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# A Simplified Analysis of the Multigrid V–Cycle as a Fast Elliptic Solver

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## Abstract

For special model problems, Fourier analysis gives exact convergence rates for the two-grid multigrid cycle and, for more general problems, provides estimates of the two-grid convergence rates via local mode analysis. A method is presented for obtaining multigrid convergence rate estimates for cycles involving more than two grids – using essentially the same analysis as for the two-grid cycle.

For the simple case of the $V$-cycle used as a fast Laplace solver on the unit square, the $k$-grid convergence rate bounds obtained by this method are sharper than the bounds predicted by the variational theory. Both theoretical justification and experimental evidence are presented.

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## 1. Introduction

Although the heuristic arguments which indicate that multigrid iterative methods lead to fast elliptic solvers are easy to understand, they cannot predict convergence rates or guarantee convergence. There exist methods which provide a means to calculate theoretical multigrid convergence rate bounds for a large class of multigrid solvers for positive definite elliptic problems. Unfortunately, the motivation and proofs of these bounds are relatively difficult. For special model problems, a simpler calculation, based on Fourier analysis, provides exact convergence rates for the two-grid algorithm, but the exact analysis becomes too complicated to carry out for more than two or three grids.

In this paper we present a technique for obtaining convergence estimates for any number of grid levels, thereby providing an easy method to verify that multigrid is a fast solver (i.e., only a fixed number of iterations will solve the problem). Although the method is strictly valid for only a few model problems and for only symmetric $V$–cycles, it provides a general insight into the details of the basic interaction between the coarse grid correction and the relaxation. These new estimates of the convergence rates for the $k$–grid problem are no more difficult to calculate than the exact convergence rates for the two–grid problem.

In Section 2 we define a simple multigrid $V$–cycle and derive a formula for the iteration matrix associated with a multigrid cycle. A brief introduction to the grid independent convergence rate bounds given by the variational theory is given in Section 3. In Section 4 it is shown that bounds on the $k$ grid convergence rate can be found in terms of the $k-1$ grid convergence rate, using the same type of calculation as for obtaining the exact two grid convergence rates. Moreover, since the two grid convergence rate can be found exactly, we obtain convergence rate bounds recursively for any number of grid levels. It is shown that these bounds are sharper than those obtained by the variational theory. In Section 5 we show the details of how the results of Section 4 can be used to obtain convergence rates for a model problem. Finally, in Section 6, we compare the convergence rate bounds predicted by our theory and the convergence rate bounds given by the variational theory. The exact three grid convergence rates are compared to our three grid convergence rate bounds.

## 2. Notation

### 2.1 Multigrid Cycle

A sequence of uniform grids is given with mesh sizes $h_k$ $(k = 1, 2, 3, ...)$, where $h_{k+1} = h_k/2$. Consider the discrete equations on the $h_k$ grid of the form

$$A_k U_k = F_k \tag{1}$$

where $A_k$ is symmetric, positive definite. Given $u_k^{(0)}$, an approximate solution to (1), the multigrid cycle MG for producing an improved approximation $u_k^{(1)}$,

$$u_k^{(1)} \leftarrow MG(k, u_k^{(0)}, F_k) \tag{2}$$

is recursively defined as follows:

If $k = 1$ solve (1) by any direct or iterative method, yielding the final result $u_k^{(1)}$. Otherwise do (A) through (E).

(A) Perform $r_1$ relaxation sweeps on (1), resulting in a new approximation $\bar{u}_k$.

(B) Transfer ("restrict") residual from grid $h_k$ to grid $h_{k-1}$.

$$F_{k-1} = I_k^{k-1}(F_k - A_k \bar{u}_k). \tag{3}$$

(C) Starting with $u_{k-1}^{(0)} = 0$, update coarse grid approximation.

$$u_{k-1}^{(1)} \leftarrow MG(k-1, u_{k-1}^{(0)}, I_k^{k-1}(F_k - A_k \bar{u}_k)). \tag{4}$$

(D) Calculate $\tilde{u}_k = \bar{u}_k + I_{k-1}^k u_{k-1}^{(1)}$, where $I_{k-1}^k$ is a suitable interpolation ("prolongation") from grid $h_{k-1}$ to grid $h_k$.

(E) Perform $r_2$ relaxation sweeps on (1), starting with $\tilde{u}_k$ and yielding the final result $u_k^{(1)}$.

This cycle is called $V-$ cycle or $V(r_1, r_2)$.

Note: The relaxations done in steps (A) and (D) need not be the same.

### 2.2 Error Analysis

As in the analysis of the classical fixed point linear iterative solvers, we are interested in analyzing the "iteration matrix" associated with the multigrid process. If the error after the $i$-th iteration can be written in terms of the error before the $i$-th iteration as:

$$U_k - u_k^{(i)} = B_k(U_k - u_k^{(i-1)}) \tag{5}$$

then $B_k$ is the iteration matrix. We are interested in the convergence factor of the $V-$ cycle, $\epsilon_k$, given by

$$\begin{aligned} \epsilon_k &= inf\{\epsilon : \|U_k - MG(k, v_k, F_k)\| \leq \epsilon \|U_k - v_k\| \quad \forall v_k \in H_k\} \\ &= inf\{\epsilon : \|B_k w_k\| \leq \epsilon \|w_k\| \quad \forall w_k \in H_k\} \\ &= \|B_k\| \end{aligned} \tag{6}$$

where $\|\cdot\|$ is some norm defined on $H_k$, the space of all vectors defined on the grid $h_k$.

To find an expression for the iteration matrix corresponding to a $V-$ cycle, the steps (A)-(E) are rewritten in terms of the errors (*true solution − current approximation*). Given an initial error, $e_k^{(0)} = U_k - u_k^{(0)}$, the iteration matrix is defined recursively as follows:

If $k = 1$, then $B_1 = 0$ (the error is eliminated completely on the coarsest grid). Otherwise

$$e_k^{(1)} = B_k e_k^{(0)}, \tag{7}$$

where $e_k^{(1)}$, the error of the improved approximation $u_k^{(1)}$, is given by steps $(A')$-$(E')$:

$(A')$ After relaxation:

$$\bar{e}_k = (G_k)^{r_1} e_k^{(0)},$$

where $G_k$ is the iteration matrix of the (pre-)relaxation.

$(B')$ After residual transfer:

$$U_{k-1} = A_{k-1}^{-1} I_k^{k-1} A_k \bar{e}_k.$$

$(C')$ After computing coarse grid correction, since $e_{k-1}^{(0)} = U_{k-1}$:

$$e_{k-1}^{(1)} = B_{k-1} e_{k-1}^{(0)}.$$

$(D')$ After interpolating correction to fine grid:

$$\begin{aligned} \tilde{e}_k^{(1)} &= \bar{e}_k - I_{k-1}^k (e_{k-1}^{(0)} - e_{k-1}^{(1)}) \\ &= \bar{e}_k - I_{k-1}^k (I - B_{k-1}) e_{k-1}^{(0)}. \end{aligned}$$

$(E')$ After relaxation:

$$e_k^{(1)} = (E_k)^{r_2} \tilde{e}_k^{(1)},$$

where $E_k$ is the iteration matrix of the (post-)relaxation.

Therefore for $1 < k \leq m$,

$$B_k = (E_k)^{r_2}(I - I_{k-1}^k(I - B_{k-1})A_{k-1}^{-1}I_k^{k-1}A_k)(G_k)^{r_1} \tag{8}$$

and

$$B_1 = 0.$$

## 3. Variational Theory

### 3.1 Notation

Consider the Euclidean spaces $H_k = I\!R^{n_k}$, $k = 1, 2, \cdots, m$ and two types of inner products on these spaces, one type is the usual discrete $L^2$ inner product in $d$-dimensions defined as

$$\langle u_k, v_k \rangle_k = h_k^d \sum_i (u_k)_i (v_k)_i$$

and the other is the "energy" or "operator" inner product (associated with an operator $L_k$) defined as

$$(u_k, v_k)_k = \langle u_k, L_k v_k \rangle_k. \tag{9}$$

The norms induced by these inner products are denoted by:

$$\|u_k\|_k = \langle u_k, u_k \rangle_k^{1/2}$$

and

$$|\!|\!| u_k |\!|\!|_k = (u_k, u_k)_k^{1/2}.$$

Adjoints relative to the $\langle \cdot, \cdot \rangle$ inner products are denoted by $(\cdot)^T$ and adjoints relative to the $(\cdot, \cdot)$ inner products are denoted by $(\cdot)^*$.

### 3.2 Bounds Independent of the Number of Levels

The best available multi-level convergence rate bounds come from the analysis in the "variational" setting, i.e., where it is assumed that:

a. The intergrid transfer operators are related by:

$$I_k^{k-1} = (I_{k-1}^k)^T$$

and

b. the "Galerkin" choice of the coarse grid operators is assumed:

$$A_{k-1} = I_k^{k-1} A_k I_{k-1}^k.$$

For the complete theory for general domains and $W-$ as well as $V-$cycles, see [3]. For positive definite symmetric elliptic p.d.e.'s in the unit square, typical $V-$cycle convergence rate bounds which are independent of the number of coarse grids are summarized below. For a more complete discussion of the types of convergence proofs, see [6].

As in the standard variational multigrid analysis, see e.g., [4], [5] and [1], define two operators, $S_k$ and $T_k$:

$$S_k = I_{k-1}^k A_{k-1}^{-1} I_k^{k-1} A_k \tag{10}$$

and

$$T_k = I - S_k. \tag{11}$$

Then $S_k$ and $T_k$ are orthogonal projections in the $(\cdot,\cdot)$ inner product. Specifically:

1. $S_k S_k = S_k$, $T_k T_k = T_k$ and $T_k S_k = S_k T_k = 0$.

2. For any $u_k$,

$$u_k = T_k u_k + S_k u_k$$

and since $S_k^* = S_k$ and $T_k^* = T_k$ (but $S_k^T \neq S_k$ and $T_k^T \neq T_k$, in general), we also have

$$\|u_k\|_k^2 = \|S_k u_k\|_k^2 + \|T_k u_k\|_k^2. \tag{12}$$

**Note:** If $u_k = S_k u_k$, then $u_k \in range(I_{k-1}^k)$, i.e. it is representable on the $h_{k-1}$ grid. If $v_k = T_k v_k$, then $I_k^{k-1} A_k v_k = 0$, i.e. $v_k \in nullspace(I_k^{k-1} A_k)$ and if $v_k$ corresponds to an error, its residual cannot be distinguished from the null vector on the $h_{k-1}$ grid.

Let $\epsilon_k$ be the operator norm convergence factor of the $k$ grid $V-$cycle, see equation (6).

**Theorem 1** *If there exist $\beta_1, \beta_2 > 0$, independent of $k$, such that for every $u_k$ in $H_k$ and for all $k = 2, 3, \cdots$,*

$$\|(G_k)^{r_1} u_k\|_k^2 + \beta_1 \|T_k (G_k)^{r_1} u_k\|_k^2 \leq \|u_k\|_k^2 \tag{13}$$

*and*

$$\|(E_k)^{r_2} u_k\|_k^2 + \beta_2 \|T_k (E_k)^{r_2} u_k\|_k^2 \leq \|u_k\|_k^2 \tag{14}$$

*then for every $k = 2, 3, \cdots$,*

$$\epsilon_k \leq (1+\beta_1)^{-1/2} (1+\beta_2)^{-1/2}. \tag{15}$$

**Proof.** See [3].

In [3], Theorem 1 is proved using recursive formulas of the form:

$$\epsilon_j = \sup_{0 \le t \le 1} \left( \frac{t + \epsilon_{j-1}(1-t)}{1 + \beta t} \right) \tag{16}$$

where $j \ge 2$.

Although these formulas appear to give grid dependent bounds, notice that $\epsilon_j = 1/(1+\beta)$, independent of $j$. Therefore these formulas give the same convergence rate bound for the two grid cycle as for an arbitrary number of grids. They cannot give sharper intermediate estimates for a specific number of grids. In the next section we show that, under special assumptions, one can do better than this.

## 4. Multi–Level Analysis

From equations (8) and (11), the two grid iteration matrix, $B_2$, is:

$$B_2 = (E_2)^{r_2}\ T_2\ (G_2)^{r_1}. \tag{17}$$

Under certain assumptions about $E_2$, $I_2^1$, $I_1^2$ and $G_2$, it is feasible to explicitly calculate the spectral radius of $B_2$ and hence find exact asymptotic convergence rates for the two grid algorithm. Section 5 contains the details of the calculation of the spectral radius of operators of the slightly more general form $(E_k)^{r_2}(T_k - \alpha S_k)(G_k)^{r_1}$ where $\alpha$ is a scalar. Notice that the two grid operator is of this form with $\alpha = 0$. In general, the $k$-grid $V$–cycle iteration matrix is given by

$$B_k = (E_k)^{r_2}(T_k + I_{k-1}^k B_{k-1} A_{k-1}^{-1} I_k^{k-1} A_k)(G_k)^{r_1} \tag{18}$$

and is not of this form unless, for instance, $B_{k-1}$ is a constant multiple of the identity (i.e., all error components on the $k-1st$ grid are uniformly reduced). It is possible, however, to obtain estimates of the $k$ grid convergence rates by replacing $B_{k-1}$ by the constant $\|B_{k-1}\|_{k-1}$ times the identity.

For convenience, fix the number of relaxations, $r_1$, and define

$$\bar{G}_j = (G_j)^{r_1}.$$

We assume the $A_j$'s are symmetric and that $A_j$, $\bar{G}_j$ and $\bar{E}_j$ commute.

**Theorem 2** *If $r_1 = r_2$ and for every $j = 1, 2, \cdots, k$, $E_j = G_j^*$, then $B_j$ is positive semi-definite and*

$$\|\bar{G}_j^* T_j \bar{G}_j\|_j \le \|B_j\|_j \le \|\bar{G}_j^*(T_j + \|B_{j-1}\|_{j-1} S_j)\bar{G}_j\|_j. \tag{19}$$

**Proof.**

Since $I_j^{j-1} = (I_{j-1}^j)^T$, $T_j = (T_j)^*$ and $S_j = (S_j)^*$, each $B_j$ is symmetric with respect to the energy inner product ($B_j = (B_j)^*$). Each $B_j$ is also positive semi-definite. This is most easily seen by induction. $B_2$ is positive semi-definite since $(B_2 w_2, w_2)_2 = \|T_2 \bar{G}_2 w_2\|_2^2$ for every $w_2 \in H_2$. Assume $B_{j-1}$ is positive semi-definite for some $j > 2$. From formula (18),

$$(B_j w_j, w_j)_j = (\bar{G}_j^* T_j \bar{G}_j w_j, w_j)_j + (\bar{G}_j^* I_{j-1}^j B_{j-1} A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j, w_j)_j.$$

Since $(T_j)^2 = T_j$ for arbitrary $w_j \in H_j$, we can write

$$(\bar{G}_j^* T_j \bar{G}_j w_j, w_j)_j = (T_j \bar{G}_j w_j, T_j \bar{G}_j w_j)_j$$

and, similarly, since $I_j^{j-1} = (I_{j-1}^j)^T$,

$$(\bar{G}_j^* I_{j-1}^j B_{j-1} A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j, w_j)_j = (B_{j-1}(A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j), (A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j))_{j-1}.$$

The right hand side of the previous equation is non-negative by the inductive assumption and thus

$$\begin{aligned} (B_j w_j, w_j)_j &= (T_j \bar{G}_j w_j, T_j \bar{G}_j w_j)_j + (B_{j-1}(A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j), (A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j))_{j-1} \\ &\geq \|T_j \bar{G}_j w_j\|_j^2 \geq 0. \end{aligned} \tag{20}$$

Therefore $B_j$ is positive semi-definite.

We can also use equation (20) to obtain the lower bound for the $\|B_j\|_j$'s in equation (19).

To prove the upper bound in equation (19), let $w_j$ be an arbitrary vector in $H_j$. By the definition of $\|B_{j-1}\|_{j-1}$ and $S_j$,

$$\begin{aligned} (\bar{G}_j^* I_{j-1}^j B_{j-1} A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j, w_j)_j &= (B_{j-1} A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j, A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j)_{j-1} \\ &\leq \|B_{j-1}\|_{j-1} (A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j, A_{j-1}^{-1} I_j^{j-1} A_j \bar{G}_j w_j)_{j-1} \\ &= \|B_{j-1}\|_{j-1} (\bar{G}_j^* S_j \bar{G}_j w_j, w_j)_j. \end{aligned}$$

Therefore,

$$(B_j w_j, w_j)_j \leq ((\bar{G}_j^* T_j \bar{G}_j + \|B_{j-1}\|_{j-1} \bar{G}_j^* S_j \bar{G}_j) w_j, w_j)_j. \tag{21}$$

This proves the upper bound in equation (19) since, for any symmetric matrix, $D_j$,

$$\|D_j\|_j = \sup_{\substack{w_j \in H_j \\ \|w_j\| \neq 0}} \frac{(D_j w_j, w_j)_j}{(w_j, w_j)_j}.$$

Let $\gamma_1 = 0$ and define $\gamma_k$ recursively as

$$\gamma_k = \|\bar{G}_k^*(T_k + \gamma_{k-1}S_k)\bar{G}_k\|_k. \tag{22}$$

The next theorem asserts that each $\gamma_k$ is a bound on the $k$-grid convergence rate.

**Theorem 3** *For every $k \geq 2$,*

$$\|B_k\|_k \leq \gamma_k.$$

**Proof.**

From the definition of the $\gamma$'s, we see that

$$\|B_2\|_2 = \gamma_2$$

If $\|B_{j-1}\|_{j-1} \leq \gamma_{j-1}$, then for each $u_j \in H_j$,

$$\begin{aligned}
(B_j u_j, u_j)_j &\leq (\bar{G}_j^*(T_j + \|B_{j-1}\|_{j-1}S_j)\bar{G}_j u_j, u_j)_j \\
&= \|T_j\bar{G}_j u_j\|_j^2 + \|B_{j-1}\|_{j-1}\|S_j\bar{G}_j u_j\|_j^2 \\
&\leq \|T_j\bar{G}_j u_j\|_j^2 + \gamma_{j-1}\|S_j\bar{G}_j u_j\|_j^2 \\
&= (\bar{G}_j^*(T_j + \gamma_{j-1}S_j)\bar{G}_j u_j, u_j)_j
\end{aligned}$$

and therefore $\|B_j\|_j \leq \gamma_j$.

In order to compare these bounds with the theoretical bounds of Theorem 3.1, we define a constant, $\beta$, independent of the number of grid levels, such that

$$\beta = \inf_{k>1}\left(\inf_{\substack{u_k \in H_k \\ T_k\bar{G}_k u_k \neq 0}}\left(\frac{\|u_k\|_k^2 - \|\bar{G}_k u_k\|_k^2}{\|T_k\bar{G}_k u_k\|_k^2}\right)\right).$$

From this definition we see that, for every $j > 1$,

$$\|\bar{G}_j u_j\|_j^2 + \beta\|T_j\bar{G}_j u_j\|_j^2 \leq \|u_j\|_j^2 \tag{23}$$

or equivalently,

$$\|T_j\bar{G}_j u_j\|_j^2 + \frac{1}{1+\beta}\|S_j\bar{G}_j u_j\|_j^2 \leq \frac{1}{1+\beta}\|u_j\|_j^2. \tag{24}$$

**Note:** The constant $\beta$ is given for some common relaxation methods and for piecewise (bi)linear finite elements (i.e., (bi)linear interpolation for finite differences) in [3]. Equation (23) holds for $\beta = 0$ whenever the iteration matrix of the relaxation, $\bar{G}_j$, is a contraction. The convergence rate of the V–cycle can be shown to be bounded away from one only when equation (23) holds for a positive $\beta$.

**Theorem 4** *For all $k \geq 1$,*

$$\gamma_k \leq 1/(1+\beta). \tag{25}$$

**Proof.**

Equation (25) holds for $\gamma_1$ since $\gamma_1 = 0$ and $\beta$ is non-negative.

Assume $\gamma_{j-1} \leq 1/(1+\beta)$ for $j \geq 2$. For any $w_j \in H_j$, equation (24) implies that

$$\begin{aligned}(\bar{G}_j^*(T_j + \gamma_{j-1}S_j)\bar{G}_j w_j, w_j)_j &= (\bar{G}_j^* T_j \bar{G}_j w_j, w_j)_j + \gamma_{j-1}(\bar{G}_j^* S_j \bar{G}_j w_j, w_j)_j \\ &\leq \frac{1}{1+\beta}(w_j, w_j)_j.\end{aligned}$$

Since $\bar{G}_j^*(T_j + \gamma_{j-1}S_j)\bar{G}_j$ is symmetric, then $\gamma_j \leq 1/(1+\beta)$ by definition (22).

Finally, we note that this new technique has another useful feature. If the $k$-grid convergence rate is known, then the $k+1$-grid rate can be estimated. Thus, if the convergence rate of a V-cycle for a certain number of grids is known, it is possible to predict the effect of adding one additional grid.

In the next section we show how to calculate the convergence rate bounds, $\gamma_k$.

## 5. Details of the Fourier Analysis

For certain model problems, it is possible to calculate exact convergence rates of the two grid $V$–cycle. The same techniques can be used to compute the $k$-grid convergence rate bounds of Theorem 4 for the symmetric $V$-cycle.

The Fourier mode $exp(i\underline{\theta} \cdot \underline{x}/h_k)$ on level $k$ appears as the mode $exp(i2\underline{\theta} \cdot \underline{x}/h_{k-1})$ on level $k-1$ since $h_k = .5h_{k-1}$. Therefore, on level $k-1$ it coincides with every Fourier mode $exp(i\underline{\theta'} \cdot \underline{x}/h_k)$ for which $\underline{\theta'} = \underline{\theta}(mod\pi)$. Thus restriction operators introduce coupling between each lower mode $\underline{\theta}$ and its $(2^d - 1)$ high-frequency harmonics $\{\underline{\theta'} : \pi/2 \leq |\underline{\theta'}| \leq \pi, \underline{\theta'} = \underline{\theta}(mod\pi)\}$. Interpolation introduces coupling among the same modes. This is summarized in the following two formulas

$$I_{k-1}^k exp(i\underline{\theta} \cdot \underline{x}/h) = \sum_{\underline{\theta'} = \underline{\theta}(mod\pi)} \hat{I}_{k-1}^k(\underline{\theta'}) exp(i\underline{\theta} \cdot \underline{x}/h) \tag{26}$$

and

$$I_k^{k-1} exp(i\underline{\theta}' \cdot \underline{x}/h) \quad = \quad \hat{I}_k^{k-1}(\underline{\theta}') exp(i\underline{\theta} \cdot \underline{x}/h). \quad \underline{\theta}' = \underline{\theta}(mod\pi). \tag{27}$$

$\hat{I}_k^{k-1}(\underline{\theta})$ and $\hat{I}_{k-1}^k(\underline{\theta})$ are called the symbols of $I_k^{k-1}$ and $I_k^{k-1}$ respectively.

Since the fine and coarse grid operators, as well as the relaxation, do not introduce coupling of more Fourier modes we conclude that the two level cycle has the $2^d$ harmonic modes as its invariant subspace. Hence the convergence properties of the multigrid cycle can be studied by looking only at the interactions of each set of $2^d$ Fourier components.

Interpolation and restriction can be represented in matrix form as

$$\tilde{I}_{k-1}^k(\underline{\theta}) = \begin{bmatrix} \hat{I}_{k-1}^k(\underline{\theta}^1) \\ \vdots \\ \hat{I}_{k-1}^k(\underline{\theta}^{2^d}) \end{bmatrix}_{2^d \times 1} \tag{28}$$

and

$$\tilde{I}_k^{k-1}(\underline{\theta}) = \begin{bmatrix} \hat{I}_k^{k-1}(\underline{\theta}^1) & \dots & \hat{I}_k^{k-1}(\underline{\theta}^{2^d}) \end{bmatrix}_{1 \times 2^d}. \tag{29}$$

The fine grid operator is represented as

$$\tilde{A}_k(\underline{\theta}) = \begin{bmatrix} \hat{A}_k(\underline{\theta}^1) & & \\ & \ddots & \\ & & \hat{A}_k(\underline{\theta}^{2^d}) \end{bmatrix}_{2^d \times 2^d}, \tag{30}$$

the coarse grid operator as

$$\tilde{A}_{k-1}(2\underline{\theta}) = \begin{bmatrix} \hat{A}_{k-1}(2\underline{\theta}^1) \end{bmatrix}_{1 \times 1}, \tag{31}$$

and the relaxation matrix as

$$\tilde{G}_k(\underline{\theta}) = \begin{bmatrix} g(\underline{\theta}^1) & & \\ & \ddots & \\ & & g(\underline{\theta}^{2^d}) \end{bmatrix}_{2^d \times 2^d}. \tag{32}$$

In terms of these matrices the two level cycle is represented by the following $2^d \times 2^d$ matrix

$$\tilde{B}_k(\underline{\theta}) = \left[\tilde{G}_k(\underline{\theta})\right]^{r2} \left[I - \tilde{I}_{k-1}^k(\underline{\theta})(\tilde{A}_{k-1}(2\underline{\theta}))^{-1} \tilde{I}_k^{k-1}(\underline{\theta}) \tilde{A}_k(\underline{\theta})\right] \left[\tilde{G}_k(\underline{\theta})\right]^{r1}. \tag{33}$$

Take the simple case of the $V$-cycle used as a fast Laplace solver on the unit square. A few simple relaxation methods (such as Richardson or damped Jacobi) can be easily analyzed for this

particular problem simply because the Fourier modes are eigenvectors of the relaxation. For this 2 dimensional problem, the two grid convergence rates are obtained by computing the largest (in absolute value) eigenvalue over all the $4 \times 4$ matrices given by formula (33). The bounds for the convergence rates for more than two grids are computed in exactly the same manner, inserting a constant in formula (33), preceding the interpolation symbol, in order to obtain the $\gamma_k$ of Section 4.

For completeness we include some examples of some operators and the elements in their symbols. If $-A_k^{(5)}$ corresponds to the five point Laplacian, given by the stencil

$$A_k^{(5)} = \frac{1}{h_k^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_{h_k}, \tag{34}$$

then

$$\hat{A}_k^{(5)}(\underline{\theta}) = \frac{4(\sin^2(\theta_1/2) + \sin^2(\theta_2/2))}{h_k^2}, \tag{35}$$

where $\underline{\theta} = (\theta_1, \theta_2)$.

For bi-linear interpolation and the residual restriction operator which is its adjoint (usually called full weighting), the elements of the symbol matrices are given by

$$\hat{I}_{k-1}^k(\underline{\theta}) = (\frac{1+\cos\theta_1}{2})(\frac{1+\cos\theta_2}{2}), \tag{36}$$

and

$$\hat{I}_k^{k-1}(\underline{\theta}) = (\frac{1+\cos\theta_1}{2})(\frac{1+\cos\theta_2}{2}). \tag{37}$$

A simple relaxation, damped Richardson (which is the same as damped Jacobi in simple cases we consider), the elements of $\tilde{G}_k(\underline{\theta})$ are given by

$$g(\underline{\theta}) = 1 - 2\omega c \hat{A}_k(\underline{\theta}) \tag{38}$$

where $c^{-1} = \sup_{\underline{\theta}} \hat{A}_k(\underline{\theta})$ and $0 < \omega < 1$.

When the coarse grid operators are determined "variationally", i.e., $A_{k-1} = I_k^{k-1} A_k I_{k-1}^k$, then

$$\hat{A}_{k-1}(2\underline{\theta}) = \sum_{i=1}^{4} \hat{I}_{k-1}^k(\underline{\theta}^i) \hat{A}_k(\underline{\theta}^i) \hat{I}_k^{k-1}(\underline{\theta}^i). \tag{39}$$

If $A_k^{(9)}$ is a nine point discrete Laplacian, given by the stencil

$$A_k^{(9)} = \frac{1}{3h_k^2} \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}_{h_k}, \tag{40}$$

then

$$\hat{A}_k^{(9)}(\underline{\theta}) = \frac{8 - 2\cos(\theta_1) - 2\cos(\theta_2) - 4\cos(\theta_1)\cos(\theta_2)}{3h_k^2}. \tag{41}$$

If $A_{k-1}^{(9)}$ is given by the corresponding stencil on the $h_{k-1}$ grid:

$$A_{k-1}^{(9)} = \frac{1}{3h_{k-1}^2}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}_{h_{k-1}}, \tag{42}$$

then

$$\hat{A}_{k-1}^{(9)}(\underline{\theta}) = \frac{8 - 2\cos(2\theta_1) - 2\cos(2\theta_2) - 4\cos(2\theta_1)\cos(2\theta_2)}{3h_{k-1}^2}. \tag{43}$$

This coarse grid operator, though corresponding to the same stencil as the fine grid operator, is in fact also the "variational" coarse grid operator when using bi-linear interpolation and full weighting. This is convenient when trying to use the variational theory but trying to keep the symbols of the discrete operators simple.

## 6. Results and Comparisons

In [2], it is observed that the variationally derived bounds are too pessimistic, at least for typical parameters in a multigrid V–cycle when using a modest number of grids. Although in Section 4 we prove that the $k$-grid bounds are smaller than the grid independent bounds of the variational theory, it is not known, in general, how much better the new bounds are. The limit of the bounds, $\gamma_k$, as $k$ goes to infinity, does not always degrade to the theoretical rate of $1/(1+\beta)$. For a modest number of grids, the new estimates are much closer to the actual convergence rates.

Moreover, we note that this new technique has another useful feature. If the $k$-grid convergence rate is known, then the $k+1$-grid rate can be estimated. Thus it is easy to predict the effect of adding an extra grid to a given multigrid cycle.

Consider the simplest case of a one dimensional problem given by

$$-\frac{d^2u}{dx^2} = f \qquad \text{on } (0,1) \tag{44}$$

$$u(0) = a\,; \qquad u(1) = b$$

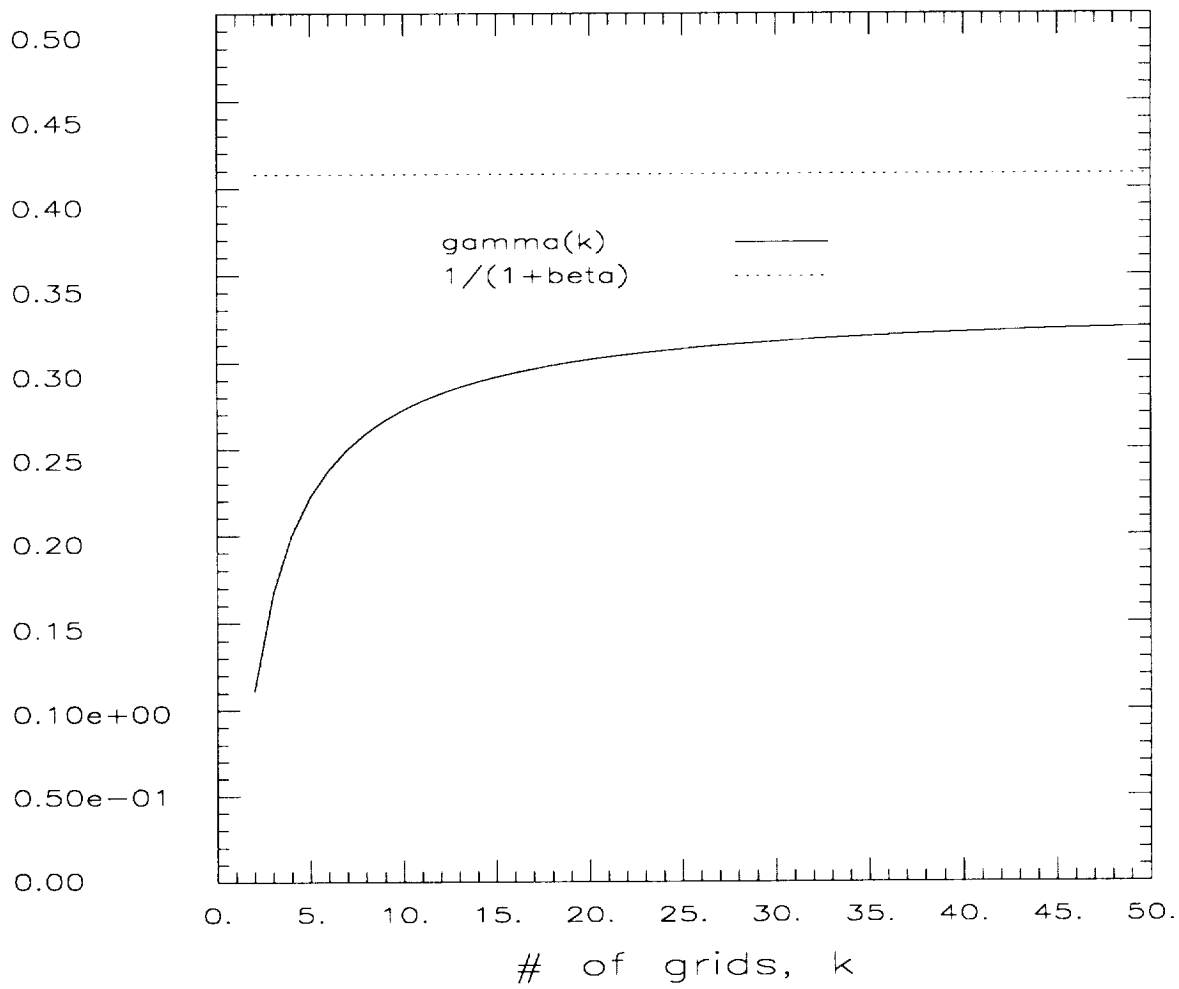


Figure 1: The convergence rate bounds for the one dimensional model problem

Define a $V$-cycle using the standard three point discretization of the second derivative, linear interpolation and damped Jacobi (with $\omega = 2/3$) for the relaxation. The convergence rate bounds predicted in Section 4 for a $V(1,1)$ cycle (one pre- and one post-relaxation) are given in Figure 1. The $\gamma_k$ are shown as a function of the number of grid levels, $k$. In contrast, the variational theory gives a convergence rate bound equal to 0.408, see [2].

Tables 1 and 2 compare the exact three grid convergence rates (computed with a three grid Fourier analysis) to the bounds predicted by our theory. For the optimal $\omega$, $\omega = 2/3$, the bounds stay close for $r_1 = r_2 = 1, \cdots, 4$.

| $r_1$ | $r_2$ | $\omega = 1/2$ | $\omega = 2/3$ | $\omega = 3/4$ |
|---|---|---|---|---|
| 1 | 1 | .2665 | .1655 | .2499 |
| 2 | 2 | .1086 | .0826 | .0903 |
| 3 | 3 | .0743 | .0562 | .0539 |
| 4 | 4 | .0570 | .0430 | .0392 |

Table 1: One-dimensional exact three grid convergence rates of $V(r_1, r_2)$ using damped Jacobi with damping parameter $\omega$

| $r_1$ | $r_2$ | $\omega = 1/2$ | $\omega = 2/3$ | $\omega = 3/4$ |
|---|---|---|---|---|
| 1 | 1 | .2667 | .1667 | .2499 |
| 2 | 2 | .1161 | .0887 | .0984 |
| 3 | 3 | .0804 | .0610 | .0582 |
| 4 | 4 | .0617 | .0466 | .0425 |

Table 2: One-dimensional estimated three grid convergence rates of $V(r_1, r_2)$ using damped Jacobi with damping parameter $\omega$

Finally, for Poisson's equation with Dirichlet boundary conditions on the unit square, we compare our bounds to the asymptotic convergence rates seen experimentally. Using the grid sizes indicated in the first column of table 3, we ran experiments using a damped Jacobi relaxation, the nine-point discretization of the Laplacian, $A_k^{(9)}$ as given in Section 5 and bilinear interpolation and full weighting. Starting with a random initial error, the $V(1,1)$ multigrid cycle was used, rescaling the error after every cycle in order to see the asymptotic convergence rate. The center column contains the bounds given by our method. The grid independent bound given by the variational theory is 0.40, see [3].

| grid sizes | our bounds | worst case-experimental |
|---|---|---|
| $h = (1/4, 1/2)$ | .110 | .110 |
| $h = (1/8, 1/4, 1/2)$ | .217 | .211 |
| $h = (1/16, 1/8, 1/4, 1/2)$ | .258 | .241 |
| $h = (1/32, 1/16, 1/8, 1/4, 1/2)$ | .286 | .246 |
| $h = (1/8, 1/4)$ | .206 | .206 |
| $h = (1/16, 1/8, 1/4)$ | .254 | .239 |
| $h = (1/32, 1/16, 1/8, 1/4)$ | .284 | .245 |
| $h = (1/16, 1/8)$ | .238 | .238 |
| $h = (1/32, 1/16, 1/8)$ | .275 | .244 |

Table 3: Comparison of bounds with actual rates for two-dimensional Laplacian (9 pt. stencil) using damped Jacobi with damping parameter .75 and $r = 1$. The grid independent bound is 0.40.

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**16. Abstract**

For special model problems, Fourier analysis gives exact convergence rates for the two-grid multigrid cycle and, for more general problems, provides estimates of the two-grid convergence rates via local mode analysis. A method is presented for obtaining multigrid convergence rate estimates for cycles involving more than two grids -- using essentially the same analysis as for the two-grid cycle.

For the simple case of the V-cycle used as a fast Laplace solver on the unit square, the k-grid convergence rate bounds obtained by this method are sharper than the bounds predicted by the variational theory. Both theoretical justification and experimental evidence are presented.

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