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A GENERAL MODEL FOR ATTITUDE DETERMINATION ERROR ANALYSIS

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ABSTRACT

This paper gives an overview of a comprehensive approach to filter and dynamics modeling for attitude determination error analysis. The models presented include both batch least-squares and sequential attitude estimation processes for both spin-stabilized and three-axis stabilized spacecraft. The discussion includes a brief description of a dynamics model of strapdown gyros, but it does not cover other sensor models. Model parameters can be chosen to be *solve-for parameters*, which are assumed to be estimated as part of the determination process, or *consider parameters*, which are assumed to have errors but not to be estimated. The only restriction on this choice is that the time evolution of the consider parameters must not depend on any of the solve-for parameters. The result of an error analysis is an indication of the contributions of the various error sources to the uncertainties in the determination of the spacecraft solve-for parameters. The model presented in this paper gives the uncertainty due to errors in the *a priori* estimates of the solve-for parameters, the uncertainty due to *measurement noise*, the uncertainty due to *dynamic noise* (also known as *process noise* or *plant noise*), the uncertainty due to the consider parameters, and the overall uncertainty due to all these sources of error.

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1. INTRODUCTION

Spacecraft attitude determination involves estimating the orientation of a spacecraft relative to inertial space, based on measurements from onboard sensors. Attitude determination error analysis is the computation of the attitude determination accuracy obtainable with sensor data of prescribed error characteristics, without processing real or simulated sensor data. This analysis takes into account the presence of certain errors in modeling the sensors and the attitude motion of the spacecraft [Wertz].

This paper gives an overview of a comprehensive approach to filter and dynamics modeling for attitude determination error analysis. The models presented include both batch least-squares and sequential attitude estimation processes for both spin-stabilized and three-axis stabilized spacecraft. Model parameters can be chosen to be *solve-for parameters*, which are assumed to be estimated as part of the determination process, or *consider parameters*, which are assumed to have errors but not to be estimated. The only restriction on this choice is that the time evolution of the consider parameters must not depend on any of the solve-for parameters. Great freedom is also allowed in specifying sensor types and measurement scheduling.

The result of an error analysis is an indication of the contributions of the various error sources to the uncertainties in the determination of the spacecraft solve-for parameters. The model presented in this paper gives the uncertainty due to errors in the *a priori* estimates of the solve-for parameters, the uncertainty due to *measurement noise*, the uncertainty due to *dynamic noise* (also known as *process noise* or *plant noise*), the uncertainty due to the consider parameters, and the overall uncertainty due to all these sources of error. This approach was developed as part of the mathematical specification of algorithms for the computer-based Attitude Determination Error Analysis System (ADEAS) [Nicholson].

2. DYNAMICS MODEL

The *state vector* $\boldsymbol{x}$ is an N-dimensional vector of parameters that completely characterizes the system. For spacecraft attitude determination, the state vector includes spacecraft attitude parameters and sensor calibration parameters. The state vector is assumed to evolve in time according to the dynamics model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t) + \boldsymbol{u}(t) \tag{2-1}$$

where the dynamic noise $\boldsymbol{u}(t)$ is a Gaussian white noise process with mean and covariance given by

$$E[\boldsymbol{u}(t)] = 0 \qquad \text{and} \qquad E[\boldsymbol{u}(t)\boldsymbol{u}^T(t')] = Q\,\delta(t - t') \tag{2-2}$$

with $E[\ldots]$ denoting the expectation value. In this equation Q is the NxN *dynamic noise spectral density matrix* and $\delta(t - t')$ denotes the Dirac delta, or unit impulse, function. The state vector includes all the parameters needed to compute $\boldsymbol{x}$, even though some of these parameters may have zero derivative.

The true value of the state vector is never exactly known, but can only be estimated. The *state estimate vector* $\boldsymbol{x}^*(t)$ evolves in time according to

$$\dot{\boldsymbol{x}}^*(t) = \boldsymbol{f}(\boldsymbol{x}^*(t), t)\,. \tag{2-3}$$

The *state error vector*, given by

$$\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*(t) \tag{2-4}$$

is assumed to always remain small, so linear error analysis techniques can be used. Then, to first order,

$$\Delta\dot{x}(t) = \dot{x}(t) - \dot{x}^*(t) = f(x(t), t) - f(x^*(t), t) + u(t) \approx (\partial f/\partial x)(t)\, \Delta x(t) + u(t). \tag{2-5}$$

Integrating this formally gives

$$\Delta x(t) = \Phi(t, t')\, \Delta x(t') + \psi(t, t') \tag{2-6}$$

where the *state transition matrix* $\Phi(t, t')$ is the solution of the differential equation

$$\dot{\Phi}(t, t') = (\partial f/\partial x)(t)\, \Phi(t, t') \tag{2-7a}$$

with the initial condition

$$\Phi(t', t') = I_N = \text{the NxN identity matrix} \tag{2-7b}$$

and the *random excitation vector* $\psi(t, t')$ is given by the integral

$$\psi(t, t') = \int_{t'}^{t} \Phi(t, t'')\, u(t'')\, dt''. \tag{2-8}$$

It follows from equations (2-7) and (2-8) that the transition matrix obeys the *group property*

$$\Phi(t, t') = \Phi(t, t'')\, \Phi(t'', t') \tag{2-9}$$

and that the random excitation vector obeys the relation

$$\psi(t, t') = \Phi(t, t'')\, \psi(t'', t') + \psi(t, t''). \tag{2-10}$$

Equations (2-2) and (2-8) give the relationship

$$E[\psi(t, t'')\psi^T(t'', t')] = 0 \qquad \text{for } t \geq t'' \geq t'. \tag{2-11}$$

The estimation computations require the *random excitation covariance matrix*

$$D(t, t') \equiv E[\psi(t, t')\psi^T(t, t')] = \int_{t'}^{t} \Phi(t, t'')\, Q\, \Phi^T(t, t'')\, dt'', \tag{2-12}$$

which equations (2-10) and (2-11) show to obey the relation

$$D(t, t') = \Phi(t, t'')\, D(t'', t')\, \Phi^T(t, t'') + D(t, t''). \tag{2-13}$$

2.1 Spin-Stabilized Spacecraft Dynamics Model

For spin-stabilized spacecraft, the attitude matrix $A_{BI}(t)$ which transforms vectors from an inertial frame I to the spacecraft body frame B is given as the product

$$A_{BI}(t) = A_{BL}(t)\, A_{LI}(t) \tag{2-14}$$

where the subscript L denotes an intermediate frame in which the total spacecraft angular momentum vector $\mathbf{L}$ is oriented along the positive z-axis. The matrix $A_{LI}(t)$ is given in terms of the right ascension

$\alpha(t)$ and declination $\delta(t)$ of the angular momentum vector as

$$A_{LI}(t) = A_2(\pi/2 - \delta)\, A_3(\alpha) \tag{2–15}$$

where $A_i(\theta)$ denotes a rotation by angle θ about axis i. The matrix $A_{BL}(t)$ is parameterized by a 3-1-3 Euler axis sequence as

$$A_{BL}(t) = A_3(\psi)\, A_1(\theta)\, A_3(\phi)\,. \tag{2–16}$$

For torque-free motion of an axially-symmetric rigid body $\alpha(t)$, $\delta(t)$, and $\theta(t)$ are constant, and

$$\dot{\phi}(t) = \omega_I(t) \tag{2-17a}$$
$$\dot{\psi}(t) = \omega_p(t), \tag{2-17b}$$

where $\omega_I(t)$ and $\omega_p(t)$ are the *inertial nutation rate* and *body nutation rate*, respectively [Wertz].

The state vector $x(t)$ for the spin-stabilized case is

$$x(t) = [\alpha(t),\ \delta(t),\ \phi(t),\ \theta(t),\ \psi(t),\ \omega_I(t),\ \omega_p(t),\ x_m{}^T(t)]^T, \tag{2–18}$$

where x_m is a μ-dimensional vector of measurement parameters depending on the sensor complement of the spacecraft being modeled. We assume that the measurement parameters are constant and that any deviations of the dynamics from torque-free motion of an axially symmetric rigid body can be approximated by independent white noise processes $u_\alpha(t)$, $u_\delta(t)$, $u_\phi(t)$, $u_\theta(t)$, $u_\psi(t)$, $u_I(t)$, and $u_p(t)$. The equations of motion for spin-stabilized spacecraft give the dynamics model

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_\mu{}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_\mu{}^T \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0_\mu{}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_\mu{}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0_\mu{}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_\mu{}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_\mu{}^T \\ 0_\mu & 0_\mu & 0_\mu & 0_\mu & 0_\mu & 0_\mu & 0_\mu & 0_{\mu x \mu} \end{bmatrix} x(t) \ + \begin{bmatrix} u_\alpha(t) \\ u_\delta(t) \\ u_\phi(t) \\ u_\theta(t) \\ u_\psi(t) \\ u_I(t) \\ u_p(t) \\ 0_\mu \end{bmatrix} \tag{2-19}$$

where 0_μ is a μ-dimensional vector of zeros and $0_{\mu x \mu}$ is a $\mu x \mu$ matrix of zeros. Since the dynamics model for spin-stabilized spacecraft is linear in the state vector, the state error vector $\Delta x(t)$ obeys an equation of the same form as equation (2-19). Thus the state transition matrix, as defined by equation (2-7), is

$$\Phi(t, t') = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 1 & 0 & 0 & \Delta t & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & 1 & 0 & \Delta t & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0}_\mu^T \\ \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & I_\mu \end{bmatrix} \tag{2-20}$$

where

$$\Delta t \equiv t - t' . \tag{2-21}$$

The inverse of the state transition matrix is

$$\Phi^{-1}(t, t') = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 1 & 0 & 0 & -\Delta t & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & 1 & 0 & -\Delta t & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0}_\mu^T \\ \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & I_\mu \end{bmatrix} \tag{2-22}$$

and the random excitation covariance matrix is

$$D(t, t') = \begin{bmatrix} Q_\alpha & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & Q_\delta & 0 & 0 & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & Q_\phi + (1/3)Q_\zeta(\Delta t)^2 & 0 & 0 & (1/2)Q_\zeta \Delta t & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & Q_\theta & 0 & 0 & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & Q_\psi + (1/3)Q_p(\Delta t)^2 & 0 & (1/2)Q_p \Delta t & \mathbf{0}_\mu^T \\ 0 & 0 & (1/2)Q_\zeta \Delta t & 0 & 0 & Q_\zeta & 0 & \mathbf{0}_\mu^T \\ 0 & 0 & 0 & 0 & (1/2)Q_p \Delta t & 0 & Q_p & \mathbf{0}_\mu^T \\ \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & \mathbf{0}_\mu & 0_{\mu x \mu} \end{bmatrix} \Delta t , \tag{2-23}$$

where Q_α is defined by

$$E[u_\alpha(t)u_\alpha(t')] = Q_\alpha\,\delta(t - t') \tag{2-24}$$

with similar relations for Q_δ, Q_ϕ, Q_θ, Q_ψ, Q_ζ, and Q_p.

2.2 Three-Axis Stabilized Spacecraft Dynamics Model

For three-axis stabilized spacecraft, the attitude matrix $A_{BI}(t)$ is given as the product

$$A_{BI}(t) = A_{BR}(t)\,A_{RI}(t) \tag{2-25}$$

where the subscript R denotes a reference frame, which can be, for example, Earth-pointing, Sun-pointing, or inertial. The inertial-to-reference matrix $A_{RI}(t)$ for any reference system is computed from the reference vectors defining that system. The nominal spacecraft attitude with respect to the reference frame evolves over time according to

$$\dot{A}_{BR}(t) = -\,\tilde{\omega}_{BR}(t)\,A_{BR}(t)\,, \tag{2-26}$$

where $\tilde{\omega}_{BR}(t)$ is the $3x3$ antisymmetric matrix

$$\tilde{\omega}_{BR}(t) \equiv \begin{bmatrix} 0 & -[\omega_{BR}(t)]_z & [\omega_{BR}(t)]_y \\ [\omega_{BR}(t)]_z & 0 & -[\omega_{BR}(t)]_x \\ -[\omega_{BR}(t)]_y & [\omega_{BR}(t)]_x & 0 \end{bmatrix} \tag{2-27}$$

defined from the column vector $\omega_{BR}(t)$ containing the components in the body frame of the spacecraft angular velocity relative to the reference frame. The nominal attitude profile is used for determining measurement geometry, sensor line-of-sight occultation, and related effects.

The attitude error is defined in terms of a three-component *attitude error vector* $\Delta\theta(t)$, whose components are the small rotations about each of the spacecraft body axes that would align the true body axes with the estimates of these axes. In terms of the true attitude $A_{BR}(t)$ relative to the reference frame and the estimate $A_{BR}{}^*(t)$ of this attitude,

$$A_{BR}{}^*(t) \approx [I_3 + \Delta\tilde{\theta}(t)]\,A_{BR}(t), \tag{2-28}$$

where I_3 is the 3x3 identity matrix and the antisymmetric matrix $\Delta\tilde{\theta}(t)$ is defined similarly to equation (2-27).

The true attitude relative to inertial space evolves according to

$$\dot{A}_{BI}(t) = -\,\tilde{\omega}_{BI}(t)\,A_{BI}(t)\,, \tag{2-29}$$

where $\omega_{BI}(t)$ is the column vector of components in the body frame of the spacecraft angular velocity relative to inertial space. Similarly, the estimated attitude relative to inertial space evolves according to

$$\dot{A}_{BI}^*(t) = -\tilde{\omega}_{BI}^*(t)\, A_{BI}^*(t)\,, \tag{2-30}$$

where $\omega_{BI}^*(t)$ is the column vector of estimates of $\omega_{BI}(t)$. These equations form the basis of the attitude error propagation, since this is assumed to be based on information obtained from gyros, which provide the estimates $\omega_{BI}^*(t)$ of the angular rates relative to inertial space. The attitude estimate relative to inertial space is related to the estimate relative to the reference frame by the analog of equation (2-25):

$$A_{BI}^*(t) = A_{BR}^*(t)\, A_{RI}(t). \tag{2-31}$$

Then from equations (2-25) - (2-31) we have

$$\begin{aligned} d(\Delta\tilde{\theta})/dt &= d(A_{BR}^*\, A_{BR}^T)/dt = d(A_{BI}^*\, A_{BI}^T)/dt = \dot{A}_{BI}^*(t)\, A_{BI}^T(t) + A_{BI}^*(t)\, \dot{A}_{BI}^T(t) \\ &= -\tilde{\omega}_{BI}^*(t)\, A_{BI}^*(t)\, A_{BI}^T(t) + A_{BI}^*(t)\, A_{BI}^T(t)\, \tilde{\omega}_{BI}(t) \\ &= -\tilde{\omega}_{BI}^*(t)\, [I_3 + \Delta\tilde{\theta}(t)] + [I_3 + \Delta\tilde{\theta}(t)]\, \tilde{\omega}_{BI}(t). \end{aligned} \tag{2-32}$$

We now define the angular velocity measurement error vector by

$$\Delta\omega_{BI}(t) \equiv \omega_{BI}^*(t) - \omega_{BI}(t) \tag{2-33}$$

and assume that its components are small. Then, to first order in $\Delta\omega_{BI}$ and $\Delta\theta$

$$d(\Delta\tilde{\theta})/dt = -\tilde{\omega}_{BI}(t)\, \Delta\tilde{\theta}(t) + \Delta\tilde{\theta}(t)\, \tilde{\omega}_{BI}(t) - \Delta\tilde{\omega}_{BI}(t), \tag{2-34}$$

which is, in vector form

$$\Delta\dot{\theta}(t) = -\tilde{\omega}_{BI}(t)\, \Delta\theta(t) - \Delta\omega_{BI}(t)\,. \tag{2-35}$$

The angular velocity measurement errors arise from gyro errors, and a general model for these errors gives [Nicholson]

$$\Delta\omega_{BI}(t) = \Delta b(t) + \Omega(t)\, \Delta k - \tilde{\omega}_{BI}(t)\, \Delta\varepsilon - u_\theta(t) \tag{2-36}$$

where $\Delta b(t)$ is a vector of first-order Markov processes representing the *gyro drift rate biases,* Δk is a vector of constant *gyro scale factor errors,* $\Delta\varepsilon$ is a vector of constant *gyro misalignment errors,* $u_\theta(t)$ is a vector of white-noise processes representing the *gyro drift rate noise*, and

$$\Omega(t) \equiv \mathrm{diag}\, [\omega_{BI}^T(t)\,], \tag{2-37}$$

which means that $\Omega(t)$ is the diagonal matrix with the components of $\omega_{BI}(t)$ as the diagonal elements. The drift rate bias vector is assumed to evolve according to

$$\Delta\dot{b}(t) = -\Delta b(t)/\tau + u_b(t), \tag{2–38}$$

where τ is the correlation time of the Markov processes and $u_b(t)$ is a vector of white-noise processes representing the *gyro drift rate ramp noise*. The white noise processes $u_\theta(t)$ and $u_b(t)$ have means and covariances given by

$$E[u_\theta(t)] = 0\,, \qquad E[u_\theta(t)u_\theta^T(t')] = Q_\theta\,\delta(t - t') \tag{2-39a}$$

$$E[u_b(t)] = 0\,, \qquad E[u_b(t)u_b^T(t')] = Q_b\,\delta(t - t') \tag{2-39b}$$

$$\text{and} \qquad E[u_\theta(t)u_b^T(t')] = 0, \tag{2-39c}$$

where Q_θ and Q_b are 3X3 symmetric, non-negative-definite matrices that are assumed to be constant. This gyro error model is a generalization of the model in [Lefferts] to include scale factor and misalignment errors.

The state error vector for the three-axis stabilized case is

$$\Delta x(t) = [\Delta\theta^T(t), \Delta b^T(t), \Delta k^T, \Delta\varepsilon^T, \Delta x_m{}^T(t)]^T \tag{2–40}$$

where Δx_m is the error in a μ-dimensional vector of measurement parameters depending on the sensor complement of the spacecraft being modeled, as in the spin-stabilized case. The time evolution of this vector is given, using the above models, by

$$\Delta\dot{x}(t) = \begin{bmatrix} -\tilde{\omega}_{BI}(t) & -I_3 & -\Omega(t) & \tilde{\omega}_{BI}(t) & 0_{3x\mu} \\ 0_{3x3} & -I_3/\tau & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x\mu} \end{bmatrix} \Delta x(t) + \begin{bmatrix} u_\theta(t) \\ u_b(t) \\ 0_3 \\ 0_3 \\ 0_\mu \end{bmatrix} \tag{2-41}$$

where 0_3 is a 3-dimensional vector of zeros and 0_{jxk} is a jXk matrix of zeros. The state transition matrix, as defined by equation (2-7) is then

$$\Phi(t, t') = \begin{bmatrix} \Phi_{\theta\theta}(t, t') & \Phi_{\theta b}(t, t') & \Phi_{\theta k}(t, t') & I_3 - \Phi_{\theta\theta}(t, t') & 0_{3x\mu} \\ 0_{3x3} & \Phi_{bb}(t, t') & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & I_3 & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & I_3 & 0_{3x\mu} \\ 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & I_\mu \end{bmatrix} \tag{2-42}$$

where

$$\Phi_{\theta b}(t, t') = - \int_{t'}^{t} \Phi_{\theta\theta}(t, t'') \, exp[- (t'' - t')/\tau] \, dt'' \tag{2-43a}$$

$$\Phi_{\theta k}(t, t') = - \int_{t'}^{t} \Phi_{\theta\theta}(t, t'') \, \Omega(t'') \, dt'' \tag{2-43b}$$

$$\Phi_{bb}(t, t') = I_3 \, exp[- (t - t')/\tau] \, . \tag{2-43c}$$

The *attitude error propagation matrix* $\Phi_{\theta\theta}(t, t')$ is given by the differential equation

$$\dot{\Phi}_{\theta\theta}(t, t') = - \tilde{\omega}_{BI}(t) \, \Phi_{\theta\theta}(t, t') \tag{2-44a}$$

with the initial condition

$$\Phi_{\theta\theta}(t', t') = I_3 \, . \tag{2-44b}$$

The form of equation (2-44a) is identical to that of equation (2-29) for the attitude matrix $A_{BI}(t)$. Thus $\Phi_{\theta\theta}(t, t')$ must also act as a transition matrix for the attitude:

$$A_{BI}(t) = \Phi_{\theta\theta}(t, t') \, A_{BI}(t'), \tag{2-45}$$

or

$$\Phi_{\theta\theta}(t, t') = A_{BI}(t) \, A_{BI}^{T}(t'). \tag{2-46}$$

Equations (2-43) reduce to quadrature after substitution of equation (2-46), where $A_{BI}(t)$ is given in terms of the nominal attitude profile by equation (2-25). The matrix $\Omega(t)$, which is needed to evaluate equation (2-43b), is also given in terms of the nominal profile by the following argument. The integral is broken up into time steps of length Δt, chosen to keep integration errors below a specified tolerance [Nicholson]. The contribution of the interval between t and $t + \Delta t$ requires the matrix $\Omega \, \Delta t$, where Ω denotes the average value of $\Omega(t)$ over the time interval. This matrix has the same elements, rearranged by row and column, as the matrix $\tilde{\omega}_{BI} \Delta t$, where ω_{BI} denotes the time average of $\omega_{BI}(t)$ over the interval. This is given in terms of the result of integrating equation (2-29) over the interval, and ignoring terms of higher than first order in Δt ;

$$A_{BI}(t + \Delta t) \approx [\, I_3 - \tilde{\omega}_{BI} \Delta t \,] \, A_{BI}(t), \tag{2-47}$$

or

$$\tilde{\omega}_{BI} \Delta t = (1/2)[A_{BI}(t) \, A_{BI}^{T}(t + \Delta t) - A_{BI}(t + \Delta t) \, A_{BI}^{T}(t)]. \tag{2-48}$$

Since the submatrix $\Phi_{\theta\theta}(t, t')$ is seen from equation (2-46) to be orthogonal, the inverse of the state transition matrix is given by

$$\Phi^{-1}(t, t') = \begin{bmatrix} \Phi_{\theta\theta}^T & -\Phi_{\theta\theta}^T\Phi_{\theta b}\Phi_{bb}^{-1} & -\Phi_{\theta\theta}^T\Phi_{\theta k} & I_3 - \Phi_{\theta\theta}^T & 0_{3x\mu} \\ 0_{3x3} & \Phi_{bb}^{-1} & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & I_3 & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & I_3 & 0_{3x\mu} \\ 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & I_\mu \end{bmatrix} \quad (2\text{-}49)$$

where the time arguments of the submatrices, which have been omitted for compactness, are the same as the arguments of the full matrix, and

$$\Phi_{bb}^{-1}(t, t') = I_3 \exp[(t - t')/\tau]. \quad (2\text{–}50)$$

The random excitation covariance matrix is

$$D(t, t') = \begin{bmatrix} D_{\theta\theta}(t, t') & D_{\theta b}(t, t') & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ D_{\theta b}^T(t, t') & D_{bb}(t, t') & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x3} & 0_{3x\mu} \\ 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x3} & 0_{\mu x\mu} \end{bmatrix} \quad (2\text{-}51)$$

where

$$D_{\theta\theta}(t, t') = \int_{t'}^{t} [\Phi_{\theta\theta}(t, t'')\, Q_\theta\, \Phi_{\theta\theta}^T(t, t'') + \Phi_{\theta b}(t, t'')\, Q_b\, \Phi_{\theta b}^T(t, t'')]\, dt'' \quad (2\text{-}52a)$$

$$D_{\theta b}(t, t') = \int_{t'}^{t} \Phi_{\theta b}(t, t'')\, Q_b\, \Phi_{bb}(t, t'')\, dt'' \quad (2\text{-}52b)$$

$$D_{bb}(t, t') = \int_{t'}^{t} \Phi_{bb}(t, t'')\, Q_b\, \Phi_{bb}(t, t'')\, dt''. \quad (2\text{-}52c)$$

with Q_θ and Q_b given by equations (2-39).

3. ESTIMATION AND COVARIANCE ANALYSIS

A filter produces state estimates based on information obtained from measurements made at discrete times. Let y_i be an n_i-dimensional vector of measurement values obtained at time t_i. Measurements are related to the state vector by the following measurement model:

$$y_i = g_i(x(t_i)) + v_i \tag{3-1}$$

where v_i is a Gaussian white noise process with mean and covariance given by

$$E[v_i] = 0 \tag{3-2a}$$

$$E[v_i v_i^T] = R_i \tag{3-2b}$$

$$E[v_i v_j^T] = 0 \qquad \text{for } i \neq j. \tag{3-2c}$$

The functions g_i are assumed to be known functions of imprecisely known arguments. Therefore, it is possible to compute predicted measurement values by

$$y_i^* = g_i(x^*(t_i)) \tag{3-3}$$

The measurement residual between the actual and computed measurements is then

$$\Delta y_i = y_i - y_i^* = g_i(x(t_i)) - g_i(x^*(t_i)) + v_i \approx G_i \, \Delta x(t_i) + v_i \tag{3-4}$$

where $$G_i \equiv \partial g_i / \partial x(t_i) \tag{3-5}$$

and Δx is assumed to be small.

It is usually not necessary to estimate all of the state parameters. Therefore, a filter may produce estimates for a set of *solve-for parameters* which are a subset of the state parameters. The filter does not account for the remaining state parameters, which are called *consider parameters* since they contain uncertainties that are considered in the error analysis. The state error vector is thus partitioned as follows:

$$\Delta x(t) = \begin{bmatrix} \Delta s(t) \\ \Delta c(t) \end{bmatrix} \tag{3-6}$$

where $\Delta s(t) \equiv$ solve-for parameter error vector
$\Delta c(t) \equiv$ consider parameter error vector.

The random excitation vector, the state transition matrix and the random excitation covariance matrix have similar partitionings:

$$\psi(t, t') = \begin{bmatrix} \psi_S(t, t') \\ \psi_C(t, t') \end{bmatrix} \tag{3-7a}$$

$$\Phi(t, t') = \begin{bmatrix} \Phi_{ss}(t, t') & \Phi_{sc}(t, t') \\ 0 & \Phi_{cc}(t, t') \end{bmatrix} \tag{3-7b}$$

and

$$D(t, t') = \begin{bmatrix} D_{ss}(t, t') & D_{sc}(t, t') \\ D_{sc}^T(t, t') & D_{cc}(t, t') \end{bmatrix} . \tag{3-7c}$$

The error propagation equation (2-6) can then be rewritten as

$$\Delta s(t) = \Phi_{ss}(t, t')\Delta s(t') + \Phi_{sc}(t, t')\Delta c(t') + \psi_s(t, t') \tag{3-8}$$

$$\Delta c(t) = \Phi_{cc}(t, t')\Delta c(t') + \psi_c(t, t') . \tag{3-9}$$

The partitioning used in equations (3-6) to (3-9) is not the same as the partitioning of the state vector used in section 2. The two partitionings are related by row and column interchanges, depending on the selection of solve-for and consider parameters. The zero in the state transition matrix in equation (3-7b) reflects an assumption that the time evolution of the consider parameters does not depend on any of the solve-for parameters. This restriction assures that solve-for parameter errors do not induce additional consider parameter errors during propagation. In the case of the three-axis stabilized case discussed in section 2.2 this means that it is impossible to solve for any gyro parameters without also solving for the attitude. Work is continuing on removing this restriction from the model.

There are four basic contributions to the total solve-for parameter error:

$$\Delta s(t) = \Delta s_a(t) + \Delta s_n(t) + \Delta s_c(t) + \Delta s_u(t) \tag{3-10}$$

where $\Delta s_a(t) \equiv$ the error at time t due to an *a priori* error at the *epoch time* t_o

$\Delta s_n(t) \equiv$ the error due to measurement noise

$\Delta s_c(t) \equiv$ the error at time t due to consider parameter errors at time t_o

$\Delta s_u(t) \equiv$ the error due to dynamic noise.

Substituting equation (3-10) into equation (3-8), and using equation (3-9), gives

$$\Delta s_a(t) = \Phi_{ss}(t, t')\Delta s_a(t') \tag{3-11a}$$

$$\Delta s_n(t) = \Phi_{ss}(t, t')\Delta s_n(t') \tag{3-11b}$$

$$\Delta s_c(t) = \Phi_{ss}(t, t')\Delta s_c(t') + \Phi_{sc}(t, t')\Phi_{cc}(t', t_o)\Delta c(t_o) \tag{3-11c}$$

$$\Delta s_u(t) = \Phi_{ss}(t, t')\Delta s_u(t') + \Phi_{sc}(t, t')\psi_c(t', t_o) + \psi_s(t,t'). \tag{3-11d}$$

The function of a full estimation system is to determine an estimate $s^*(t)$ given measurements y_i. Error analysis, however, does not require the actual computation of an estimate, but determines how good an estimate would be if it were produced in a given situation. This is done by computing the *estimation covariance matrix* defined by

$$P(t) \equiv E[\Delta s(t) \Delta s^T(t)]. \tag{3-12}$$

The covariance matrix $P(t)$ provides a statistical measure of how good an estimate could be produced at time t of a given scenario. We assume that at the *epoch time* t_o the solve for error $\Delta s(t_o)$ and the consider error $\Delta c(t_o)$ are uncorrelated. If all the various error sources are also initially uncorrelated, then by equations (3-11) they remain uncorrelated at all times. Thus, substituting equation (3-10) into equation (3-12) gives

$$P(t) = P_a(t) + P_n(t) + P_c(t) + P_u(t) \tag{3-13}$$

where

$$P_a(t) \equiv E[\Delta s_a(t) \Delta s_a{}^T(t)] \tag{3-14a}$$

$$P_n(t) \equiv E[\Delta s_n(t) \Delta s_n{}^T(t)] \tag{3-14b}$$

$$P_c(t) \equiv E[\Delta s_c(t) \Delta s_c{}^T(t)] \tag{3-14c}$$

$$P_u(t) \equiv E[\Delta s_u(t) \Delta s_u{}^T(t)]\ . \tag{3-14d}$$

In addition to providing a solve-for parameter estimate, an estimation system will generally also compute an estimate P^* of the estimation covariance P. Since the true *a priori* error and noise covariance matrices may not be known, the estimation system must use *assumed* values for the covariances of these error sources. Further, the estimation filter, by definition, does not account for consider parameter errors. Therefore, there are three basic contributions to P^*:

$$P^*(t) = P_a{}^*(t) + P_n{}^*(t) + P_u{}^*(t) \tag{3-15}$$

where

$P_a{}^*(t) \equiv$ the covariance contribution at time t induced by the assumed *a priori* covariance

$P_n{}^*(t) \equiv$ the covariance contribution induced by the assumed measurement noise covariance

$P_u{}^*(t) \equiv$ the covariance contribution induced by the assumed dynamic noise covariance

If the assumed covariances do not reflect the actual values (the filter is *mistuned*) then there will be some covariance contribution due to residual *a priori* error, measurement noise and dynamic noise. Thus

$$P(t) = P^*(t) + P_c(t) + \Delta P_a(t) + \Delta P_n(t) + \Delta P_u(t) \tag{3-16}$$

where

$$\Delta P_a(t) = P_a(t) - P_a{}^*(t) \tag{3-17a}$$

$$\Delta P_n(t) = P_n(t) - P_n{}^*(t) \tag{3-17b}$$

$$\Delta P_u(t) = P_u(t) - P_u{}^*(t)\ . \tag{3-17c}$$

Note that these matrices may not be non-negative-definite.

3.1 Batch Filter Covariance Analysis

A batch filter produces an estimate $s^*(t_o)$ at an epoch time t_o, based on a single batch of measurements y that may have been made at various times. Thus

$$y \equiv \begin{bmatrix} y_1 \\ \cdots \\ y_m \end{bmatrix}, \quad y^* \equiv \begin{bmatrix} y_1^* \\ \cdots \\ y_m^* \end{bmatrix} \quad \text{and} \quad \Delta y \equiv \begin{bmatrix} \Delta y_1 \\ \cdots \\ \Delta y_m \end{bmatrix} . \tag{3-18}$$

The batch filter produces an estimate $s^*(t_o)$ that gives the computed measurement y^* which minimizes the cost function

$$V \equiv \Delta y^T W \Delta y + \Delta s_o^{*T} W_o \Delta s_o^* \tag{3-19}$$

with

$$\Delta s_o^* \equiv s^*(t_o) - s_o^* = s^*(t_o) - s(t_o) + s(t_o) - s_o^* = \Delta s_o - \Delta s(t_o) \tag{3-20a}$$

$$\Delta s_o \equiv s(t_o) - s_o^* \tag{3-20b}$$

where $W \equiv$ positive-definite symmetric measurement weight matrix

$s_o^* \equiv$ *a priori* estimate of $s(t_o)$

$W_o \equiv$ non-negative-definite symmetric *a priori* weight matrix.

Since the batch filter determines $s^*(t_o)$, it is necessary to relate Δy to $\Delta s(t_o)$. Substituting equation (2-6) into equation (3-4), and using the partitioning of equations (3-6) and (3-7b), gives

$$\Delta y_i = G_i\,[\Phi(t_i, t_o)\Delta x(t_o) + \psi(t_i, t_o)] + v_i = F_i\,\Delta s(t_o) + C_i\,\Delta c(t_o) + U_i + v_i \tag{3-21}$$

where

$$F_i \equiv G_i \begin{bmatrix} \Phi_{ss}(t_i, t_o) \\ 0 \end{bmatrix}, \quad C_i \equiv G_i \begin{bmatrix} \Phi_{sc}(t_i, t_o) \\ \Phi_{cc}(t_i, t_o) \end{bmatrix} \quad \text{and} \quad U_i \equiv G_i\,\psi(t_i, t_o) . \tag{3-22}$$

Then

$$\Delta y = F \Delta s(t_o) + \Delta e \tag{3-23}$$

where

$$\Delta e \equiv C \Delta c(t_o) + U + v \tag{3-24a}$$

and

$$F \equiv \begin{bmatrix} F_1 \\ \cdots \\ F_m \end{bmatrix} \tag{3-24b}$$

with C, U and v defined similarly from C_i, U_i and v_i.

Substituting equations (3-20a) and (3-23) into equation (3-19) for the cost function gives

$$V = \Delta s^T(t_o)\,(W_o + F^T W F)\,\Delta s(t_o) + \Delta s^T(t_o)\,(F^T W \Delta \boldsymbol{e} - W_o \Delta s_o)$$

$$+ (\Delta \boldsymbol{e}^T W F - \Delta s_o{}^T W_o)\,\Delta s(t_o) + \Delta \boldsymbol{e}^T W \Delta \boldsymbol{e} + \Delta s_o{}^T W_o \Delta s_o$$

$$= [\Delta s(t_o) + W_n{}^{-1}(F^T W \Delta \boldsymbol{e} - W_o \Delta s_o)]^T\, W_n\, [\Delta s(t_o) + W_n{}^{-1}(F^T W \Delta \boldsymbol{e} - W_o \Delta s_o)]$$

$$- (F^T W \Delta \boldsymbol{e} - W_o \Delta s_o)^T\, W_n{}^{-1}\,(F^T W \Delta \boldsymbol{e} - W_o \Delta s_o) + \Delta \boldsymbol{e}^T W \Delta \boldsymbol{e} + \Delta s_o{}^T W_o \Delta s_o \qquad (3\text{-}25)$$

where $$W_n \equiv W_o + F^T W F\,. \qquad (3\text{-}26)$$

The matrix W_n is known as the *normal matrix*. The final equality in equation (3-25) is valid as long as W_n is nonsingular. The singularity (or ill-conditioning) of the normal matrix indicates a lack of observability of the solve-for parameters from the measurements $\boldsymbol{y}$.

If W_n is nonsingular, then it is clear from the form of equation (3-25) that V is minimized when

$$\Delta s(t_o) = -\,W_n{}^{-1}\,(F^T W \Delta \boldsymbol{e} - W_o \Delta s_o)$$

$$= -\,W_n{}^{-1}\,\{F^T W\,[C \Delta c(t_o) + \boldsymbol{U} + v] - W_o \Delta s_o\}$$

$$= \Delta s_a(t_o) + \Delta s_n(t_o) + \Delta s_c(t_o) + \Delta s_u(t_o) \qquad (3\text{-}27)$$

where $$\Delta s_a(t_o) \equiv W_n{}^{-1} W_o \Delta s_o \qquad (3\text{-}28a)$$

$$\Delta s_n(t_o) \equiv -\,W_n{}^{-1} F^T W v \qquad (3\text{-}28b)$$

$$\Delta s_c(t_o) \equiv -\,W_n{}^{-1} F^T W C \Delta c(t_o) \qquad (3\text{-}28c)$$

$$\Delta s_u(t_o) \equiv -\,W_n{}^{-1} F^T W \boldsymbol{U}\,. \qquad (3\text{-}28d)$$

The estimate $s^*(t_o)$ at the epoch time t_o may be propagated to any other time using equation (2-3). The solve-for parameter errors at these other times are given by equations (3-11), with $t' = t_o$ and with equations (3-28) as initial conditions.

Using equations (3-11a) and (3-28a) in equation (3-14a) gives the *a priori* error induced contribution to the solve-for covariance:

$$P_a(t) = \Phi_{ss}(t, t_o)\, P_a(t_o)\, \Phi_{ss}{}^T(t, t_o) \qquad (3\text{-}29)$$

where $$P_a(t_o) = W_n{}^{-1} W_o P_o W_o W_n{}^{-1} \qquad (3\text{-}30)$$

with $$P_o \equiv E[\Delta s_o \Delta s_o{}^T]\,. \qquad (3\text{-}31)$$

Using equations (3-11b) and (3-28b) in equation (3-14b) gives the measurement noise induced contribution to the solve-for covariance:

$$P_n(t) = \Phi_{ss}(t, t_o)\, P_n(t_o)\, \Phi_{ss}^T(t, t_o) \tag{3-32}$$

where $$P_n(t_o) = W_n^{-1} F^T W R W F W_n^{-1} \tag{3-33}$$

with $$R \equiv E[vv^T]\,. \tag{3-34}$$

Using equations (3-11c) and (3-28c) in equation (3-14c) gives the consider parameter induced contribution to the solve-for covariance:

$$P_c(t) = (\partial s/\partial c)(t)\, E[\Delta c(t_o)\Delta c^T(t_o)]\, (\partial s/\partial c)^T(t) \tag{3-35}$$

where $$(\partial s/\partial c)(t) \equiv -\, \Phi_{ss}(t, t_o)\, W_n^{-1} F^T W C + \Phi_{sc}(t, t_o)\,. \tag{3-36}$$

The computation of the dynamic noise contribution P_u is complicated by the fact that the U in equation (3-28d) is correlated with the $\psi_s(t, t_o)$ term introduced by the propagation equation (3-11d). Using equations (3-11d) and (3-28d) in equation (3-14d) gives

$$\begin{aligned} P_u(t) = {} & \Phi_{ss}(t, t_o)\, P_u(t_o)\, \Phi_{ss}^T(t, t_o) - \Phi_{ss}(t, t_o)\, W_n^{-1} F^T W\, E[U\psi_s^T(t, t_o)] \\ & - E[\psi_s(t, t_o) U^T]\, W F W_n^{-1} \Phi_{ss}^T(t, t_o) + D_{ss}(t, t_o) \end{aligned} \tag{3-37}$$

where $$P_u(t_o) = W_n^{-1} F^T W E[UU^T] W F W_n^{-1}\,. \tag{3-38}$$

From equation (3-22) we have

$$E[UU^T] = \begin{bmatrix} G_1 D(t_1, t_o) G_1^T & \cdots & G_1 D'(t_1, t_m) G_m^T \\ \cdots & \cdots & \cdots \\ G_m D'(t_m, t_1) G_1^T & \cdots & G_m D(t_m, t_o) G_m^T \end{bmatrix} \tag{3-39a}$$

$$E[U\psi_s^T(t, t_o)] = \begin{bmatrix} G_1 D'_s(t_1, t) \\ \cdots \\ G_m D'_s(t_m, t) \end{bmatrix} \tag{3-39b}$$

where
$$\begin{aligned} D'(t_i, t_j) &\equiv E[\psi(t_i, t_o)\psi^T(t_j, t_o)] = [D'_s(t_i, t_j), D'_c(t_i, t_j)] \\ &= \Phi(t_i, t_j)\, D(t_j, t_o) = \Phi(t_i, t_o)\, \Phi^{-1}(t_j, t_o)\, D(t_j, t_o) \qquad \text{for } t_i \ge t_j \\ &= D(t_i, t_o)\, \Phi^T(t_j, t_i) = [\Phi(t_j, t_o)\, \Phi^{-1}(t_i, t_o)\, D(t_i, t_o)]^T \qquad \text{for } t_j \ge t_i\,. \end{aligned} \tag{3-40}$$

The last equality on the first line of equation (3-40) indicates a partitioning of $D'(t_i, t_j)$ into submatrices $D'_s(t_i, t_j)$ and $D'_c(t_i, t_j)$, and the equalities on the last two lines follow from equations (2-9) to (2-11).

A *minimum variance* batch estimator produces solve-for parameter estimates with minium covariance due to noise sources known to the filter [Sorenson, Wertz]. The weights for such a filter are chosen as follows:

$$W = R^{*-1} \quad \text{and} \quad W_o = P_o^{*-1} \tag{3-41}$$

where $R^* \equiv$ an assumed value for the measurement noise covariance

$P_o^* \equiv$ an assumed value for the *a priori* error covariance.

The estimated covariance at the epoch time $P^*(t_o)$ is obtained by substituting equations (3-41) into equations (3-30) and (3-33), and assuming that $R = R^*$ and $P_o = P_o^*$, giving

$$P^*(t_o) = P_a^*(t_o) + P_n^*(t_o) = W_n^{-1}(W_o + F^TWF)W_n^{-1} = W_n^{-1} \tag{3-42}$$

with

$$P_a^*(t_o) = W_n^{-1}W_oW_n^{-1} \tag{3-43a}$$
$$P_n^*(t_o) = W_n^{-1}F^TWFW_n^{-1}\,. \tag{3-43b}$$

Note that the $P_u^*(t_o) = 0$ because the batch filter does not account for dynamic noise at all. The covariance estimate is propagated to other times by using equations (3-29) and (3-32), which give

$$P^*(t) = \Phi_{ss}(t, t_o)\, P^*(t_o)\, \Phi_{ss}^T(t, t_o)\,. \tag{3-44}$$

Using equations (3-30), (3-33), (3-41) and (3-43) in equations (3-17) gives the residual covariance contributions:

$$\Delta P_a(t_o) = W_n^{-1}W_o(P_o - P_o^*)W_oW_n^{-1} \tag{3-45a}$$
$$\Delta P_n(t_o) = W_n^{-1}F^TW(R - R^*)WFW_n^{-1} \tag{3-45b}$$
$$\Delta P_u(t_o) = P_u(t_o)\,. \tag{3-45c}$$

The matrices propagate in the same manner as P_a, P_n and P_u, respectively.

3.2 Sequential Filter Covariance Analysis

A sequential filter produces an estimate $s^*(t)$ based on measurements taken at discrete times $t_i \le t$. Between the measurement times t_i, the state estimate $x^*(t)$ is propagated using equation (2-3). At each time t_i, the solve-for parameters are updated based on the propagated state $x^*(t_i)$ and the measurements y_i. Typically, this update has the following form:

$$s^*(t_i) = s^*(t_i-) + K_i\Delta y_i \tag{3-46}$$

where $s^*(t_i)$ and $s^*(t_i-)$ denote estimates of the solve-for parameters immediately after and immediately before incorporating the information contained in the measurements at time t_i. The gain matrix K_i determines how much the propagated state is corrected, based on the measurement residuals Δy_i.

The estimation error immediately after an update is

$$\Delta s(t_i) = s(t_i) - s^*(t_i) = s(t_i) - s^*(t_i-) - K_i \Delta y_i = \Delta s(t_i-) - K_i \Delta y_i \tag{3-47}$$

since the true state is continuous at t_i. Substituting equation (3-4) for Δy_i and using the partitioning of equation (3-6) gives

$$\begin{aligned} \Delta s(t_i) &= \Delta s(t_i-) - K_i \,[G_i \Delta x(t_i-) + v_i] \\ &= (I - K_i G_{si})\, \Delta s(t_i-) - K_i \,[G_{ci} \Delta c(t_i-) + v_i] \end{aligned} \tag{3-48}$$

where G_i has been partitioned as

$$G_i = [G_{si},\ G_{ci}]\ . \tag{3-49}$$

Substituting equation (3-10) into equation (3-48), and using equation (3-9), gives update equations for each of the contributions to the total solve-for error:

$$\Delta s_a(t_i) = (I - K_i G_{si})\, \Delta s_a(t_i-) \tag{3-50a}$$

$$\Delta s_n(t_i) = (I - K_i G_{si})\, \Delta s_n(t_i-) - K_i v_i \tag{3-50b}$$

$$\Delta s_c(t_i) = (I - K_i G_{si})\, \Delta s_c(t_i-) - K_i G_{ci} \Phi_{cc}(t_i\, , t_o)\, \Delta c(t_o) \tag{3-50c}$$

$$\Delta s_u(t_i) = (I - K_i G_{si})\, \Delta s_u(t_i-) - K_i G_{ci} \Psi_c(t_i, t_o)\ . \tag{3-50d}$$

Each of these error contributions may be propagated individually between measurement times using equations (3-11), with the initial conditions:

$$\Delta s_a(t_o) = \Delta s_o \tag{3-51a}$$

$$\Delta s_n(t_o) = \Delta s_c(t_o) = \Delta s_u(t_o) = 0\ , \tag{3-51b}$$

where Δs_o is defined in equation (3-20b).

Using equation (3-11a) in equation (3-14a) gives the propagation equation for the *a priori* error induced contribution to the solve-for covariance:

$$P_a(t) = \Phi_{ss}(t, t_i)\, P_a(t_i)\, \Phi_{ss}^T(t, t_i) \quad \text{for } t_i \le t < t_{i+1} \tag{3-52}$$

where

$$P_a(t_o) = P_o \tag{3-53}$$

with the *a priori* covariance P_o defined in equation (3-31). Substituting equation (3-50a) into equation (3-14a) gives the update equation:

$$P_a(t_i) = (I - K_i G_{si})\, P_a(t_i-)\, (I - K_i G_{si})^T\ . \tag{3-54}$$

Using equation (3-11b) in equation (3-14b) gives the propagation equation for the measurement noise induced contribution to the solve-for covariance:

$$P_n(t) = \Phi_{ss}(t, t_i)\, P_n(t_i)\, \Phi_{ss}^T(t, t_i) \quad \text{for } t_i \le t < t_{i+1} \tag{3-55}$$

where $$P_n(t_o) = 0 \tag{3-56}$$

Substituting equation (3-50b) into equation (3-14b) gives the update equation:

$$P_n(t_i) = (I - K_i G_{si})\, P_n(t_i\text{-})\, (I - K_i G_{si})^T + K_i R_i K_i^T \tag{3-57}$$

with R_i defined by equation (3-2b).

The consider parameter induced contribution to the covariance can be most easily expressed in terms of the partial derivative $(\partial s/\partial c)(t)$ implicitly defined by

$$\Delta s_c(t) = (\partial s/\partial c)(t)\, \Delta c(t_o)\,. \tag{3-58}$$

Substituting this into equation (3-11c) gives the propagation equation:

$$(\partial s/\partial c)(t) = \Phi_{ss}(t, t_i)\, (\partial s/\partial c)(t_i) + \Phi_{sc}(t, t_i)\Phi_{cc}(t_i, t_o) \quad \text{for } t_i \le t < t_{i+1} \tag{3-59}$$

where $$(\partial s/\partial c)(t_o) = 0\,. \tag{3-60}$$

Substituting equation (3-58) into equation (3-50c) gives the update equation:

$$(\partial s/\partial c)(t_i) = (I - K_i G_{si})\, (\partial s/\partial c)(t_i\text{-}) - K_i G_{ci} \Phi_{cc}(t_i, t_o)\,. \tag{3-61}$$

From equations (3-14c) and (3-58), the consider parameter contribution to the solve-for covariance is then

$$P_c(t) = (\partial s/\partial c)(t)\, E[\Delta c(t_o) \Delta c^T(t_o)]\, (\partial s/\partial c)^T(t)\,. \tag{3-62}$$

As in the case of a batch filter, the dynamic noise contribution is more complicated to compute than the other contributions. Substituting equation (3-11d) into equation (3-14d) and using equation (2-11) gives:

$$\begin{aligned} P_u(t) = {} & \Phi_{ss}(t, t_i)\, P_u(t_i)\, \Phi_{ss}^T(t, t_i) + \Phi_{ss}(t, t_i)\, P_{uc}(t_i)\, \Phi_{sc}^T(t, t_i) \\ & + \Phi_{sc}(t, t_i)\, P_{uc}^T(t_i)\, \Phi_{ss}^T(t, t_i) + \Phi_{sc}(t, t_i)\, D_{cc}(t_i, t_o)\, \Phi_{sc}^T(t, t_i) \\ & + D_{ss}(t, t_i) \end{aligned} \tag{3-63}$$

for $t_i \le t < t_{i+1}$, where

$$P_u(t_o) = 0 \tag{3-64}$$

and

$$P_{uc}(t) \equiv E[\Delta s_u(t) \psi_c^T(t, t_o)] \tag{3-65}$$

and the random excitation covariance D is partitioned as in equation (3-7c). It follows from equations

(2-10), (3-7a) and (3-7b) that

$$\psi_c(t,t_o) = \Phi_{cc}(t,t_i)\, \psi_c(t_i,t_o) + \psi_c(t,t_i)\,. \tag{3-66}$$

Using this and equations (2-11) and (3-11d) in equation (3-65) gives the equation for propagating $P_{uc}(t)$;

$$P_{uc}(t) = \Phi_{ss}(t, t_i)\, P_{uc}(t_i)\, \Phi_{cc}^T(t, t_i) + \Phi_{sc}(t, t_i)\, D_{cc}(t_i, t_o)\, \Phi_{cc}^T(t, t_i) + D_{sc}(t, t_i) \tag{3-67}$$

for $t_i \leq t < t_{i+1}$, where

$$P_{uc}(t_o) = 0\,. \tag{3-68}$$

From equations (3-14d), (3-50d) and (3-65), the update equations for $P_u(t)$ and $P_{uc}(t)$ are:

$$P_u(t_i) = (I - K_iG_{si})\, P_u(t_i\text{-})\, (I - K_iG_{si})^T - (I - K_iG_{si})\, P_{uc}(t_i\text{-})\, G_{ci}^TK_i^T - K_iG_{ci}\, P_{uc}^T(t_i\text{-})\, (I - K_iG_{si})^T + K_iG_{ci}\, D_{cc}(t_i,t_o)\, G_{ci}^TK_i^T \tag{3-69a}$$

$$P_{uc}(t_i) = (I - K_iG_{si})\, P_{uc}(t_i\text{-}) - K_iG_{ci}\, D_{cc}(t_i,t_o)\,. \tag{3-69b}$$

A *Kalman filter* is a sequential filter which produces solve-for parameter estimates with minimum covariance due to noise sources known to the filter [Gelb, Lefferts]. In addition to the solve-for parameter estimates, a Kalman filter maintains an estimate P^* of the solve-for parameter covariance, and uses this to compute an optimal gain K_i at each time t_i. The covariance estimate P^* is given by algorithms similar to those for P, with the full state error vector replaced by the solve-for parameter error vector. The resulting propagation equation for P^* is

$$P^*(t) = \Phi_{ss}(t, t_i)\, P^*(t_i)\, \Phi_{ss}^T(t, t_i) + D_{ss}^*(t, t_i) \quad \text{for } t_i \leq t < t_{i+1} \tag{3-70}$$

where the matrix D_{ss}^* is the estimate of the random excitation covariance used by the filter. It is based on an assumed spectral density Q_{ss}^* of the dynamic noise on the solve-for parameters:

$$D_{ss}^*(t, t_i) \equiv \int_{t_i}^{t} \Phi_{ss}(t, t'')\, Q_{ss}^*\, \Phi_{ss}^T(t, t'')\, dt''. \tag{3-71}$$

The update equation for the covariance estimate is

$$P^*(t_i) = (I - K_iG_{si})\, P^*(t_i\text{-})\, (I - K_iG_{si})^T + K_iR_i^*K_i^T \tag{3-72}$$

where $\quad R_i^* \equiv$ an assumed value for the measurement noise covariance

and the *Kalman gain* is given by [Gelb, Lefferts]

$$K_i = P^*(t_i\text{-})\, G_i^T\, [G_iP^*(t_i\text{-})G_i^T + R_i^*]^{-1}\,. \tag{3-73}$$

Substituting equation (3-15) into equation (3-70), gives the following propagation equations for the component contributions to P^*:

$$P_a^*(t) = \Phi_{ss}(t, t_i)\, P_a^*(t_i)\, \Phi_{ss}^T(t, t_i) \tag{3-74a}$$

$$P_n^*(t) = \Phi_{ss}(t, t_i)\, P_n^*(t_i)\, \Phi_{ss}^T(t, t_i) \tag{3-74b}$$

$$P_u^*(t) = \Phi_{ss}(t, t_i)\, P_u^*(t_i)\, \Phi_{ss}^T(t, t_i) + D_{ss}^*(t, t_i) \tag{3-74c}$$

for $t_i \leq t < t_{i+1}$, with initial conditions

$$P_a^*(t_o) = P_o^* \equiv \text{an assumed value for the } a\ priori \text{ error covariance.} \tag{3-75a}$$
$$P_n^*(t_o) = P_u^*(t_o) = 0\,. \tag{3-75b}$$

Substituting equation (3-15) into equation (3-72), gives the corresponding update equations:

$$P_a^*(t_i) = (I - K_i G_{si})\, P_a^*(t_i-)\, (I - K_i G_{si})^T \tag{3-76a}$$
$$P_n^*(t_i) = (I - K_i G_{si})\, P_n^*(t_i-)\, (I - K_i G_{si})^T + K_i R_i^* K_i^T \tag{3-76b}$$
$$P_u^*(t_i) = (I - K_i G_{si})\, P_u^*(t_i-)\, (I - K_i G_{si})^T\,. \tag{3-76c}$$

A Kalman filter will produce an estimate with the minimum covariance P^* due to the *assumed* covariances P_o^*, R_i^* and Q_{ss}^*. If the filter is mistuned, the true covariance will *not* be minimized. Using equations (3-52), (3-55), (3-63) and (3-74) in equations (3-17) gives propagation equations for the residual covariance contributions:

$$\Delta P_a(t) = \Phi_{ss}(t, t_i)\, \Delta P_a(t_i)\, \Phi_{ss}^T(t, t_i) \tag{3-77a}$$

$$\Delta P_n(t) = \Phi_{ss}(t, t_i)\, \Delta P_n(t_i)\, \Phi_{ss}^T(t, t_i) \tag{3-77b}$$

$$\begin{aligned} \Delta P_u(t) = {} & \Phi_{ss}(t, t_i)\, \Delta P_u(t_i)\, \Phi_{ss}^T(t, t_i) + \Phi_{ss}(t, t_i)\, P_{uc}(t_i)\, \Phi_{sc}^T(t, t_i) \\ & + \Phi_{sc}(t, t_i)\, P_{uc}^T(t_i)\, \Phi_{ss}^T(t, t_i) + \Phi_{sc}(t, t_i)\, D_{cc}(t_i, t_o)\, \Phi_{sc}^T(t, t_i) \\ & + D_{ss}(t, t_i) - D_{ss}^*(t, t_i) \end{aligned} \tag{3-77c}$$

for $t_i \leq t < t_{i+1}$, where

$$\Delta P_a(t_o) = P_o - P_o^* \tag{3-78a}$$
$$\Delta P_n(t_o) = \Delta P_u(t_o) = 0\,. \tag{3-78b}$$

Using equations (3-54), (3-57), (3-69a) and (3-76) in equations (3-17) gives update equations for the residual covariance contributions:

$$\Delta P_a(t_i) = (I - K_i G_{si})\, \Delta P_a(t_i-)\, (I - K_i G_{si})^T \tag{3-79a}$$
$$\Delta P_n(t_i) = (I - K_i G_{si})\, \Delta P_n(t_i-)\, (I - K_i G_{si})^T + K_i\, (R_i - R_i^*)\, K_i^T \tag{3-79b}$$
$$\begin{aligned} \Delta P_u(t_i) = {} & (I - K_i G_{si})\, \Delta P_u(t_i-)\, (I - K_i G_{si})^T - (I - K_i G_{si})\, P_{uc}(t_i-)\, G_{ci}^T K_i^T \\ & - K_i G_{ci}\, P_{uc}^T(t_i-)\, (I - K_i G_{si})^T + K_i G_{ci}\, D_{cc}(t_i, t_o)\, G_{ci}^T K_i^T\,. \end{aligned} \tag{3-79c}$$

4. CONCLUSIONS

Error analysis can be crucial during mission design, providing assistance in the specification of a sensor complement and a calibration plan, possibly requiring a set of scheduled attitude maneuvers, to deliver the pointing accuracy necessary to satisfy the mission objectives. Error analysis is also necessary to determine what level of ground-based processing will be needed to meet high-accuracy attitude determination requirements. Thus, to ensure the achievement of mission objectives, it is critical that the analyst produce accurate estimates of determination uncertainties, especially the often-underestimated contributions of process noise and consider parameter errors. In this paper we have presented a general, comprehensive approach to filter and dynamics modeling for spacecraft attitude determination error analysis.

The model is general in that it allows great freedom in specifying orbit geometry, sensor types, measurement scheduling and parameter selection. Further, it covers both spin-stabilized and three-axis stabilized spacecraft, with process noise appropriate to the two types of stabilization, and both batch least-squares and sequential attitude estimation processes. This paper does not include models of sensors, with the exception of a model for strapdown gyros used for dynamics model replacement in the three-axis stabilized case. However, the only restriction on sensor modeling is that the measurement noise must be additive.

The model is comprehensive in that it considers all the major sources of error in the determination process. The model gives the separate contributions to the solve-for parameter uncertainty arising from errors in the *a priori* estimates of the solve-for parameters, from measurement noise, from process noise, and from consider parameter uncertainties, as well as the overall uncertainty due to all these sources of error. This allows the analyst to judge the importance of various sources of error, and make informed recommendations to reduce the effect of the largest contributors.

The analysis of the effect of dynamics errors in the batch estimation case is particularly important, since batch filters generally do not account for this source of error. Indeed, for both the batch and the sequential cases, the model carefully separates the estimation covariance based on *true* sources of error from the estimation covariance based on sources of error *assumed* by the filter. This gives the analyst the ability to study *mistuned* filters. While the concept of tuning is primarily associated with sequential filters, the presentation here makes it clear that it may also be an important consideration in the batch case.

The model for attitude determination error analysis presented here was developed as part of the mathematical specification of algorithms for the computer-based Attitude Determination Error Analysis System. This software system incorporates the dynamics model presented in this paper for three-axis stabilized spacecraft, a simplified dynamics model for spin-stabilized spacecraft, slightly simplified batch and sequential filter models and a wide variety of sensor models, including digital and analog sun sensors, scanning and fixed-head star trackers, gimballed line-of-sight sensors, horizon sensors, and magnetometers. The Attitude Determination Error Analysis System is currently undergoing acceptance testing, and will be an important component of the institutional flight support software of the Goddard Space Flight Center Flight Dynamics Division when this testing has been successfully completed.

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