COMPUTATION OF ORBITS USING TOTAL ENERGY

by

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1.0 SUMMARY

The computation of orbits can be done more efficiently by the use of any of several new formulations (Reference 1, 2, 3, 4, 5) of the perturbed two body problem which consider the total energy of the orbital system as one of the dependent variables. The total energy is the osculating two body energy plus the potential energy due to perturbing masses. The use of the total energy as the dependent variable instead of the two body energy is a relatively new idea (Reference 1). The advantage of using total energy arises from the fact that the more perturbing potential energy that is accounted for in the total energy variable, the more nearly constant is the total energy. In fact, except for dissipative forces such as drag, the only reason for the total energy not being constant is the rotation or revolution of the perturbing mass. This near constancy of the total energy has the effect of inhibiting error growth during numerical solution (Reference 1). This paper will present the results of an application of total energy formulation (Reference 2) to the problem of the precise computation of orbits.

2.0 INTRODUCTION

The differential equation of motion of the perturbed two-body problem can be expressed as,

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = \underline{F} = \underline{P} - \frac{\partial V}{\partial \underline{r}} \tag{1}$$

where $\underline{r}$ is the position vector of one of the bodies relative to the other. The perturbations, those derivable from a potential $\partial V/\partial \underline{r}$, as well as other forces $\underline{P}$, are included in the total perturbation $\underline{F}$. •

The total energy element formulations (References 1, 2, 3, 4, 5) of the perturbed two-body problem are developed such that $\partial V/\partial t$ is used as well as $\partial V/\partial \underline{r}$. The perturbations are split into those derivable from a potential and those which are included in the perturbation $\underline{P}$. The perturbation $\underline{P}$ normally includes non-conservative perturbations, but it can also include perturbations derivable from a potential when convenient. For the total energy formulations, the right-hand sides of all differential equations, except that for the total energy, include the perturbation factors $\underline{P}$ and $\partial V/\partial \underline{r}$. The total energy differential equation has the form

$$\dot{h} = \dot{\underline{r}} \cdot \underline{P} + \frac{\partial V}{\partial t} \tag{2}$$

where h is the total energy,

$$h = \frac{1}{2} \dot{\underline{r}} \cdot \dot{\underline{r}} - \frac{\mu}{r} + V(\underline{r},t) \tag{3}$$

Equation (2) is derived by taking the time derivative of equation (3) and substituting equation (1) into this result to eliminate $\ddot{\underline{r}}$. Note that this differential equation includes the perturbations $\underline{P}$ and $\partial V/\partial t$, but does not include $\partial V/\partial \underline{r}$.

There are three options available in the total energy formulations depending upon the way in which the perturbations derivable from a potential are used in the differential equations. These options are categorized as follows:

(A) The entire perturbing potential is considered with its effect including $\partial V/\partial \underline{r}$ and $\partial V/\partial t$. This is the option which is developed and discussed in this paper.

(B) The perturbing potential can be portioned, including some of the perturbation in $\partial V/\partial \underline{r}$ and some in $\underline{P}$. This has been the approach most often used when the geopotential is the perturbation. The zonal terms have been included in $\partial V/\partial \underline{r}$, while the explicitly time dependent terms (the tesseral and sectorial terms) have been included in $\underline{P}$. This approach has been used in order to avoid the computation of $\partial V/\partial t$. The potential used is that of the zonal terms only.

(C) The perturbing potential is not considered at all. The $\partial V/\partial \underline{r}$ are included in the perturbation $\underline{P}$. The potential is set to zero.

It must be emphasized that all three options are correct. The advantage that any of the options has over the others is numerical accuracy and speed in computation. The advantage of option (B) over option (C) in accuracy and speed is considerable and is discussed at length in References 1, 2, and 4.*

In order to properly implement the differential equation (2) for the total energy as discussed in option (A), the partial derivative $\partial V/\partial t$ must be computed. This report will derive a simple formula for this computation. This formula will be developed for the general case of any perturbation derivable from a potential. Then the particular case of a geopotential perturbation acting on an Earth satellite will be used as an example to show the advantage of using $\partial V/\partial t$ in the computation.

3.0 DEVELOPMENT OF $\partial V/\partial t$

Let $\underline{r}$ be the position vector in an inertial system and let $\underline{r}_G$ be the same position vector in a system rotating with angular velocity $\underline{\omega}$ with respect to the inertial system. Then,

$$\underline{r} = \underline{r}_G \tag{4}$$

The velocity vectors are related by,

$$\dot{\underline{r}} = \dot{\underline{r}}_G + \underline{\omega} \times \underline{r}_G \tag{5}$$

In the inertial system, the potential function is expressed as an explicit function of time,

$$V = V(\underline{r},t) \tag{6}$$

* For these formulations a slightly different energy parameter α_0, where $h = -2\alpha_0$, is used and a new independent variable called fictitious time is introduced. With these changes, equation (2) becomes

$$\alpha_0' = -\frac{r}{2}\frac{\partial V}{\partial t} - \frac{1}{2}\,\underline{r}' \bullet \underline{P}$$

where $(\ )' = d(\ )/ds$ and s is the independent variable such that $dt/ds = r$.

having the total derivative

$$\frac{dV}{dt} = \frac{\partial V}{\partial \underline{r}} \cdot \dot{r} + \frac{\partial V}{\partial t} \tag{7}$$

In a properly chosen rotating system, the same potential function can be expressed as a function of position only,

$$V = V(\underline{r}_G) \tag{8}$$

having the total derivative,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \underline{r}} \cdot \dot{\underline{r}}_G \tag{9}$$

since in the rotating system the potential has no explicit dependence on time, $\partial V(\underline{r}_G)/\partial t = 0$.

Using equations (4) and (5), equation (7) becomes

$$\frac{dV}{dt} = \frac{\partial V}{\partial \underline{r}_G} \cdot (\dot{\underline{r}}_G + \underline{\omega} \times \underline{r}_G) + \frac{\partial V}{\partial t} \tag{10}$$

Comparing equations (9) and (10), we obtain

$$\frac{\partial V}{\partial t} = - \frac{\partial V}{\partial \underline{r}_G} \cdot \underline{\omega} \times \underline{r}_G \tag{11}$$

Note that to this point, we have not considered any particular potential function. The result, equation (11), can be applied under proper conditions to the case where V represents the perturbing geopotential function or to the case of a lunar, solar, or planetary perturbation on a satellite.

4.0 APPLICATION TO THE GEOPOTENTIAL

We now consider the case of the perturbing geopotential which can be divided into two parts,

$$V(\underline{r},t) = V_Z(\underline{r}) + V_T(\underline{r},t) \tag{12}$$

where $\underline{r}$ is expressed in an inertial system having one axis normal to the Earth equatorial plane and the other two orthogonal axes in the equatorial plane. The portion of the perturbing geopotential $V_Z(\underline{r})$ arising from the zonal terms has no explicit time dependence. The portion of the perturbing geopotential $V_T(\underline{r},t)$ arising from the sectorial and tesseral terms are explicitly dependent upon time.

For the case of a perturbing geopotential equation (11) can be reduced further. Define the rotating system X_G, Y_G, Z_G such that Z_G is in a direction normal to the Earth equatorial plane and the X_G and Y_G axes lie in the Earth equatorial plane and are fixed in the Earth. The zonal portion of the perturbing geopotential is

$$V_Z = -\frac{\mu}{r}\sum_{n=2}^{\infty} C_{n,o}\left(\frac{a_e}{r}\right)^n P_n\,(Z_G/r)$$

where, $C_{n,o}$ are the zonal coefficients
a_e is the equatorial radius of the Earth
P_n is the n^{th} degree Legendre polynomial which is a function of Z_G/r.

Now, $\underline{r}_G = \hat{i}_G X_G + \hat{j}_G Y_G + \hat{k}_G Z_G\,, \qquad r = |\underline{r}_G|$

where $\hat{i}_G, \hat{j}_G, \hat{k}_G$ are unit vectors along the X_G, Y_G, Z_G axes. The partial derivative $\partial V_Z/\partial \underline{r}_G$ has the form,

$$\frac{\partial V_Z}{\partial \underline{r}_G} = f_1(r,Z_G)\underline{r}_G + f_2(r,Z_G)\hat{k}_G$$

Note that since $\underline{\omega} = \omega\hat{k}_G$,

$$\frac{\partial V_Z}{\partial \underline{r}_G} \cdot \underline{\omega} \times \underline{r}_G = 0.$$

Thus, the zonal part V_Z of the perturbing geopotential does not contribute to $\partial V/\partial t$. Equation (11) becomes

$$\frac{\partial V}{\partial t} = - \frac{\partial V_T}{\partial \underline{r}_G} \bullet \underline{\omega} \times \underline{r}_G . \tag{13}$$

Consider the first two options given in Section 2.0 for the formulation of the differential equation (2) for the total energy.

4.1 Option A - All zonal, sectorial, and tesseral terms of the perturbing geopotential are included in the potential and hence in the total energy.

Let $\quad V = V_Z(\underline{r}) + V_T(\underline{r})$

then $\quad \dfrac{\partial V}{\partial \underline{r}} = \dfrac{\partial V_Z}{\partial \underline{r}} + \dfrac{\partial V_T}{\partial \underline{r}}$

also $\quad \underline{P} = 0$

and from equation (13), we compute $\partial V/\partial t$.

Further, since $\underline{r} = \underline{r}_G$, we can express equation (2) as

$$\dot{h}_A = \frac{\partial V}{\partial t} = - \frac{\partial V_T}{\partial \underline{r}} \bullet \underline{\omega} \times \underline{r} . \tag{14}$$

4.2 Option B - Only the zonal terms of the perturbing geopotential are included in the potential and hence in the total energy.

Let $\quad V = V_Z(\underline{r})$

then $\quad \dfrac{\partial V}{\partial \underline{r}} = \dfrac{\partial V_Z}{\partial \underline{r}}$

and $\quad \dfrac{\partial V}{\partial t} = 0$

The sectorial and tesseral terms are considered to be in the perturbation $\underline{P}$,

$$\underline{P} = -\frac{\partial V_T}{\partial \underline{r}}$$

and the total energy differential equation (2) becomes,

$$\dot{h}_B = \dot{\underline{r}} \cdot \underline{P} = -\frac{\partial V_T}{\partial \underline{r}} \cdot \dot{\underline{r}} \tag{15}$$

4.3 Comparison of Options A and B:

Both equations (14) and (15) depend directly upon the factor $\partial V_T/\partial \underline{r}$, which is a small term depending only upon the sectorial and tesseral terms. But we also observe that for Option B, $\dot{h}_B$ is proportional to the inertial velocity, $\dot{\underline{r}}$, whereas for Option A, $\dot{h}_A$ is proportional to the component ($\underline{\omega}$ x $\underline{r}$) of the inertial velocity which arises from the rotation of the axes fixed in the Earth.

For near Earth satellite orbits,

$$\left| \underline{\omega} \times \underline{r} \right| < \left| \dot{\underline{r}} \right| \tag{16}$$

and also

$$\left| \dot{h}_A \right| < \left| \dot{h}_B \right|$$

In fact, if the Earth were not rotating ($\omega = 0$), then $\dot{h}_A$ would be zero. For satellite orbits which are at large distances from the Earth, the inequality (16) does not always hold. However, at large distances from the Earth, the perturbing geopotential is not as significant as perturbations due to the Sun or Moon. The global region for which the inequality (16) holds is complicated and depends upon the semi-major axis, the eccentricity, and the true anomaly (or angular position) of the satellite trajectory.

For near circular orbits, it can be shown that the ratio

$$\dot{h}_A / \dot{h}_B = \frac{\omega}{n}\left(1 - (Z/r)^2\right)^{1/2}$$

where n is the mean motion of the satellite and the factor $\left(1 - (Z/r)^2\right)$ is always less than unity. For orbits within the geosynchronous distance, the inequality (16) holds since

$$n > \omega$$

For near Earth orbits,

$$\frac{\omega}{n} \approx \frac{1}{16}$$

and so

$$\dot{h}_A / \dot{h}_B < \frac{1}{16}$$

The formulations of the perturbed two-body problem discussed in References 1, 2, and 4 are in effect perturbed harmonic oscillators having frequencies which are dependent upon the total energy. The use of the full geopotential as shown in Option A in the computation of the total energy causes $\dot{h}_A$ to be small. Thus, h_A is nearly constant and the resulting frequency of the perturbed oscillator equations is nearly constant. Options A and B as well as Option C are also compared in Table I.

5.0 NUMERICAL RESULTS

The numerical effect of using the full geopotential as in Option A is shown in Figure 1. A near circular orbit was propagated for ten days first using Option A and then using Option B. This computation was done using the KSUR12 total energy formulation (Reference 2) and the RK4(5) variable step numerical integrator (Reference 6). The geopotential model used was the complete GEM-L2 (Reference 7). The results of these computations were compared to a reference trajectory computed with very high precision as given in Reference 8 and originally provided in Reference 9. Figure 1 shows the RSS of the position vector of Options A and B, with each compared to the reference.

Option B (using only the zonal part of the geopotential in the total energy) required an average of 59.4 variable steps per revolution with a maximum error of 25 meters. Option A (using the full geopotential in the total energy) required an average of 45.2 variable steps per revolution with a maximum error of about 8 meters. The two options are also compared on Figure 1 using 30 fixed steps per revolution. Option B showed a rapidly growing error reaching 25 meters after 4 days and still diverging. Option A reached a maximum error of about 15 meters after 10 days.

6.0 REFERENCES

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Table I. Comparison of Options A, B, and C for perturbing geopotential.

	Option A	Option B	Option C
Perturbing Geopotential (V)	$V_Z + V_T$	V_Z	0
Total Energy (h) equation (3)	$\frac{1}{2}\dot{\underline{r}}\cdot\dot{\underline{r}} - \frac{\mu}{\underline{r}} + V_Z + V_T$	$\frac{1}{2}\dot{\underline{r}}\cdot\dot{\underline{r}} - \frac{\mu}{\underline{r}} + V_Z$	$\frac{1}{2}\dot{\underline{r}}\cdot\dot{\underline{r}} - \frac{\mu}{r}$
Perturbation $\left(\frac{\partial V}{\partial \underline{r}}\right)$	$\frac{\partial V_Z}{\partial \underline{r}} + \frac{\partial V_T}{\partial \underline{r}}$	$\frac{\partial V_Z}{\partial \underline{r}}$	0
Perturbation (P)	0	$-\frac{\partial V_T}{\partial \underline{r}}$	$-\left(\frac{\partial V_Z}{\partial \underline{r}} + \frac{\partial V_T}{\partial \underline{r}}\right)$
Derivative of total energy ($\dot{h}$) equation (2)	$-\frac{\partial V_T}{\partial \underline{r}} \cdot \underline{\omega} \times \underline{r}$	$-\frac{\partial V_T}{\partial \underline{r}} \cdot \dot{\underline{r}}$	$-\left(\frac{\partial V_Z}{\partial \underline{r}} + \frac{\partial V_T}{\partial \underline{r}}\right) \cdot \dot{\underline{r}}$

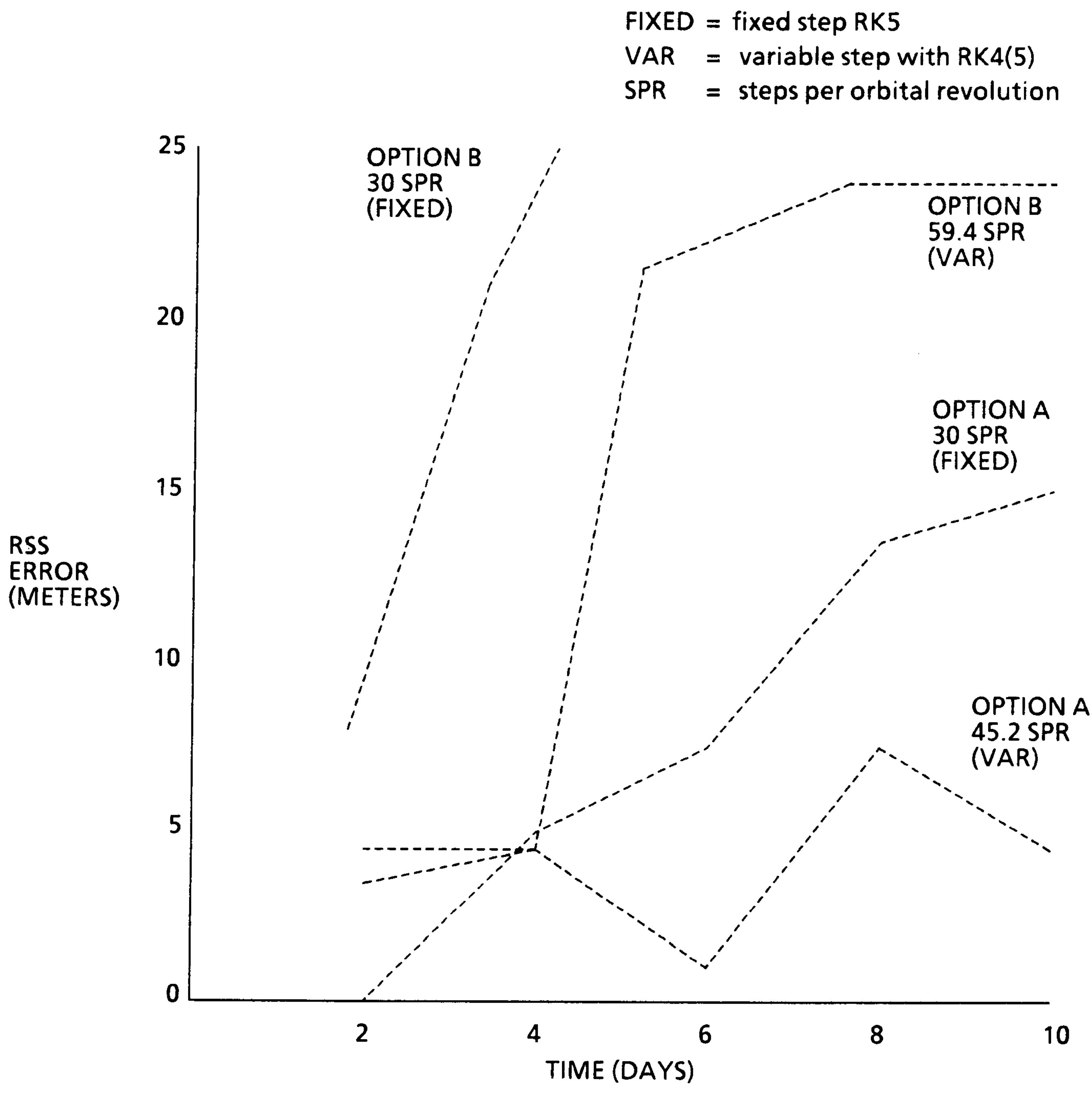

Figure 1. Comparison of RSS errors for near Earth trajectories computed using only the zonal terms of the perturbing geopotential in the total energy (Option B) and using the full perturbing geopotential in the total energy (Option A).