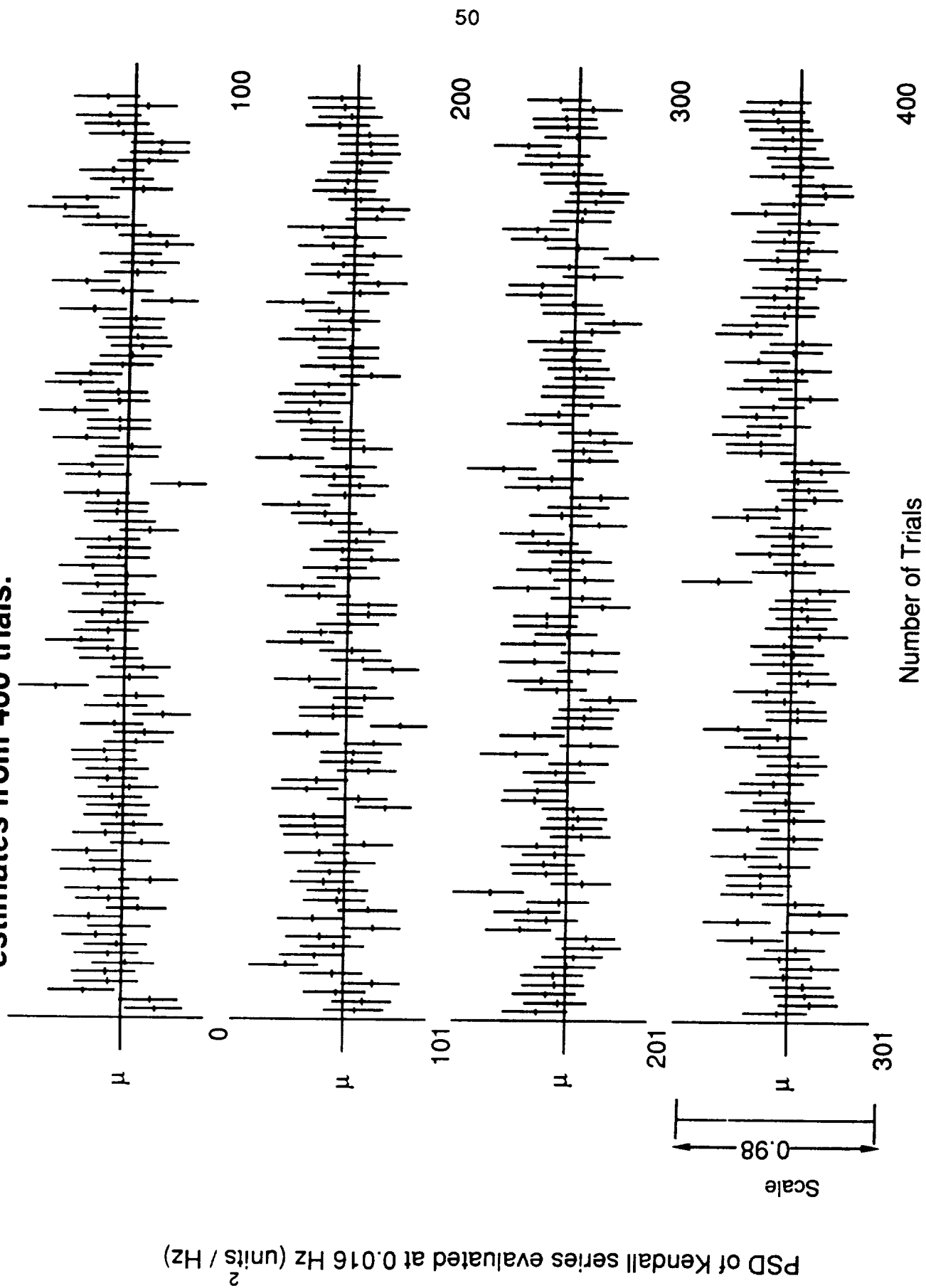


Figure 5-D Eighty percent confidence intervals of 1000 DOF spectral estimates from 400 trials.

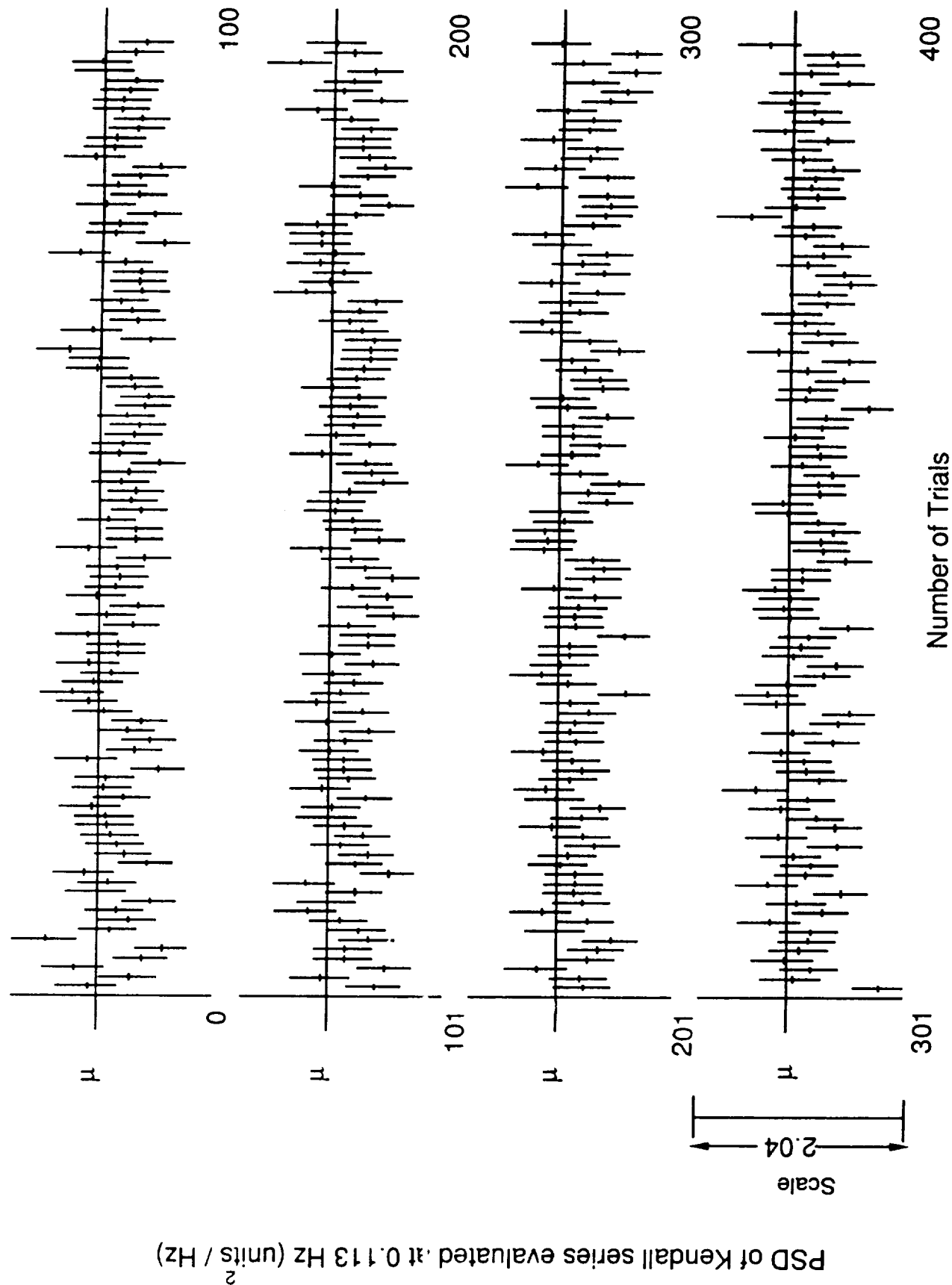


Figure 6 Power Spectrum of NASA/MDHC 500E Acoustic Data

AZIMUTHAL DIRECTIVITY ANGLE 0 0
POLAR DIRECTIVITY ANGLE 75 9
ANALYSIS BW 12 2 HZ
ENSEMBLE AVERAGED SPECTRUM N=30

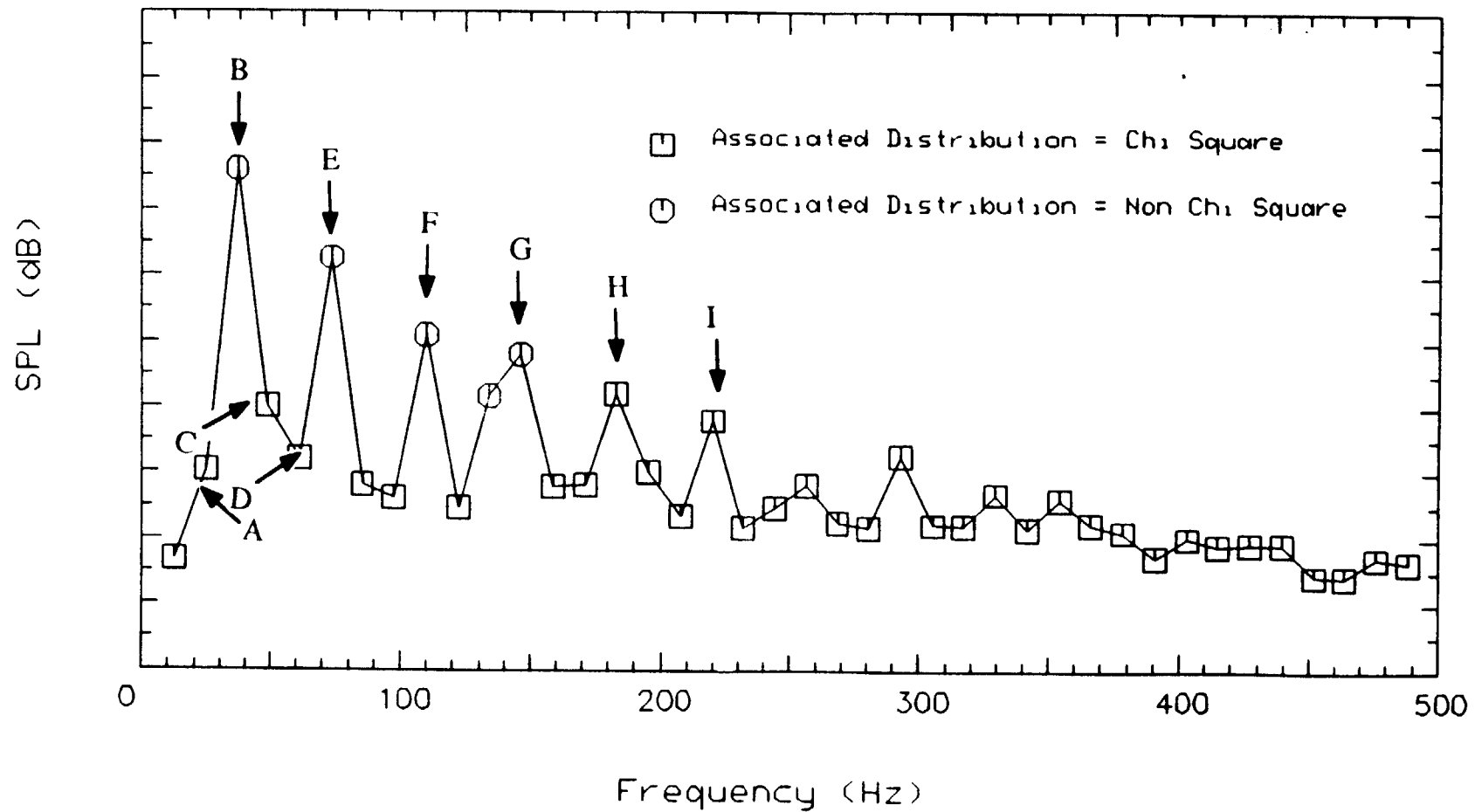
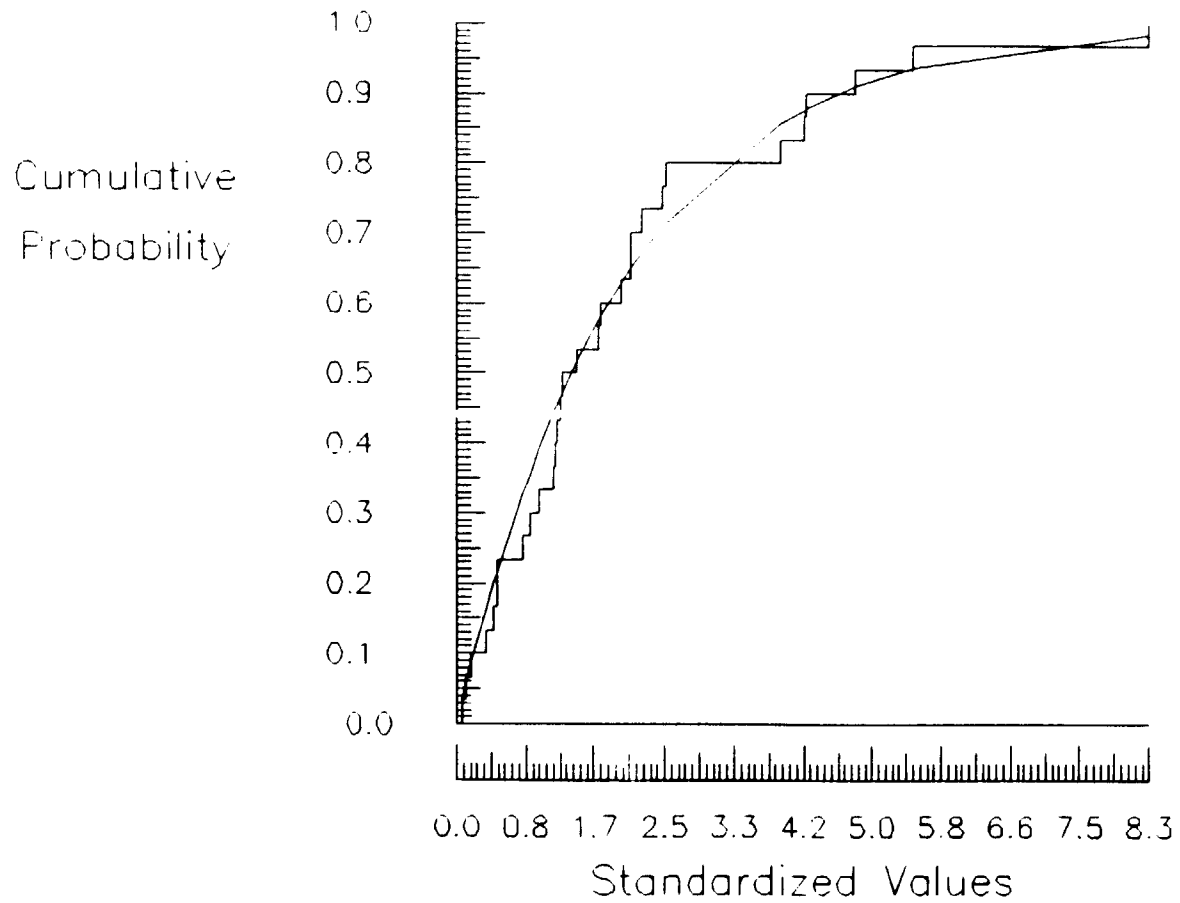


Figure 7-A Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 24.4 Hertz.



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Figure 7-B Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 36.6 Hertz.

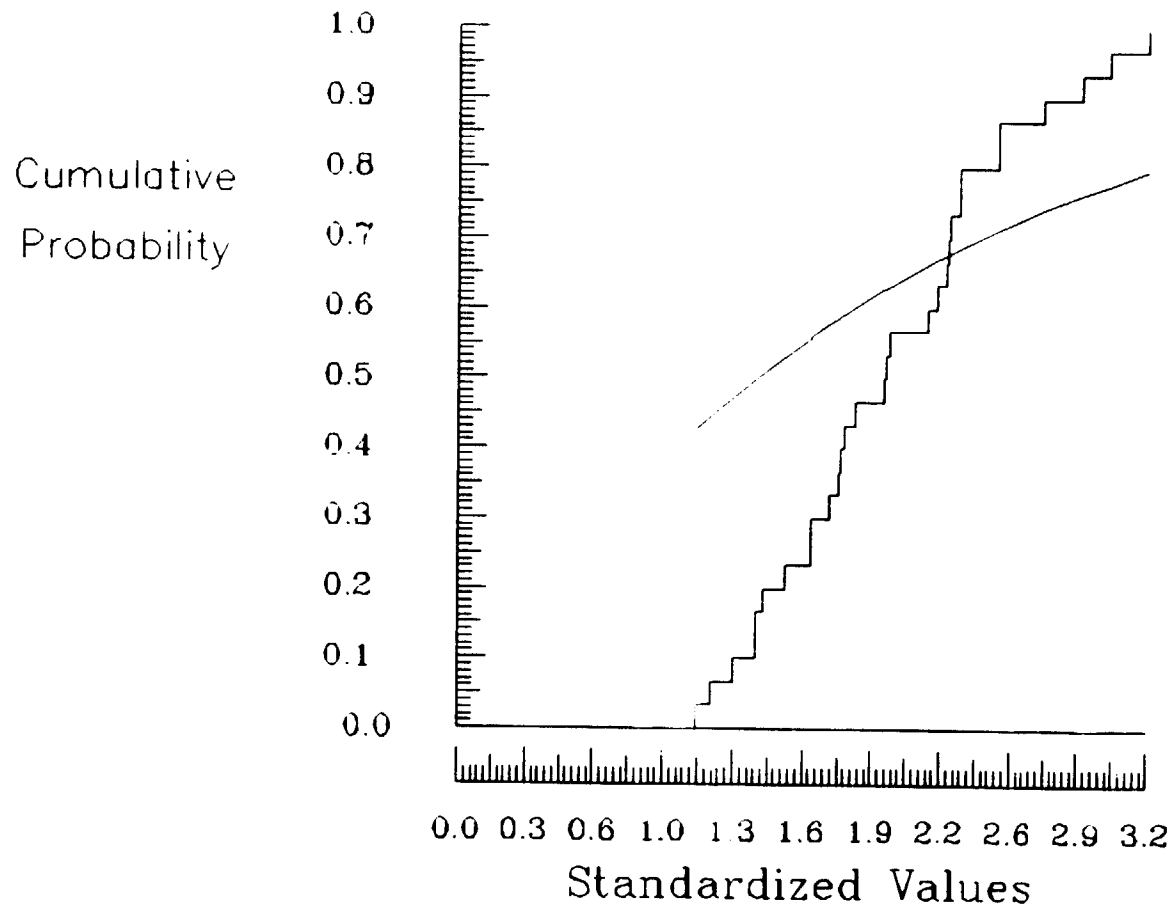


Figure 7-C Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 48.8 Hertz.

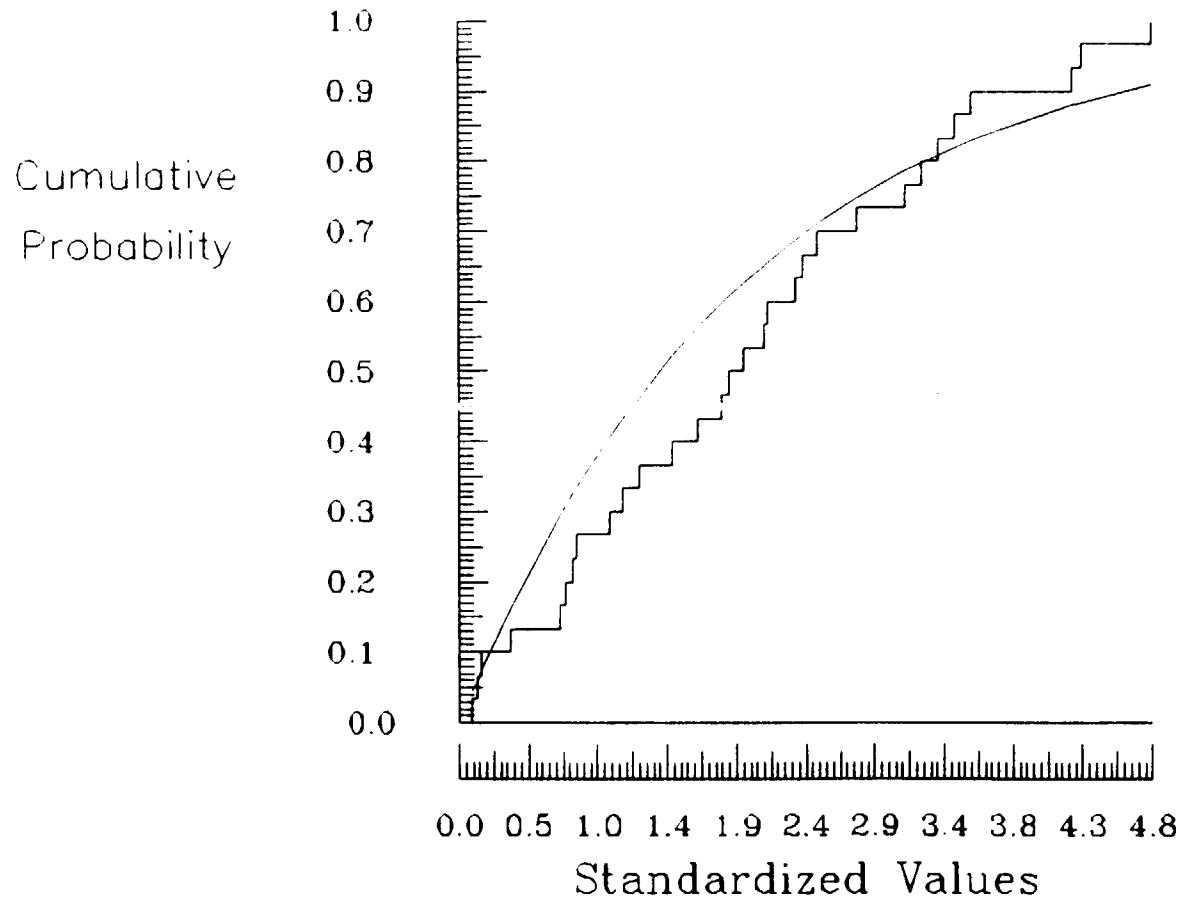


Figure 7-D Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 61.0 Hertz.

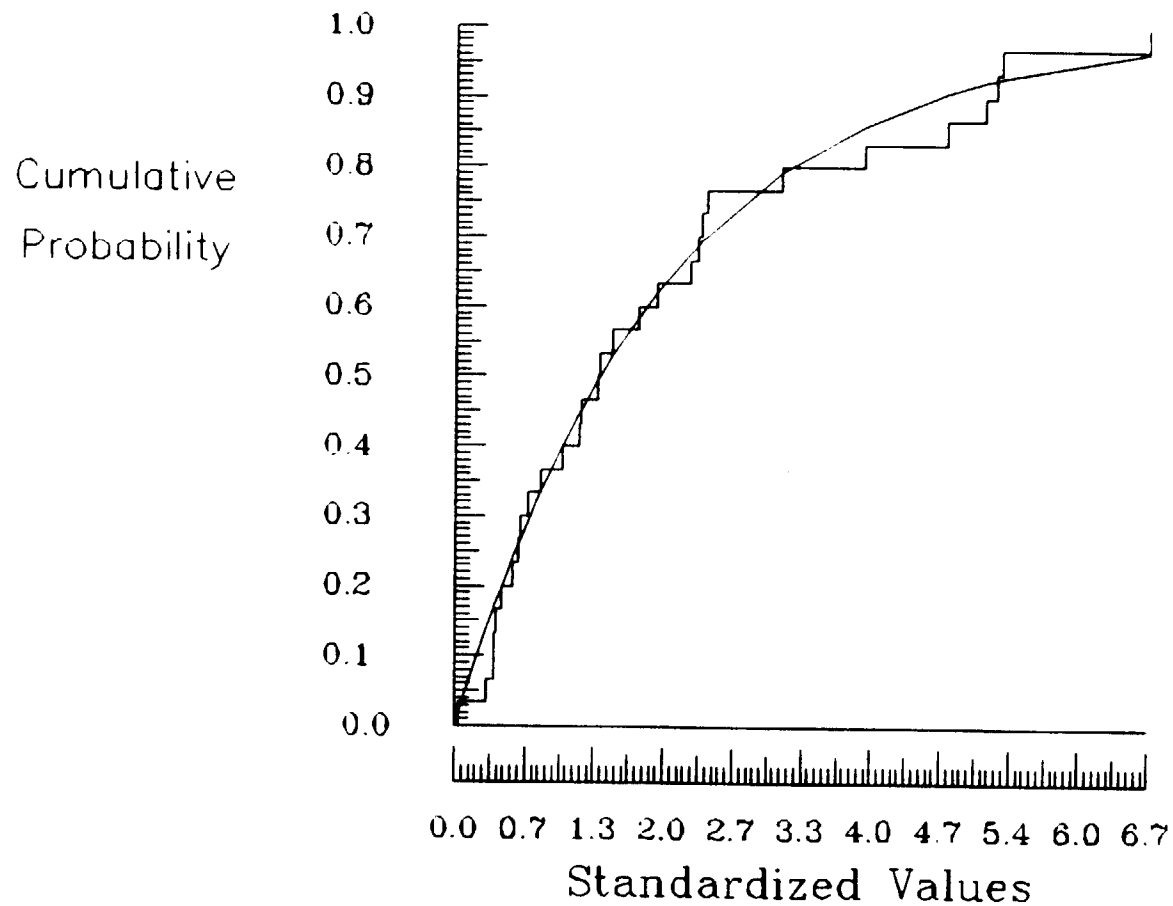


Figure 7-E Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 73.2 Hertz.

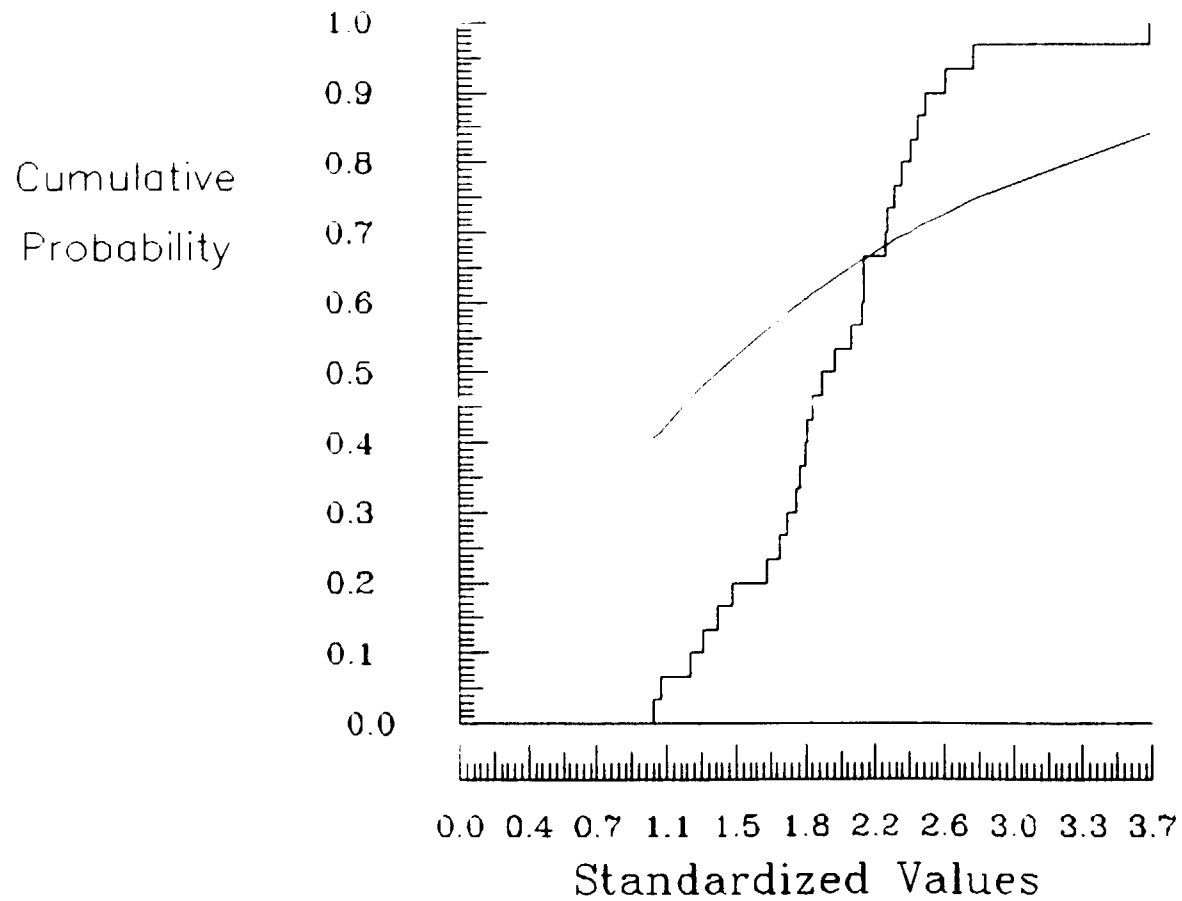


Figure 7-F Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 109.9 Hertz.

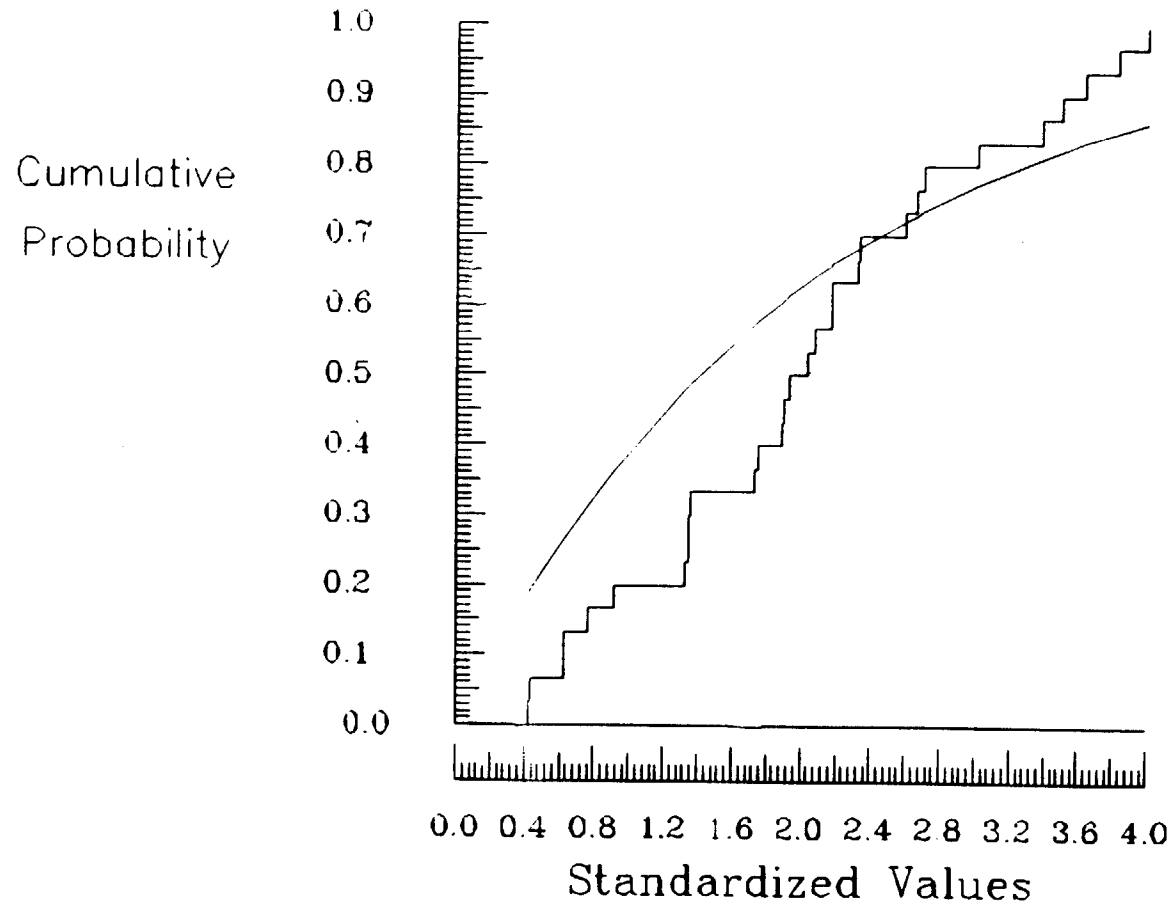


Figure 7-G Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 146.5 Hertz.

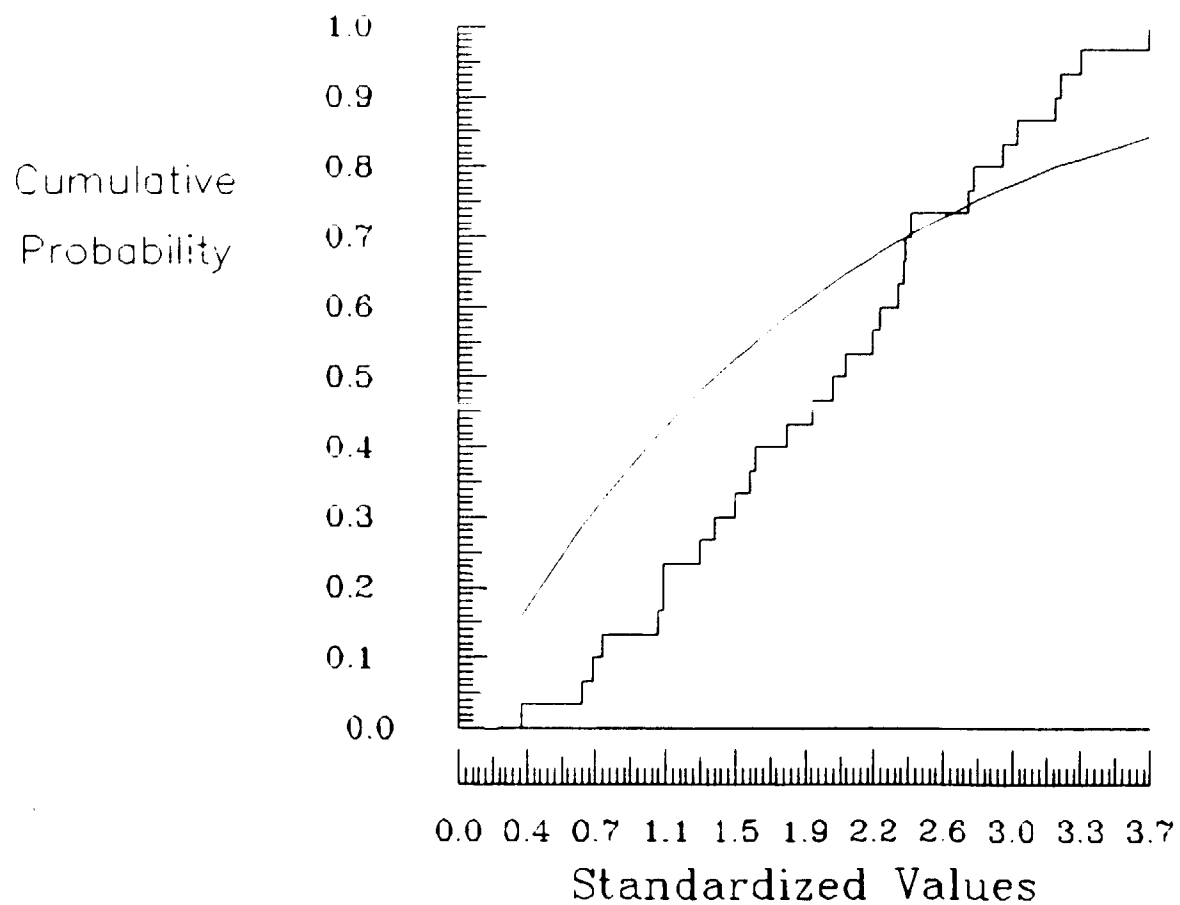


Figure 7-H Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 183.1 Hertz.

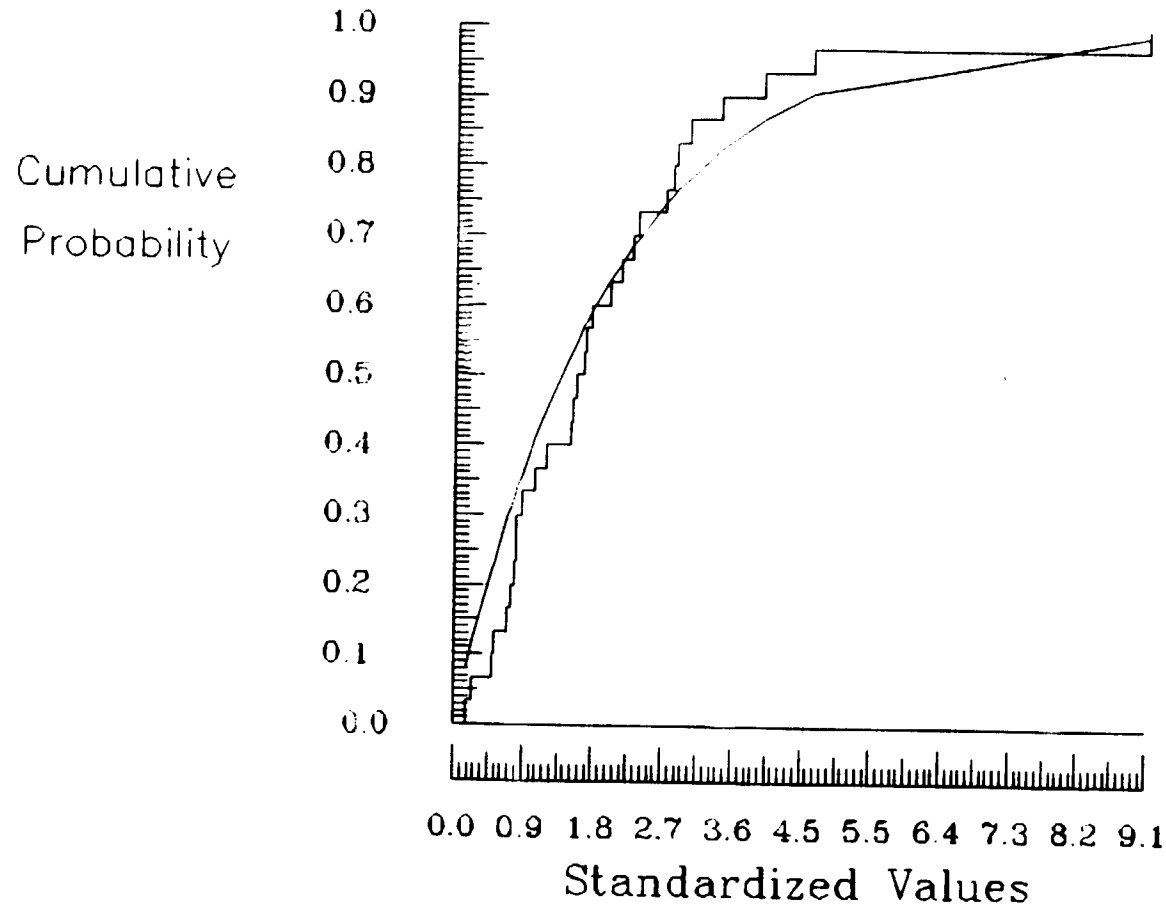
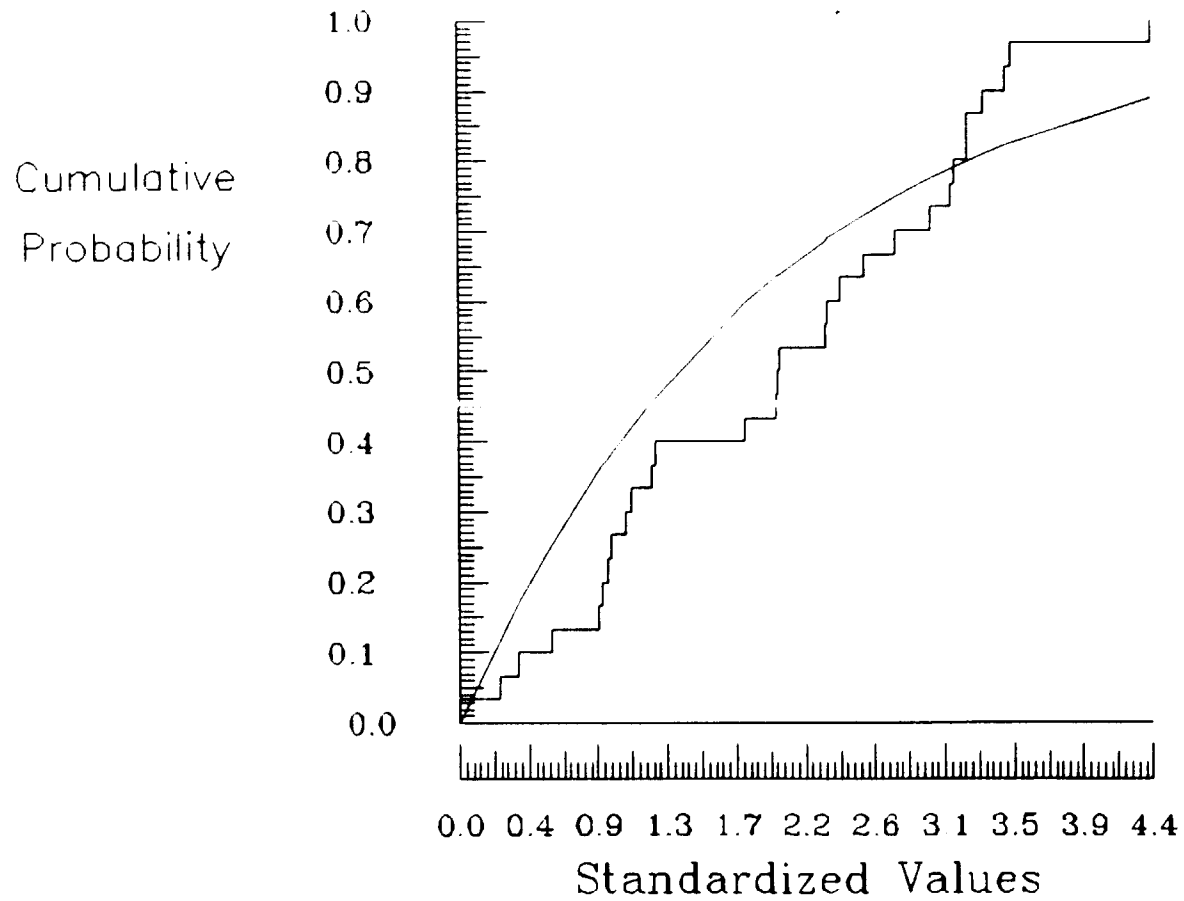


Figure 7-1 Cumulative probability distribution functions for theoretical Chi-square distribution with 2 degrees of freedom and a distribution of 30 spectral estimates from 500E data evaluated at 219.7 Hertz.





Report Documentation Page

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15. Supplementary Notes Langley Technical Monitor: Robert A. Golub ORIGINAL PAGE IS OF POOR QUALITY					
16. Abstract The validity of applying chi-square based confidence intervals to far-field acoustic flyover spectral estimates was investigated. Simulated data, using a Kendall series and experimental acoustic data from the NASA/McDonnell Douglas 500E acoustics test, were analyzed. Statistical significance tests to determine the equality of distributions of the simulated and experimental data relative to theoretical chi-square distributions were performed. Bias and uncertainty errors associated with the spectral estimates were easily identified from the data sets. A model relating the uncertainty and bias errors to the estimates resulted, which aided in determining the appropriateness of the chi-square distribution based confidence intervals. Such confidence intervals were appropriate for the nontonally associated frequencies of the experimental data but were inappropriate for the tonally associated estimate distributions. The inappropriateness at the tonally associated frequencies was indicated by the presence of bias error and nonconformity of the distributions to the theoretical chi-square distribution. A technique for determining appropriate confidence intervals at the tonally associated frequencies was suggested.					
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