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3-D INELASTIC ANALYSIS METHODS FOR HOT SECTION COMPONENTS - BEST 3D CODE[1]

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The goal of the Advanced Formulation Development portion of the Inelastic Analysis Methods contract is the development of an alternative stress analysis tool, distinct from the finite element method, applicable to the engineering analysis of gas turbine engine structures. The boundary element method was selected for this development effort on the basis of its already demonstrated applicability to a variety of geometries and problem types characteristic of gas turbine engine components. This paper describes briefly major features of the BEST3D computer program and outlines some of the significant developments carried out as part the Inelastic Methods Contract.

BEST3D OVERVIEW

BEST3D (Boundary Element Stress Technology - Three Dimensional) is a general purpose three-dimensional structural analysis program utilizing the boundary element method. The method has been implemented for very general three-dimensional geometries, for elastic, inelastic and dynamic stress analysis. Although the feasibility of many of the capabilities provided had been demonstrated in a number of individual research efforts, the present code is the first in which they have been made available for large scale problems in a single code. In addition, important basic advances have been made in a number of areas, including the development and implementation of a variable stiffness plasticity algorithm, the incorporation of an embedded time algorithm for elastodymanics and the extensive application of particular solutions within the boundary element method. Major features presently available in the BEST3D code include:

- Very general geometry definition, including the use of doubly curved isoparametric surface elements and volume cells, with provision of full substructuring capability

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- General capability for the definition of complex, time-dependent boundary conditions
- Capability for nonlinear analysis using a variety of algorithms, solution proceedures and constitutive models
- A very complete elastodynamic capability including provision for free vibration, forced response and transient analysis

BEST3D has been successfully implemented on a variety of computers, including the IBM 3090, various CRAY models and the Hewlett-Packard HP9000.

The analytical basis and numerical implementation of the boundary element method for the major problem types considered are very briefly reviewed in the next two sections. Particular attention is devoted to the variable stiffness plasticity algorithm and the time-embedded elastodynamic algorithm. Full details of both the analysis and implementation may be found in references 1, 2 and 3.

QUASI-STATIC ANALYSIS

By making use of the reciprocal work theorem, the governing differential equations for a three-dimensional (homogeneous) structure under combined thermal, mechanical and body force loadings can be converted to an integral equation written on the surface of the structure. This integral equation is:

$$c_{ij} u_i = \int_S (G_{ij} t_i - F_{ij} u_i)\, dS + \int_V (G_{ij} f_i + \beta W_j T)\, dV \tag{1}$$

where T = temperature, $W_j = T_{ikj}\delta_{ik}$, β = coefficient of thermal expansion, T_{ikj} = the stress, σ_{ik}, due to a point force system, e_j, and G_{ij}, T_{ikj} and F_{ij} are defined reference 1. The equation

$$\sigma_{ij} = \int_S (D_{ijk} t_k - S_{ijk} u_k)\, dS + \int_V (T_{ijk} t_k + M_{ij} T)\, dV \tag{2}$$

allows calculation of stresses at any interior point where they are required. A similar equation for interior displacements can be obtained by setting $c_{ij} = \delta_{ij}$ in (1).

In a purely elastic problem BEM stress analysis can be carried out entirely on the boundary of the structure. Once a physically reasonable set of boundary conditions has been prescribed, (1) can, in principle, be solved for all of the remaining boundary displacements and tractions.

It is generally impossible to solve (1) exactly for real structures and loading conditions. Suitable approximations of the boundary geometry, displacements and tractions must be used in order to reduce (1) to a system of algebraic equations. The present version of BEST3D models boundary geometry and boundary values of field quantities using linear and/or quadratic isoparametric shape functions. The surface integrals in (1) are then evaluated numerically using product Gaussian quadrature rules. The numerical implementation of the BEM is discussed in detail in textbooks (ref. 4), as well as in references 1 and 2.

In the case of inelastic analysis, the volume integrals in (1) cannot be calculated *a priori*, since they require knowledge of inelastic strain, which is itself a part of the solution. In this case equations (1), (2) and the inelastic material model can be regarded as a coupled system of nonlinear equations. In the numerical implementation of the BEM (2) is used to calculate the stresses at interior points, and the nonlinear material model is then used to evaluate inelastic strain. Since the volume integrals of inelastic strain vanish except in regions of nonlinear material response, approximations of geometry and field quantities are required only where nonlinearity is expected. In the original version of BEST3D, strain variation in the interior was represented using isoparametric volume cells, with the solution carried out using a relatively standard iteration proceedure. More recently, a new approach has been developed which exploits certain features of the constitutive relationships involved. The unknown nonlinear terms in the interior are now defined as scalar variables. A new direct numerical solution scheme comparable to the variable stiffness method used in finite element analysis has been developed and implemented, avoiding the requirement for an iterative solution.

For a standard elasto-plastic flow problem the evolution of plastic flow is governed by:

$$F(\sigma_{ij}, h) = 0 \tag{3}$$

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial F}{\partial \sigma_{ij}} \tag{4}$$

These equations together with the consistency relations (i.e., the stress point must remain on a newly developing yield surface characterized by a change in the hardening parameter $\dot{h}$) leads to an expression for the unknown plastic flow factor $\dot{\lambda}$ as:

$$\dot{\lambda} = L^{\sigma}_{ij}\, \dot{\sigma}_{ij} \tag{5}$$

where

$$L^{\sigma}_{ij} = \frac{1}{H}\frac{\partial F}{\partial \sigma_{ij}}$$

$$H = -\left(\frac{\partial F}{\partial \epsilon^p_{mn}} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial \epsilon^p_{mn}}\right)\frac{\partial F}{\partial \sigma_{mn}}$$

It should be noted that L^{σ}_{ij} depends upon the current state variable, not on the incremental quantities.

However, the relationship given by (5) does not exist for ideal plasticity, as H vanishes for zero hardening. This can be avoided by reformulating the above expression in terms of strain increments:

$$\dot{\lambda} = L^{\epsilon}_{ij}\dot{\epsilon}_{ij} \tag{6}$$

where

$$L^{\epsilon}_{ij} = \frac{1}{H'}\frac{\partial F}{\partial \sigma_{ij}}$$

$$H' = \frac{\partial F}{\partial \sigma_{kl}} D_{klmn}\frac{\partial F}{\partial \sigma_{mn}} - \left(\frac{\partial F}{\partial \epsilon^p_{kl}} + \frac{\partial F}{\partial h}\frac{\partial F}{\partial \epsilon^p_{kl}}\right)\frac{\partial f}{\partial \sigma_{kl}}$$

where D_{ijkl} is the elastic constitutive tensor. It is evident that H' does not vanish for zero hardening (ideal plasticity).

The basic boundary element formulation for an inelastic body undergoing infinitesimal strain is given by:

$$c_{ij}\dot{u}_i(\xi) = \int_\Gamma [G_{ij}(x,\xi)\dot{t}_i(x) - F_{ij}(x,\xi)\dot{u}_i(x)]\, d\Gamma + \int_\Omega B_{ijk}(x,\xi)\, \dot{\sigma}^0_{ij}\, d\Omega \tag{7}$$

The stress rates at an interior point ξ are obtained from equation (7) via the strain-displacement relations and the constitutive relationships ($\dot{\sigma}_{ij} = D^e_{ijkl}\dot{\epsilon}_{kl} - \dot{\sigma}^0_{ij}$) as

$$\begin{aligned} \dot{\sigma}_{ij}(\xi) &= \int_\Gamma [G^\sigma_{ijk}(x,\xi)\dot{t}_i(x) - F^\sigma_{ijk}(x,\xi)\dot{u}_i(x)]\, d\Gamma \\ &\quad + \int_\Omega B^\sigma_{ipjk}(x,\xi)\, \dot{\sigma}^0_{ip}(x)\, d\Omega + J^\sigma_{ipjk}\, \dot{\sigma}^0_{ip}(\xi) \end{aligned} \tag{8}$$

where the kernel functions have been defined in the reference 2.

In equation (8) the volume integral must be evaluated in the sense of $(\Omega - D)$ with the limit taken as $D \to 0$, where D is a spherical exclusion of small arbitrary radians with ξ as its center. The term J^σ is the jump term derived from the analytical treatment of the integral over D. It is of considerable interest to note that the value of J^σ is independent of the size of the exclusion D, provided the initial stress distribution is locally homogeneous, i.e. uniform over its volume.

The evaluation of strains and stresses at boundary points can be accomplished by considering the equilibrium of the boundary segment and utilizing constitutive and kinematic equations. The stresses and global derivatives of the displacements which lead to strains at a point ξ can be obtained from the following set of coupled equations:

$$\begin{aligned} \dot{\sigma}_{ij}(\xi) - (\Delta\delta_{ij}\dot{u}_{k,k}(\xi) + \mu(\dot{u}_{i,j}(\xi) + \dot{u}_{j,i}(\xi))) &= -\dot{\sigma}^0_{ij}(\xi) \\ \dot{\sigma}_{ij}(\xi)n_j(\xi) &= \dot{t}_i(\xi) \\ \frac{\partial \xi_k}{\partial \eta_l}\frac{\partial \dot{u}_i(\xi)}{\partial \xi_k} &= \frac{\partial \dot{u}_i(\xi)}{\partial \eta_l} \end{aligned} \tag{9}$$

where η_i are a set of local axes at the field point ξ.

All the above nonlinear formulations include initial stresses in the governing equations which are not known *a priori* and, therefore, are solved by using iterative procedures. A non-iterative direct solution procedure is made feasible in this work by reducing the number of unknowns in the governing equations by utilizing certain features of the incremental theory of plasticity expressed by equations (3) to (6). The initial stresses σ^0_{ij} appearing in equations (7) to (9) can be expressed in the context of an elastoplastic deformation as:

$$\dot{\sigma}^0_{ij} = K_{ij}\dot{\lambda} \tag{10}$$

where $K_{ij} = D^e_{ijkl}\dfrac{\partial F}{\partial \sigma_{kl}}$.

Substituting (5) and (10) in equations (7) and (8) we can obtain:

$$c_{ij}\dot{u}_i(\xi) = \int_\Gamma [G_{ij}(x)\dot{t}_i(x) - F_{ij}(x,\xi)\dot{u}_i(x)]\, d\Gamma \tag{11}$$

and

$$\begin{aligned} \dot{\lambda}(\xi) &= L^\sigma_{jk}(\xi) \int_\Gamma [G^\sigma_{ijk}(x,\xi)\dot{t}_i(x) - F^\sigma_{ijk}(x,\xi)\dot{u}_i(x)]\, d\Gamma \\ &\quad + L^\sigma_{jk}(\xi) \int_\Omega B^\sigma_{ipjk}(x,\xi)\; K_{ip}(x)\; \dot{\lambda}(x)\, d\Omega \\ &\quad + L^\sigma_{jk}\; J^\sigma_{ipjk}\; K_{ip}(\xi)\; \dot{\lambda}(\xi) \end{aligned} \tag{12}$$

Equations (11) and (12) can be solved simultaneously to evaluate the unknown values of displacements, traction rates and the scalar variable $\dot{\lambda}$.

The equations for the boundary nodes (9) are similarly transformed to express them in terms of the scalar variable $\dot{\lambda}$ using equations (5) and (10).

TRANSIENT STRESS ANALYSIS

The direct boundary integral formulation for a general, transient, elastodynamic problem can be constructed by combining the fundamental point force solution of the governing equations (Stokes' solution) with Graffi's dynamic reciprocal theorem. Details of this construction can be found in Banerjee and Butterfield (ref. 4). For zero initial conditions and zero body forces, the boundary integral formulation for transient elastodynamics reduces to:

$$c_{ij}(\underline{\xi})u_i(\underline{\xi},T) = \int_S [G_{ij}(\underline{x},\underline{\xi},T) * t_i(\underline{x},T) - F_{ij}(\underline{x},\underline{\xi},T) * u_i(\underline{x},T)]\, dS(x) \tag{13}$$

where

$$G_{ij} * t_i = \int_0^T G_{ij}(\underline{x},T;\underline{\xi},\tau)\; t_i(x,\tau)\, d\tau \tag{14}$$

$$F_{ij} * u_i = \int_0^T F_{ij}(\underline{x},T;\underline{\xi},\tau)\; u_i(x,\tau)\, d\tau \tag{15}$$

are Reimann convolution integrals and $\underline{\xi}$ and $\underline{x}$ are the space positions of the receiver (field point) and the source (source point). The fundamental solutions G_{ij} and F_{ij} are the displacements and tractions at a point x and at a time T due to a unit force vector acting at a point $\underline{\xi}$ at a time τ. Equation (13) represents an exact formulation involving integration over the surface as well as the time history. It should also be noted that this is an implicit time-domain formulation because the response at time T is calculated by taking into account the history of surface tractions and displacements up to and including the time T. Furthermore, equation (13) is valid for both regular and unbounded domains.

Once the boundary solution is obtained, the stresses at the boundary nodes can be calculated without any integration by using the scheme described for the static case. For calculating displacements at interior points equation (13) can be used with $c_{ij} = \delta_{ij}$ and the interior stresses can be obtained from

$$\sigma_{ij}(\underline{\xi}, T) = \int_S [G^{\sigma}_{ijk}(\underline{x}, \underline{\xi}, T) * t_i(\underline{x}, T) - F^{\sigma}_{ijk}(\underline{x}, \underline{\xi}, T) * u_i(\underline{x}, T)] \, dS(x) \tag{16}$$

The functions G^{σ}_{ij} and F^{σ}_{ij} in the above equation are derived from the Stokes' solution by differentiation.

In the initial version of BEST3D, constant time stepping was used to obtain the transient dynamic response. It was found to be more effective to use a linear time variation of u and t on the boundaries. In this case:

$$u_i(\underline{x}, \tau) = \sum_{n=1}^{N} [\overline{M}_1 \, u_i^{n-1}(\underline{x}) + \overline{M}_2 \, u_i^{n}(\underline{x})] \tag{17}$$

$$t_i(\underline{x}, \tau) = \sum_{n=1}^{N} [\overline{M}_1 \, t_i^{n-1}(\underline{x}, \tau) + \overline{M}_2 \, t_i^{n}(\underline{x})] \tag{18}$$

where $\overline{M}_1$ and $\overline{M}_2$ are the time functions, and are of the form:

$$\overline{M}_1 = \frac{\tau_n - \tau}{\Delta T} \phi_n(\tau) \tag{19}$$

$$\overline{M}_2 = \frac{\tau - \tau_{n-1}}{\Delta T} \phi_n(\tau) \tag{20}$$

For illustration purposes, consider the boundary integral equation for the first time step, i.e.

$$c_{ij} u_i(\xi, T_1) - \int_{T_0}^{T_1} \int_S [G_{ij} t_i - F_{ij} u_i] \, dS \, d\tau = 0 \tag{21}$$

The time integration in equation (21) by utilizing (18) is done analytically. After the usual numerical integration and assembly process, the resulting system equation is of the form:

$$[A_2^1][X^1] - [B_2^1][Y_1] + [A_1^1][X^0] - [B_1^1][Y^0] = 0 \tag{22}$$

where:

- A and B are matrices related to the unknown and known field quantities, respectively:
- X and Y are the vectors of unknown and known field quantities, respectively:
- for X and Y the superscript denotes the time:
- for A and B the superscript denotes the time step at which they are calculated, and the subscript denotes the local time nodes (1 or 2) during that time-stepping interval.

Since all the unknowns at time $T = 0$ are assumed to be zero, equation (22) reduces to:

$$[A_2^1][X^1] = [B_2^1][Y^1] + [B_1^1][Y^0] \tag{23}$$

For second time step, the assembled system equation has the form

$$[A_2^1][X^2] - [B_2^1][Y^2] + [A_1^1][X^1] - [B_1^1][Y^1] = -[A_2^2][X^1] + [B_2^2][Y^1] - [A_1^2][X^0] + [B_1^2][Y^0] \tag{24}$$

As in the constant time variation scheme, only the matrices on the right hand side of equation (24) need be evaluated. However, one needs to integrate and assemble four matrices at each time step as compared to two in the case of constant time variation. This can be done with only a small increase in computing time by integrating all the kernels together and then assembling all the matrices together. Equation 24 can be rearranged such that:

$$[A_2^1][X^2] = [B_2^1][Y^2] - [A_1^1 + A_2^1][X^1] - [B_1^1 + B_2^2][Y^1] + [B_1^2][Y^1] \tag{25}$$

In the above equation, all the quantities on the right hand side are known. Therefore, the unknown vector X^2 at time T_2 can be obtained by solving the above equation.

Thus, for the present case, the boundary integral equation (25) can be written in discretized form as:

$$[A_2^1][X^N] - [B_2^1][Y^N] = -\sum_{n=2}^{N}[[A_2^n + A_1^{n-1}][X^{N-n+1}] - [B_2^n + B_1^{n-1}][Y^{N-n+1}] + [B_1^n][Y^0]] \tag{26}$$

or

$$[A_2^1][X^N] = [B_2^1][Y^N] + [R^N] \tag{27}$$

It is of interest to note that, if time interpolation functions $\overline{M}_1$ and $\overline{M}_2$ are replaced by $\overline{M}_1 = \overline{M}_2 = 0.5\phi_n(\tau)$, the time stepping scheme for linear variation can be used for the case of constant variation with averaging between the local time nodes.

SYMBOLS

δ_{ij}	Kronecker delta symbol
u_i, t_i	boundary displacements and tractions
G_{ij}	displacement point load solution
F_{ij}	traction kernel derived from G_{ij}
f_i	mechanical body forces
T	temperature
β	coefficient of thermal expansion
$B_{ijk}, T_{ijk}, D_{ijk}, S_{ijk}$	higher order kernels derived from G_{ij}
S	surface of three-dimensional structure
V	interior of a three-dimensional structure
σ_{ij}	stress tensor
ϵ_{ij}	strain tensor

c_{ij}	jump terms in boundary integral equation
$\dot{\lambda}$	plastic flow factor
$F(\sigma_{ij}, h)$	yield function
h	hardening parameter
$\cdot$, same as $\dot{\sigma}_{ij}$	time derivative
superscript ^{p}, as in ϵ^p_{ij}	plastic component
superscript ^{e}, as in ϵ^p_{ij}	elastic component
D^e_{ijkl}	elastic constitutive tensor
x, y, z	points in three-dimensional space
superscript $^\circ$, as in σ°_{ij}	initial stress (or strain)
B^σ_{ijkl}	higher order kernel derived from G_{ij}
λ, μ	Lamé constants
t, τ	denote time in dynamic analysis
$*$, as in $G_{ij} * t_i$	time convolution
M_1, M_2	shape functions for time variation
$[A^j_i], etc.$	coefficient matrices in time domain solution

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