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AN EXAMPLE OF A HARMONIZABLE PROCESS WHOSE SPECTRAL DOMAIN IS NOT COMPLETE

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ABSTRACT

The question of completeness of the spectral domain of harmonizable processes has been open for some years. We give an example of a harmonizable process whose spectral domain is not complete. This shows that a recent result of M. M. Rao which claims the completeness of all such spectral domains is false.

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1. INTRODUCTION. The completeness of the spectral domain of stationary processes is well known. This result has played an important role in the development of prediction theory of these processes. The completeness of the spectral domain of other classes of processes is equally important. This completeness question for the class of harmonizable processes has been open for many years. It was always felt by the authors working in the field that by going from the stationary processes to harmonizable ones the completeness may be lost, see for example the work of Cambanis [1] and Cramer [2]. However, recently M.M. Rao in [7] has claimed that the spectral domain of any harmonizable process is complete. This was also mentioned in the article [3] by D.K. Chang and M.M. Rao. Doubt about the completeness of $L^2(F)$ was raised by one of us in a recent review which appeared in 1987 in Zbl. Math. (cf. Zbl. #616.60009).

The main purpose of this note is to give an example of a harmonizable process whose spectral domain is not complete. This will show that the proofs pertaining to the completeness property given in [7] and [3] are in error. This will also put to rest any speculation about this important question.

2. PRELIMINARIES. In this section we give some preliminary results concerning harmonizable processes, their spectral bimeasures, and integration with respect to bimeasures. For mor information the reader is refered to the articles of Rozanov [8], Niemi [6] and Chang & Rao [3]

Let $L_0^2(P)$ be the space of all complex random variables X on a Probability space (Ω, Σ, P) for which EX=0 and $E|X|^2 < \infty$, E stands for expectation operator. Suppose that the stochastic process $\{X_t : t \in R = \text{real numbers}\} \subset L_0^2(P)$ admits a harmonic representation

(1)
$$X_t = \int_R e^{it\theta} Z(d\theta), \quad t \in R,$$

where $Z(\cdot)$ is a σ-additive $L_0^2(P)$ - valued measure. Let $F(A,B) = E\, Z(A)\overline{Z(B)}$, be the bimeasure induced by $Z(\cdot)$. One can see that $F(\cdot, A)$ and $F(A, \cdot)$ are scalar measures for each $A \in \Sigma$ and that $F(\cdot,\cdot)$ is of bounded semi-variation. (See for example [8]). Any process $\{X_t, t \in R\}$ with a representation as in (1) is called weakly harmonizable, it is called strongly harmonizable if in addition $F(\cdot,\cdot)$ is of bounded variation. It is called stationary if $F(\cdot,\cdot)$ is concentrated on the diagonal $\lambda = \theta$ of R^2. In all these cases F is called the spectral measure of the process. In case that our bimeasure $F(\cdot,\cdot)$ is of bounded variation it can be extended to a Radon measure on R^2 and hence for any two bounded measureable f and g the integral

$$\int_R\int_R f(\theta)\overline{g(\lambda)}\, F(d\theta, d\lambda)$$

can be defined as a Lebesque integral and the covariance of the corresponding strongly harmonizable process $\{X_t\}$ has the representation

$$(2) \qquad EX_t\overline{X}_s = \int_R\int_R e^{it\theta - is\lambda}\, F(d\theta, d\lambda), \quad t,s \in R$$

When our process $\{X_t\}$ is weakly harmonizable the spectral measure $F(\cdot,\cdot)$ is of bounded semi-variation and the representation (2) still holds. However in this case the integral must be given a new interpretation in the following sense due to Morse and Transue [5]. The Morse-Transue integration with respect to bimeasures and related problems are treated in detail in the comprehensive article of D.K. Chang and M.M. Rao [3]. The following definitions and Theorem are taken from [3].

DEFINITION. Let (Ω_i, Σ_i) $i = 1,2$ be measurable spaces and $F(\cdot,\cdot)$ be a bimeasure on $\Sigma_1 \times \Sigma_2$. If the function $f_i: \Omega_i \longrightarrow c$ is Σ_i measurable, then (f_1, f_2) is said to be strictly F-integrable, provided the following two conditions hold: (a) f_1 is $F(\cdot, B)$ - integrable for each $B \in \Sigma_2$, and f_2 is $F(A,\cdot)$ - integrable for each $A \in \Sigma_1$, that $F_1^M : A \longrightarrow \int_M f_2(\lambda)\, F(A, d\lambda)$ and $F_2^N : B \longrightarrow \int_N f_1(\theta)\, F(d\theta, B)$ are scalar measures on Σ_1 and Σ_2 respectively;

(b) f_1 is F_1^M - integrable, for each $M \in \Sigma_2$, and f_2 is F_2^N - integrable, for each $N \in \Sigma_1$, and that

(3) $$\int_N f_1(\theta)\, F_1^M(d\theta) = \int_M f_2(\lambda)\, F_2^N(d\lambda).$$

The common value in (3) is denoted by $\int_N \int_M f_1(\theta)\, f_2(\lambda)\, F(d\theta,\, d\lambda)$.

DEFINITION. Let $\{X_t\ ,\ t \in R\}$ be a weekly harmonizable process, with spectral measure $F(\cdot,\cdot)$. ($F(\cdot,\cdot)$ is now a bimeasure on $\mathcal{B}\times\mathcal{B}$, $\mathcal{B}$ being the Borel sets of R). Then following [3] and [7] the spectral domain of $\{X_t,\ t \in R\}$, denoted by $L^2(F)$, is defined to be

$$L^2(F) = \{f\colon R \longrightarrow C \mid (f,f) \text{ is strictly } F\text{- integrable}\}.$$

For any stochastic measure Z on $\mathcal{B}$, $L_1(Z)$ stands for the set of all functions $f : R \longrightarrow C$ for which the integral $\int_R f(\theta)\, Z(d\theta)$ exists (in the Donford and Schwartz sense [4]).

The following theorem proved by Chang and Rao in [3] (cf. pp. 34-44) is used to show that our example constructed in the next section has the desired property.

THEOREM. Let $\{X_t\}$ be a weakly harmonizable process with stochastic measure Z as in (1) and spectral measure $F(\cdot,\cdot)$ as in (2). Then $L^2(F) = L^1(Z)$ and for any two functions f and g in this space we have $\int_R \int_R f(\theta)\, \overline{g(\lambda)}\, F(d\theta,\, d\lambda) = E\left(\int_R f(\theta)\ Z\ (d\theta)\ \overline{\int_R g(\lambda)\ Z\ (d\lambda)}\right)$.

(Beware that the integral in the left hand side is in the sense of Morse - Transue). In this case $L^2(F)$ can be considered as an inner product space with $(f,g)_F = \int_R \int_R f(\theta)\, \overline{g(\lambda)} F(d\theta,\, d\lambda)$ and norm $||f||_F^2 = (f,f)_F$.

3. EXAMPLE. Before we proceed to present our example of a harmonizable process whose spectral domain is not complete we need to prove a Lemma which is essential for establishing our counter example.

For any stationary stochastic process $\{Y_n,\ n \in Z = \text{integers}\}$, its time domain is the subspace $H_y(+\infty) = \overline{SP}\ \{Y_k\ :\ k \in Z\}$, its past subspace is the subspace $H_y(n) = \{Y_k\ :\ k \le n\}$, and its remote past is the subspace $H_y(-\infty) = \bigcap_n Hy(n)$. The stationary sequence $\{Y_n,\ n \in Z\}$ is called nondeterministic if

$Y_n \notin H_y(n-1)$ for some and hence every n. It is called purely nondeterministic if $H_y(-\infty) = 0$.

LEMMA. Let $\{Y_n, n \in Z\}$ be a stationary stochastic process which is nondeterministic but is not purely nondeterministic. Then there exists a nonzero vector V in the remote past of $H_y(-\infty)$ of the process which cannot be expressed as a series of the form.

(4) $$V = \sum_{k=0}^{\infty} a_k Y_{-k}.$$

PROOF. Suppose not, i.e., suppose any nonzero vector in $H_y(-\infty)$ has a series representation as in (4). Take a nonzero vector V in $H_y(-\infty)$. Then we can write $$V = \sum_{k=t}^{\infty} a_k Y_{-k}, \quad \text{with } a_t \neq 0,$$

and hence

(5) $$U^{t+1} V = \sum_{k=t}^{\infty} a_k Y_{t+1-k} = a_t Y_1 + \sum_{k=0}^{\infty} a_{k+t+1} Y_{-k},$$

where U is the usual shift operator on $H_y(+\infty)$ associated with our stationary process. On the other hand since $U^{t+1}V$ itself is in $H_y(-\infty)$ we can write

(6) $$U^{t+1}V = \sum_{k=0}^{\infty} b_k Y_{-k}$$

Comparing (5) and (6) we get $a_t Y_1 = \sum_{k=0}^{\infty} c_k Y_{-k}$, which since $a_t \neq 0$ implies

$$Y_1 = \sum_{k=0}^{\infty} \frac{c_k}{a_t} Y_{-k}.$$

This contradicts our nondeterministic assumption. Hence the proof of the Lemma is complete.

Now we are ready to give our counter example mentioned before.

Take a stationary stochastic process $\{Y_n : n \in Z\} \subset L_0^2(f)$ which is nondeterministic but not purely nondeterministic. Take any sequence λ_i, $i \in Z$, of positive numbers which is summable, i.e. $\sum_{i=1}^{\infty} \lambda_i < \infty$. Define a stochastic measure Z on Borel subsets of R which is concentrated on the positive integers I by

$$Z(\{i\}) = \lambda_i Y_{-i}, \quad i \in I.$$

Consider the harmonizable process $\{X_t,\ t \in R\} \subset L_0^2(P)$ given by $X_t = \int_R e^{it\theta} Z(d\theta)$ and its spectral bimeasure which is induced by Z, i.e.,

$$F(A,B) = E\, Z(A)\, \overline{Z(B)}.$$

We claim that the corresponding spectral domain $L^2(F)$ in this case is not complete.

Verification. By our Lemma there exists a nonzero vector in $H_y(-\infty)$ which does not have a series representation as in (4). Take one such vector V. Since V is clearly in $H_y(0)$ there exists a sequence $\Sigma\, a_k^n\, Y_{-k} = V_n$ of finite linear combination of Y_k's; $k \leq 0$ which converges to V in $L_0^2(P)$. We can write

$$v_n = \int_R f_n(\theta) Z(d\theta)$$

where the nonzero functions f_n are defined on positive integers with $f_n(k) = a_k^n$. By our Theorem in section 2 we have

$$||f_n - f_m||_F = ||\, v_n - v_m|| \;.$$

Now since v_n converges to v and hence is Cauchy so is f_n. However this particular sequence f_n of functions in $L^2(F)$ does not converge to any element f in $L^2(F)$. Because otherwise another application of the Theorem in section 2 shows that f is in $L^1(Z)$ and

$$||f_n - f||_F = ||\int_R (f_n - f)\, dZ\, || = ||V_n - \int_R f dZ|| \;.$$

Thus we see that V_n also converges to $\int_R f dz$. So

$$V = \int_R f\, dZ = \sum_{i=0}^{\infty} f(i)\, Z(\{i\}) = \sum_{i=0}^{\infty} f(i)\lambda_i\, Y_{-i},$$

which contradicts our choice of V.

REMARK 1. Our example shows that the main result of [7] claiming the completeness of the spectral domain of any multivariate weakly harmonizable process X_t is false even for a univariate strongly harmonizable process.

REMARK 2. We feel that the error in [7] occurs in lines 8 and 9 of the second column of page 4612, where the existence of a "certain projection onto a subspace" is asserted and a reference to page 33 of [9] is made to support it. In view of the results established in this note the results in the later part of [3] which are based on the completeness of $L^2(F)$ deserves reinvestigation.

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