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EULER SOLVERS FOR TRANSONIC APPLICATIONS

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1. INTRODUCTION

The 1980s may well be called the Euler era of applied aerodynamics. Computer codes based on discrete approximations of the Euler equations are now routinely used to obtain solutions of transonic flow problems in which the effects of entropy and vorticity production are significant. Such codes can even predict separation from a sharp edge, owing to the inclusion of artificial dissipation, intended to lend numerical stability to the calculations but at the same time enforcing the Kutta condition.

One effect not correctly predictable by Euler codes is the separation from a smooth surface, and neither is viscous drag; for these we need some form of the Navier-Stokes equations. It, therefore, comes as no surprise to observe that the Navier-Stokes era has already begun before Euler solutions have been fully exploited. Moreover, most numerical developments for the Euler equations are now constrained by the requirement that the techniques introduced, notably artificial dissipation, must not interfere with the new physics added when going from an Euler code to a full Navier-Stokes approximation.

In order to appreciate the contributions of Euler solvers to the understanding of transonic aerodynamics, it is useful to review the components of these computational tools. Space discretization, time- or pseudo-time marching and boundary procedures are their essential constituents, to be discussed in Sections 2-4. The subject of grid generation and, in particular, grid adaptation to the solution, is worthy of a separate review and will be touched upon only where relevant; the influence of computer architecture on the choice of discretization is covered similarly. Section 5 rounds off with a list of unanswered questions and an outlook for the near future.

2. SPACE DISCRETIZATION

While finite-element discretizations are gaining ground, the majority of codes for inviscid compressible flow adhere to the finite-volume formulation. Two classes of finite-volume codes must be distinguished: those based on cell-centered data that represent cell averages of the conserved state quantities (refs. 1-3), and those based on cell-vertex data representing point samples of the state quantities (refs. 4-6). Cell-vertex schemes have been the lesser studied but bear the promise of a greater accuracy for a given grid (ref. 7), especially if the grid is unstructured. To appreciate the difference between the two sorts of data, consider the integral form of the Euler equations in two dimensions:

$$\frac{\partial u_{i,j}}{\partial t} = -\frac{1}{V_{i,j}} \oint_{B_{i,j}} (f dy - g dx).$$

On the left-hand side, we see the time derivative of the state vector u, averaged over the cell volume $V_{i,j}$; the right-hand side shows the boundary integral of the normal flux. The right-hand side is called the residual, at least if a steady solution is sought. In order to compute it, we only need data on the cell boundary; therefore, providing cell-vertex data is very efficient. In contrast, if cell-averaged data are given, boundary data must be obtained by interpolation. On the other hand, the left-hand side of (1) shows that cell-averaged data are the right choice for bookkeeping in time, i.e. when computing transient flows. An approach in which the discrete solution is described by a combination of cell-vertex and cell-boundary data is only known for one-dimensional flow (ref. 8).

Among schemes based on cell averages, one may again distinguish two approaches to the problem of finding boundary fluxes. In the "projection-evolution" approach (refs. 8, 9), boundary data are obtained by interpolation on both sides of a cell interface; the two state vectors then merge into one single flux vector by an "approximate Riemann solver" (ref. 10), which more or less describes the interaction of two fluid cells at their interface. Almost all upwind-biased schemes follow this format. In the projection or interpolation phase, non-oscillatory interpolation guarantees the absence of numerical oscillations in the final discrete solution (refs. 9, 11). The latest development in interpolation is the reconstruction of discontinuous solutions (refs. 12, 13).

The other approach (ref. 1) is to compute at each interface a straight flux average, leading to central differencing, and two different dissipative terms, one to stabilize the solution against pattern instabilities (zebra, checkerboard), the other one to help out near shock waves. The "artificial viscosity" approach has been shown to contain the same ingredients as the "projection-evolution" approach, but implemented differently (refs. 9, 10). For a comparison, see figure 1.

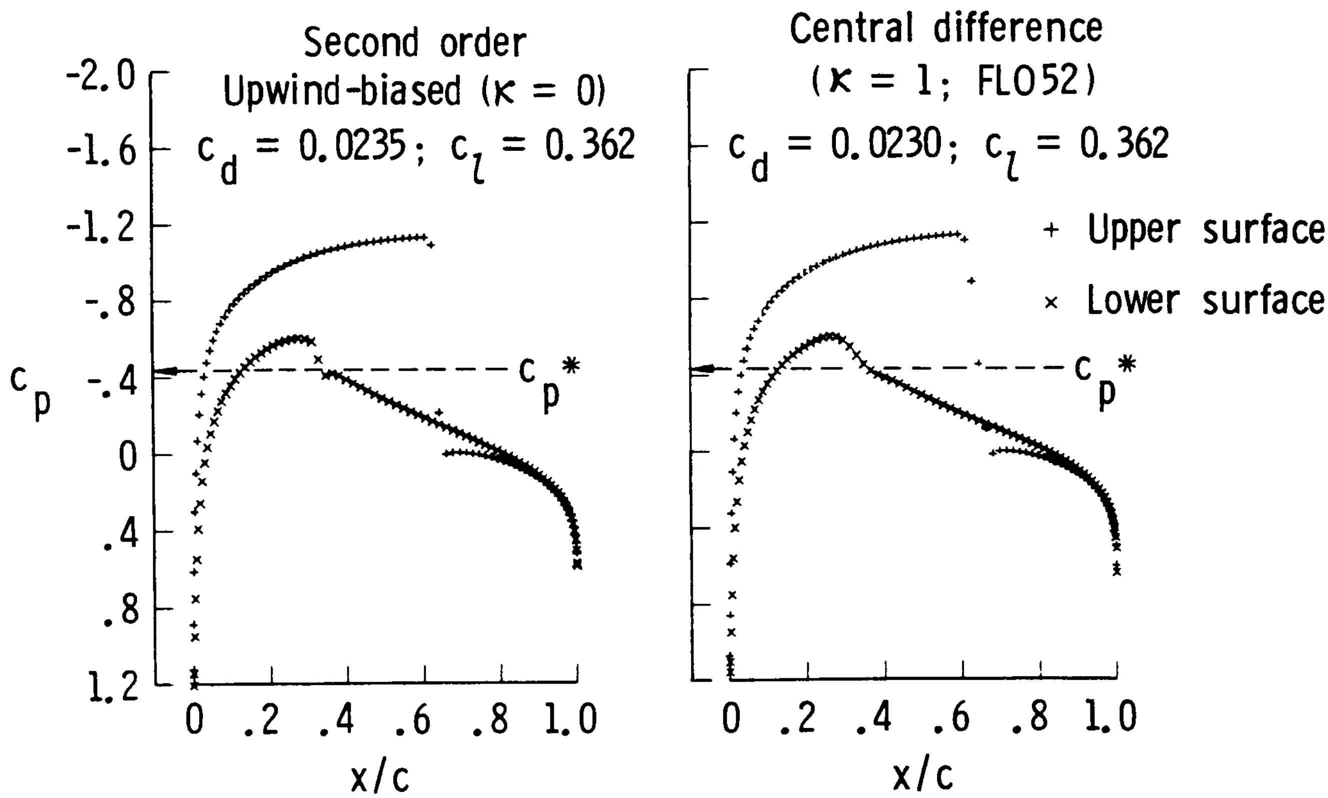

Figure 1. Pressure-coefficient distributions on a NACA 0012 airfoil for $M_\infty = 0.80$, $\alpha = 1.25$ deg, computed with codes based on upwind-biased (left) and central differencing (right). From ref. 2.

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For cell-vertex schemes, a theory of monotone interpolation has not yet been developed, so artificial viscosity is still the only instrument to smooth numerical solutions. It is worth noticing that cell-vertex schemes are less prone to pattern instabilities than cell-average schemes (ref. 12).

No discrete solution can exist without specifying a computational grid. Structured grids as building blocks in a strategy of domain decomposition (ref. 14) seem to be in the lead here, with nested grid refinement (ref. 6) as a welcome accessory technique. Fully unstructured grids, such as traditionally used in the finite-element method, form the alternative (ref. 15) (see figure 2); to retain sufficient accuracy, use of cell-vertex schemes is mandatory (see figure 3).

3. MARCHING IN TIME OR TO A STEADY STATE

Unlike the transient-flow problems of high-energy physics and astrophysics, problems of time-dependent transonic flow are of a gentler nature, the time dependence usually arising in the form of a slow oscillation about some mean state (ref. 16). The length of the characteristic time suggests the use of time steps greater than permitted by explicit time-accurate schemes, so implicit methods have been favored in this area. More generally speaking, the most effective methods are those that are also used to march to a steady state. More rapidly varying flows, such as encountered in turbo- machinery (rotor-stator interaction, ref. 17), require the full temporal resolution of an explicit marching scheme.

The effort spent on developing time-accurate marching methods for aerospace applications is scant, which clearly illustrates that this field is dominated by steady-flow problems. Still in use is MacCormack's predictor-corrector central-difference scheme (ref. 18); an upwind-biased non-oscillatory predictor-corrector scheme was presented and tested in (ref. 19). These schemes are examples of a "package deal": space and time discretization are inseparable. A more general strategy is to use a multi-stage Runge-Kutta-type scheme for the time integration, matched with the chosen spatial differencing operator to form a stable overall method.

Interesting enough, multi-stage Runge-Kutta methods have predominantly been used to march to steady solutions (ref. 1), without regard to their potential time accuracy. When developing Runge-Kutta methods for problems of time-dependent transonic flow, the same standards of accuracy and robustness must apply as those that led to the Piecewise Parabolic Method (PPM) (ref. 20) and other schemes suited equally well for smooth flow as for shocked flow. A breakthrough in this respect are the recently derived Total Variation Diminishing (TVD) multi-stage schemes (ref. 21), which preserve the TVD property of the spatial operator while advancing in time.

Regarding methods for marching to a steady state, explicit and implicit methods have been going up and down in popularity as on a wheel of fortune. Some of these changes were driven by developments in computer technology, others by advancement in numerical analysis. In the mid-seventies, Approximate Factorization (AF) emerged as an efficient method for solving steady inviscid problems (ref. 22). This is a relative of Alternating-Direction Implicit methods and requires considerable storage for maximum efficiency, namely, storage of the block-LU decomposition resulting from a line-inversion (this decomposition may be "frozen" for many iteration cycles). Jameson et al. (ref. 1) avoided storage problems by developing an explicit marching strategy based on a multi-stage time-discretization, use of "local" time-steps (constant Courant number rather than constant time step for the whole grid), residual smoothing and enthalpy

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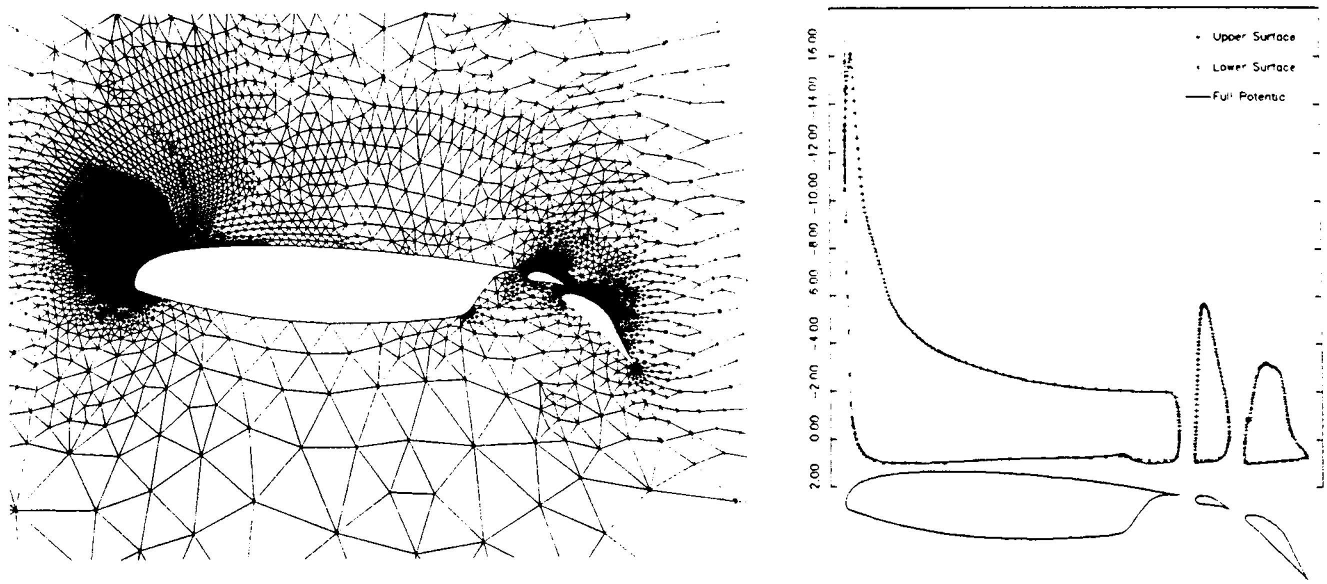

Figure 2. Unstructured, adaptively refined grid (left) and corresponding pressure-coefficient distributions (right) for a multi-element airfoil obtained with a cell-vertex scheme. Also included are results obtained with a finite-difference approximation of the full potential equation. From ref. 15.

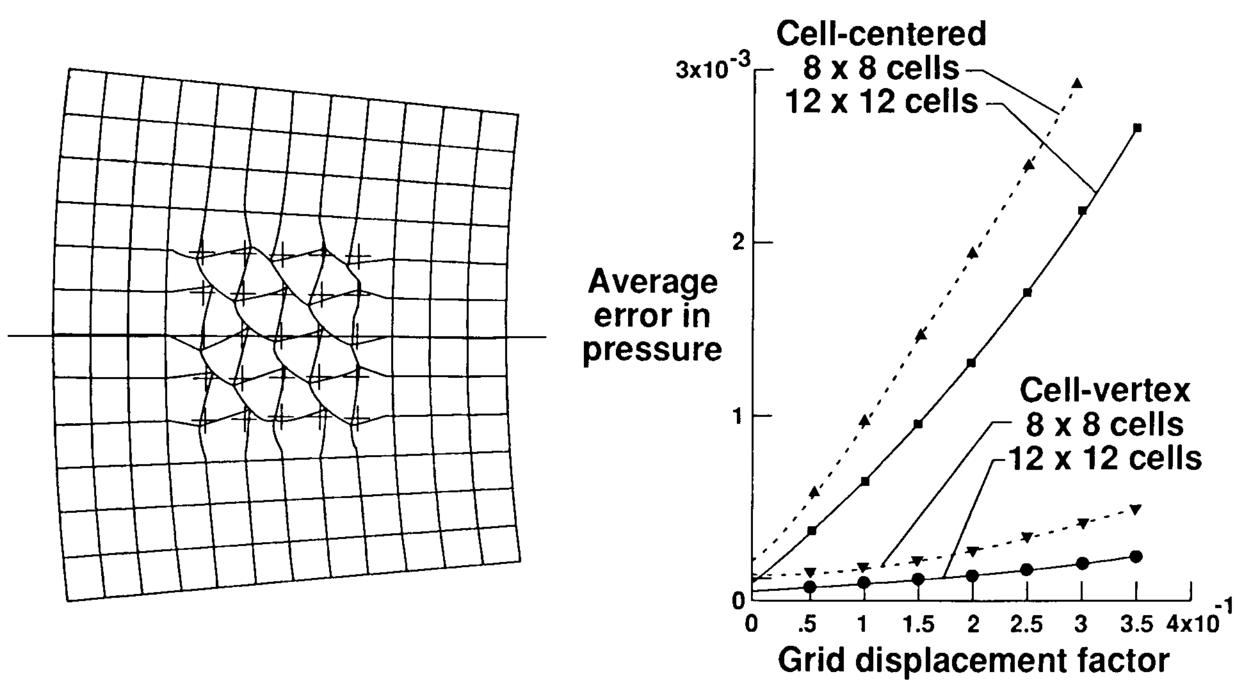

Figure 3. Grid distortion (left) and associated numerical error (right) in the computation of subsonic Ringleb flow with a cell-centered and a cell-vertex scheme. From ref. 7.

damping (using the fact that the specific total enthalpy becomes uniform in the steady state). Both AF and multi-stage marching schemes were combined with spatial discretizations based on central differencing and explicit artificial viscosity.

Next, the introduction of upwind-biased fluxes in schemes for finding steady Euler solutions (refs. 23, 24) led to a revival of the classical relaxation schemes such as Gauss-Seidel and line/Gauss-Seidel. It turns out that these methods, developed for finding solutions of elliptic equations, or steady solutions of parabolic equations, are the perfect match to upwind residual approximations, owing to the inherent dissipativity of the latter. On a scalar computer, Alternating Line/Gauss-Seidel (ALGS) relaxation easily outperforms AF (ref. 25), while requiring the same computational effort (two block-tridiagonal systems solved per iteration); see figure 4a.

Almost at the same time, the introduction of vector computing again reversed the order of preference among the known relaxation methods: although it requires considerably more iterations, AF outperforms ALGS when comparing CPU times, because it vectorizes better (ref. 25); see figure 4b. Such a radical changing of the guards suggests a search for new methods that exploit vector arithmetics even more strongly. This is not a trivial job, as there is something unnatural about the combination of vectorization and hyperbolicity. Hyperbolic equations model the propagation of signals moving in a continuum of directions, which is rather well imitated by a series of Gauss-Seidel sweeps in alternate discrete directions. The very fact of sequential dependence in Gauss-Seidel updating prohibits code vectorization to a great extent. It is worth mentioning here that recently it was proved that Symmetric Line/Gauss-Seidel relaxation (SLGS) is not unconditionally stable for upwind-biased Euler residuals (ref. 26). This accounts for some earlier, unexplained non-convergence of numerical results (ref. 23).

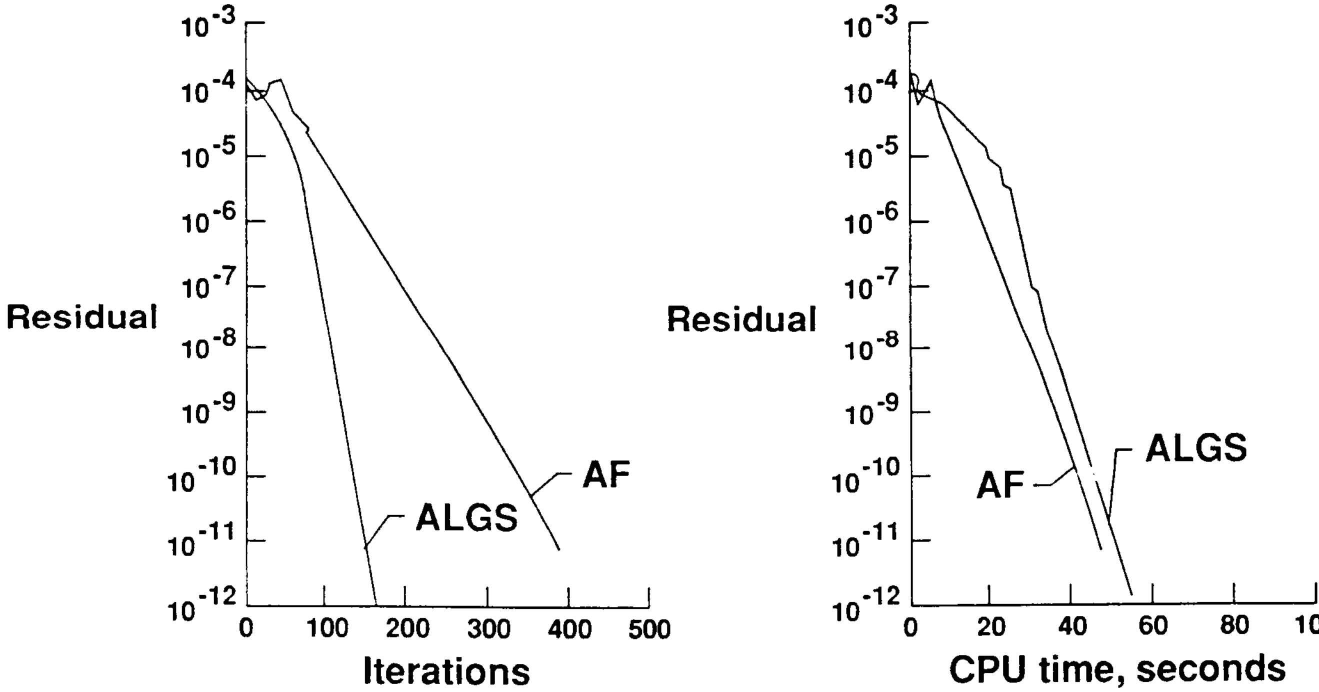

Figure 4. A comparison of iteration strategies for the computation of a transonic flow (NACA 0012, $M_\infty = 0.8$, $\alpha = 1.25$ deg, 161x41 C-mesh, first-order upwind differencing). Left(a): convergence of Alternating-Line Gauss-Seidel (ALGS) relaxation and Approximate Factorization (AF) in terms of number of iterations; right(b): in terms of CPU seconds on a Cyber 205 vector computer. From ref. 25.

Meanwhile, multigrid relaxation, a fully matured technique for solving elliptic equations, has been shown to be a valuable accessory technique for removing long-wave components in distributions of Euler residuals (refs. 27-29). This, again, is causing a shift of interest from implicit to explicit relaxation methods, with the boom still to come. For a better understanding, it should be said that the basic marching scheme in a multigrid strategy must be a good relaxation scheme only for the short-wave components in a residual distribution; such a scheme is called a "smoother." In a discrete distribution, wave lengths scale with the cell size, so when going to coarser and coarser grids, the shortest wave that can be represented on the grid eventually becomes as long as the largest scale in the problem. Explicit marching schemes for the Euler equations can be designed to damp just the shortest waves without going beyond the stable range of the time-step, and therefore seem to be the perfect match to multigrid relaxation. In contrast, implicit schemes, such as line relaxation, will also attack long one-dimensional waves, which would seem unnecessary in a multigrid framework.

The highest possible achievement of a multigrid scheme is the so-called "multigrid convergence": convergence in a fixed number of multigrid cycles, regardless of the cell size of the finest grid. Such convergence has actually been realized in solving a simple channel-flow problem (refs. 30, 31); see figure 5. The fly in the ointment is that classical multigrid has also been shown *not* to work in very similar inviscid problems. This is caused by loss of information during coarsening when long waves in one direction are coupled to short waves in another direction, a situation that can arise when the grid is aligned with a stratified flow over an appreciable distance, as in channel flow (ref. 32), or when it is strongly stretched (ref. 33). Under these circumstances, one-

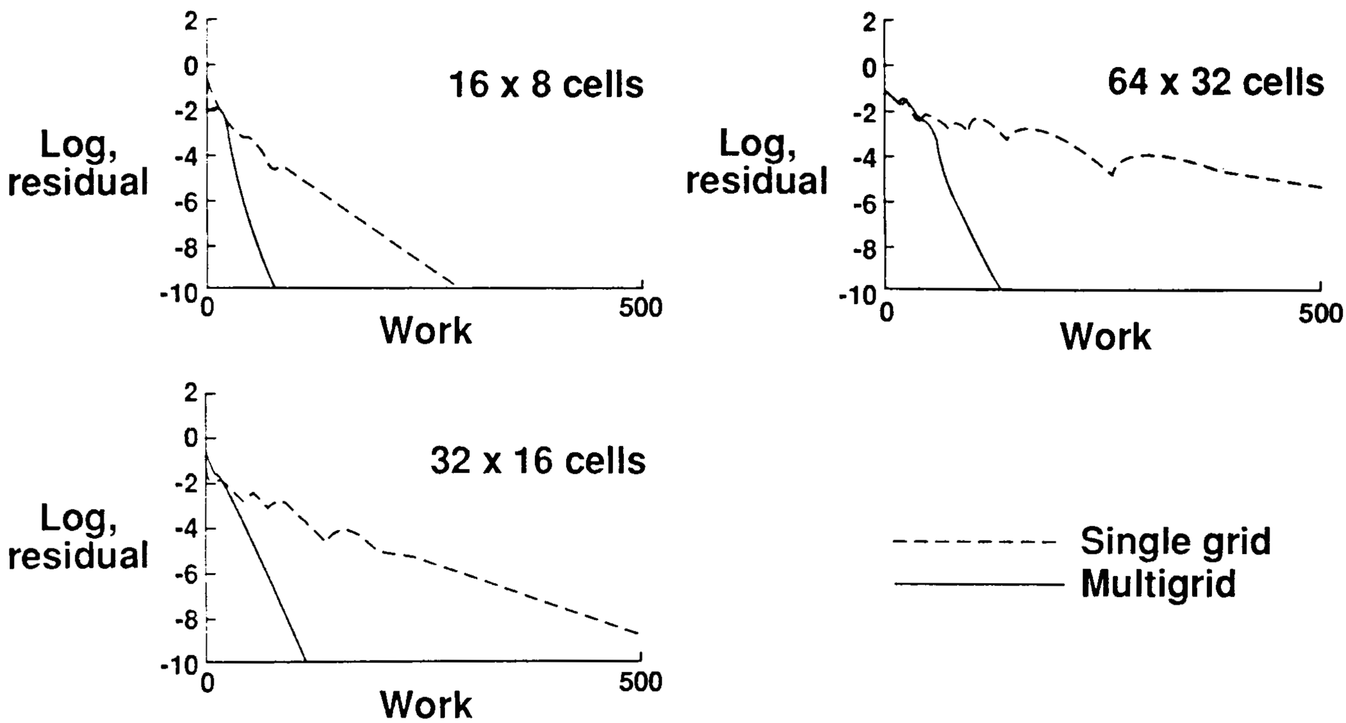

Figure 5. Convergence histories for the computation of the flow at $M_\infty = 0.85$ over a circular bump (thickness .042 of chord) in a channel, using first-order upwind differencing and multigrid relaxation on three different grids. In all cases, the slower convergence is for the single-grid scheme; the total work for the multigrid scheme stays constant for grids on which the flow features are sufficiently resolved. From ref. 30. (Copyright ©1985. Academic Press, Inc.)

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dimensional coarsening rather than multidimensional coarsening should be considered. It turns out that the need for semi-coarsening was already demonstrated in an early multigrid application to the potential equations (ref. 34); a full utilization of semi-coarsening has recently been proposed and analyzed by Mulder (ref. 35). The latter method has an appreciable degree of parallelism (yet to be exploited) and shows convergence rates for subsonic, transonic and supersonic channel flows between .3 and .4 per multigrid cycle.

With multigrid relaxation finally outgrowing its "elliptic" origin and becoming robust for the Euler equations, a new wave of advanced explicit algorithms is awaited with impatience. These algorithms do away with models of coordinate-wise wave propagation on which all present discretizations, including the upwind-biased ones, are based; instead, discrete models of the infinite variety of multidimensional wave motions are adopted. The basic concepts have been formulated (refs. 36, 37), but it is at present not clear how to incorporate these into robust marching methods. Yet, this development promises significant gains in accuracy, efficiency (see figure 6), and generality, while retaining programming simplicity and matching the computer architectures of today and tomorrow.

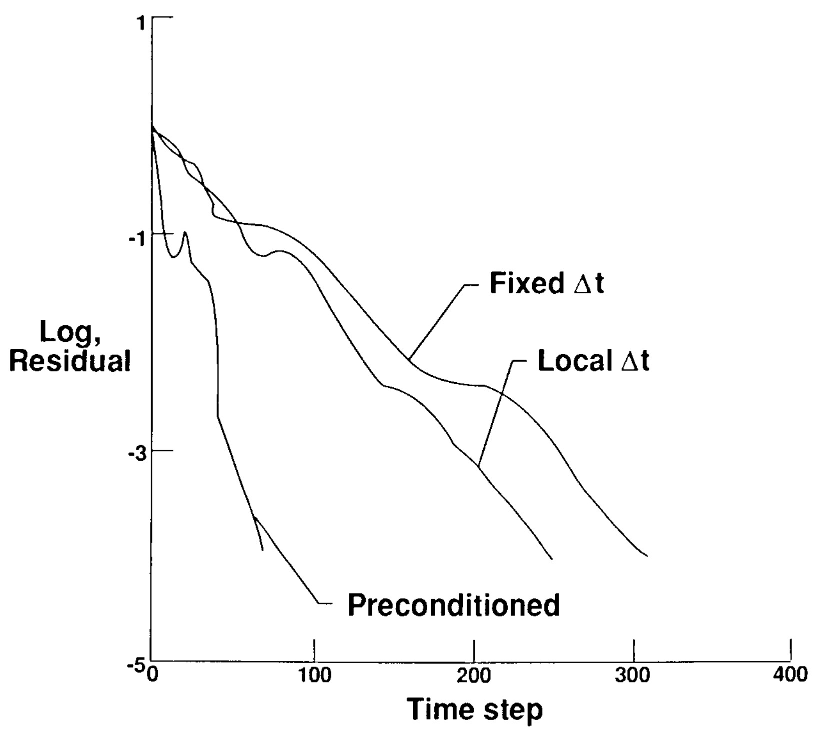

Figure 6. Effect of preconditioning on the computation of one-dimensional transonic nozzle flow with a first-order upwind scheme. Shown are the convergence histories resulting from the use of (1) the same time-step in all cells, (2) the maximum time-step in each cell (the so-called "local" time-step) and (3) the maximum time-step for each characteristic wave in each cell. The latter technique requires decomposition of the residual into wave contributions, which is needed anyway for upwind differencing, and leads to significant savings.

In the area of implicit methods, on the other hand, the tendency is to move toward greater complexity. With in-core storage capacity orders of magnitude larger than a decade ago, genuine Newton methods have been formulated and implemented using direct rather than iterative solution techniques for the large linear systems arising in the process (ref. 38); see figure 7. These methods are almost competitive with relaxation methods for the number of unknowns typically encountered in two-dimensional calculations, and a lot more robust. The main contribution to the complexity is the derivation of the full Jacobian of today's sophisticated numerical flux functions such as Roe's (ref. 39). If progress is to be made in this direction, automation of the algebra of differentiation, or, alternatively, reliance on numerical differentiation, seems to be inevitable. Any increase in generality, such as applicability on unstructured or locally refined grids, will involve a major programming effort.

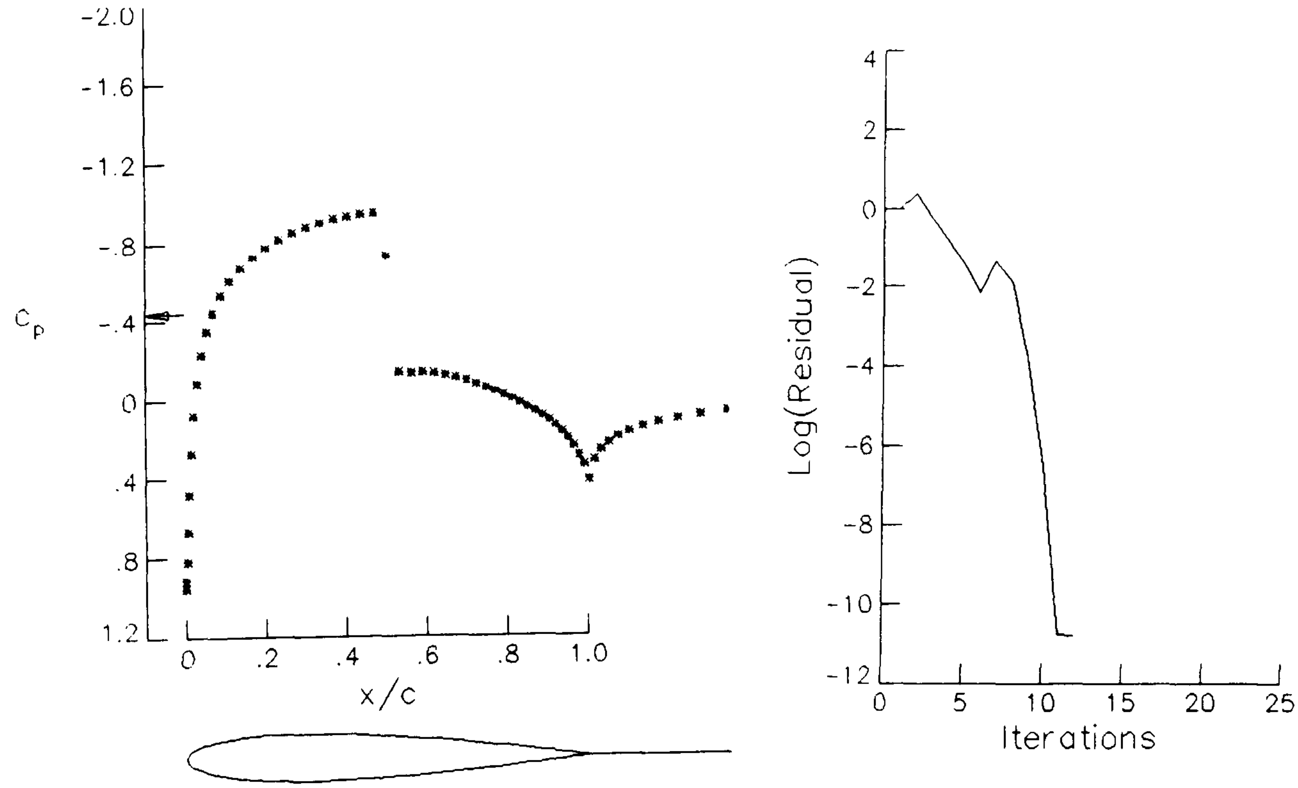

Fig. 3a.

Fig. 3b.

Results using Ordering A (M = 0.8, α =0)

160 X 32 GRID M = .800 ALPHA = 0.000 CD, CL = .01103 0.0000
MIN VALUE= -.94) MAX= .9535

Figure 7. Pressure-coefficient results and convergence history of a transonic-airfoil computation (NACA 0012, $M_\infty = 0.80$, $\alpha = 0$ deg) with a full Newton method. Note the quadratic convergence of the residual in the final phase. From ref. 38.

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The abundance of core memory in modern computers is also a stimulus for the development of marching methods based on multiple iterates. Most classical iteration schemes make due with information exclusively from iteration level k when advancing to level k+1, thus ignoring the information contained in the first k-1 iterates. Obviously, this information might give clues as to the best way to proceed beyond level k. Since each iterate of a numerical solution can be represented by a large vector, and the aim of iterative methods is to make this vector converge to some unknown final value, this subject is commonly referred to as "convergence acceleration of vector sequences" (ref. 40). The label suggests that the convergence of the vectors can be studied without in-depth knowledge of the physics represented by the vectors. This is probably as true as the statement that all CFD problems are similar to solving Laplace's equation on a rectangle. Some recent applications of acceleration methods like GMRES (Generalized Minimum Residual Method), MPE (Minimum Polynominal Extrapolation), reduced-rank extrapolation and Wynn's ϵ-algorithm (see figure 8) can be found in references 41-45.

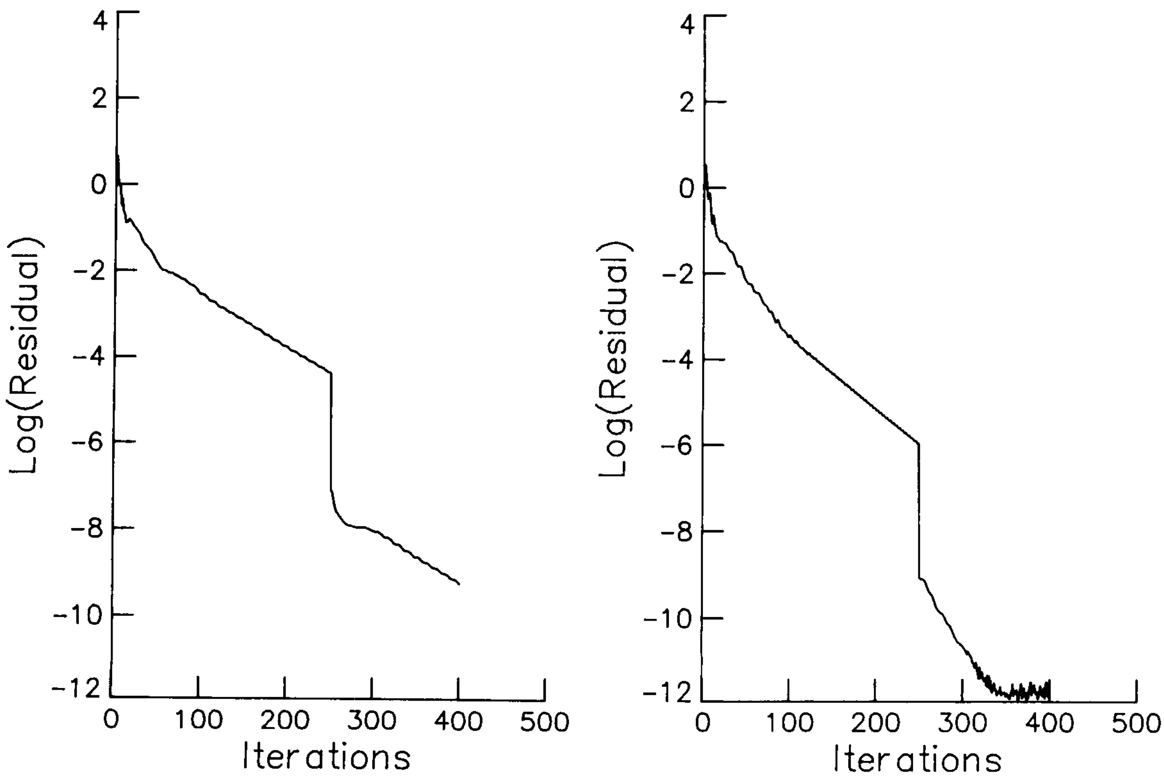

Figure 8. Convergence acceleration for two transonic-airfoil computations by the use of Wynn's ϵ-algorithm. The residual decreases sharply upon application of the acceleration technique after 250 iterations. Left: Korn airfoil, $M_\infty = 0.75$, $\alpha = 0$ deg; right: NACA 0012, $M_\infty = 0.80$, $\alpha = 1.25$ deg. From ref. 45.

One aspect not addressed at all in these papers is the inherent nonlinearity of the CFD problem to be solved. When solving problems of steady transonic flow there actually are two separate but related questions:

1) how to efficiently march from an arbitrary initial guess to within the range of attraction of the steady solution;

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2) how to quickly converge to the steady solution from a nearby state.

Vector-sequencing strategies address the second problem, which is the easier of the two owing to the validity of linearization. The only current methodology that also addresses the first problem is full multigrid.

4. BOUNDARY PROCEDURES

How to derive non-reflective far-field boundary conditions has been pretty well understood since the appearance of a key paper by Engquist and Majda (ref. 46). For the Euler equations, a safe technique is to discretize the characteristic equations for waves moving outward normal to the boundary; supplemental conditions regarding the far field may be provided in the form of free-stream values. Greater accuracy, faster convergence and a smaller computational domain are the benefits of a more accurate description of the far field, based on various kinds of expansions of the solution (see, e.g. ref. 47). The most remarkable reduction in the computational domain has been demonstrated by Ferm and Gustafsson (ref. 48); see figure 9.

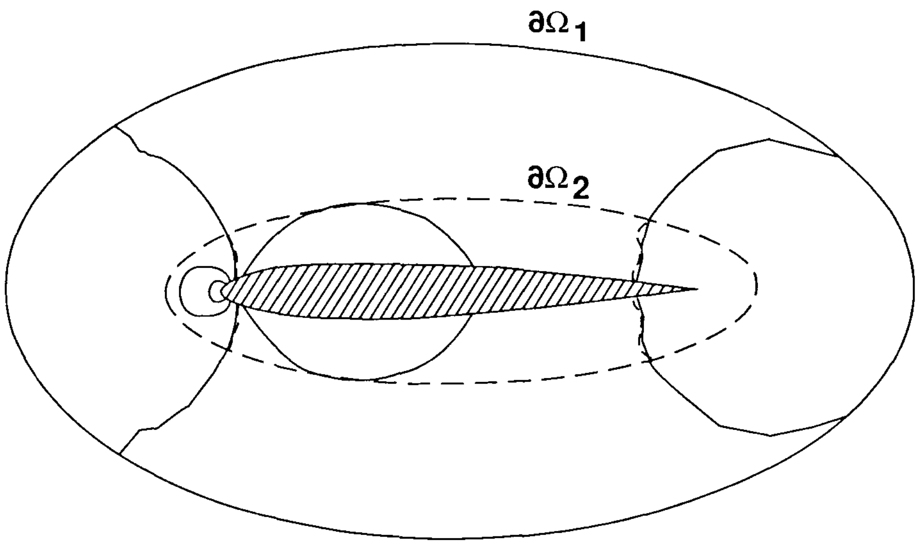

Figure 9. Isobars for the incompressible flow over NACA 0012 airfoil, computed with Ferm and Gustafsson's (ref. 48) "fundamental boundary conditions." Solid-line contours are from the solution on the larger domain ($\partial\Omega_1$), dashed lines are for the smaller domain ($\partial\Omega_2$).

5. CONCLUSIONS

During the past decade, Euler solvers have come of age. Accuracy and efficiency of today's finite-volume Euler codes are sufficient to deliver detailed two-dimensional and useful three-dimensional flow solutions.

The accuracy achieved is somewhat surprising, as the ingredients for the spatial discretiza-

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tions are almost exclusively based on a one-dimensional analysis, applied coordinate-wise. The insufficiency of this strategy shows up, for instance, as a loss of resolution of shock waves and shear waves oblique to the grid. Keeping in mind the even higher resolution required for a Navier-Stokes solution, it appears necessary to develop truly multi-dimensional numerical building blocks for use in future Euler codes.

Independently, the development of grid-adaptation techniques will support solutions of high accuracy at a reasonable cost. To comply with complicated geometries, the use of unstructured triangular and tetrahedral meshes is gaining ground; this in turn is stimulating the development of cell-vertex-based spatial discretizations, which are relatively insensitive to grid deformations.

On the efficiency front, multigrid relaxation and vector-sequence convergence acceleration are the two auxiliary techniques that bear the most promise of making three-dimensional calculations feasible and robust. The explicit marching schemes on which these techniques build, however, remain to be optimized as regards their capacity to smooth short waves and to overcome stiffness of the equations. Again, a truly multi-dimensional approach is needed.

Among implicit methods, Newton's method with full Jacobian evaluation and direct solution of the linearized system, is now competitive in obtaining two-dimensional flow fields; at present, the extension to three dimensions does not seem feasible.

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