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NUMERICAL STUDIES OF IDENTIFICATION IN NONLINEAR DISTRIBUTED PARAMETER SYSTEMS[1]

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Abstract. An abstract approximation framework and convergence theory for the identification of first and second order nonlinear distributed parameter systems developed previously by the authors and reported on in detail elsewhere are summarized and discussed. The theory is based upon results for systems whose dynamics can be described by monotone operators in Hilbert space and an abstract approximation theorem for the resulting nonlinear evolution system. The application of the theory together with numerical evidence demonstrating the feasibility of the general approach are discussed in the context of the identification of a first order quasi-linear parabolic model for one dimensional heat conduction/mass transport and the identification of a nonlinear dissipation mechanism (i.e. damping) in a second order one dimensional wave equation. Computational and implementational considerations, in particular, with regard to supercomputing, are addressed.

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1. Introduction

In this paper we report on the results of our efforts in the area of approximation for the identification of nonlinear distributed parameter systems. The central focus of the present paper is our essential features of the underlying theory. The theoretical basis (i.e. convergence analysis) for the computational results to be presented below has been treated in detail in two of our earlier papers, [3] and [5]. In [3] and [5] we developed nonlinear analogs of the general abstract approximation framework for the identification of linear systems given by Banks and Ito in [2]. Inverse problems for first order nonlinear evolution equations are handled in [3], while [5] is concerned with second order systems. We note that in another earlier paper, [4], we have developed an approximation framework for the estimation of parameters in nonautonomous nonlinear distributed systems. Although the general approach taken in [4] differs somewhat from the ideas in [3] and [5], and from those to be discussed here, the results presented there are certainly related, and at present, remain our only means of dealing with either linear or nonlinear nonautonomous systems.

In the next section we briefly review some abstract functional analytic existence, uniqueness, regularity, and approximation results for nonlinear evolution equations in Banach spaces. In section 3 we consider first order systems, define the class of inverse problems and evolution systems with which we shall be dealing and sketch the relevant approximation and convergence theory. We then consider an example involving a quasi-linear model for heat conduction and present the results (inclusing those aspects related to supercomputing) of our computational study and numerical investigations. In the fourth section we treat inverse problems for second order systems. In particular we consider the estimation or identification of nonlinear damping or dissipation mechanisms in distributed parameter models for mechanical systems. A brief fifth section contains some concluding remarks.

2. Nonlinear Evolution Equations in Banach Spaces-Existence, Uniqueness, and Approximation Results

We consider quasi-autonomous, in general nonlinear, initial value problems of the form

$$\dot{x}(t) + Ax(t) \ni f(t), \quad 0 < t \leq T, \tag{2.1}$$

$$x(0) = x^0 \tag{2.2}$$

set in a Banach space X with norm $|\cdot|_X$. In (2.1), (2.2) above we assume that $T > 0, x^0 \ \varepsilon \ \overline{Dom(A)} = \overline{\{x \ \varepsilon X : Ax \neq \phi\}}, f \ \varepsilon \ L_1(0,T;X)$, and that for some $\omega \ \varepsilon \ \mathbf{R}$ the operator $A + \omega I : X \to 2^X$ is m-accretive. In other words, that for some $\omega \ \varepsilon \ \mathbf{R}$ we have (i) $|x_1 - x_2|_X \leq |(1+\lambda\omega)(x_1 - x_2) + \lambda(y_1 - y_2)|_X$ for every $x_1, x_2 \ \varepsilon \ Dom(A)$, $y_1 \ \varepsilon \ Ax_1, y_2 \ \varepsilon \ Ax_2$, and $\lambda > 0$, and (ii) $\mathcal{R}(I + \lambda(A+\omega I)) \equiv \bigcup_{x \varepsilon Dom(A)} (I + \lambda(A + \omega I))x = X$ for some $\lambda > 0$. We note that $A + \omega I$ m-accretive implies that for each $\lambda > 0$ the resolvent of $A + \omega I$ at $\lambda, J(\lambda; A+\omega I) : X \to X$, a single valued, everywhere defined, nonexpansive, nonlinear operator on X, can be defined by $J(\lambda; A + \omega I) = (I + \lambda(A + \omega I))^{-1}$.

A nonlinear evolution system on a subset $\Omega \subset X$ is a two parameter family of nonlinear operators, $\{U(t,s) : 0 \leq s \leq t \leq T\}$, on Ω satisfying $U(t,s)\varphi \ \varepsilon \ \Omega, U(s,s)\varphi = \varphi$ and $U(t,s)U(s,r)\varphi = U(t,r)\varphi$ for every $\varphi \ \varepsilon \ \Omega$ and $0 \leq r \leq s \leq t \leq T$ with the mapping $(s,t) \to U(t,s)\varphi$ continuous from the triangle $\Delta = \{(s,t) : 0 \leq s \leq t \leq T\} \subset \mathbf{R}^2$ into X for each $\varphi \ \varepsilon \ \Omega$. A strongly continuous function $x : [0,T] \to X$ is said to be a strong solution to the initial value problem (2.1), (2.2) if $x(0) = x^0$, it is absolutely continuous on compact subintervals of (0,T), differentiable almost everywhere, and satisfies $f(t) - \dot{x}(t) \ \varepsilon \ Ax(t)$ for almost every $t \ \varepsilon \ (0,T)$.

It can be shown (see [7], [9], and [3]) that under the assumptions on A, f, and x^0 made above, a unique nonlinear evolution system $\{U(t,s) : 0 \leq s \leq t \leq T\}$, on $\overline{Dom(A)}$ with the following properties can be constructed.

(i) $|U(t,s)\varphi - U(t,s)\psi|_X \leq e^{\omega(t-s)}|\varphi - \psi|_X$, for all $\varphi, \psi \ \varepsilon \ \overline{Dom(A)}$ and $0 \leq s \leq t \leq T$;

(ii) $|U(s+t,s)\varphi - U(r+t,r)\varphi|_X \leq 2\int_0^t e^{\omega(t-\tau)}|f(\tau+s) - f(\tau+r)|_X d\tau$, for all $\varphi \in \overline{Dom(A)}$ and all $t > 0$ such that $s+t, r+t \leq T$;

(iii) If the initial value problem (2.1), (2.3) has a strong solution x, then $x(t) = U(t,s)x(s)$, for $0 \leq s \leq t \leq T$;

The strongly continuous function x given by $x(t) = U(t,0)x^0, t \in [0,T]$, is referred to as the unique mild, generalized, or integral (see [6]) solution to (2.1), (2.2). It is immediately clear from (iii) above that when the initial value problem (2.1), (2.2) admits a strong solution, it and the mild solution coincide.

The abstract approximation result which will play a fundamental role in the discussions to follow, is given in Theorem 2.1 below. The theorem we state here is similar in spirit to other related approximation results for nonlinear evolution equations - in particular, those that can be found in [8] and [10]. The proof of Theorem 2.1 is given in [3]. In the statement of Theorem 2.1 and elsewhere, for sets $S_n, n = 0,1,2,...$ we use the notation $\lim\limits_{n\to\infty} S_n \supset S_0$ to mean that given any $s_0 \in S_0$, there exists $s_n \in S_n$, n = 1,2,... for which $\lim\limits_{n\to\infty} s_n = s_0$.

THEOREM 2.1. *For each $n \in \mathbf{Z}^+ = \{1,2,3,...\}$ let X_n be a closed linear subspace of X, let $f_n \in L_1(0,T;X_n)$, and let $A_n : X_n \to 2^{X_n}$ be an operator on X_n with $A_n + \omega I$ m-accretive. Suppose that there exists a function $g \in L_1(0,T)$ for which $|f_n(t)|_X \leq g(t)$, and that $\lim\limits_{n\to\infty} \overline{Dom(A_n)} \supset \overline{Dom(A)}$. Suppose further that*

(i) $\lim\limits_{n\to\infty} f_n(t) = f(t)$, *a.e.* $t \in (0,T)$,

and that

(ii) $\lim\limits_{n\to\infty} J(\lambda; A_n+\omega I)\varphi_n = J(\lambda; A+\omega I)\varphi$ *for each $\varphi \in X$ whenever $\varphi_n \in X_n$ with $\lim\limits_{n\to\infty} \varphi_n = \varphi$, for some $\lambda > 0$.*

Then if $\{U_n(t,s) : 0 \leq s \leq t \leq T\}$ is the evolution system on $\overline{Dom(A_n)}$ generated by A_n and f_n, we have

$$\lim_{n\to\infty} U_n(t,s)\varphi_n = U(t,s)\varphi$$

uniformly in s and t for $(s,t) \in \Delta$, for each $\varphi \in \overline{Dom(A)}$ and $\varphi_n \in \overline{Dom(A_n)}$ with $\lim\limits_{n\to\infty} \varphi_n = \varphi$.

3. THE IDENTIFICATION OF FIRST ORDER SYSTEMS WITH DYNAMICS GOVERNED BY MONOTONE OPERATORS ON HILBERT SPACE

Let H be a real Hilbert space with inner product denoted by $< \cdot,\cdot >$ and corresponding norm $|\cdot|$, and let V be a real reflexive Banach space with norm $||\cdot||$. We shall assume that V is densely and continuously embedded in H. It follows therefore that $V \hookrightarrow H \hookrightarrow V^*$ and that there exists a constant $\mu > 0$ for which $||\varphi||_* \leq \mu|\varphi|$, for all $\varphi \in H$ and $|\varphi| \leq \mu||\varphi||$, for all $\varphi \in V$ where $||\cdot||_*$ denotes the usual uniform operator norm on V^*. We shall also use $< \cdot,\cdot >$ to denote the natural extension of the H inner product to the duality pairing between V and V^*. Let Q and Z be metric spaces and let Q be a fixed, nonempty, sequentially compact subset of $\mathcal{Q}$.

For each $q \in Q$ let $\mathfrak{A}(q) : V \to V^*$ be a single valued, everywhere defined, hemicontinuous (see [6]), in general nonlinear, operator from V into V^* satisfying the following conditions.

(A) (Continuity): For each $\varphi \in V$, the map $q \to \mathfrak{A}(q)\varphi$ is continuous from $Q \subset \mathcal{Q}$ into V^*.

(B) (Equi-V-montonicity): There exist constants $\omega \in \mathbf{R}$ and $\alpha > 0$, both independent of $q \in Q$, for which

$$< \mathfrak{A}(q)\varphi - \mathfrak{A}(q)\psi, \varphi - \psi > + \omega|\varphi - \psi|^2 \geq \alpha||\varphi - \psi||^2,$$

for every $\varphi, \psi \in V$.

(C) (Equi-boundedness): There exists a constant $\beta > 0$, independent of $q \varepsilon Q$, for which

$$\|\mathfrak{A}(q)\varphi\|_{*} \leq \beta(\|\varphi\| + 1),$$

for every $\varphi \varepsilon V$.

For each $q \varepsilon Q$ define the operator $A(q) : Dom(A(q)) \subset H \rightarrow H$ to be the restriction of the operator $\mathfrak{A}(q)$ to the set $Dom(A(q)) = \{\varphi \varepsilon V : \mathfrak{A}(q)\varphi \varepsilon H\}$. It can be shown (see [3]) that $A(q)$ is densely defined (i.e. that $\overline{Dom(A(q))} = H$) and that $A(q) + \omega I$ is m-accretive. Let $T > 0$ and for each $q \varepsilon Q$ let $f(\cdot\ ; q) \varepsilon L_1(0, T; H)$ and let $u^0(q) \varepsilon H$. We assume that the mapping $q \rightarrow u^0(q)$ is continuous from $Q \subset \mathcal{Q}$ into H and that the mapping $q \rightarrow f(t; q)$ is continuous from $Q \subset \mathcal{Q}$ into H for almost every $t \varepsilon (0, T)$. Also, for every $z \varepsilon Z$, let $u \rightarrow \tilde{\Phi}(u; z)$ be a continuous map from C(0,T;H) into $\mathbf{R}^+$. We consider the abstract parameter identification problem given by:

(ID) Given observations $z \varepsilon Z$, determine parameters $\bar{q} \varepsilon Q$ which minimize the functional

$$\Phi(q) = \tilde{\Phi}(u(q); z)$$

where for each $q \varepsilon Q$ $u(q) = u(\cdot\ ; q)$ is the mild solution to the initial value problem in H given by

$$\dot{u}(t) + A(q)u(t) = f(t; q), \quad 0 < t \leq T \tag{3.1}$$

$$u(0) = u^0(q). \tag{3.2}$$

Recalling the discussion in section 2 and our remarks above, it is clear that for each $q \varepsilon Q, A(q)$ and $f(\cdot\ ; q)$ generate a nonlinear evolution system $\{U(t, s; q) : 0 \leq s \leq t \leq T\}$ on H with the mild solution to the initial value problem (3.1), (3.2) given by

$$u(t; q) = U(t, 0; q)u^0(q), \quad 0 \leq t \leq T.$$

We develop an abstract Galerkin based approximation theory for problem (ID). For each n = 1,2,... let H_n be a finite dimensional subspace of H with $H_n \subset V$ for all n. We let $P_n : H \rightarrow H_n$ denote the orthogonal projection of H onto H_n with respect to the inner product $< \cdot, \cdot >$ and we make the standing assumption

(D) $$\lim_{n \rightarrow \infty} \|P_n\varphi = \varphi\| = 0, \qquad \text{for each } \varphi \varepsilon V.$$

It is clear that assumption (D) together with the dense and continuous embedding of V in H yield that the P_n tend strongly to the identity on H as well, as $n \rightarrow \infty$.

For each $q \varepsilon Q$ we define the Galerkin approximation $A_n(q), n = 1, 2, ...,$ to A(q) in the usual manner. That is, for $\varphi_n \varepsilon H_n$ we set $A_n(q)\varphi_n = \psi_n$ where ψ_n is the unique element in H_n (guaranteed to exist by the Riesz Representation Theorem applied to H_n) satisfying $< \mathfrak{A}(q)\varphi_n, \chi_n > = < \psi_n, \chi_n >$, $\chi_n \varepsilon H_n$. We set $f_n(\cdot\ ; q) = P_n f(\cdot\ ; q) \varepsilon L_1(0, T; H_n)$ and $u_n^0(q) = P_n u^0(q) \varepsilon H_n$. With these definitions, it is not difficult to argue that for each n = 1,2,... and each $q \varepsilon Q$, $A_n(q)$ and $f_n(\cdot\ ; q)$ generate a nonlinear evolution system, $\{U_n(t, s; q) : 0 \leq s \leq t \leq T\}$ on H_n. Thus we consider the sequence of approximating identification problems given by:

(ID_n) Determine parameters $\bar{q}_n \varepsilon Q$ which minimize

$$\Phi_n(q) = \tilde{\Phi}(u_n(q); z)$$

where for each $q \varepsilon Q, u_n(q) = u_n(\cdot\ ; q)$ is the mild solution to the initial value problem in H_n given by

(3.3) $$\dot{u}_n(t) + A_n(q)u_n(t) = f(t;q),\ 0 < t \le T$$

(3.4) $$u_n(0) = u_n^0(q).$$

The mild solution, $u_n(q)$, to (3.3), (3.4) is given by $u_n(t;q) = U_n(t,0;q)u_n^0(q),\ 0 \le t \le T$.

We may summarize the existence and convergence theory for solutions to problem (ID_n) given in [3] as follows. Let $\{q_n\}_{n=1}^{\infty}$ be a sequence in Q with $\lim_{n\to\infty} q_n = q_0\ \varepsilon\ Q$. Assumption (D) and the continuity of the map $q \to u^0(q)$ imply $\lim_{n\to\infty} u_n^0(q_n) = u^0(q_0)$ in H. Similarly we have $\lim_{n\to\infty} f_n(t;q_n) = f(t;q_0)$ in H for almost every $t\ \varepsilon\ (0,T)$ with $|f_n(\cdot\ ;q_n)|$ dominated by an L_1 function which is independent of n. Assumption (D) also implies that $\lim_{n\to\infty} \overline{Dom(A_n(q_n))} = \lim_{n\to\infty} H_n \supset H = \overline{Dom(A(q_0))}$, and that for each $\lambda > 0$, $\lim_{n\to\infty} J(\lambda; A_n(q_n) + \omega I)\varphi_n = J(\lambda; A(q_0) + \omega I)\varphi$ in H whenever $\varphi_n\ \varepsilon\ H_n$ with $\lim_{n\to\infty} \varphi_n = \varphi\ \varepsilon\ H$. Thus, Theorem 2.1 yields

(3.5) $$\lim_{n\to\infty} u_n(q_n) = \lim_{n\to\infty} U_n(\cdot,0;q_n)u_n^0(q_n) = U(\cdot,0;q_0)u^0(q_0) = u(q_0),$$

in C(0,T;H). Similar arguments can be used to demonstrate that if $\{q_m\}_{m=1}^{\infty} \subset Q$ with $\lim_{m\to\infty} q_m = q_0$, then

(3.6) $$\lim_{m\to\infty} u_n(q_m) = \lim_{m\to\infty} U_n(\cdot,0;q_m)u_n^0(q_m) = U_n(\cdot,0;q_0)u_n^0(q_0) = u_n(q_0)$$

in C(0,T;H) for each n = 1,2,...

The compactness of Q, the continuity assumption on $\tilde{\Phi}$, and (3.6) are sufficient to conclude that for each n = 1,2,..., problem (ID_n) admits a solution $\bar{q}_n\ \varepsilon\ Q$. Since $\{\bar{q}_n\}_{n=1}^{\infty} \subset Q$ and Q compact, a convergent subsequence, $\{\bar{q}_{n_j}\}_{j=1}^{\infty}$, may be extracted from $\{\bar{q}_n\}_{n=1}^{\infty}$. If $\bar{q} = \lim_{j\to\infty} \bar{q}_{n_j}$, then for each $q\ \varepsilon\ Q$ (3.5) implies

(3.7) $$\begin{aligned}
\Phi(\bar{q}) &= \tilde{\Phi}(u(\bar{q});z) = \tilde{\Phi}(\lim_{j\to\infty} u_{n_j}(\bar{q}_{n_j});z) \\
&= \lim_{j\to\infty} \tilde{\Phi}(u_{n_j}(\bar{q}_{n_j});z) = \lim_{j\to\infty} \Phi_{n_j}(\bar{q}_{n_j}) \\
&\le \lim_{j\to\infty} \Phi_{n_j}(q) = \lim_{j\to\infty} \tilde{\Phi}(u_{n_j}(q);z) \\
&= \tilde{\Phi}(\lim_{j\to\infty} u_{n_j}(q);z) = \tilde{\Phi}(u(q);z) \\
&= \Phi(q)
\end{aligned}$$

and consequently that $\bar{q}$ is a solution to problem (ID).

When the admissible parameter set Q is also infinite dimensional (when, for example, as is frequently the case, the unknown parameters to be identified are elements in a function space) it must be discretized as well. When this is in fact the case, the theory presented above requires the following modification. For each m = 1,2,... let $I^m : Q \subset \mathcal{Q} \to \mathcal{Q}$ be a continuous map with range $Q^m = I^m(Q)$ in a finite dimensional space and with the property that $\lim_{m\to\infty} I^m(q) = q$, uniformly on Q. We consider the doubly indexed sequence of approximating identification problems (ID_n^m) where for each n and m (ID_n^m) is defined to be problem (ID_n) with Q replaced by Q^m. It can be shown that each of these problems admits a solution $\bar{q}_n^m\ \varepsilon\ Q^m$, and that the sequence $\{\bar{q}_n^m\}$ will have a

$\mathcal{Q}$-convergent subsequence, $\{\bar{q}_{n_k}^{m_j}\}$, with limit in Q. If $\lim_{j,k\to\infty} \bar{q}_{n_k}^{m_j} = \bar{q} \;\varepsilon\; Q$, then $\bar{q}$ can be shown to be a solution to problem (ID). The problems (ID_n^m) involve the minimization of functionals over compact subsets of Euclidean space subject to finite dimensional state space constraints. Consequently they may be solved using standard computational techniques.

We illustrate the application of our theory with an example involving the identification of a quasi-linear model for one dimensional heat flow (see [13], [14]). We consider the quasi-linear parabolic partial differential equation

$$\frac{\partial u}{\partial t}(t,x) - \frac{\partial}{\partial x}\left\{q\left(\frac{\partial u}{\partial x}(t,x)\right)\frac{\partial u}{\partial x}(t,x)\right\} = f(t,x), \qquad t > 0, 0 < x < 1,$$

together with the Dirichlet boundary conditions

$$u(t,0) = 0,\; u(t,1) = 0,\; t > 0$$

and initial data

$$u(0,x) = u^0(x),\;\; 0 < x < 1.$$

We assume that $u^0 \;\varepsilon\; L_2(0,1), f \;\varepsilon\; L_1(0,T;L_2(0,1))$, and that $q \;\varepsilon\; C_B(\mathbf{R})$, the space consisting of all bounded continuous functions defined on the entire real line and endowed with the usual supremum metric which we shall denote by $d_\infty(\cdot,\cdot)$. We assume further that q satisfies

$$(q(\theta)\theta - q(\eta)\eta)(\theta - \eta) \geq \alpha|\theta - \eta|^2,\; \theta,\eta \;\varepsilon\; \mathbf{R} \tag{3.8}$$

for some $\alpha > 0$. (We note that if q is differentiable on $\mathbf{R}$, then the Mean Value Theorem implies that the condition $q'(\theta)\theta + q(\theta) \geq \alpha > 0,\; \theta \;\varepsilon\; \mathbf{R}$, is sufficient to conclude that condition (3.8) holds.) To apply our framework we set $H = L_2(0,1)$ endowed with the standard inner product and norm, and set $V = H_0^1(0,1)$ with norm $\|\cdot\|$ given by $\|\phi\| = \left(\int_0^1 |D\phi(x)|^2 dx\right)^{\frac{1}{2}}$. In this case we have $V^* = H^{-1}(0,1)$ and $V \hookrightarrow H \hookrightarrow V^*$ with the embeddings dense and continuous ($\mu = 1$). We take $\mathcal{Q} = C_B(\mathbf{R}), Z = C(0,T;L_2(0,1))$, and for given fixed values of $\alpha_0, \rho_0, \sigma_0, \theta_0 > 0$ we take Q to be the $\mathcal{Q}$-closure of the set

$$\{q \;\varepsilon\; C_B(\mathbf{R}) : q(\theta) = q(-\theta),\; |q(\theta)| \leq \rho_0,\; |q'(\theta)| \leq \sigma_0,\; q'(\theta)\theta + q(\theta) \geq \alpha_0,\; for\ \theta \;\varepsilon\; \mathbf{R},\; q(\theta) = constant\ for\ |\theta| \geq \theta_q\ for\ some\ numbers\ \theta_q\ satisfying\ 0 \leq \theta_q \leq \theta_0\}.$$

A straight forward application of the Arzelá-Ascoli Theorem reveals that Q is a sequentially compact subset of $C_B(\mathbf{R})$. If for each $q \;\varepsilon\; Q$ we define the operator $\mathfrak{A}(q) : V \to V^*$ by

$$<\mathfrak{A}(q)\varphi,\psi> = \int_0^1 q(D\varphi(x))D\varphi(x)D\psi(x)dx,\; \varphi,\psi \;\varepsilon\; V,$$

then it is not difficult to show that assumptions (A), (B), and (C) are satisfied. Let $\{t_i\}_{i=1}^{\nu}$ with $0 \leq t_1 < t_2 \cdots < t_\nu \leq T$ be given, and for each $z \;\varepsilon\; Z$ define the least squares performance index $\tilde{\Phi} : C(0,T;L_2(0,1)) \to \mathbf{R}^+$ by

$$\tilde{\Phi}(v;z) = \sum_{i=1}^{\nu} \int_0^1 |v(t_i,x) - z(t_i,x)|^2 dx. \tag{3.9}$$

We consider the parameter estimation problem (ID) with $Q, \tilde{\Phi}, \mathfrak{A}(q), f$, and u^0 as defined above.

For each n = 1,2,... let $H_n = \text{span}\{\phi_n^j\}_{j=1}^{n-1}$ where ϕ_n^j is the j-th linear B-spline on the interval [0,1] defined with respect to the uniform mesh $\{0, 1/n, 2/n, ..., 1\}$. That is

$$\phi_n^j(x) = \begin{cases} 0 & 0 \le x \le \frac{j-1}{n} \\ nx - j + 1 & \frac{j-1}{n} \le x \le \frac{j}{n} \\ j + 1 - nx & \frac{j}{n} \le x \le \frac{j+1}{n} \\ 0 & \frac{j+1}{n} \le x \le 1, \end{cases} \tag{3.10}$$

$j = 1, 2, \cdots, n-1$. Clearly $H_n \subset V = H_0^1(0,1), n = 1, 2, ...$ Let $P_n : H \to H_n$ denote the orthogonal projection of $L_2(0,1)$ onto H_n with respect to the usual L_2-inner product. Standard approximation results for interpolatory splines (see [15]) can be used to argue that assumption (D) is satisfied.

We discretize the admissible parameter set as follows. For $m \; \varepsilon \; \mathbf{Z}^+$ and $q \; \varepsilon \; Q$ set

$$(I^m q)(\theta) = \sum_{j=0}^{m} q(j\theta_q/m)\psi_j^m(|\theta|\,; \theta_q), \quad \theta \; \varepsilon \; \mathbf{R}$$

where the $\psi_j^m(\cdot \; ; \theta_q)$, j = 0,1,2,...,m are the standard linear B-splines on the interval $[0, \theta_q]$ defined with respect to the uniform mesh $\{0, \theta_q/m, 2\theta_q/m, ..., \theta_q\}$ and then extended to a continuous function on the entire positive real line via $\psi_j^m(\theta; \theta_q) = \psi_j^m(\theta_q; \theta_q)$, $\theta \ge \theta_q$. Using the Peano Kernel Theorem (see [12]) it can be argued that

$$d_\infty\left(I^m q, q\right) = \sup|I^m q - q| \;\le \frac{1}{2}\frac{\theta_0}{m}\sigma_0, \quad q \; \varepsilon \; Q,$$

and consequently that $\lim_{m\to\infty} I^m q = q$ in $C_B(\mathbf{R})$, uniformly in q for $q \; \varepsilon \; Q$.

For $q^m \; \varepsilon \; Q^m = I^m(Q)$ the finite dimensional initial value problem (3.3), (3.4) takes the form

$$M_n\dot{w}_n(t) + K_n(w_n(t); q^m) = F_n(t), \quad 0 < t \le T \tag{3.11}$$

$$M_n w_n(0) = w_n^0 \tag{3.12}$$

where $w_n(t) \; \varepsilon \; \mathbf{R}^{n-1}$, M_n is the $(n-1) \times (n-1)$ - Gram matrix whose (i,j)-th entry is given by $M_n^{i,j} = < \varphi_n^i, \varphi_n^j >, F_n(t)$ and w_n^0 are the $(n-1)$-vectors whose i-th elements are given by $F_n^i(t) = < f_n(t,\cdot), \varphi_n >$ and $w_n^{0i} = < u^0, \varphi_n^i >$, respectively, and $K_n(\cdot \; ; q^m) : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ is given by

$$K_n^i(v; q^m) = \begin{cases} nq^m(nv^1)v^1 - nq^m(n\{v^2 - v^1\})\{v^2 - v^1\}, & i = 1 \\ nq^m(n\{v^i - v^{i-1}\})\{v^i - v^{i-1}\} & \\ -nq^m(n\{v^{i+1} - v^i\})\{v^{i+1} - v^i\}, & i = 2, 3, ..., n-2 \\ nq^m(n\{v^{n-1} - v^{n-2}\})\{v^{n-1} - v^{n-2}\} & \\ \qquad +nq^m(-nv^{n-1})v^{n-1}, & i = n-1 \end{cases}$$

for $v \; \varepsilon \; \mathbf{R}^{n-1}$.

If $q^m \; \varepsilon \; Q^m$ is given by $q^m(\theta) = \sum_{j=0}^{m} q_j^m \psi_j^m(|\theta|; \theta_{q^m})$, for $\theta \; \varepsilon \; \mathbf{R}$, solving the identification problem (ID_n^m) involves the choosing of parameters $(q_0^m, q_1^m, ..., q_m^m, \theta_{q^m})^T$ from a compact subset of $\mathbf{R}^{m+2}$ so as to minimize the functional $\Phi(q^m) = \tilde{\Phi}(u_n(q^m); z)$ where $u_n(q^m)$ is given by $u_n(t; q^m) =$

$\sum_{j=1}^{n-1} w_n^j(t;q^m)\varphi_n^j$, $t \in [0,T]$ with $w_n(\cdot\ ;q^m)$ the solution to the initial value problem (3.11), (3.12) in $\mathbf{R}^{n-1}$ corresponding to q^m.

In order to actually test our scheme we let $q^*(\theta) = a(1 - .5e^{-b\theta^2})$ for $\theta \in \mathbf{R}$ with $a = .9$ and $b = .5$ (note that $\theta Dq^*(\theta) + q^*(\theta) \geq .45$ for $\theta \in \mathbf{R}$ and consequently that condition (3.8) is satisfied by q^*) and set $z(t,x) = 5e^{-t}(x - x^3)$, $x \in [0,1], t > 0$. Then setting

$$f(t,x) = \frac{\partial z}{\partial t}(t,x) - \frac{\partial}{\partial x}\left\{q^*\left(\frac{\partial z}{\partial x}(t,x)\right)\frac{\partial z}{\partial x}(t,x)\right\},$$

$x \in [0,1], t > 0$, and $u^0(x) = 5(x - x^3), x \in [0,1]$, we used our scheme to attempt to estimate q^* based upon the observations $\{z(.5j,\cdot)\}_{j=1}^{12}$.

All integrals that had to be computed numerically (i.e. some of the L_2 inner products and the integral appearing in the definition of the least-squares performance index (3.9)) were computed using a composite two point Gauss-Legendre quadrature rule on [0,1]. For each n and m the IMSL implementation (routine ZXSSQ) of the Levenberg-Marquardt algorithm, an iterative steepest descent/Newton's method hybrid, was used to solve the finite dimensional nonlinear least-squares minimization problem (ID_n^m). For a given choice of the parameters q^m, the initial value problem (3.11), (3.12) was solved in each iteration using the IMSL routine DGEAR with the stiff option operative. As an initial guess for q^* we took $q^0(\theta) = 1$, for $\theta \in \mathbf{R}$ with $\theta_{q^0} = 4$.

All computations were carried out on a Cray X-MP/48 at the San Diego Supercomputer Center. Standard coding techniques which permit optimal vectorization were used whenever possible. These included the nesting of loops with the largest ranges the deepest, and the separation of vectorizable and non-vectorizable code into different loops. In general, we observed that in the absence of vectorization, the Cray was able to run our codes in approximately 1% of the time that it took an IBM 3081. Vectorization on the Cray then yielded an additional speed-up factor of 17. Representative results that we obtained for various values of n and m are shown in Figures 3.1 - 3.3 below.

The CPU times on the Cray for these runs ranged from about 3 seconds for n = 8, m = 1 to about 180 seconds for n = 20, m = 4. The value of the performance index was reduced from $\Phi_n(q^0) \approx 10^{-2}$ to $\Phi_n(\bar{q}_n^m) \approx 10^{-4}$. We solved the problem (ID_n^m) unconstrained. That is we did not enforce the constraints in the definition of Q which render it and $I^m(Q) = Q^m$ compact. Thus it was not surprising that, as we have seen before in the case of linear system identification, for each n, the inherent ill-posedness of the problem of identifying functional parameters began to cause difficulties as m was increased (see [1]). We were, to a certain degree, able to mitigate these instability effects with the introduction of Tikhonov regularization (see [11]). However, at least from a qualitative point of view, this is probably unnecessary since we seem to obtain reasonably good estimates with m relatively small. It is worth noting that we have also tested our approach on the much simpler problem of identifying constant parameters (for example, the estimation of the parameters a and b in the definition of q^*). In these tests it performed superbly with convergence to the true values of the parameters as $n \to \infty$ immediately apparent. Finally, we also tested our scheme using discrete or sampled rather than distributed observations in the spatial variable although strictly speaking these examples can not be treated with our theory. With measurements taken at only one spatial point, x = .58 (i.e. with the observations $\{z(.5j,.58)\}_{j=1}^{12}$), the scheme's performance remained essentially unchanged from that observed with distributed observations. We note that the existing theory for the case of linear dynamics (see [2]) can handle spatially discrete measurements. An extension of these results to the nonlinear case is currently being investigated but at present remains an open question.

4. The Identification of Nonlinear Damping in Second Order Systems

In this section we consider the identification of nonlinear dissipation mechanisms in abstract infinite dimensional second order elastic systems. In our treatment below we assume that the

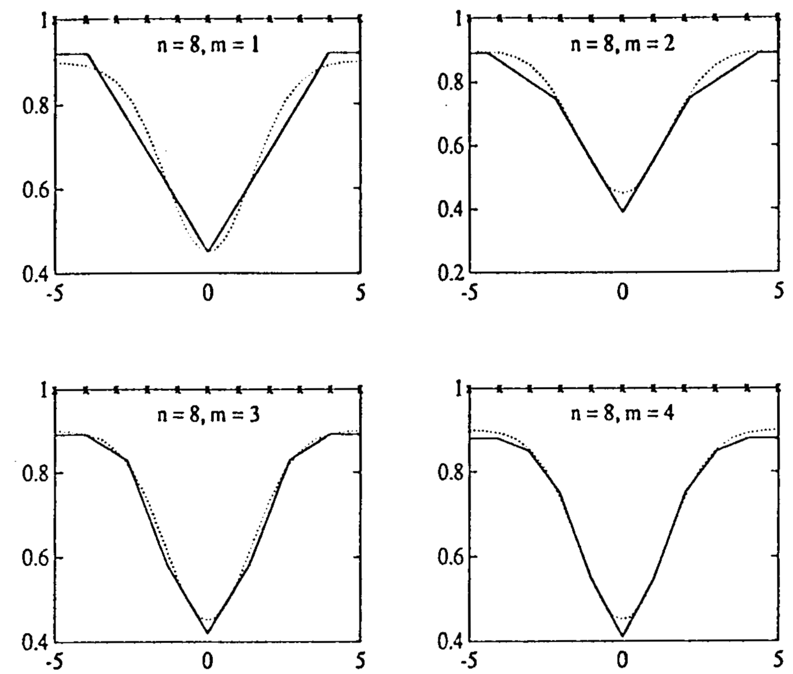

Figure 3.1 Legend: q^* $\cdots\cdots$, $\bar{q}_n^m$ $-----$, $q^0 - \times - \times - \times$.

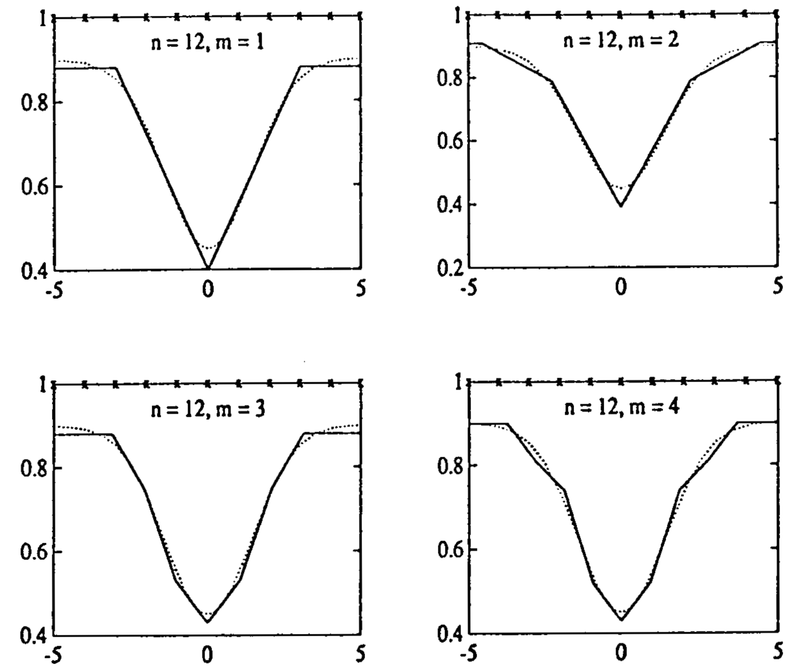

Figure 3.2 Legend: q^* $\cdots\cdots$, $\bar{q}_n^m$ $-----$, $q^0 - \times - \times - \times$.

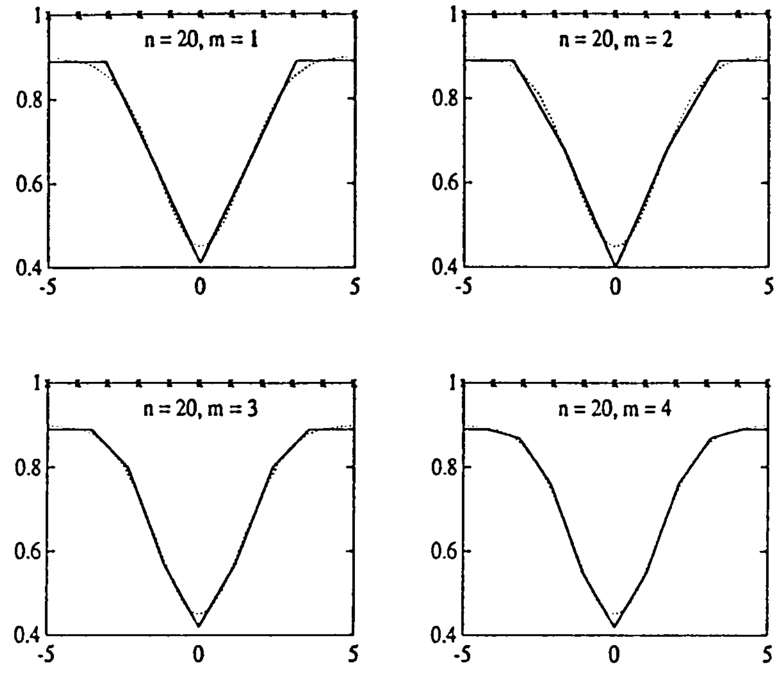

Figure 3.3 Legend: $q^* \cdots\cdots$, $\bar{q}_n^m$ — — — — —, $q^0 - \times - \times - \times$.

stiffness operator is linear. However we note that a similar approach can be used to identify a nonlinear stiffness operator in the presence of linear damping. We are currently looking into the extension of our theory to systems which involve both nonlinear stiffness and damping. A more detailed presentation along with proofs for the theoretical results we summarize below can be found in [5].

Let the spaces $H, V, V^*, \mathcal{Q}$, and Z, and the set Q be as they were defined for abstract first order systems at the beginning of section 3. For each $q \in Q$ let the operator $\mathfrak{A}(q) \in \mathcal{L}(V, V^*)$ satisfy the following conditions:

(A1) (Symmetry) For all $\varphi, \psi \in V$ $<\mathfrak{A}(q)\varphi, \psi>=<\varphi, \mathfrak{A}(q)\psi>$;

(A2) (Continuity) For each $\varphi \in V$ the map $q \rightarrow \mathfrak{A}(q)\varphi$ is continuous from $Q \subset \mathcal{Q}$ into V^*;

(A3) (Equi-V-Coercivity) There exist constants $\omega \in \mathbf{R}$ and $\alpha > 0$, both independent of $q \in Q$ for which $<\mathfrak{A}(q)\varphi, \varphi> +\omega|\varphi|^2 \geq \alpha||\varphi||^2$, for all $\varphi \in V$ and $q \in Q$;

(A4) (Equi-Boundedness) The operators $\mathfrak{A}(q)$ are uniformly bounded in q for $q \in Q$. That is, there exists a constant $\beta > 0$, independent of $q \in Q$, for which $||\mathfrak{A}(q)||_* \leq \beta||\varphi||$, for all $\varphi \in V$;

Also, for each $q \in Q$ let the operator $\mathfrak{B}(q) : Dom(\mathfrak{B}(q)) \subset V \rightarrow 2^{V^*}$ satisfy the folling conditions:

(B1) (Domain Uniformity) $Dom(\mathfrak{B}(q)) = Dom(\mathfrak{B})$ is independent of q for $q \in Q$, and $0 \in Dom(\mathfrak{B})$;

(B2) (Continuity) For each $\varphi \in Dom(\mathfrak{B})$ and $\psi(q) \in \mathfrak{B}(q)\varphi$ the map $q \rightarrow \psi(q)$ is continuous from $Q \subset \mathcal{Q}$ into V^*;

(B3) (Maximal Monotonicity) For all $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathfrak{B}_q \equiv \{(\varphi, \psi) \in V \times V^* : \varphi \in Dom(\mathfrak{B}), \psi \in \mathfrak{B}(q)\varphi\}$ we have $<\psi_1 - \psi_2, \varphi_1 - \varphi_2> \geq 0$ with $\mathfrak{B}_q$ not properly contained in any other subset of $V \times V^*$ for which this monotonicity condition holds;

(B4) (Equi-Boundedness) The operators $\mathfrak{B}(q)$ map V-bounded subsets of $Dom(\mathfrak{B})$ into subsets of V^* which are uniformly bounded in q for $q \in Q$. That is, if S is a V-bounded subset of $Dom(\mathfrak{B})$, the set $\mathfrak{B}(q)S$ is V^*-bounded, uniformly in q for $q \in Q$.

Let $T > 0$ and for each $q \in Q$ let $u^0(q) \in V$, $u^1(q) \in H$, and $f(\cdot\ ; q) \in L_1(0, T; H)$. We assume that the mappings $q \rightarrow u^0(q), q \rightarrow u^1(q)$, and $q \rightarrow f(t; q)$ are continuous from $Q \subset \mathcal{Q}$ into V, H, and

H respectively, for almost every $t \in (0,T)$. For every $z \in Z$ let $(u,v) \to \tilde{\Phi}(u,v;z)$ be a continuous mapping from $C(0,T;V \times H)$ into $\mathbf{R}^+$. The identification problem, which we shall again denote by (ID), takes the form:

(ID) Given observations $z \in Z$, determine parameters $\bar{q} \in Q$ which minimize the functional

$$\Phi(q) = \tilde{\Phi}(u(q), \dot{u}(q); z)$$

where for each $q \in Q, u(q) = u(\cdot\ ; q)$ is the mild solution to the initial value problem

$$\ddot{u}(t) + \mathfrak{B}(q)\dot{u}(t) + \mathfrak{A}(q)u(t) \ni f(t;q),\ 0 < t \le T \tag{4.1}$$

$$u(0) = u^0(q),\ \dot{u}(0) = u^1(q). \tag{4.2}$$

To make the notion of a mild solution to a second order initial value problem of the form (4.1), (4.2) precise, we rely on a reformulation as an equivalent first order system in a product space and then apply the abstract theory outlined in section 2. For each $q \in Q$ define the Hilbert space $X_q = V \times H$ with inner product $(\cdot,\cdot)_q$ given by

$$((\varphi_1,\psi_1),(\varphi_2,\psi_2))_q = < \mathfrak{A}(q)\varphi_1, \varphi_2 > + \omega < \varphi_1, \varphi_2 > + < \psi_1, \psi_2 > . \tag{4.3}$$

We denote the corresponding induced norm on X_q by $|\cdot|_q$. We note that our assumptions on the operators $\mathfrak{A}(q)$ guarantee that (4.3) indeed defines an inner product on $V \times H$ and that the Banach spaces $\{X_q, |\cdot|_q\}$ are norm equivalent, uniformly in q for $q \in Q$, to the Banach space $X = V \times H$ endowed with the standard product topology induced by the norm $|(\varphi,\psi)|_X = (\|\varphi\|^2 + |\psi|^2)^{\frac{1}{2}}$. For each $q \in Q$ define the operator $A(q) : Dom(A(q)) \subset X_q \to 2^{X_q}$ by

$$A(q)(\varphi,\psi) = (-\psi, \{\mathfrak{A}(q)\varphi + \mathfrak{B}(q)\psi\} \cap H),$$

for $(\varphi,\psi) \in Dom(A(q)) = \{(\varphi,\psi) \in V \times H : \psi \in V, \{\mathfrak{A}(q)\varphi + \mathfrak{B}(q)\psi\} \cap H \neq \phi\}$. It can be shown (see [5]) that there exists a $\gamma > 0$, independent of $q \in Q$, for which the operator $A(q) + \gamma I$ is m-accretive on $Dom(A(q)) \subset X_q$ for each $q \in Q$. Also, for each $q \in Q$ define $F(\cdot\ ;q) \in L_1(0,T;X_q)$ by $F(t;q) = (0, f(t;q))$, for almost every $t \in (0,T)$ and set $x^0(q) = (u^0(q), u^1(q)) \in X_q$. It follows that for every $q \in Q$, A(q) and $F(\cdot\ ;q)$ generate a nonlinear evolution system $\{U(t,s;q) : 0 \le s \le t \le T\}$ on $\overline{Dom(A(q))} \subset X_q$ satisfying conditions (i) - (iii) given in section 2. Henceforth we shall assume that $x^0(q) \in \overline{Dom(A(q))}$ for each $q \in Q$, and by a mild solution to the second order initial value problem (4.1), (4.2) we shall mean the V-continuous function $u(\cdot\ ;q)$ given by the first component of the X_q (or X)-continuous function $x(\cdot\ ;q) = U(\cdot,0;q)x^0(q)$. We shall take $\dot{u}(\cdot\ ;q)$ to be the H-continuous second component of $x(\cdot\ ;q)$.

We note that if assumption (B4) is strengthened to the condition that the operators $\mathfrak{B}(q)$ map H-bounded subsets into V^*-bounded subsets, uniformly in q for $q \in Q$, it can be argued (see [5]) that $\overline{Dom(A(q))} = X_q = X$. In addition, since conditions (A3) and (A4) imply that Dom $(\mathfrak{A}(q)) = \{\varphi \in V : \mathfrak{A}(q)\varphi \in H\}$ is dense in V (see [16]) it is clear that the operators A(q) will also be densely defined when the set $\{\varphi \in V : \mathfrak{B}(q)\varphi \in H\}$ is dense in H. In particular this will in fact be the case for all of the standard linear dissipation mechanisms - for example, air ($\mathfrak{B} \sim I$), so called structural ($\mathfrak{B} \sim \mathfrak{A}^{\frac{1}{2}}$), and Kelvin-Voigt viscoelastic ($\mathfrak{B} \sim \mathfrak{A}$) damping.

With the existence and uniqueness of mild solutions on X_q now demonstrated for each $q \in Q$, the q-uniform norm equivalence of X_q and X will allow us to subsequently ignore the q-dependence of the state spaces and to develop our approximation theory and convergence results on the q-independent space X.

Once again, as was the case with first order systems, our approximation theory is of Galerkin type. For each n = 1,2,... let H_n be a finite dimensional subspace of H with $H_n \subset V$. Let $P_n; H \rightarrow H_n$ denote the orthogonal projection of H onto H_n with respect to the standard inner product on $H, < \cdot, \cdot >$, and we assume that $P_n(Dom(\mathfrak{B})) \subset Dom(\mathfrak{B})$, for all n. We also again assume that condition (D) given in section 3 is satisfied. For each n = 1,2,... and each $q \in Q$ let $\mathfrak{A}_n(q) \in \mathcal{L}(H_n)$ and $\mathfrak{B}_n(q) : Dom(\mathfrak{B}_n) \subset H_n \rightarrow 2^{H_n}$ denote the Galerkin approximations to $\mathfrak{A}(q)$ and $\mathfrak{B}(q)$ respectively. That is for $\varphi_n \in H_n, \mathfrak{A}_n(q)\varphi_n = \psi_n$ where ψ_n is the unique element in H_n which satisfies $< \mathfrak{A}(q)\varphi_n, \chi_n > = < \varphi_n, \chi_n >, \chi_n \in H_n$, and for $\varphi_n \in Dom(\mathfrak{B}_n) = Dom(\mathfrak{B}) \cap H_n$, $\mathfrak{B}_n(q)\varphi_n = \{\psi_n \in H_n :< \psi, \chi_n > = < \psi_n, \chi_n >, \chi_n \in H_n \text{ for some } \psi \in \mathfrak{B}(q)_n\}$. We set $f_n(\cdot\ ;q) = P_n f(\cdot\ ;q) \in L_1(0,T;H_n), u_n^0(q) = P_n u^0(q)$, and $u_n^1(q) = P_n u^1(q)$, and consider the sequence of approximating parameter identification problems given by:

(ID_n) Determine parameters $\bar{q}_n \in Q$ which minimize the functional

$$\Phi_n(q) = \tilde{\Phi}(u_n(q), \dot{u}_n(q); z)$$

where for each $q \in Q, u_n(q) = u_n(\cdot\ ;q)$ is the mild solution to the initial value problem in H_n given by

$$\ddot{u}_n(t) + \mathfrak{B}_n(q)\dot{u}_n(t) + \mathfrak{A}_n(q)u_n(t) \ni f_n(t;q), \quad 0 < t \leq T \tag{4.4}$$

$$u_n(0) = u_n^0(q), \quad \dot{u}_n(0) = u_n^1(q). \tag{4.5}$$

We again use the theory in section 2 to define what is meant by a mild solution to the second order initial value problem (4.4), (4.5) in H_n. For each n = 1,2,... let $X_n = H_n \times H_n$, and for each $q \in Q$ define the operator $A_n(q) : Dom(A_n) \subset X_n \rightarrow 2^{X_n}$ by

$$A_n(q)(\varphi_n, \psi_n) = (-\psi_n, \mathfrak{A}_n(q)\varphi_n + \mathfrak{B}_n(q)\psi_n)$$

for $(\varphi_n, \psi_n) \in Dom(A_n) = H_n \times Dom(\mathfrak{B}_n)$. Set $F_n(\cdot\ ;q) = (0, f_n(\cdot\ ;q)) \in L_1(0,T;X_n)$ and $x_n^0(q) = (u_n^0(q), u_n^1(q))$. We assume $u_n^1(q) \in \overline{Dom(\mathfrak{B}_n)}$ so that $x_n^0(q) \in \overline{Dom(A_n)}$. We define the mild solution to (4.4), (4.5) to be the first component of the function $x_n(\cdot\ ;q) = U_n(\cdot, 0; q)x_n^0(q) \in C(0,T; X_n)$ where $\{U_n(t,s;q) : 0 \leq s \leq t \leq T\}$ is the nonlinear evolution system on $\overline{Dom(A_n)}$ generated by $A_n(q)$ and $F_n(\cdot\ ;q)$. That such an evolution system in fact exists can be argued as it was for the corresponding infinite dimensional second order system using the definitions of the operators $A_n(q)$ and $\mathfrak{B}_n(q)$, and the function $f_n(\cdot\ ;q)$ (see [5]). The function $\dot{u}_n(\cdot\ ;q)$ is obtained from the second component of $x_n(\cdot\ ;q)$.

By using condition (D) to argue resolvent convergence, i.e., that for each $\lambda > 0$ sufficiently large, $\lim_{n\rightarrow\infty} J(\lambda; A_n(q_n) + \tilde{\omega}I)(\phi_n, \psi_n) = J(\lambda; A(q_0) + \tilde{\omega}I)(\phi, \psi)$ in X for some $\tilde{\omega} \in \mathbf{R}$ whenever $(\varphi, \psi) \in X$ and $(\varphi_n, \psi_n) \in X_n$ with $\lim_{n\rightarrow\infty} (\varphi_n, \psi_n) = (\varphi, \psi)$ and $q_n, q_0 \in Q$ with $\lim_{n\rightarrow\infty} q_n = q_0$, we are able to apply Theorem 2.1 to obtain that $\lim_{n\rightarrow\infty} u_n(q_n) = u(q_0)$ in C(0,T; V) and $\lim_{n\rightarrow\infty} \dot{u}_n(q_n) = \dot{u}(q_0)$ in C(0,T;H) whenever $q_n, q_0 \in Q$ with $\lim_{n\rightarrow\infty} q_n = q_0$. A continuous dependence result analogous to (3.6) can also be obtained in this fashion. Then using estimates in the spirit of those given in (3.7) we find that solutions $\bar{q}_n$ to the problem (ID_n) exist and that the sequence $\{\bar{q}_n\}_{n=1}^{\infty}$ admits a Q-convergent subsequence, $\{\bar{q}_{n_j}\}_{j=1}^{\infty}$, with $\lim_{n\rightarrow\infty} \bar{q}_{n_j} = \bar{q}$ and $\bar{q}$ a solution to problem (ID). The discretization of the admissible parameter set Q can be carried out, and a subsequent convergence theory established exactly as they were in section 3 for first order systems. A complete and detailed discussion of the results for second order inverse problems which we have summarized above can be found in [5].

To illustrate the application of our approach we consider an inverse problem involving the identification of a nonlinear damping functional in a one dimensional wave equation. Let $\mathcal{Q} = C_B(\mathbf{R})$ endowed with the usual supremum norm, and let Q be the $\mathcal{Q}$-closure of the set

$$\{q \,\varepsilon\, \mathcal{Q} : q(\theta) = -q(-\theta),\ \theta q(\theta) \geq 0,\ \textit{for}\ \theta \,\varepsilon\, \mathbf{R}, |q(\theta)| = q(\theta_0),\\ \textit{for}\ |\theta| \geq \theta_0, q \,\varepsilon\, H^1(-\theta_0,\theta_0),\ |q|_{H^1(-\theta_0,\theta_0)} \leq K_0,\\ q'(\theta) \geq 0,\ \textit{for a.e.}\ \theta \,\varepsilon\, (-\theta_0,\theta_0)\}$$

where θ_0 and K_0 are given positive constants. It is not difficult to show that Q is a compact subset of $\mathcal{Q}$.

For each $q \,\varepsilon\, Q$ we consider the one dimensional wave equation with nonlinear damping given by

$$\frac{\partial^2 u}{\partial t^2}(t,x) + q(\frac{\partial u}{\partial t}(t,x)) - \frac{\partial}{\partial x}\left(E(x)\frac{\partial u}{\partial x}(t,x)\right) = f(t,x), \quad t > 0, \quad 0 < x < 1$$

with boundary and initial conditions

$$u(t,0) = 0, \quad u(t,1) = 0, \quad t > 0,$$

$$u(0,x) = u^0(x),\ \tfrac{\partial u}{\partial t}(0,x) = u^1(x),\ 0 < x < 1$$

where $E \,\varepsilon\, L_\infty(0,1)$ with $E(x) \geq E_0 > 0$, a.e. $x \,\varepsilon\, (0,1), f \,\varepsilon\, L_2((0,T)\times(0,1)), u^0 \,\varepsilon\, H_0^1(0,1)$, and $u^1 \,\varepsilon\, L_2(0,1)$ are given. We set $H = L_2(0,1), V = H_0^1(0,1)$ and $V^* = H^{-1}(0,1)$. The operator $\mathfrak{A} \,\varepsilon\, \mathcal{L}(V,V^*)$ given by

$$< \mathfrak{A}\varphi,\psi >= \int_0^1 E(x)D\varphi(x)D\psi(x)dx, \quad \varphi,\psi \,\varepsilon\, H_0^1(0,1),$$

is easily shown to satisfy conditions (A1) - (A4). For each $q \,\varepsilon\, Q$ we define the operator $\mathfrak{B}(q) : V \to V^*$ via

$$< \mathfrak{B}(q)\varphi,\psi >= \int_0^1 q(\varphi(\theta))\psi(\theta)d\theta, \quad \varphi,\psi \,\varepsilon\, H_0^1(0,1).$$

(Note that in this case we in fact have $\mathcal{R}(\mathfrak{B}(q)) \subset H$.) With the set Q as it has been defined above, it is clear that conditions (B1)-(B4) are satisfied and moreover that $\mathfrak{B}(q)$ maps H-bounded subsets of V into V^*-bounded subsets, uniformly in q for $q \,\varepsilon\, Q$.

We take the observation space Z to be $\times_{i=1}^{\nu}\left\{\mathbf{R}^\ell \times L_2(0,1)\right\}$ and a weighted least-squares performance index, $\tilde{\Phi}$, of the form

$$\tilde{\Phi}(\varphi,\psi;z) = \sum_{i=1}^{\nu}\left\{\rho_i \sum_{j=1}^{\ell}|\varphi(t_i,x_j) - z_{i,j}^1|^2 + \sigma_i \int_0^1 |\psi(t_i,x) - z_i^2(x)|^2 dx\right\} \tag{4.6}$$

for $\varphi \,\varepsilon\, C(0,T;V), \psi \,\varepsilon\, C(0,T;H)$, and $z = ((z_1^1,z_1^2),(z_2^1,z_2^2),...,(z_\nu^1,z_\nu^2)) \,\varepsilon\, Z$ with $\rho_i,\sigma_i \geq 0,\ i = 1,2,...,\nu,\ 0 < t_1 < t_2 < \cdots < t_\nu \leq T$, and $0 < x_1 < x_2 < \cdots < x_\ell < 1$.

As in our first order example, we employ linear spline based state approximation. For each n = 1,2,... let $H_n = \text{span}\{\varphi_n^j\}_{j=1}^{n-1}$ where the φ_n^j are the standard linear B-splines on the interval [0,1] defined with respect to the uniform mesh $\{0, 1/n, 2/n, ..., 1\}$ as given by (3.10). Let $P_n : H \to H_n$ denote the orthogonal projection of $L_2(0,1)$ onto H_n with respect to the standard L_2-inner product. We again use linear interpolating splines to discretize the admissible parameter set Q. For $m \; \varepsilon \; \mathbf{Z}^+$ and $q \; \varepsilon \; Q$ set

$$(4.7) \qquad (I^m q)(\theta) = \sum_{j=1}^{m} q(j\theta_q/m)\psi_j^m(|\theta|;\theta_q)sgn(\theta)$$

for $\theta \; \varepsilon \; \mathbf{R}$, where the $\psi_j^m(\cdot;\theta_q)$ are as they were defined in section 3, and θ_q is that number in $(0, \theta_0)$ for which $|q(\theta)| = q(\theta_q)$, $|\theta| \geq \theta_q$. (Note that in this case the lower limit of the sum in (4.7) is 1 rather than 0 since $q \; \varepsilon \; Q$ implies q(0) = 0.) We again have that condition (D) is satisfied and that $\lim_{m\to\infty} I^m q = q$, uniformly in q for $q \; \varepsilon \; Q$. We set $Q^m = I^m(Q)$.

For H_n and Q^m as defined above, the finite dimensional initial value problem (4.4), (4.5) takes the form

$$(4.8) \qquad M_n\ddot{w}_n(t) + C_n(\dot{w}_n(t);q^m) \; + \; K_n w_n(t) = F_n(t), \; 0 < t \leq T,$$

$$(4.9) \qquad M_n w_n(0) = w_n^0, \; M_n\dot{w}_n(0) \; = \; w_n^1,$$

where the $(n-1) \times (n-1)$ matrix M_n and the (n-1)-vectors $f_n(t)$ and w_n^0 are as they were defined in section 3, w_n^1 is the (n-1)-vector whose i-th component is given by $w_n^{1i} \; = \; < u^1, \varphi_n^i >$, K_n is the $(n-1) \times (n-1)$ matrix whose (i,j)-th entry is given by $K_n^{i,j} =< E\varphi_n^i, \varphi_n^j >$, and $C_n(\cdot \; ; q^m) : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ is given by

$$C_n^i(v;q^m) = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \{nx - i + 1\} \, q^m\left(\{nx - i\}\left\{v^i - v^{i-1}\right\} + v^i\right) dx$$

$$+ \int_{\frac{i}{n}}^{\frac{i+1}{n}} \{i + 1 - nx\} \, q^m\left(\{nx - i\}\left\{v^{i+1} - v^i\right\} + v^i\right) dx,$$

$i = 1, 2, \cdots, n-1$, for $v \; \varepsilon \; \mathbf{R}^{n-1}$ with $v^0, v^n \equiv 0$. If $w_n(\cdot \; ; q^m)$ is the solution to the second order initial value problem (4.8), (4.9) corresponding to $q^m \; \varepsilon \; Q^m$, then $u_n(t;q^m) = \sum_{j=1}^{n-1} w_n^j(t;q^m)\varphi_n^j$ and $\dot{u}_n(t;q^m) = \sum_{j=1}^{n-1} \dot{w}_n^j(t;q^m)\varphi_n^j$, for $t \; \varepsilon \; [0,T]$. If $q^m \; \varepsilon \; Q^m$ is given by $q^m(\theta) = \sum_{j=1}^{m} q_j^m \psi_j^m(|\theta|;\theta_{q^m}) \; sgn(\theta)$, $\theta \; \varepsilon \mathbf{R}$, the identification problem (ID_n^m) becomes one of determining parameters $(\bar{q}_1^m, ..., \bar{q}_m^m, \bar{\theta}_{q^m})$ in some compact subset of $\mathbf{R}^{m+1}$ which minimize $\Phi_n(q^m) \; = \; \tilde{\Phi}_n(u_n(q^m), \dot{u}_n(q^m); z)$.

In order to actually test our scheme, we set

$$q^*(\theta) = \begin{cases} \beta|\theta|^\alpha sgn(\theta) & -\theta_{q^*} \leq \theta \leq \theta_{q^*} \\ \beta|\theta_{q^*}|^\alpha sgn(\theta) & |\theta| \geq \theta_{q^*} \end{cases}$$

with $\beta = .15, \alpha = 2$, and $\theta_{q^*} = 2.5$. With

$$y(t,x) = \{3cos(\frac{1}{3}\pi t) \; + \; 2sin(\frac{1}{2}\pi t)\}sin\pi x$$

for $t > 0$ and $x \; \varepsilon \; [0,1]$, we set

$$f(t,x) = \frac{\partial^2 y}{\partial t^2}(t,x) + q^*(\frac{\partial y}{\partial t}(t,x)) - E\frac{\partial^2 y}{\partial x^2}(t,x),$$

with E $= 1$, $u^0(x) = y(0,x) = 3sin\pi x$, and $u^1(x) = \frac{\partial y}{\partial t}(0,x) = \pi \sin \pi x$, for $t > 0$ and $x \; \varepsilon \; [0,1]$. For observations upon which to base our fit, we took $z = \{(z^1_{i,1}, z^2_i)\}_{i=1}^{10}$ with $z^1_{i,1} = y(.5i, .12)$ and $z^2_i(x) = \frac{\partial y}{\partial t}(.5i, x)$, $x \; \varepsilon \; [0,1]$, $i = 1,2,...,10$. As an initial guess we set

$$q^0(\theta) = \begin{cases} .6\theta & -1.2 \le \theta \le 1.2 \\ .6sgn(\theta) & |\theta| \ge 1.2. \end{cases}$$

The weights $\{\rho_i\}_{i=1}^{10}$ and $\{\sigma_i\}_{i=1}^{10}$ in the performance index (4.6) were all set equal to 1.

Using the same computational techniques and resources (both hardware and software) that we used for the first order example described in the previous section, we obtained the results plotted in Figures 4.1 - 4.3 below. The CPU times on the Cray X-MP/48 for these runs ranged from 84.96 seconds for n = 8, m = 3 to 1032.78 seconds for n = 20, m = 3. When n = 20, the value of the performance index Φ_n was reduced from $\Phi_n\,(q^0) \approx 6.0 \times 10^{-2}$ to $\Phi_n(\bar{q}^m_n) \approx 2.0 \times 10^{-3}$. For other values of n, the reduction in Φ_n was less pronounced. In this particular example we found (and it is apparent from the figures) that truly satisfactory results could not be obtained until n was chosen sufficiently large. However, as is clear from Figure 4.3 the scheme performed extremely well when n = 20. Once again, as in the case of a first order system, we found that although our theory does not apply, similar results could be obtained using a performance index involving spatially discrete measurements of velocity. As expected, since the problems (ID^m_n) were solved unconstrained (i.e. the compactness assumption on Q, and therefore Q^m for each m, were not enforced) the presence of instabilities became apparent for each n with m sufficiently large.

5. Concluding Remarks

In this paper we have summarized the theoretical framework for the identification of nonlinear distributed parameter systems which we have developed elsewhere ([3] and [5]), and, more importantly, have for the first time provided numerical evidence that our approach is indeed feasible and in fact performs well. In the case of second order systems, while our focus here has been on the identification of nonlinear damping in systems with linear stiffness, our theory is easily modified to handle the estimation of a nonlinear stiffness operator in the presence of linear damping. We are currently studying the extension of our results to second order systems which simultaneously involve both nonlinear stiffness and damping. Further numerical studies involving supercomputing are also presently underway. In addition to continuing our efforts using simulation data, we intend to test our schemes using experimental data in the near future.

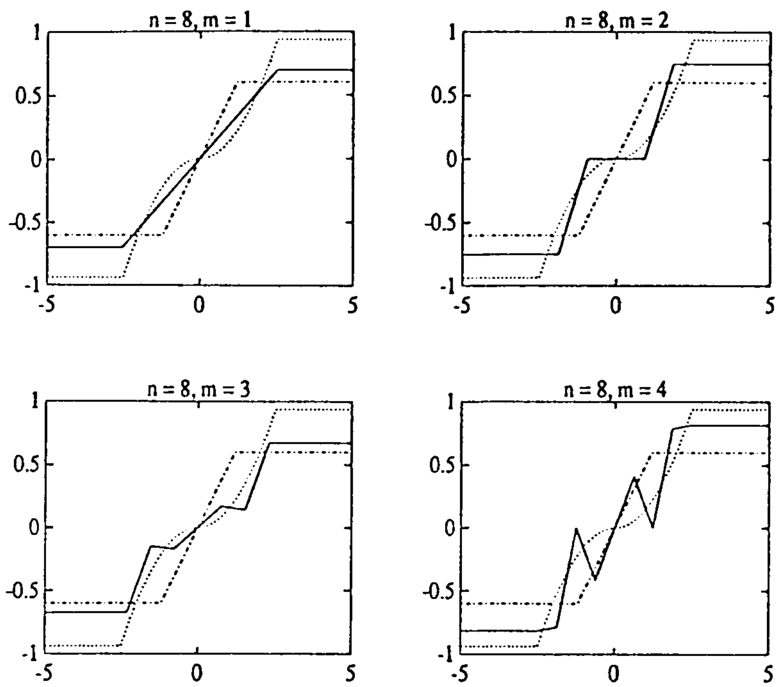

Figure 4.1 Legend: q^* ······, $\bar{q}_n^m$ — — — —, q^0 — · — · — ·.

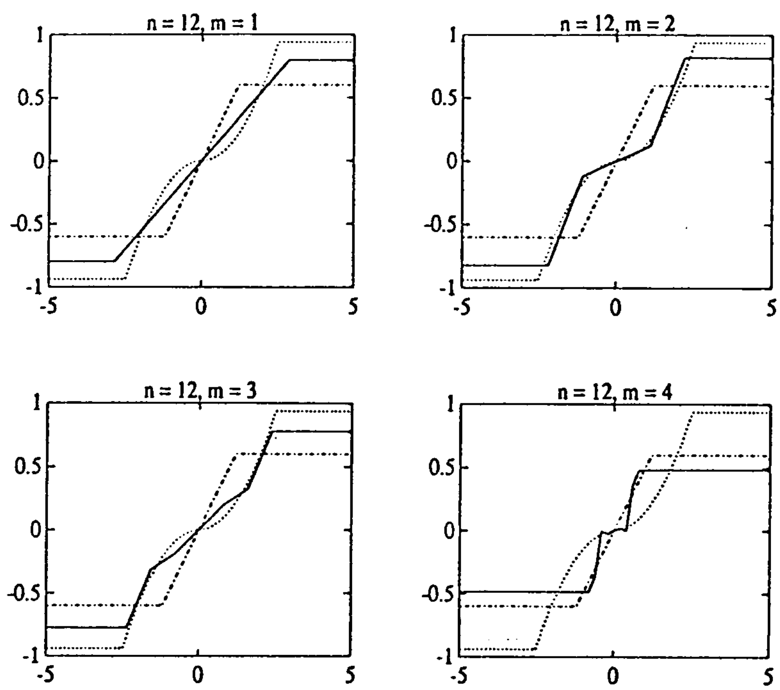

Figure 4.2 Legend: q^* ······, $\bar{q}_n^m$ — — — —, q^0 — · — · — ·.

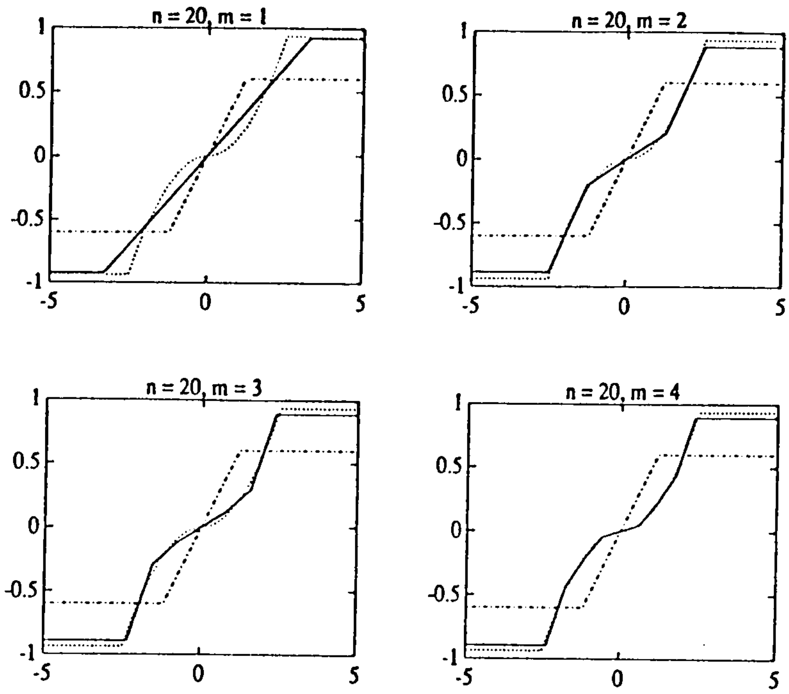

Figure 4.3 Legend: q^* ······ , $\bar{q}_n^m$ — — — —, q^0 — · — · — ·.

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Final Report

16. Abstract

An abstract approximation framework and convergence theory for the identification of first and secon order nonlinear distributed parameter systems developed previously by the authors and reported on in detail elsewhere are summarized and discussed. The theory is based upon results for systems whose dynamics can be described by monotone operators in Hilbert space an an abstract approximation theorem for the resulting nonlinear evolution system. The application of the theory together with numerical evidence demonstrating the feasibility of the general approach are discussed in the context of the identification of a first order quasi-linear parabolic model for one dimensional heat conduction/mass transport and the identification of a nonlinear dissipation mechanism (i.e. damping) in a second order one dimensional wave equation. Computational and implementational considerations, in particular, with regard to supercomputing, are addressed.

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