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# A General Formalism for Phase Space Calculations

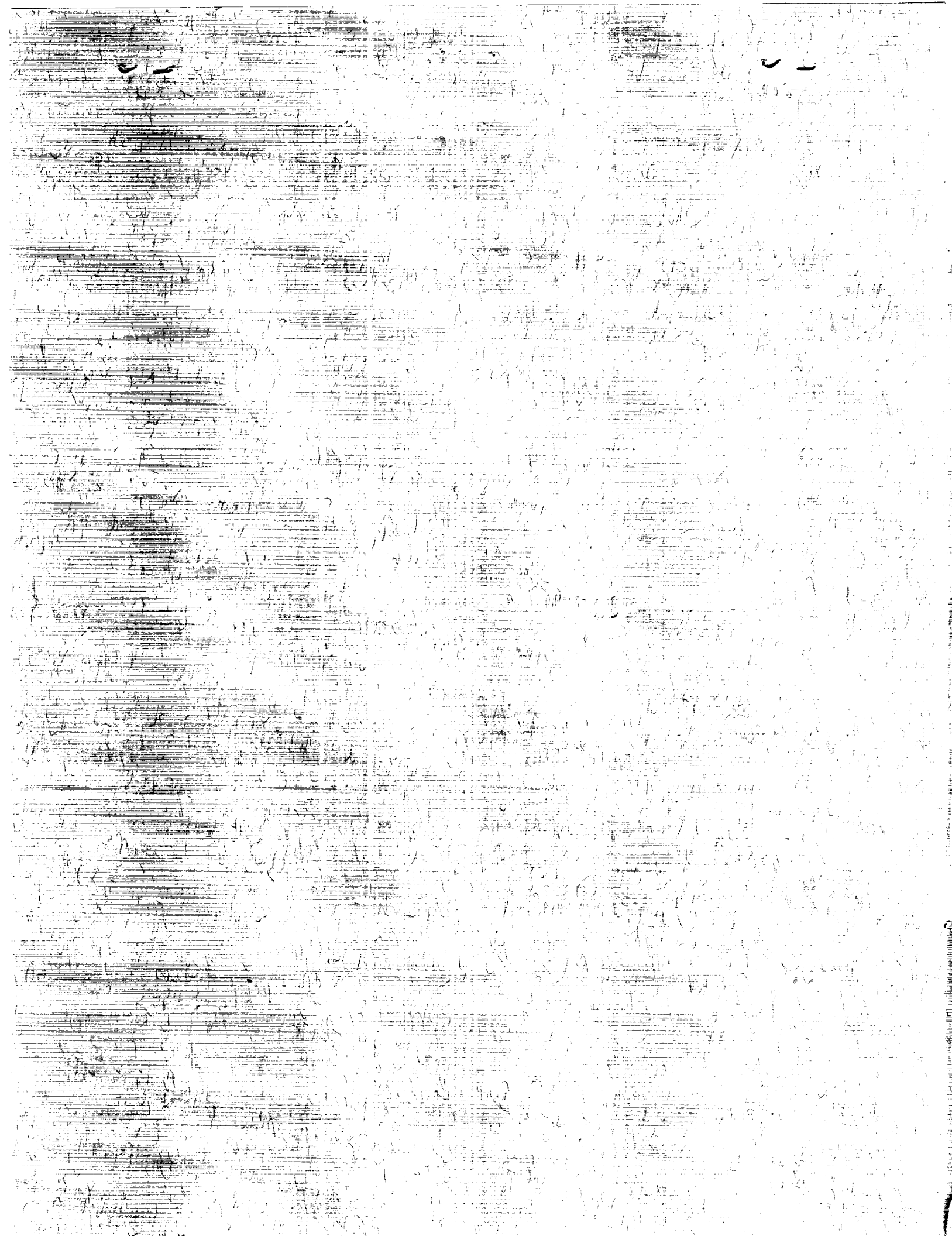
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# A General Formalism for Phase Space Calculations

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## Introduction

In the upcoming era of permanent occupation of space by humans, the problem of adequate protection from cosmic rays assumes greater and greater importance. An extensive research program is currently underway (ref. 1) to address this concern. In particular, this program is concerned with the calculation of cross sections describing the interaction of galactic cosmic ray nuclei with target nuclei.

In the present work, we wish to show how to set up the general formulas for these cross sections (once the T-matrix elements are known) and how to obtain and handle the appropriate normalization volume elements which occur in the phase space and wave functions. More importantly, however, we wish to show how to calculate the correct phase space factors for two- and three-body final states. We shall present these calculations for both Lorentz-invariant and noninvariant phase spaces (ref. 2) and shall also show the equivalence of the two approaches that use energy derivatives and energy delta functions.

The purpose of this report is to present the results of diverse phase space calculations in a consistent and unified manner. All too often the results of phase space calculations are presented in different ways depending on who is the author and what particular reaction is being studied. Thus, when one tries to calculate phase space factors for a particular reaction, it is often almost impossible to use the results of another author. Thus, a general formalism for the evaluations of phase space factors is needed that can be applied to any particular reaction. The specific results of other authors can be obtained through the use of our general results.

## Cross Section

In scattering problems (as opposed to bound state problems) one introduces the concept of cross section  $\sigma$ , defined as (refs. 3-6)

$$\sigma = \frac{w}{F_i} \quad (1)$$

where  $w$  is the transition rate and  $F_i$  is the incident flux defined as the number of particles per unit area

## Noninvariant Phase Space

The noninvariant phase space element (or density of states) is defined as (ref. 2)

per unit time, that is,

$$F_i = n_i \rho_i v \quad (2)$$

where  $n_i$  is the number of incident particles,  $\rho_i$  is the incident probability density (i.e., per unit volume, so that  $n_i \rho_i$  is the number of incident particles per unit volume), and  $v$  is the incident velocity. Frauenfelder (ref. 3) points out that in most calculations the number of incident particles is normalized to one particle per volume  $V$ , that is,

$$\int \Phi_i^*(\mathbf{r}) \Phi_i(\mathbf{r}) dV = 1 \quad (3)$$

Thus we shall take

$$n_i = 1 \quad (4)$$

Now in quantum mechanics,

$$\rho_i = \Phi_i^* \Phi_i = \frac{1}{V} \quad (5a)$$

consistent with one particle per volume  $V$ , whereas in classical mechanics the probability is unity always so that again

$$\rho_i = \frac{1}{V} \quad (5b)$$

Thus,

$$F_i = \frac{v}{V} \quad (6)$$

and the cross section can be written (ref. 2, p. 12)

$$\sigma = \frac{V}{v} w \quad (7)$$

where the transition rate is given by

$$w = \frac{2\pi}{\hbar} \int |T_{ki}|^2 dN_{ips} \quad (8)$$

where  $T_{ki}$  is the appropriate matrix element and  $dN_{ips}$  represents the phase space elements. The units and volume normalization conventions are explained in appendixes A and B.

$$\rho_N \equiv dN_{\text{ips}}(E; p_1, p_2, \dots, p_N) \equiv \left[ \frac{V}{(2\pi\hbar)^3} \right]^{N-1} \left( \prod_{i=1}^N d^3 p_i \right) \delta \left( E - \sum_{i=1}^N E_i \right) \delta^3 \left( \mathbf{p} - \sum_{i=1}^N \mathbf{p}_i \right) \quad (9)$$

(This definition of  $dN_{\text{ips}}$  differs slightly from that given on p. 388 of ref. 7). Frauenfelder (ref. 3) writes this as

$$\rho_N = \left[ \frac{V}{(2\pi\hbar)^3} \right]^{N-1} \frac{d}{dE} \prod_{i=1}^{N-1} d^3 p_i \quad (10)$$

where  $E$  is the total energy of the final-state particles. In this version, the  $\delta(E - \sum_{i=1}^N E_i)$  has been replaced with  $d/dE$  and the  $\delta^3(\mathbf{p} - \sum_{i=1}^N \mathbf{p}_i)$  has been taken into account by using  $\prod_{i=1}^{N-1}$  in equation (10) rather than the  $\prod_{i=1}^N$  in equation (9). Although Frauenfelder's version in equation (10) gives the same results as equation (9), we do not recommend its use because the inclusion of energy conservation is very obscure (being much more transparent and automatic in eq. (9)) and elementary errors are all too easy to make, such as bringing kinematic factors outside the momentum integrals when they should remain inside. We recommend using equation (9) to evaluate phase space factors, and we shall illustrate how to use it in the following sections.

## Two-Body Phase Space

The two-body density is

$$\begin{aligned} dN_{\text{ips}}(E; p_1, p_2) &= \frac{V}{(2\pi\hbar)^3} \int d^3 p_1 d^3 p_2 \delta(E - E_{12}) \delta^3 \left( \mathbf{p} - \sum_{i=1}^2 \mathbf{p}_i \right) \\ &= \frac{V}{(2\pi\hbar)^3} \int d^3 p_1 \delta(E - E_{12}) \\ &= \frac{V}{(2\pi\hbar)^3} \iint p_1^2 dp_1 \delta(E - E_{12}) d\Omega_1 \end{aligned} \quad (11)$$

with

$$E_{12} \equiv E_1 + E_2 \quad (12)$$

From appendix C, we have

$$dp_1 = \frac{1}{p_1 c^2} \frac{E_1 E_2'}{E_1 + E_2'} dE_{12} \quad (\text{CM}_{12}) \quad (13)$$

which is valid in the center of mass of particles 1 and 2 ( $\text{CM}_{12}$ ). Note that  $E_2'$  is identical to  $E_2$ , but is now defined in a certain way as detailed in appendix C. Thus, equation (11) becomes

$$dN_{\text{ips}}(E; p_1, p_2) = \frac{V}{(2\pi\hbar)^3} \iint \frac{p_1}{c^2} \frac{E_1 E_2'}{E_1 + E_2'} \delta(E - E_{12}) dE_{12} d\Omega_1 \quad (14a)$$

$$= \frac{V}{(2\pi\hbar)^3} \frac{1}{c^2} \int p_1 \frac{E_1 E_2'}{E_1 + E_2'} d\Omega_1 \quad (14b)$$

(Note the concerns raised in appendix D). However, there is something perplexing about equation (14b). We have supposedly integrated over  $dp_1$ ; yet  $p_1$  is still a variable in equations (14), both explicitly and also in  $E_1$  and  $E_2'$ . This is resolved by considering the following. Energy conservation is required in equations (14) through  $\delta(E - E_{12})$ . (Momentum conservation is also included through  $\delta^3(\mathbf{p} - \sum_{i=1}^2 \mathbf{p}_i)$ .) In the CM frame, this implies that  $p_1^2 = p_2^2$ , as shown in appendix C.



Thus we need to express  $p_1$  as a function of  $E_{12}$ , that is,

$$p_1 = p_1(E_{12}) \quad (15a)$$

and then because

$$E_1 = E_1(p_1) \quad (15b)$$

$$E'_2 = E'_2(p_1) \quad (15c)$$

we have

$$E_1 = E_1(E_{12}) \quad (15d)$$

$$E'_2 = E'_2(E_{12}) \quad (15e)$$

Thus all terms in the integrand of equation (14a) are functions of  $E_{12}$ , and the  $\delta(E - E_{12})$  can then be incorporated nicely.

Consider the nonrelativistic limit in the  $CM_{12}$  frame. Then because  $p_1^2 = p_2^2$ , we have

$$E_{12} = \frac{p_1^2}{2m_1} + \frac{p_1^2}{2m_2} = \frac{p_1^2}{2\mu} \quad (CM_{12}) \quad (16)$$

so that

$$p_1 = p_1(E_{12}) = \sqrt{2\mu E_{12}} \quad (17)$$

where  $\mu$  is the reduced mass of particles 1 and 2. Consequently,

$$E_1 = \frac{p_1^2}{2m_1} = \frac{\mu E_{12}}{m_1} \quad (18a)$$

$$E'_2 = \frac{p_1^2}{2m_2} = \frac{\mu E_{12}}{m_2} \quad (18b)$$

so that

$$p_1 \frac{E_1 E'_2}{E_1 + E'_2} = \frac{\sqrt{2}}{M} (\mu E_{12})^{3/2} \quad (19)$$

where  $M$  is the total mass of particles 1 and 2. Consequently, equation (14a) becomes

$$\begin{aligned} dN_{\text{Nips}}(E; p_1, p_2)_{\text{NR}} &= \frac{V}{(2\pi\hbar)^3} \frac{1}{c^2} \frac{\sqrt{2}}{M} \mu^{3/2} \iint (E_{12})^{3/2} dE_{12} \delta(E - E_{12}) d\Omega_1 \\ &= \frac{V}{(2\pi\hbar)^3} \frac{1}{c^2} \frac{\sqrt{2}}{M} (\mu E)^{3/2} \int d\Omega_1 \end{aligned} \quad (20)$$

where  $E$  is the total initial (or final) energy and NR denotes the nonrelativistic limit.

An important comment is in order. Why write the perplexing equation (14b) when things worked out so clearly in equations (15) to (20)? Why not follow the latter procedure always? The answer is that this latter method worked only because we have a simple, analytic expression, equation (19), for the kinematic factor. This is generally not the case. It might often be more convenient to leave  $dN_{\text{Nips}}$  in the form of equation (14b) and, in conjunction, separately write the very complicated expressions for  $p_1(E_{12})$ ,  $E_1(E_{12})$ , and  $E'_2(E_{12})$  which are to be inserted into equation (14b).

This is illustrated by considering the *relativistic* evaluation of the two-body kinematic factor considered in appendix E.

### Three-body Phase Space

The three-body density is

$$\begin{aligned}
dN_{\text{ips}}(E; p_1, p_2, p_3) &= \frac{V^2}{(2\pi\hbar)^6} \int d^3p_1 d^3p_2 d^3p_3 \delta(E - E_{123}) \delta^3\left(\mathbf{p} - \sum_{i=1}^3 \mathbf{p}_i\right) \\
&= \frac{V^2}{(2\pi\hbar)^6} \int d^3p_1 d^3p_2 \delta(E - E_{123}) \\
&= \frac{V^2}{(2\pi\hbar)^6} \iint p_1^2 dp_1 \delta(E - E_{123}) d\Omega_1 d^3p_2
\end{aligned} \tag{21}$$

with

$$E_{123} \equiv E_1 + E_2 + E_3 \tag{22}$$

From appendix F we have

$$dp_1 = \frac{1}{p_1 c^2} \frac{E_1 E'_3}{E'_3 + (1 + p_2/p_1 \cos \theta_{12}) E_1} dE_{123} \quad (\text{CM}_{123}) \tag{23}$$

where  $\theta_{12}$  is the angle between particles 1 and 2. Thus equation (21) becomes

$$dN_{\text{ips}}(E; p_1, p_2, p_3) = \frac{V^2}{(2\pi\hbar)^6} \iint \frac{p_1}{c^2} \frac{E_1 E'_3}{E'_3 + (1 + p_2/p_1 \cos \theta_{12}) E_1} \delta(E - E_{123}) dE_{123} d\Omega_1 d^3p_2 \tag{24}$$

Evaluation of the kinematic factor gives

$$p_1 = p_1(E_{123}, p_2) \tag{25a}$$

and then because

$$E_1 = E_1(p_1) \tag{25b}$$

$$E'_3 = E'_3(p_1, p_2) \tag{25c}$$

we have

$$E_1 = E_1(E_{123}, p_2) \tag{25d}$$

$$E'_3 = E'_3(E_{123}, p_2) \tag{25e}$$

The  $p_2$  that appears here is consistent with the integral over  $d^3p_2$  in equation (24).

Evaluating the three-body phase space in the nonrelativistic limit, we have

$$E_{123} = \frac{p_1^2}{2\mu_{13}} + \frac{p_2^2}{2\mu_{23}} + \frac{p_1 p_2 \cos \theta_{12}}{m_3} \tag{26}$$

where

$$\mu_{ij} \equiv \frac{m_i m_j}{m_i + m_j} \tag{27}$$

Solving for  $p_1$  in equation (26), we have eliminated the variable  $p_1$  through

$$p_1 = p_1(E_{123}, p_2) = \frac{1}{\mu_{13}} \left[ -\frac{p_2 \cos \theta_{12}}{m_3} \pm \sqrt{\left(\frac{p_2 \cos \theta_{12}}{m_3}\right)^2 - \frac{2}{\mu_{13}} \left(\frac{p_2^2}{2\mu_{23}} - E_{123}\right)} \right] \quad (28)$$

which is analogous to equations (17) and (E17). One now simply substitutes equation (28) into equations (25b) and (25c) and these results into equation (24). That is, one follows exactly the same procedure that we used for the two-body case in equations (18) through (20). Clearly the final result is an algebraic monstrosity and is best handled by computer code. One must be very careful about the signs for the square root in equation (28).

We consider the relativistic three-body problem in appendix G. Ultrarelativistic kinematics are presented in appendix H for both two-body and three-body problems.

### Dalitz Plots

The Dalitz plot refers to three-body decays and tells us the kinematical distribution of particles 1 and 2 due to phase space alone. Any deviation from the phase space diagram is due to variables in the T-matrix. An excellent intuitive discussion of Dalitz plots is given by Leon (ref. 4, pp. 145-147) and Pilkuhn (ref. 8, p. 160).

The three-body density given in equation (21) is written now as

$$dN_{\text{ips}}(E; p_1, p_2, p_3) = \frac{V^2}{(2\pi\hbar)^6} \int p_1^2 dp_1 d\Omega_1 \int p_2^2 dp_2 d\Omega_2 \delta(E - E_{123}) \quad (29)$$

Recalling that  $\theta_{12}$  is the angle between particles 1 and 2, we can write (ref. 4, p. 57; ref. 9, p. 99; ref. 10, p. 329)

$$dN_{\text{ips}}(E; p_1, p_2, p_3) = \frac{V^2}{(2\pi\hbar)^6} 8\pi^2 \iint p_1^2 p_2^2 dp_1 dp_2 d(\cos \theta_{12}) \delta(E - E_{123}) \quad (30)$$

In equation (21) we replaced  $dp_1$  with a function of  $dE_{123}$ . In equation (30) we replace  $d(\cos \theta_{12})$  with a function of  $dE_{123}$  and replace  $dp_1$  and  $dp_2$  with functions of  $dE_1$  and  $dE_2$ , respectively. From appendix D, it is evident that

$$dp_1 = \frac{E_1}{p_1} dE_1 \quad (31a)$$

$$dp_2 = \frac{E_2}{p_2} dE_2 \quad (31b)$$

However, equation (F4) gives

$$d(\cos \theta_{12}) = \frac{E_{123}}{p_1 p_2 c^2} dE_{123} \quad (32)$$

so that

$$\begin{aligned} dN_{\text{ips}}(E; p_1, p_2, p_3) &= \frac{V^2}{(2\pi\hbar)^6} \frac{8\pi^2}{c^2} \iint E_1 E_2 E_{123} dE_1 dE_2 dE_{123} \delta(E - E_{123}) \\ &= \frac{V^2}{(2\pi\hbar)^6} \frac{8\pi^2}{c^2} E \int E_1 E_2 dE_1 dE_2 \end{aligned} \quad (33)$$

This, of course, is not the usual form of the Dalitz phase space, which is written using  $dL_{\text{ips}}$  where the product  $E_1 E_2$  in equation (33) does not appear. (See the subsequent section on the Lorentz-invariant form of Dalitz plots.)

## Recurrence Relations

The density of  $N$ -body final states can often be simplified in terms of products of two-body densities. Consider, for example, the reaction  $NN \rightarrow N\Delta \rightarrow NN\pi$ , where  $N$  represents a nucleon,  $\pi$  a pion, and  $\Delta$  an isobar. Normally we would simply write the three-body density  $dNips(E, p_N, p_N, p_\pi)$ , especially if the square of the T-matrix  $|\sum_n T_{kn} G_{ni} T_{ni}|^2$  cannot be simplified. However, if the full T-matrix simplifies to  $\sum_n |T_{kn}|^2 |G_{ni}|^2 |T_{ni}|^2$ , then the full cross section for  $NN\pi$  can be written as a product of cross sections for forming  $N\Delta$  and  $\Delta \rightarrow N\pi$ . Schematically this is

$$\sigma_{NN \rightarrow NN\pi} \approx \left| \sum_n T_{kn} G_{ni} T_{ni} \right|^2 \quad (34)$$

However, if

$$\sigma_{NN \rightarrow NN\pi} \approx \sum_n |T_{kn}|^2 |G_{ni}|^2 |T_{ni}|^2 \quad (35)$$

then

$$\sigma_{NN \rightarrow NN\pi} \approx (\sigma_{NN \rightarrow N\Delta})(\sigma_{\Delta \rightarrow N\pi}) \quad (36)$$

This equation shows intuitively that one might break the three-body factor  $dNips(E; p_N, p_N, p_\pi)$  into a product of  $dNips(E; p_N, p_\Delta)$  and  $dNips(E; p_N, p_\pi)$ . The result for three bodies is (ref. 2)

$$dNips(E; p_1, p_2, p_3) = dNips(E; p_c, p_3) dNips(E_c; p_1, p_2) dE_c \quad (37)$$

and for four bodies is

$$dNips(E; p_1, p_2, p_3, p_4) = dNips(E; p_c, p_d) dNips(E_c; p_1, p_2) dNips(E_d; p_3, p_4) dE_c dE_d \quad (38)$$

where  $dE_c$  and  $dE_d$  represent integration over the "nonobserved" particles. These equations can easily be generalized for  $N$  bodies.

Finally we note that even though the process  $NN \rightarrow N\Delta \rightarrow NN\pi$ , where we have a well-defined intermediate state, may be occurring, the T-matrix may not break up as in equation (35), and so the best method would be not to use a recurrence relation, but rather the full three-body phase spaces developed earlier in the text.

## Lorentz-Invariant Phase Space

We shall now repeat all the results thus far obtained and write them in terms of Lorentz-invariant phase space elements. (In this section we set  $\hbar = c = 1$ .) The Lorentz-invariant phase space elements are defined as (refs. 2, 8, and 11)

$$dLips(E; p_1, p_2, \dots, p_N) \equiv (2\pi)^{4-3N} \left( \prod_{i=1}^N \frac{d^3 p_i}{2E_i} \right) \delta^4 \left( p - \sum_{i=1}^N p_i \right) \quad (39)$$

This definition is  $(2\pi)^4$  times the definition of reference 12. Note that the essential difference between  $dNips$  (eq. (9)) and  $dLips$  is the appearance of  $d^3 p_i$  in  $dNips$  as opposed to  $d^3 p_i / E_i$  in  $dLips$ . This, in fact, makes the expressions for  $dLips$  somewhat simpler.

### Two-Body Phase Space

The two-body density is

$$dLips(E; p_1, p_2) = \frac{1}{4(2\pi)^2} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta(E - E_{12}) \delta^3(\mathbf{p} - \mathbf{p}_{12}) \quad (40)$$

with

$$\mathbf{p}_{12} = \mathbf{p}_1 + \mathbf{p}_2 \quad (41)$$

Equation (40) becomes

$$d\text{Lips}(E; p_1, p_2) = \frac{1}{4(2\pi)^2} \int \frac{p_1}{E_{12}} \delta(E - E_{12}) dE_{12} d\Omega_1 \quad (42)$$

analogous to equation (14a). In the nonrelativistic limit this is

$$d\text{Lips}(E; p_1, p_2)_{\text{NR}} = \frac{1}{4(2\pi)^2} \sqrt{\left(\frac{2\mu}{E}\right)} \int d\Omega_1 \quad (43)$$

analogous to equation (20). The relativistic expression is obtained by substituting equation (E17) into equation (42) and is much simpler than the  $d\text{Nips}$  expression. We give it as

$$d\text{Lips}(E; p_1, p_2)_{\text{Rel}} = \frac{1}{4(2\pi)^2} \frac{1}{2E^2} \sqrt{[E^2 - (m_1 + m_2)^2][E^2 - (m_1 - m_2)^2]} \int d\Omega_1 \quad (44)$$

### Three-Body Phase Space

The three-body density is

$$d\text{Lips}(E; p_1, p_2, p_3) = \frac{1}{8(2\pi)^5} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \delta(E - E_{123}) \delta^3(\mathbf{p} - \mathbf{p}_{123}) \quad (45)$$

with

$$\mathbf{p}_{123} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad (46)$$

Equation (45) becomes

$$d\text{Lips}(E; p_1, p_2, p_3) = \frac{1}{8(2\pi)^5} \int \frac{p_1}{E_2 E'_3 + (1 + p_2/p_1 \cos \theta_{12}) E_1} \delta(E - E_{123}) dE_{123} d\Omega_1 d^3 p_2 \quad (47)$$

and evaluation of the kinematic factor proceeds by substituting equation (28) into equation (47) for the nonrelativistic case. The relativistic case is considered in appendix G.

### Dalitz Plots

The Dalitz plot form of the three-body density is treated in equations (29) through (33) for  $d\text{Nips}$ . The Lorentz-invariant form is evaluated in exactly the same manner. The result is particularly simple

$$d\text{Lips}(E; p_1, p_2, p_3) = \int dE_1 dE_2 \quad (48)$$

which is the usual form (refs. 4, 8, and 12) for the Dalitz plot.

### Recurrence Relations

The Lorentz-invariant recurrence relations are considered in references 2, 8, and 12. The result for three bodies is

$$d\text{Lips}(E; p_1, p_2, p_3) = \frac{1}{(2\pi)} d\text{Lips}(E; p_c, p_3) d\text{Lips}(E_c; p_1, p_2) dE_c \quad (49)$$

and for four bodies (ref. 2) is

$$d\text{Lips}(E; p_1, p_2, p_3, p_4) = \frac{1}{(2\pi)^2} d\text{Lips}(E; p_c, p_d) d\text{Lips}(E_c; p_1, p_2) d\text{Lips}(E_d; p_3, p_4) dE_c dE_d \quad (50)$$

## **Concluding Remarks**

Methods of constructing phase space factors for two- and three-body densities have been presented for use in nuclear collision studies. Included in the discussion are methods for obtaining Lorentz-invariant ones. The factoring of higher order phase space factors into products of lower order ones was also presented for use in simplifying nuclear cross section calculations.

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## Appendix A

### Units

Consider the units of the quantities in equations (5) through (8). The units of  $T_{ki}$  are seen from

$$\begin{aligned} T_{ki} &= \langle k|t|i\rangle \\ &= \int \Psi_k^* t \Psi_i dV \end{aligned} \quad (\text{A1})$$

and using box normalization as in

$$\Psi_i = \frac{1}{\sqrt{V}} e^{ikr} \quad (\text{A2})$$

we see that the volume terms in equation (A1) cancel so that the units of  $T_{ki}$  are Energy. (Note that instead of  $T_{ki}$ , we can have terms like  $T_{kn}G_{ni}T_{ni}$  where  $G$  is Green's function with units (Energy) $^{-1}$  which cancel with the units of  $T_{ni}$ , so that  $T_{kn}G_{ni}T_{ni}$  has the same units as  $T_{ki}$ , namely Energy). Looking at equation (9), we see that the units of dNips are always (Energy) $^{-1}$ , no matter how many bodies are in the final state. Thus equation (8) can be written schematically as

$$w \propto \frac{c}{\hbar} (\text{Energy})^2 \frac{1}{\text{Energy}} \quad (\text{A3})$$

which gives the correct units for  $w$  of (Time) $^{-1}$ . Clearly then  $\sigma$ , in equation (7), has units of (Length) $^2$ .

In the above discussion, we have considered  $\Psi$  as a one-body wave function with normalization  $1/\sqrt{V}$ . However, in a reaction  $\Psi_i$  is typically a two-body state (beam plus target) and  $\Psi_k$  an  $N$ -body state. Consider a two-body final state. In second quantized form the (composite) two-body potential operator is (ref. 13)

$$T = \frac{1}{2} \sum_{ij} \sum_{kl} \langle ij|t|kl\rangle a_i^+ a_j^+ a_l a_k \quad (\text{A4})$$

$\begin{matrix} i \neq j \\ k \neq l \end{matrix}$

where

$$\langle ij|t|kl\rangle = \iint \Phi_i^*(\mathbf{r}_1) \Phi_j^*(\mathbf{r}_2) t(\mathbf{r}_1, \mathbf{r}_2) \Phi_k(\mathbf{r}_1) \Phi_l(\mathbf{r}_2) d^3r_1 d^3r_2 \quad (\text{A5})$$

so that obviously the units of  $\langle ij|t|kl\rangle$  are still Energy. The operators  $a$  and  $a^+$  have no units, so that  $T$  has units of Energy. When we take composite matrix elements as in nucleus-nucleus collisions, for example,  $\langle P_o T_o | T | P_o T_o \rangle$ , we eventually obtain  $\langle P_o T_o | T | P_o T_o \rangle$  in terms of  $\langle k_p(P) l_T(T) | t | k_p(P) l_T(T) \rangle$  (see eq. (36) of ref. 13) so that the units of  $\langle P_o T_o | T | P_o T_o \rangle$  are still obviously Energy. What happens when more than two particles are present in the final state as in the reaction  $NN \rightarrow N\Delta \rightarrow NN\pi$  (ref. 14)? The matrix element is of the same form  $T_{kn}G_{ni}T_{ni}$ , which can be written schematically as

$$\begin{aligned} &[\langle N | (\langle N | \langle \pi | T | \Delta \rangle) | N \rangle] \\ &\times [\langle N | \langle \Delta | G | N \rangle | N \rangle] [\langle N | \langle \Delta | T | N \rangle | N \rangle] \end{aligned} \quad (\text{A6})$$

From our consideration of  $\langle ij|t|kl\rangle$ , which we found to have Energy units, it is clear that  $G_{ni}$  and  $T_{ni}$  still have units of (Energy) $^{-1}$  and Energy, respectively. The only concern is the extra  $1/\sqrt{V}$  units carried by the newly created state  $\langle \pi |$ . These units, however, are canceled by extra units carried in the pion-nucleon interaction.

Let us summarize the ideas presented above. No matter how many bodies are present,  $T_{ki}$  always has units of Energy. If  $T_{ki}$  is replaced by terms such as  $T_{kn}G_{ni}T_{ni}$ , it still has units of Energy, again no matter for how many bodies. The units of dNips are always Energy. Consequently  $w$ , as defined in equation (8), always has units of (Time) $^{-1}$  and  $\sigma$  units of (Length) $^2$ .

## Appendix B

### Volume Normalization

This appendix represents a continuation of the concerns raised in appendix A, where we established that our methods resulted in a consistent system of units. Here we are interested in the explicit cancellation of the volume factors  $V$  appearing in the expressions for  $\sigma$  (eq. (7)),  $dN_{\text{ips}}$  (eq. (9)), and the wave function  $\Psi$  (eq. (A1)). One might question why one should worry about volume factors  $V$  canceling when the units have already been established as correct. The answer lies in how one actually takes limits in box normalization.

Consider the cross section expression from equations (7) and (8)

$$\sigma = \frac{V}{v} \frac{2\pi}{\hbar} \int |T_{ki}|^2 dN_{\text{ips}} \quad (\text{B1})$$

where (ref. 3)

$$dN_{\text{ips}} \propto V^{N-1} \quad (\text{For } N \geq 2) \quad (\text{B2a})$$

$$dN_{\text{ips}} \propto V \quad (\text{For } N = 1) \quad (\text{B2b})$$

We write  $\sigma$  schematically as

$$\sigma = (V|i\rangle^2)(\langle k|^2 V^{N-1}) \quad (\text{B3})$$

Now consider a one-body initial state and one-body final state. Then

$$|i\rangle^2 \propto 1/V \quad (\text{B4a})$$

$$\langle k|^2 \propto 1/V \quad (\text{B4b})$$

so that combining equations (B2b), (B4), and (B3), we see that the volume terms cancel. Now for a two-body state we do not write  $|i\rangle^2 \propto 1/V^2$  because we use a two-body center-of-mass wave function with normalization  $1/\sqrt{V}$ , so that equations (B4a) and (B4b) are still valid for the two-body center-of-mass reduced wave function. From equation (B2a), it is clear that the volume terms again cancel.

Let us be a little more specific for the two-body case. Consider nucleus-nucleus scattering and write

$$|i\rangle = |\Phi\rangle|P_o\rangle|T_o\rangle \quad (\text{B5})$$

where  $|i\rangle$  represents the total initial state;  $|\Phi\rangle$  represents the wave function for the motion of the overall CM and is proportional to  $1/\sqrt{V}$ ; and  $|P_o\rangle$  and  $|T_o\rangle$

represent the internal states of the projectile and target, respectively, each with their own normalizations, say  $1/\sqrt{V_p}$  and  $1/\sqrt{V_T}$ . Assume that the T-matrix separates as

$$t = t_{\Phi} t_p t_T \quad (\text{B6})$$

so that

$$\langle k|t|i\rangle = \langle \Phi_k|t_{\Phi}|\Phi_i\rangle \langle P_k|t_p|P_o\rangle \langle T_k|t_T|T_o\rangle \quad (\text{B7})$$

Now for example

$$\langle P_k|t_p|P_o\rangle \propto \frac{1}{\sqrt{V_p}} \frac{1}{\sqrt{V_p}} \Phi_k^* t_p \Phi_o dV_p \quad (\text{B8})$$

so that all the  $V_p$  volume terms cancel when the integrals are performed (so that taking the limit as  $V_p$  approaches  $\infty$  provides no difficulties). Let us conclude then that all matrix elements connected with the internal states do not display  $V$  factors explicitly. However, for the CM states in  $\langle \Phi_k|T_{\Phi}|\Phi_i\rangle$ , the cross section  $\sigma$  is defined in terms of the normalization of  $|\Phi_i\rangle$ , which is why the  $V$  factor appears explicitly in equations (5) to (7). Clearly the  $1/\sqrt{V}$  factor of  $|\Phi_i\rangle$  (when squared) always cancels the  $V$  factor of equation (7) because of the way they were defined originally. Similarly (in the two-body case), the  $1/\sqrt{V}$  factor of  $|\Phi_k\rangle$  (when squared) cancels the  $V$  factor from the two-body  $dN_{\text{ips}}$  because again the  $dN_{\text{ips}}$  volume factor refers to the normalization of  $|\Phi_k\rangle$ .

Now consider a two-body initial state with an  $N$ -body final state. Clearly  $|\Phi_i\rangle$  volume factors cancel with the factor in equation (7) as before. The  $N$ -body final state is written

$$|k\rangle = |2\text{-body CM}\rangle |N-2 \text{ created particles}\rangle \quad (\text{B9})$$

so that the final state has  $2 + (N - 2)$ , that is,  $N$  particles, in the final state. However,  $|2\text{-body CM}\rangle$  carries only a  $1/\sqrt{V}$  factor (as in eq. (B4b)), whereas the created particles each carry a  $1/\sqrt{V}$  factor. (The two-body CM is like a single reduced particle as explained previously.) Thus,

$$|k\rangle \propto \frac{1}{\sqrt{V}} \left( \frac{1}{\sqrt{V}} \right)^{N-2} \quad (\text{B10})$$



so that

$$\langle k|^2 \propto \frac{1}{V^{N-1}} \quad (\text{B11})$$

which always cancels the  $V^{N-1}$  from  $dN_{\text{ips}}$ , as it should because of the way  $dN_{\text{ips}}$  is constructed.

Finally we consider  $T_{ki}$  replaced by terms like  $T_{kn}G_{ni}T_{ni}$ . The  $\langle k|$  and  $|i\rangle$  normalizations cancel  $dN_{\text{ips}}$  and  $F_i$ , respectively, exactly as described above. Any extra  $1/\sqrt{V}$  terms arising should be canceled by the integrals arising in the matrix elements

exactly as described for equation (B7). We summarize as follows: the normalization of  $|i\rangle$  (eq. (B4a)) must be written explicitly as  $1/\sqrt{V}$  in order to cancel the normalization of  $F_i$ . Similarly the normalization in  $|k\rangle$  (eq. (B11)) must be written explicitly to cancel  $V^{N-1}$  in  $dN_{\text{ips}}$ . All other normalizations (such as internal states or terms arising in  $T_{kn}G_{ni}T_{ni}$ ) are canceled as in the integrals (eq. (B8)). The apparent inconsistency (i.e.,  $|i\rangle$  and  $\langle k|$  not canceling in integrals) is due to our original definitions of  $F_i$  and  $dN_{\text{ips}}$  in terms of  $|i\rangle$  and  $|k\rangle$ .

## Appendix C

### Derivation of Equations (14)

The total energy of particles 1 and 2 is

$$\begin{aligned} E_{12} &\equiv E_1 + E_2 \\ &= \sqrt{(p_1c)^2 + (m_1c^2)^2} \\ &\quad + \sqrt{(p_2c)^2 + (m_2c^2)^2} \end{aligned} \quad (\text{C1})$$

However  $p_1$  and  $p_2$  are not independent variables because momentum is conserved. In the center-of-mass frame of particles 1 and 2,

$$\mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 \quad (\text{CM}_{12}) \quad (\text{C2})$$

so that

$$p_1^2 = p_2^2 \quad (\text{CM}_{12}) \quad (\text{C3})$$

Thus,

$$E_{12} = \sqrt{(p_1c)^2 + (m_1c^2)^2} + \sqrt{(p_1c)^2 + (m_2c^2)^2} \quad (\text{C4})$$

so that  $E_{12}$  is a function of only  $p_1$ , that is,

$$E_{12} = E_{12}(p_1) \quad (\text{C5})$$

Thus,

$$\frac{dE_{12}}{dp_1} = \frac{\partial E_{12}}{\partial p_1} = p_1c^2 \frac{E_1 + E'_2}{E_1E'_2} \quad (\text{CM}_{12}) \quad (\text{C6})$$

It is absolutely essential here to realize that we are now using

$$E'_2 \equiv \sqrt{(p_1c)^2 + (m_2c^2)^2} \quad (\text{C7})$$

and not the form of  $E_2$  in equation (C1). That is,  $E'_2$  in equation (C6) is only a shorthand for equation (C7). We have emphasized this with the prime on  $E'_2$ , which is identical to  $E_2$  except that it is to be written as in equation (C7).

Note also that equation (C6) is obtained from the Jacobians in appendix I (eqs. (I5) and (I6)) as

$$dp_1 = \frac{\partial p_1}{\partial E_{12}} dE_{12} \quad (\text{C8})$$

or

$$dp_1 = \left( \frac{\partial E_{12}}{\partial p_1} \right)^{-1} dE_{12} \quad (\text{C9})$$

## Appendix D

### Some Technical Points

We raise some technical mathematical details now and illustrate them with reference to the two-body problem. They can be immediately generalized to the full N-body integral in equation (9).

In equation (11) we had

$$\begin{aligned} & \int d^3 p_1 d^3 p_2 \delta(E - E_{12}) \delta^3 \left( \mathbf{p} - \sum_{i=1}^2 \mathbf{p}_i \right) \\ &= \int d^3 p_1 \delta(E - E_{12}) \end{aligned} \quad (\text{D1})$$

where the momentum-conserving delta function eliminated the integral over  $d^3 p_2$ . Note that momentum conservation is thereby imposed on the right-hand side (RHS) of equation (D1).

Energy and momentum are related via

$$E_1 = \sqrt{(p_1 c)^2 + (m_1 c^2)^2} \quad (\text{D2})$$

so that

$$p_1 dp_1 = E_1 dE_1 \quad (\text{D3})$$

Consequently

$$\begin{aligned} \int d^3 p_1 \delta(E - E_{12}) &= \iint p_1^2 dp_1 \delta(E - E_{12}) d\Omega_1 \\ &= \iint p_1 E_1 \delta(E - E_{12}) dE_1 d\Omega_1 \end{aligned} \quad (\text{D4})$$

The subject of the present appendix is the following question: Why is it not correct to let  $\delta(E - E_{12})$  eliminate the  $dE_1$  integral? Then we would obtain

$$\iint p_1 E_1 \delta(E - E_{12}) dE_1 d\Omega_1 \approx \int p_1 E_1 d\Omega_1 \quad (\text{D5})$$

After all, aren't we just repeating the procedure used in equation (D1)?

This question is very subtle, but absolutely vital in obtaining correct results. The incorrect result in

equation (D5) is radically different from the correct result in equations (14). The answer follows.

In the left-hand side (LHS) of equation (D1), all the variables are totally independent. In the RHS of equation (D1), they are not independent, but are constrained by momentum conservation. If the energy delta function

$$\delta(E - E_{12}) = \delta(E - E_1 - E_2) \quad (\text{D6})$$

were such that  $E_1$  and  $E_2$  are completely independent variables, then the procedure in equation (D5) would be correct. However,  $E_1$  and  $E_2$  are not independent; they are constrained by momentum conservation in equation (D1). In fact in the center-of-mass frame

$$\begin{aligned} \delta(E - E_1 - E_2) &= \delta \left( E - \sqrt{(p_1 c)^2 + (m_1 c^2)^2} \right. \\ &\quad \left. - \sqrt{(p_1 c)^2 + (m_2 c^2)^2} \right) \end{aligned} \quad (\text{D7})$$

so that the combination

$$\begin{aligned} & \int \delta \left( E - \sqrt{(p_1 c)^2 + (m_1 c^2)^2} \right. \\ & \quad \left. - \sqrt{(p_1 c)^2 + (m_2 c^2)^2} \right) dE_1 \neq 1 \end{aligned} \quad (\text{D8})$$

Thus the correct procedure (with reference to eq. (9)) is always

1. Let the  $\delta^3(\mathbf{p} - \sum_{i=1}^N \mathbf{p}_i)$  freely kill a  $d^3 p_i$  integral.
2. Then make sure that  $\delta(E - \sum_{i=1}^N E_i)$  kills an integral of the form  $d(\sum_{i=1}^N E_i)$  and not  $dE_i$ .

Forcing item 2 to occur always guarantees the correct kinematic factors.

## Appendix E

### Two-Body Relativistic Kinematic Factor

We define the metric tensor

$$\mathbf{g}_{\mu\nu} = \mathbf{g}^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (\text{E1})$$

which transforms contravariant 4-vectors  $A^\mu$  to covariant vectors  $A_\mu$  via

$$A_\mu = \mathbf{g}_{\mu\nu} A^\nu \quad (\text{E2})$$

Now

$$A^\nu \equiv \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} = (A^0, \mathbf{A}) \quad (\text{E3})$$

so that

$$A_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} = \begin{bmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{bmatrix} \quad (\text{E4})$$

and so (ref. 8, p. 2)

$$A_\mu = (A^0, -\mathbf{A}) \quad (\text{E5})$$

The scalar product of two 4-vectors is therefore (ref. 8, p. 2)

$$\begin{aligned} A \cdot B &= A_\mu B^\mu = A_\mu \mathbf{g}^{\mu\nu} B_\nu \\ &= A^0 B^0 - \mathbf{A} \cdot \mathbf{B} \end{aligned} \quad (\text{E6})$$

Another way to obtain the above result is simply to use complex Minkowski space

$$A = (A^0, i\mathbf{A}) \quad (\text{E7})$$

but this has the disadvantage of hiding the true nature of the space-time metric (ref. 15).

The energy-momentum 4-vector is

$$p_\mu = (p^0, -\mathbf{p}c) = (E, -\mathbf{p}c) \quad (\text{E8a})$$

or again we would use

$$p = (E, i\mathbf{p}c) \quad (\text{E8b})$$

The magnitude of all 4-vectors are Lorentz-invariant quantities, so that

$$p^2 = p \cdot p = p_\mu p^\mu = E^2 - (|\mathbf{p}c|^2) = (mc^2)^2 \quad (\text{E9})$$

Now let us consider two-body kinematics. Aguilar-Benitez et al. (ref. 12, p. 58) consider two- and three-body kinematics from the point of view of a single particle decaying into two or three bodies. They consider

the kinematics in the rest frame of the decaying particle. However, in our case we have a reaction process where two particles in the initial state react to form two or three particles in the final state. However, because the overall center-of-mass (CM) frame is equivalent to the rest frame of the decaying particle, we can take over the results of reference 12 with minor modifications.

Let us define the total final 4-momentum of the final-state particles 1 and 2 as

$$p_{12} = (E_{12}, -\mathbf{p}_{12}c) \quad (\text{E10a})$$

and the invariant mass is

$$(M_{12}c^2)^2 = p_{12}^2 \quad (\text{E10b})$$

The overall CM frame is defined as

$$\mathbf{p}_{12} = \mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 \quad (\text{E11a})$$

that is,

$$p_{12} = (E_{12}, \mathbf{0}) = (M_{12}c^2, \mathbf{0}) \quad (\text{E11b})$$

Thus

$$\begin{aligned} p_{12}^2 &= E_{12}^2 = (M_{12}c^2)^2 = (p_1 + p_2)^2 \\ &= p_1^2 + p_2^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 \\ &= (m_1c^2)^2 + (m_2c^2)^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2c^2) \end{aligned} \quad (\text{E12})$$

but in the CM frame

$$\mathbf{p}_1 = -\mathbf{p}_2 \quad (\text{E13})$$

$$E_2 = E_{12} - E_1 \quad (\text{E14})$$

so that

$$\begin{aligned} E_{12}^2 &= (m_1c^2)^2 + (m_2c^2)^2 + 2[E_1(E_{12} - E_1) + |\mathbf{p}_1|^2] \\ &= (m_1c^2)^2 + (m_2c^2)^2 + 2[E_1(E_{12} - E_1) + E_1^2 - (m_1c^2)^2] \end{aligned} \quad (\text{E15})$$

and thus

$$E_1 = \frac{E_{12}^2 + (m_1c^2)^2 - (m_2c^2)^2}{2E_{12}} \quad (\text{E16})$$

so that (after some algebra)

$$|\mathbf{p}_1|c = \frac{\sqrt{[E_{12}^2 - (m_1 + m_2)^2c^4][E_{12}^2 - (m_1 - m_2)^2c^4]}}{\sqrt{4E_{12}^2}} \quad (\text{E17})$$

which is analogous to the result of reference 12 (p. 58).

Equation (E17) thus allows for a solution to equations (14). Although the result is not as simple as the nonrelativistic result of equation (20), it is still a simple matter to substitute equation (E17) into equation (14b) to obtain our solution. A computer code would best handle this.

As a final point, we stress that we have not used conservation of momentum or energy in the above analysis.

## Appendix F

### Derivation of Equation (23)

The total energy of particles 1, 2, and 3 is

$$\begin{aligned} E_{123} &\equiv E_1 + E_2 + E_3 \\ &= \sqrt{(p_1 c)^2 + (m_1 c^2)^2} + \sqrt{(p_2 c)^2 + (m_2 c^2)^2} + \sqrt{(p_3 c)^2 + (m_3 c^2)^2} \end{aligned} \quad (\text{F1})$$

where, again,  $p_1$ ,  $p_2$ , and  $p_3$  are not independent variables because of conservation of momentum. In the total center-of-mass frame of particles 1, 2, and 3,

$$\mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad (\text{CM}_{123}) \quad (\text{F2})$$

so that

$$p_3^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos \theta_{12} \quad (\text{F3})$$

where  $\theta_{12}$  is the relative angle between the momentum vectors of particles 1 and 2.  
Thus

$$\begin{aligned} E_{123} &= \sqrt{(p_1 c)^2 + (m_1 c^2)^2} + \sqrt{(p_2 c)^2 + (m_2 c^2)^2} \\ &\quad + \sqrt{(p_1 c)^2 + (p_2 c)^2 + 2p_1 p_2 c^2 \cos \theta_{12} + (m_3 c^2)^2} \end{aligned} \quad (\text{F4})$$

so that  $E_{123}$  is a function of both  $p_1$  and  $p_2$ , but not  $p_3$ , that is,

$$E_{123} = E_{123}(p_1, p_2) \quad (\text{F5})$$

Therefore,

$$\frac{dE_{123}}{dp_1} = \frac{\partial E_{123}}{\partial p_1} + \frac{\partial E_{123}}{\partial p_1} \frac{dp_2}{dp_1} \quad (\text{F6})$$

but

$$\frac{dp_2}{dp_1} = 0 \quad (\text{F7})$$

because  $p_2$  and  $p_1$  are independent variables, and so

$$\frac{dE_{123}}{dp_1} = \frac{\partial E_{123}}{\partial p_1} = \frac{p_1 c^2}{E_1} + \frac{(p_1 + p_2 \cos \theta_{12}) c^2}{E'_3} \quad (\text{CM}_{123}) \quad (\text{F8})$$

where again it is essential to realize that

$$E'_3 \equiv \sqrt{(p_1 c)^2 + (p_2 c)^2 + 2p_1 p_2 c^2 \cos \theta_{12} + (m_3 c^2)^2} \quad (\text{F9})$$

Note again that

$$dp_1 = \left( \frac{\partial E_{123}}{\partial p_1} \right)^{-1} dE_{123} \quad (\text{F10})$$

using Jacobian notation of appendix I.

## Appendix G

### Three-Body Relativistic Kinematic Factor

Our aim is to obtain the relativistic version of  $p_1(E_{123}, p_2)$  given in equation (28). That is, we would like the three-body version of equations (E16) and (E17). The result is then simply inserted into equation (24).

Proceeding as in appendix E, the overall CM frame is defined as

$$\mathbf{p}_{123} = 0 = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad (\text{G1a})$$

that is,

$$p_{123} = (E_{123}, 0) = (M_{123}c^2, 0) \quad (\text{G1b})$$

which are analogous to equations (E11a) and (E11b).

Thus,

$$\begin{aligned} p_{123}^2 &= E_{123}^2 = (M_{123}c^2)^2 = (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)^2 \\ &= p_1^2 + p_2^2 + p_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 + 2(\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{p}_3 \\ &= (m_1c^2)^2 + (m_2c^2)^2 + (m_3c^2)^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2c^2) \\ &\quad + 2[(E_1 + E_2)E_3 - (\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{p}_3c^2] \end{aligned} \quad (\text{G2})$$

analogous to equation (E12), but in the overall CM frame

$$\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2) \quad (\text{G3})$$

## Appendix H

### Ultrarelativistic Kinematics

Let us evaluate the two- and three-body phase space factors in equations (14) and (24) using ultrarelativistic kinematics. Normally we have

$$E^2 = (pc)^2 + (mc^2)^2 \quad (\text{H1})$$

but in the ultrarelativistic limit

$$mc^2 \ll pc \quad (\text{H2})$$

so that

$$\begin{aligned} E_{123}^2 &= (m_1c^2)^2 + (m_2c^2)^2 + (m_3c^2)^2 \\ &\quad + 2(E_1E_2 - |\mathbf{p}_1||\mathbf{p}_2|c^2 \cos \theta_{12}) \\ &\quad + 2[(E_1 + E_2)(E_{123} - E_1 - E_2) \\ &\quad + (\mathbf{p}_1 + \mathbf{p}_2)^2c^2] \quad (\text{G4}) \\ &= (m_1c^2)^2 + (m_2c^2)^2 + (m_3c^2)^2 \\ &\quad + 2\left(E_1E_2 - \sqrt{E_1^2 - (m_1c^2)^2}|\mathbf{p}_2|c^2 \cos \theta_{12}\right) \\ &\quad + 2\left[(E_1 + E_2)(E_{123} - E_1 - E_2) \right. \\ &\quad + E_1^2 - (m_1c^2)^2 + E_2^2 - (m_2c^2)^2 \\ &\quad \left. + 2\sqrt{E_1^2 - (m_1c^2)^2}|\mathbf{p}_2|c^2 \cos \theta_{12}\right] \quad (\text{G5}) \end{aligned}$$

analogous to equation (E15).

Thus, by solving equation (G5) for  $E_1$  as a function of  $E_{123}$  and  $|\mathbf{p}_2|$ , we have equations analogous to equations (E16) and (E17). Clearly the solutions are very complicated, and we shall not solve them explicitly here.

so that

$$E = pc \quad (\text{H3})$$

is the ultrarelativistic expression relating energy and momentum (i.e., it is the expression for the photon).

For the two-body final state, in the CM frame, the analogue to equation (16) is

$$E_{12} = (p_1c) + (p_1c) = 2p_1c \quad (\text{H4})$$

so that

$$p_1 = p_1(E_{12}) = \frac{E_{12}}{2c} \quad (\text{H5})$$

Consequently

$$E_1 = p_1 c = \frac{1}{2} E_{12} \quad (\text{H6})$$

$$E_2 = \frac{1}{2} E_{12} \quad (\text{H7})$$

so that

$$p_1 \frac{E_1 E_2'}{E_1 + E_2'} = \frac{E_{12}^2}{8c} \quad (\text{H8})$$

Equations (H4) to (H8) are the analogues to equations (16) to (19). We substitute equation (H8) into equation (14a) so that the ultrarelativistic expression for  $dN_{\text{ips}}$  is

$$\begin{aligned} dN_{\text{ips}}(E; p_1, p_2)_{\text{UR}} &= \frac{V}{(2\pi\hbar)^3} \frac{1}{8c^3} \int E_{12}^2 \delta(E - E_{12}) dE_{12} d\Omega_1 \\ &= \frac{V}{(2\pi\hbar)^3} \frac{E^2}{8c^3} \int d\Omega_1 \end{aligned} \quad (\text{H9})$$

For the three-body final state, in the overall CM frame, the analogue to equation (26) is

$$E_{123} = (p_1 c) + (p_2 c) + c \sqrt{p_1^2 + p_2^2 + 2p_1 p_2 \cos \theta_{12}} \quad (\text{H10})$$

which we solve for  $p_1$  as

$$\begin{aligned} p_1 &= p_1(E_{123}, p_2) \\ &= \frac{E_{123}^2}{2p_2 c^2 (E_{123} + \cos \theta_{12} - 1)} \end{aligned} \quad (\text{H11})$$

analogous to equation (28) so that one would proceed in the same manner as described after equation (28).

## Appendix I

### Jacobian Transformations

The change of variables from  $x$  and  $y$  to  $u$  and  $v$  where

$$x = f(u, v) \quad (\text{I1})$$

$$y = g(u, v) \quad (\text{I2})$$

is accomplished through (ref. 16, p. 556)

$$\iint \phi(x, y) dx dy = \iint \phi[f(u, v), g(u, v)] \frac{\partial(x, y)}{\partial(u, v)} du dv \quad (\text{I3})$$

where the Jacobian is defined as

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad (\text{I4})$$

This may be generalized to transform any set of

differentials via (ref. 9, p. 37)

$$dx_1 dx_2 \cdots dx_n = \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(y_1, y_2, \cdots, y_n)} dy_1 dy_2 \cdots dy_n \quad (\text{I5})$$

with the Jacobian

$$\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(y_1, y_2, \cdots, y_n)} \equiv \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \quad (\text{A6})$$

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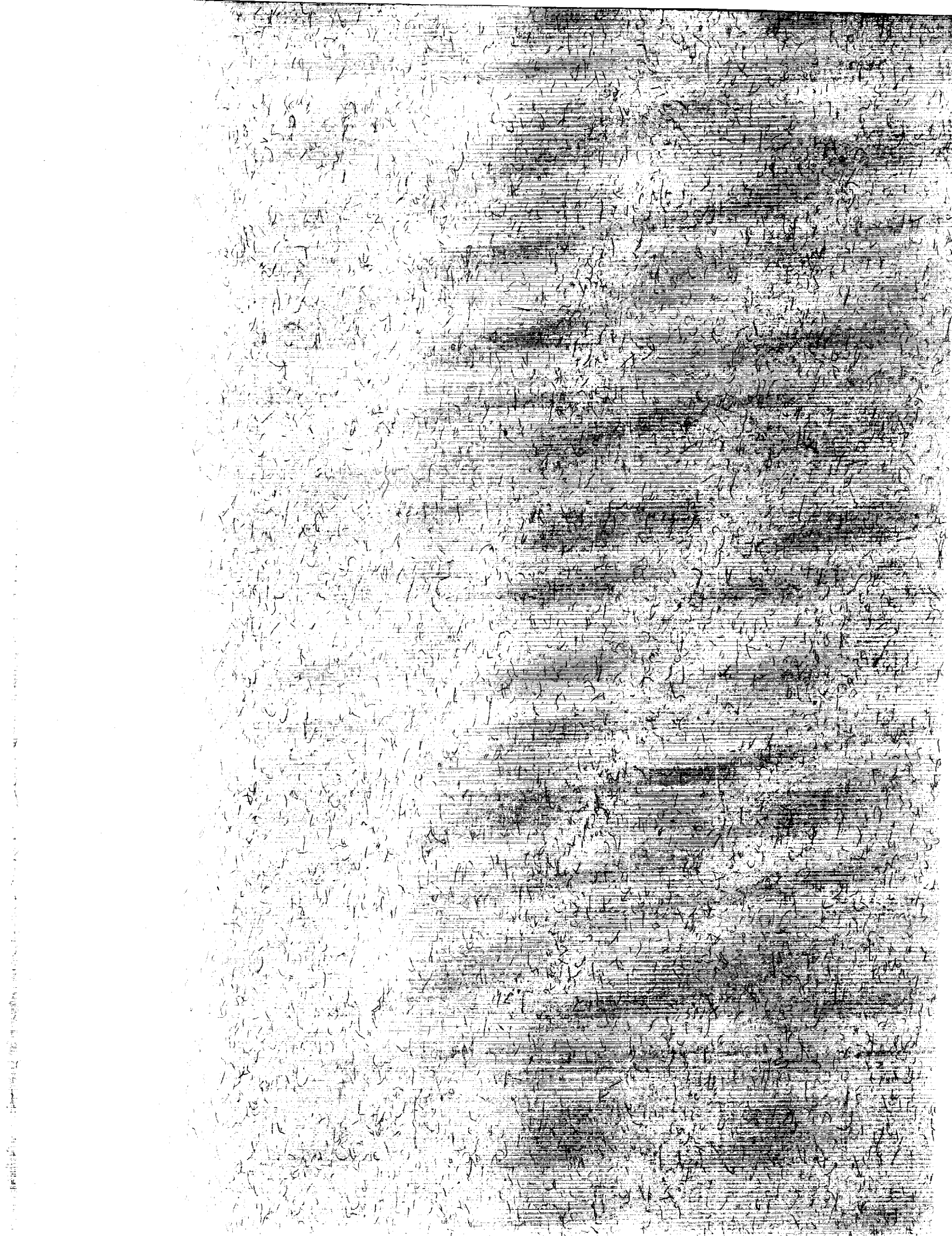
## Symbols

$A$	4-vector	$\mathbf{p}$	momentum vector, MeV/ $c$
$c$	speed of light, $3 \times 10^8$ m/sec	$\mathbf{r}$	position vector, m
$dN_{ips}$	noninvariant phase space element, MeV $^{-1}$	$t$	T-matrix operator
$dL_{ips}$	Lorentz-invariant phase space element	$T_{ni}$	T-matrix element
$E$	energy, MeV	$v$	incident velocity, m-sec $^{-1}$
$E_{12}, E_{123}$	defined in equations (12) and (22)	$V$	volume, m $^3$
$F_i$	incident flux, m $^{-2}$ -sec $^{-1}$	$w$	transition rate, sec $^{-1}$
$\mathbf{g}^{\mu\nu}$	metric tensor	$\delta$	Dirac delta function
$G_{ni}$	Green's function, MeV $^{-1}$	$\Delta$	isobar
$\hbar$	Planck's constant, $6.58 \times 10^{-22}$ MeV-sec	$\theta$	angle, radians
$ i\rangle$	initial state vector	$\mu$	reduced mass, MeV/ $c^2$
$ k\rangle$	final state vector	$\pi$	pion
$m$	mass, MeV/ $c^2$	$\rho_i$	incident probability density, m $^{-3}$
$M$	total mass, MeV/ $c^2$	$\sigma$	cross section, millibarns
$n_i$	number of incident particles	$\phi$	one-body wave function, m $^{-3/2}$
$N$	number of bodies in reaction; also nucleon	$\Phi, \Psi$	wave functions, m $^{-3/2}$
		$\Omega$	solid angle, steradian
		*	complex conjugate



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