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CRACK-FACE DISPLACEMENTS FOR EMBEDDED ELLIPTIC AND SEMI-ELLIPTICAL SURFACE CRACKS

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Abstract

Analytical expressions for the crack-face displacements of an embedded elliptic crack in infinite solid subjected to arbitrary tractions are obtained. The tractions on the crack-faces are assumed to be expressed in a polynomial form. These displacements expressions complete the exact solution of Vijayakumar and Atluri, and Nishioka and Atluri. For the special case of an embedded crack in an infinite solid subjected to uniform pressure loading, the present displacements agree with those by Green and Sneddon. The displacement equations derived in this paper were used with the finite-element alternating method (FEAM) for the analysis of a semi-elliptic surface crack in a finite solid subjected to remote tensile loading. The maximum opening displacements obtained with FEAM are compared to those with the finite-element method with singularity elements. The maximum crack opening displacements by the two methods showed good agreement.

Introduction

Damage tolerant analyses require accurate calculations of stress-intensity factors at crack tips in two-dimensional and around crack fronts in three-dimensional bodies. Experience with several crack configurations have shown that these cracks tend to grow in nearly elliptic, semi-elliptic and quarter-elliptic shape. Therefore, considerable attention has been devoted to analytical and experimental studies of these crack configurations. While considerable data exists on the stress-intensity factors for these crack configurations [1-4], very little information exists on crack-face displacements. Crack-face displacements are directly measured in an experiment, while the stress-intensity factors are deduced from the experimental observations (such as stress freezing). Therefore, a need exists for accurate analytical crack-face displacements for comparison with experimental observations for three-dimensional crack configurations. Also, crack-face displacements are needed to develop three-dimensional weight function methods [5,6].

Recent literature has shown that the three-dimensional finite-element alternating method (FEAM) is a very economical and an accurate method for obtaining stress-intensity factors for elliptical and part-elliptical crack configurations [7-10]. The FEAM uses the exact solution of an embedded elliptical crack in an infinite solid subjected to arbitrary crack-face tractions obtained by Vijakumar and Atluri [11], and Nishioka and Atluri [7,8]. While these solutions are comprehensive, the crack-face displacements are not available in references 7,8 and 11. In order to use the FEAM to calculate crack-face displacements, the exact solution of references 7,8 and 11 needs to be extended to derive the crack-face displacements for each of the polynomial pressure loadings on the crack faces.

The purpose this paper is obtain the analytical crack-face displacements for an embedded elliptic crack in an infinite elastic solid subjected to arbitrary polynomial normal and shear tractions. First embedded crack solutions in the literature are reviewed. Next the fundamental equations relevant to the embedded elliptic crack problem are summarized. Then the crack-face displacements for mode I loading are obtained. Next the crack-face displacements for the mixed-mode loadings are derived. The displacements for the special cases of uniform pressure on the crack faces are compared with those from Green and Sneddon [12] and Kassir and Sih [13]. The crack-face displacements equations presented in this paper are used in conjunction with the FEAM to calculate the maximum crack opening displacements (COD) of a semi-elliptical surface cracks in a finite thickness solid. These COD values are compared to those obtained with finite-element method using singularity elements.

Symbols

a_1, a_2 semi-major and semi-minor axes of the embedded elliptic crack

$A^{i,j}_{\alpha,m-n,n}$ magnitudes of the polynomial pressure loading on the crack faces defined by Eq. (1) and Eq. (47)

$E(\kappa)$ elliptic integral of the second kind

f_α potential functions

F_{kl} potential function defined by Eq. (7)

$K(\kappa)$ elliptic integral of the first kind

M degree of truncation of the polynomials

u_α displacement along x_α direction

x_α Cartesian coordinates with the origin at center of the elliptic crack with x_1 along the major axis and x_2 along the minor axis of the ellipse

μ shear modulus of the isotropic material of the solid

κ^2 $(a_1^2 - a_2^2)/a_1^2$

κ'^2 a_2^2/a_1^2

ν Poisson's ratio of the material

ξ_α ellipsoidal coordinates defined by Eq. (3)

$\sigma_{\alpha\beta}$ stresses in the solid

Embedded Crack Solutions

The exact solutions for an embedded crack in an infinite solid are the fundamental solutions used in alternating methods. Therefore, this section briefly reviews the classical works in the literature on embedded crack solutions. Some of the results from these references are used to build the present solution for the crack-face displacements.

Exact analytical solution for a penny-shaped crack subjected to uniform pressure loading on the crack surfaces was obtained by Sneddon [14]. Later, Green and Sneddon [12] obtained the exact solution for an embedded elliptic crack in a infinite solid subjected to uniform pressure loading. Kassir and Sih [13] obtained an exact solution for an elliptic crack subjected to shear loading on the crack faces that can be expressed by a simple trignometric functions. Shah and Kobayashi [15] derived close form solutions for an embedded elliptic crack subjected normal loading that can be described by polynomial functions up to the third degree in terms of the two Cartesian coordinates that describe the ellipse. Smith et al [16] obtained the solution for a penny-shaped crack subjected to a polynomial pressure distribution on the crack faces. Smith and Sorensen [17] obtained the solution to an embedded elliptical crack in an infinite solid subjected to shear tractions that can be described by polynomial functions up to the third degree in terms of the two Cartesian coordinates that describe the ellipse. Later, Vijayakumar and Atluri [11] and Nishioka and Atluri [7,8] obtained the exact solution for an embedded elliptic crack subjected to *arbitrary* normal and shear loadings on the crack surfaces. Note that the solutions in references 12 through 17 are special cases of this general solution. As mentioned previously, the purpose of this paper is to extend the solution in references 7,8 and 11 and derive the crack face displacements.

Fundamental Equations For Mode I Loadings

For completeness this section presents the equations that are relevant to an embedded elliptical crack in an infinite solid subjected to arbitrary tractions on the crack faces. The notation of reference 7 is followed. Some of these equations can also be found in references 11-17.

Consider an embedded elliptic crack in an infinite solid as shown in Figure 1. The crack faces are subjected to prescribed normal tractions σ_{33} which can be expressed in a polynomial form as

$$\sigma_{33}^{(0)} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{m=0}^{M}\sum_{n=0}^{m} A_{3,m-n,n}^{i,j}\; x_1^{2m-2n+i}\; x_2^{2n+j} \tag{1}$$

where i and j specify the symmetries of the load with respect to the axes of the ellipse, $A_{3,m-n,n}^{i,j}$ are constants that give the magnitude of the load, and M is the degree of truncation of the polynomial.

The stress distribution everywhere in the solid and the stress-intensity factors all along the crack border due to the prescribed pressure distribution of Eq. 1 has been obtained in references 7, 8, and 11. Here the crack opening displacements corresponding to the arbitrary pressure distribution of Eq. 1 will be obtained.

Ellipsoidal Coordinates

The elliptic crack perimeter can be described by

$$(x_1/a_1)^2 \; + \; (x_2/a_2)^2 \; = 1 \;, \quad a_1 > a_2 \tag{2}$$

The coordinates x_i $(i = 1,2,3)$ of any point in the solid may be expressed in terms of the ellipsoidal coordinates, $\xi_\alpha, (\alpha = 1,2,3)$ in the form

$$\begin{aligned} a_1^2\,(a_1^2 - a_2^2)x_1^2 &= (a_1^2+\xi_1)\,(a_1^2+\xi_2)\,(a_1^2+\xi_3) \\ a_2^2\,(a_2^2 - a_1^2)x_2^2 &= (a_2^2+\xi_1)\,(a_2^2+\xi_2)\,(a_2^2+\xi_3) \\ a_1^2\,a_2^2\,x_3^2 &= \xi_1\,\xi_2\,\xi_3 \end{aligned} \tag{3}$$

where

$$\infty > \xi_3 \geq 0 \geq \xi_2 \geq -a_2^2 \geq \xi_1 \geq -a_1^2$$

Note that the ellipsoidal coordinates ξ_α are defined by the roots of the cubic equation

$$\begin{aligned} \omega(s) &= 1 - \frac{x_1^2}{a_1^2+s} - \frac{x_2^2}{a_2^2+s} - \frac{x_3^2}{s} \\ &= P(s)/Q(s) \end{aligned} \tag{4}$$

where

$$P(s) \quad = \quad (s-\xi_1)\,(s-\xi_2)\,(s-\xi_3),$$

and

$$Q(s) \quad = \quad s\,(s+a_1^2)\,(s+a_2^2) \tag{5}$$

In the $x_3 = 0$ plane (the plane of the crack), the inside of the ellipse is given by $\xi_3 = 0$, outside of the ellipse is given by $\xi_2 = 0$, and the elliptic crack front is described by $\xi_2 = \xi_3 = 0$.

Potential Functions, Stresses, and Displacements

Using Trefftz's formulation [7,8,11] for mode I problems, a potential function f_3 is defined such that

$$f_3 \quad = \quad \sum_k \sum_l C_{3,k,l}\, F_{kl} \tag{6}$$

where

$$F_{kl} \quad = \quad \frac{\partial^{k+l}}{\partial x_1^k\, \partial x_2^l} \int_{\xi_3}^{\infty} \frac{[\,\omega(s)\,]^{k+l+1}}{\sqrt{Q(s)}}\, ds \tag{7}$$

In equation (6), $C_{3,k,l}$ are undetermined constants and $\omega(s)$ and $Q(s)$ in Eq. (7) are defined in Eqs. (4) and (5), respectively. The partial derivatives of the potential function f_3 are needed to obtain the stresses and displacements. They are denoted as follows.

$$f_{3,\beta} \quad = \quad \sum_k \sum_l C_{3,k,l}\, F_{kl,\beta} \tag{8}$$

$$f_{3,\beta\gamma} \quad = \quad \sum_k \sum_l C_{3,k,l}\, F_{kl,\beta\gamma} \tag{9}$$

$$f_{3,\beta\gamma\delta} \quad = \quad \sum_k \sum_l C_{3,k,l}\, F_{kl,\beta\gamma\delta} \tag{10}$$

where $f_{3,\beta}$ is the partial derivative of f_3 with respect to $x_\beta, (\beta = 1,2,3)$ and so on.

The stresses and displacements in the solid are given by

$$
\begin{aligned}
\sigma_{11} &= 2\,\mu\,[\,f_{3,11} + 2\,\nu\, f_{3,22} + x_3 f_{3,311}\,] \\
\sigma_{22} &= 2\,\mu\,[\,f_{3,22} + 2\,\nu\, f_{3,11} + x_3 f_{3,322}\,] \\
\sigma_{12} &= 2\,\mu\,[\,(1-2\nu)\, f_{3,12} + x_3 f_{3,312}\,] \\
\sigma_{33} &= 2\,\mu\,[\,-f_{3,33} + x_3 f_{3,333}\,] \\
\sigma_{31} &= 2\,\mu\, x_3 f_{3,313} \\
\sigma_{32} &= 2\,\mu\, x_3 f_{3,323}
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
u_1 &= (1-2\nu)\, f_{3,1} + x_3\, f_{3,31} \\
u_2 &= (1-2\nu)\, f_{3,2} + x_3\, f_{3,32} \\
u_3 &= -2(1-\nu)\, f_{3,3} + x_3\, f_{3,33}
\end{aligned}
\tag{12}
$$

where μ is the shear modulus and ν is the Poisson's ratio of the material.

Crack Boundary Conditions

The boundary conditions on the crack plane ($x_3 = 0$) are in two regions: inside and outside the elliptic crack. Inside the elliptic crack ($\xi_3 = 0$), the boundary conditions are

$$
\begin{aligned}
\sigma_{33}^{(0)} &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{m=0}^{M}\sum_{n=0}^{m} A_{3,m-n,n}^{i,j}\; x_1^{2m-2n+i}\; x_2^{2n+j}\,, \\
\sigma_{31}^{(0)} &= \sigma_{32}^{(0)} = 0
\end{aligned}
\tag{13}
$$

for

$$
(x_1/a_1)^2 + (x_2/a_2)^2 < 1, \;\; x_3 = 0.
$$

Note that the right hand side of Eq. (13) are the prescribed tractions on the crack faces.

Outside the elliptic crack ($\xi_2 = 0$), because of symmetry in this mode I problem

$$u_3 = 0 \,, \; \sigma_{31}^{(0)} = \sigma_{32}^{(0)} = 0. \tag{14}$$

for

$$(x_1/a_1)^2 + (x_2/a_2)^2 \geq 1, \;\; x_3 = 0.$$

From the stresses in Eq. (11), the shear stress boundary conditions are identically satisfied and the normal stress boundary conditions require that

$$-2\mu\, f_{3,33} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{m=0}^{M}\sum_{n=0}^{m} A_{3,m-n,n}^{i,j}\, x_1^{2m-2n+i}\, x_2^{2n+j} \tag{15}$$

Because the potential function f_3 is expressed in terms of the unknown constants $C_{3,k,l}$ and an unknown function F_{kl} (see Eq. (6)), Eq. (15) provides the relationship between $C_{3,k,l}$ and $A_{3,m-n,n}^{i,j}$. Such a relationship was obtained from Eq. (42a) of reference 7.

Equation (12) define the displacements everywhere in the solid. As the displacements at any point on the crack faces are sought, the displacements inside the ellipse ($\xi_3 = 0$) on the $x_3 = 0$ plane need to be evaluated. Because the derivatives $F_{kl,3\gamma}$ (and hence $f_{3,3\gamma}$) are finite when $x_3, \xi_3 = 0$ (see Appendix I for details), the crack-face displacements can be written as

$$\begin{aligned} u_1 &= (1-2\nu)\, f_{3,1} \\ u_2 &= (1-2\nu)\, f_{3,2} \\ u_3 &= -2(1-\nu)\, f_{3,3} \end{aligned} \tag{16}$$

Therefore, to obtain the crack-face displacements one needs the derivatives of the potential function f_3 with respect to x_β.

Mode I Crack Face Displacements

$F_{kl,\beta}$ must be evaluated to determine the partial derivatives of f_3 with respect to x_β. The derivatives of F_{kl} are obtained by differentiating Eq. (7) as

$$F_{kl,\beta} = \frac{\partial^{k_1+l_1+m_1}}{\partial x_1^{k_1}\, \partial x_2^{l_1}\, \partial x_3^{m_1}} \int_{\xi_3}^{\infty} \frac{[\omega(s)]^{k+l+1}}{\sqrt{Q(s)}}\, ds \tag{17}$$

where

$$k_1 = k + \delta_{1\beta}\ , \quad l_1 = l + \delta_{2\beta}\ , \quad m_1 = \delta_{3\beta} \tag{18}$$

and $\delta_{1\beta}, \delta_{2\beta}$, and $\delta_{3\beta}$ are the well known Kronecker deltas.

Because $\omega(\xi_3) = 0$, Eq. (17) can be written as

$$F_{kl,\beta} = \int_{\xi_3}^{\infty} \frac{\partial^{k_1+l_1+m_1}}{\partial x_1^{k_1}\, \partial x_2^{l_1}\, \partial x_3^{m_1}} [\,\omega(s)\,]^{k+l+1} \frac{ds}{\sqrt{Q(s)}} \tag{19}$$

Equation (19) needs to be evaluated in the interior of the ellipse ($\xi_3 = 0$). Substituting $\omega(s)$ from Eq. (4) into Eq. (19) and performing the integrations one obtains

$$F_{kl,\beta} = (k+l+1)! \sum_{p=0}^{k+l+1} \sum_{q=0}^{p} \sum_{r=0}^{q} \frac{(-1)^p}{(k+l+1-p)!}\, t_1\, t_2\, t_3\, t_4\, t_5\, t_6\, J_{p-q,q-r,r}(\xi_3) \tag{20}$$

where

$$t_1 = \frac{(2p-2q)!}{(p-q)!}$$

$$t_2 = \frac{(2q-2r)!}{(q-r)!}$$

$$t_3 = \frac{(2r)!}{r!}$$

$$t_4 = \frac{x_1^{2p-2q-k_1}}{(2p-2q-k_1)!}$$

$$t_5 = \frac{x_2^{2q-2r-l_1}}{(2q-2r-l_1)!}$$

$$t_6 = \frac{x_3^{2r-m_1}}{(2r-m_1)!}$$

and

$$J_{p-q,q-r,r}(\xi_3) = \int_{\xi_3}^{\infty} \frac{ds}{(a_1^2+s)^{p-q}\,(a_2^2+s)^{q-r}\, s^r\, \sqrt{Q(s)}} \tag{21}$$

Equation (21) is an elliptical integral and can be written in terms of the Jacobian elliptic functions [7,18] as

$$\begin{aligned} J_{p-q,q-r,r} &= \frac{2}{a_1^{2p+1}} \int_0^u (sn^{2p}t)\,(nd^{2q-2r}t)\,(nc^{2r}t)\,dt \\ &= \frac{2}{a_1^{2p+1}} L_{p,q-r,r} \end{aligned} \tag{22}$$

where

$$sn^2u = \frac{a_1^2}{a_1^2+\xi_3},\quad dn^2u = \frac{a_2^2+\xi_3}{a_1^2+\xi_3},\quad cn^2u = \frac{\xi_3}{a_1^2+\xi_3} \tag{23}$$

The integral $L_{p,q-r,r}$ was integrated by parts and a recurrence relation is available in reference 18 (Eq. 357.01) and reference 7 (Eq. 30). A slightly different recurrence relationship that is useful for the crack-face displacement evaluation was obtained using integration by parts as shown in Appendix II. The recurrence relation is

$$\begin{aligned} L_{p,q-r,r} &= \frac{1}{(2r-1)\kappa'^2}\left[(sn^{2p-1}u)\,(nc^{2r-1}u)\,(nd^{2q-2r-1}u)\right. \\ &\quad \left. -(2p-1)\,L_{p-1,q-r-1,r-1} - \kappa^2\,(2q-2)\,L_{p,q-r,r-1}\right] \end{aligned} \tag{24}$$

Equations (19)-(24), provide all the information necessary to evaluate the partial derivative $f_{3,\beta}$.

Partial Derivatives With Respect To x_3

Because the crack opening displacements are sought the discussion is first focused on the derivative $F_{kl,3}$. To evaluate $F_{kl,3}$, $k_1 = k$, $l_1 = l$, and $m_1 = 1$ need to be substituted in Eq. (20).

On the $x_3 = 0$ plane and outside the ellipse ($\xi_2 = 0$), all terms in equations (23) and (24) are non-zero and the term x_3^{2r-1} in Eq. (20) is identically zero because $r \geq 1$ and $x_3 = 0$. Hence, $F_{kl,3}$, and therefore u_3 are identically equal to zero outside the ellipse, as required by the boundary condition of the problem.

On the $x_3 = 0$ plane and inside the ellipse ($\xi_3 = 0$), because $cn^2u = 0$ (see Eq. (23)), the integral $L_{p,q-r,r}$ diverges. However, on this plane, x_3^{2r-1} vanishes. The term

$$x_3^{2r-1}\,L_{p,q-r,r}$$

takes an indefinite form and therefore needs to be examined when $x_3 = \xi_3 = 0$.

The term is written as

$$x_3^{2r-1} L_{p,q-r,r} |_{x_3,\xi_3=0} = \frac{1}{(2r-1)\kappa'^2} \Big[x_3^{2r-1} (sn^{2p-1}u) (nc^{2r-1}u) (nd^{2q-2r-1}u)$$

$$- (2p-1) x_3^{2r-1} L_{p-1,q-r-1,r-1} - \kappa^2 (2q-2) x_3^{2r-1} L_{p,q-r,r-1} \Big]_{x_3=\xi_3=0}$$

$$= \frac{1}{(2r-1)\kappa'^2} \Big[\left\{ \frac{x_3}{cn\, u} \right\}^{2r-1} (sn^{2p-1}u) (nd^{2q-2r-1}u)$$

$$- (2p-1) x_3^{2r-1} L_{p-1,q-r-1,r-1} - \kappa^2 (2q-2) x_3^{2r-1} L_{p,q-r,r-1} \Big]_{x_3=\xi_3=0} \tag{25}$$

As will be obvious later, the last two terms in the square brackets of Eq. (25) are zero when $x_3 = \xi_3 = 0$. Now consider the term in braces in Eq. (25). With the help of Eq. (23) one obtains

$$\begin{aligned} \left[\frac{x_3}{cn\, u} \right]^{2r-1}_{x_3=\xi_3=0} &= \left[\frac{x_3}{\sqrt{\xi_3}} \sqrt{a_1^2 + \xi_3} \right]^{2r-1}_{x_3=\xi_3=0} \\ &= a_1^{2r-1} \left[\frac{x_3}{\sqrt{\xi_3}} \right]^{2r-1}_{x_3=\xi_3=0} \end{aligned} \tag{26}$$

Noting that $\omega(\xi_3) = 0$ and utilizing equation (4), the term in the square brackets in Eq. (26) can be written as

$$\left[\frac{x_3}{cn\, u} \right]^{2r-1}_{x_3=\xi_3=0} = a_1^{2r-1} \left[1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} \right]^{\frac{2r-1}{2}} \tag{27}$$

Substituting Eq. (27) in Eq. (25) and using Eq. (23) gives

$$x_3^{2r-1} L_{p,q-r,r} |_{x_3=\xi_3=0} = \frac{1}{(2r-1)} \frac{1}{\kappa'^{2q}} \left[a_2 \left\{ 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} \right\}^{\frac{1}{2}} \right]^{2r-1} \tag{28}$$

Now consider the term $x_3^{2r-1} L_{p-1,q-r-1,r-1}$ in Eq. (25). As before, considering the product of x_3 and $nc\ u$ terms, one has

$$x_3^{2r-1}\, nc^{2(r-1)-1} u = x_3^2 \left[\frac{x_3}{cn\, u} \right]^{2r-3} \tag{29}$$

The term in Eq. (29) is identically zero when $x_3 = \xi_3 = 0$. A similar argument holds for the term $x_3^{2r-1} L_{p,q-r,r-1}$.

The partial derivative $F_{kl,3}$ when $x_3 = \xi_3 = 0$ can now be written as

$$F_{kl,3}\,|_{x_3=\xi_3=0} = (k+l+1)! \sum_{p=0}^{k+l+1} \sum_{q=0}^{p} \sum_{r=1}^{q} \frac{(-1)^p}{(k+l+1-p)!}\, t_1\, t_2\, t_3\, t_4\, t_5$$

$$\frac{1}{(2r-1)!} \cdot \frac{2}{a_1^{2p+1}} \cdot \frac{1}{(2r-1)} \cdot \frac{1}{\kappa'^{2q}} \cdot \left[a_2 \left\{1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right\}^{\frac{1}{2}}\right]^{2r-1} \tag{30}$$

Non-dimensionalizing Eq. (30) one obtains

$$F_{kl,3}\,|_{x_3=\xi_3=0} = \frac{2}{a_1^{k+1}\, a_2^{l+1}} (k+l+1)! \sum_{p=0}^{k+l+1} \sum_{q=0}^{p} \sum_{r=0}^{q} \frac{(-1)^p}{(k+l+1-p)!}\, t_1\, t_2\, t_3$$

$$(\frac{x_1}{a_1})^{2p-2q-k} \cdot (\frac{x_2}{a_2})^{2q-2r-l} \cdot \frac{1}{(2p-2q-k)!\,(2q-2r-l)!\,(2r-1)!} \cdot$$

$$\frac{1}{(2r-1)} \cdot \left\{1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right\}^{\frac{2r-1}{2}} \tag{31}$$

Note that in the above equation the valid terms in the triple summation are those whose factorial arguments are either zero or positive integers. The opening displacements u_3 are obtained as

$$u_3(x_1, x_2, 0) = -2\,(1-\nu) \sum_k \sum_l C_{3,k,l}\, F_{kl,3}|_{x_3=\xi_3=0} \tag{32}$$

The unknown constants $C_{3,k,l}$ are obtained in terms of the prescribed tractions on the crack faces by Eq.(42a) of reference 7.

The non-dimensionalized opening displacements at the center of the crack ($x_1 = x_2 = 0$) can be obtained from Eq.(32), where

$$F_{kl,3}\,|_{x_1=x_2=x_3=0} = \frac{2}{a_1^{k+1}\, a_2^{l+1}} (k+l+1)!\, \frac{(2k_2)!}{k_2!}\, \frac{(2l_2)!}{l_2!}$$

$$\sum_{r=1}^{k_2+l_2+1} \frac{(-1)^{k_2+l_2+r}}{(k_2+l_2+1-r)!} \cdot \frac{1}{r!} \cdot \frac{2r}{2r-1} \tag{33}$$

for even values of k and l , with $2k_2 = k, 2l_2 = l$, and

$$F_{kl,3}\,|_{x_1=x_2=x_3=0} = 0 \tag{34}$$

for odd values of k and l. Note that for odd values of k and l the antisymmetry of deformation requires that u_3 be equal to zero along x_1 and x_2 axes, respectively. Equations (33) and (34) were evaluated for various values of k and l and $F_{kl,3}$ is presented in Table 1.

Note that Eq. (30) through (34) are equally valid for the penny-shaped crack ($a_1 = a_2 = a$) because these equations do not contain any indefinite forms.

Parial Derivatives With Respect To x_1 and x_2

The displacements u_1 and u_2 (see Eq. (12)) require the evaluation of $f_{3,1}$ and $f_{3,2}$, and, hence, $F_{kl,1}$ and $F_{kl,2}$ with $x_3 = \xi_3 = 0$.

The partial derivative $F_{kl,1}$ and $F_{kl,2}$ are obtained by substituting

$$k_1 \;=\; k+1\;,\;\; l_1 \;=\; l\;,\;\; m_1 \;=0$$

$$k_1 \;=\; k\;,\;\; l_1 \;=\; l+1\;,\;\; m_1 \;=0 \tag{35}$$

in Eq. (20). This substitution yields terms like

$${x_3}^{2r}\; L_{p,q-r,r} \tag{36}$$

Evaluation of these terms can be performed in two parts, when $r = 0$ and when $r \geq 1$. First consider the later case. When $r \geq 1$, the term in Eq. (36) yields

$$x_3^{2r}\; nc^{2r-1}\, u \;=\; x_3\left(\frac{x_3}{cn\, u}\right)^{2r-1} \tag{37}$$

Because $(x_3/cn\, u)$ at x_3, $\xi_3 = 0$ is finite (see Eq. (27)), the term in Eq. (37) vanishes when $x_3 = 0$.

Therefore contributions to the displacements do not occur when $r \geq 1$. Now consider the term in Eq. (36) when $r = 0$.

$$\left[x_3^{2r}\; L_{p,q-r,r}\;\right]_{r=0} \;=\; L_{p,q,0} \;=\; \frac{2}{a_1^{2p+1}} \int_0^u sn^{2p}\, t\; nd^{2q}\, t\; dt \tag{38}$$

Instead of using the recurrence relation of Eq. (24), it is convenient to use (see [18] and Appendix II)

$$\int_0^u sn^{2p}\, t\; nd^{2q}\, t\; dt \;=\; \frac{1}{\kappa^{2p}} \sum_{j=0}^{p} \frac{(-1)^j\; p!}{(p-j)!j!}\; I_{2(q-j)} \tag{39}$$

where

$$I_{2m} \;=\; \int_0^u nd^{2m}\, t\; dt \tag{39a}$$

for $2m \geq 0$ and

$$I_{-2m} \;=\; G_{2m} \;=\; \int_0^u dn^{2m}\, t\; dt \tag{39b}$$

for $2m < 0$. The integrals in Eq. (39a) and (39b) can be evaluated recursively by using

$$I_{2m+2} \;=\; \frac{2m(2-\kappa^2)\, I_{2m} \;+\; (1-2m)\, I_{2m-2} \;-\; \kappa^2\; sn\, u\; cn\, u\; nd^{2m+1}\, u}{(2m+1)\,\kappa'^2} \tag{40a}$$

for $2m \geq 0$ and

$$G_{2m+2} \;=\; \frac{2m(2-\kappa^2)\, G_{2m} \;+\; (1-2m)\kappa'^2\, G_{2m-2} \;+\; \kappa^2\; sn\, u\; cn\, u\; dn^{2m-1}\, u}{(2m+1)} \tag{40b}$$

for $2m < 0$. To evaluate the integrals in Eq. (39) and (40) one needs the starting values I_0, G_0, I_{-2}, and G_2. These are obtained by setting $m = 0$ and $m = 1$ in Eqs. (39a) and (39b). This yields

$$I_0 \;=\; G_0 \;=\; u, \;\; I_{-2} \;=\; G_2 \;=\; E(u) \tag{41}$$

As the displacements on the crack faces are sought, Eqs. (39) and (40) need be evaluated when $x_3 = \xi_3 = 0$. Thus, instead of incomplete elliptic integrals, one has complete elliptic integrals (with $u = K(\kappa)$) and therefore Eqs. (40a) and (40b) reduce to

$$I_{2m+2} \;=\; \frac{2m(2-\kappa^2)\, I_{2m} \;+\; (1-2m)\, I_{2m-2}}{(2m+1)\,\kappa'^2} \tag{42a}$$

for $2m \geq 0$,

$$G_{2m+2} \;=\; \frac{2m(2-\kappa^2)\, G_{2m} \;+\; (1-2m)\kappa'^2\, G_{2m-2}}{(2m+1)} \tag{42b}$$

for $2m < 0$, and with

$$I_0 \;=\; G_0 \;=\; K(\kappa), \;\; I_{-2} \;= G_2 \;=\; E(\kappa) \tag{42c}$$

where $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind, respectively.

Therefore, the displacements u_1 and u_2 at an arbitrary point on the crack faces are given by

$$u_\gamma(x_1, x_2, 0) = (1-2\nu) \sum_k \sum_l C_{3,k,l}\, F_{kl,\gamma}|_{x_3=\xi_3=0} \tag{43}$$

where

$$F_{kl,\gamma}|_{x_3=\xi_3=0} = (k+l+1)! \sum_{p=0}^{k+l+1} \sum_{q=0}^{p} \frac{(-1)^p}{(k+l+1-p)!} \cdot \frac{(2p-2q)!}{(p-q)!} \cdot \frac{(2q)!}{q!} \cdot$$

$$\frac{x_1^{2p-2q-k_1}}{(2p-2q-k_1)!} \cdot \frac{x_2^{2q-l_1}}{(2q-l_1)!} \cdot \frac{2}{a_1^{2p+1}} \cdot \frac{1}{\kappa^{2p}} \cdot \sum_{j=0}^{p} \frac{(-1)^j\, p!}{(p-j)!\, j!}\, I_{2(q-j)} \tag{44}$$

where $\gamma = 1, 2$. Thus Eqs. (43) and (44) completely define the crack-face displacements u_1 and u_2.

This completes the analysis of crack-face displacements for mode I type of deformations.

Mixed-Mode Crack Face Displacements

For mixed-mode deformations, instead of a single potential function f_3 of Eq. (6) for the mode I case, there are three potential functions $f_\alpha, (\alpha = 1, 2, 3)$

$$f_\alpha = \sum_k \sum_l C_{\alpha,k,l}\, F_{kl} \tag{45}$$

where F_{kl} is the function defined in Eq. (7).

The displacements u_α on the $x_3 = 0$ plane are given by [7,8,11-17]

$$\begin{aligned} u_1(x_1, x_2, 0) &= -2(1-\nu)\, f_{1,3} + (1-2\nu)\, f_{3,1} \\ u_2(x_1, x_2, 0) &= -2(1-\nu)\, f_{2,3} + (1-2\nu)\, f_{3,2} \\ u_3(x_1, x_2, 0) &= -(1-2\nu)\,(f_{1,1} + f_{2,2}) - 2(1-\nu)\, f_{3,3} \end{aligned} \tag{46}$$

The first terms on the right hand side of Eq. (46) represent the displacements due to the mode II and mode III loading while the second terms represent the displacements due to mode I loading.

As in the mode I case, to evaluate the displacements one needs the partial derivatives $f_{\alpha,\beta}$ (and hence $F_{kl,\beta}$) on the $x_3 = 0$ plane and inside the ellipse ($\xi_3 = 0$). However, all the partial derivatives $F_{kl,\beta}$ are already obtained and are given by Eqs. (30) and (44). Thus for a general mixed-mode problem of an embedded elliptic crack in a infinite solid subjected to arbitrary tractions of the form

$$\sigma_{3\alpha}(x_1, x_2, 0) = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} A^{i,j}_{\alpha,m-n,n}\, x_1^{2m-2n+i}\, x_2^{2n+j} \tag{47}$$

where $\alpha = 1, 2, 3$, the displacements on the crack surfaces are given by Eq. (46). The relationships between the magnitudes of $A^{i,j}_{\alpha,m-n,n}$ and the constants $C_{\alpha,k,l}$ are already available from Eq.(42b) of reference 7 for mode II and mode III, and Eq.(42a) of reference 7 for mode I deformations.

Special Cases

Elliptic Crack

Consider an elliptic crack subjected to a uniform pressure of magnitude S on the crack faces. For this case $k = l = 0$ and $A^{0,0}_{3,0,0} = S$. Therefore, the opening displacements are computed using Eq. (32) as

$$\begin{aligned} u_3(x_1, x_2, 0) &= -2\,(1-\nu)\, C_{3,0,0}\, F_{00,3} \\ &= -2\,(1-\nu)\, C_{3,0,0} \left[\frac{-4}{a_1\, a_2} \left\{ 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} \right\}^{\frac{1}{2}} \right] \end{aligned} \tag{48}$$

The relationship between $A^{0,0}_{3,0,0}$ and $C_{3,0,0}$ is [12,15]

$$\frac{4E(\kappa)}{a_1\, a_2^2} C_{3,0,0} = \frac{1}{2\mu} A^{0,0}_{3,0,0} \tag{49}$$

or

$$C_{3,0,0} = \frac{a_1\, a_2^2}{8\,\mu\, E(\kappa)}\, S \tag{50}$$

Therefore, the opening displacements for an elliptic crack subjected to a uniform pressure S are obtained by substituting Eq. (50) into Eq. (48) and are

$$u_3(x_1, x_2, 0) = \frac{(1-\nu)}{\mu} \frac{S a_2}{E(\kappa)} \left[1 - (x_1^2/a_1^2) - (x_2^2/a_2^2) \right]^{\frac{1}{2}} \tag{51}$$

which agrees identically with those due to Green and Sneddon [12]. The maximum opening displacement is

$$u_3(0,0,0) \;=\; \frac{(1-\nu)}{\mu}\frac{Sa_2}{E(\kappa)} \tag{52}$$

Note that the total opening displacements between the crack faces is the COD and is equal to twice the value of the opening displacement $u_3(x_1, x_2, 0)$.

Now consider the displacements u_1 and u_2. The displacements are

$$u_\gamma \;=\; (1-2\nu)\, C_{3,0,0}\, F_{00,\gamma} \tag{53}$$

with $\gamma = 1, 2$. Using Eq. (44) to evaluate $F_{00,\gamma}$ one obtains

$$F_{00,1} \;=\; \frac{-4x_1}{a_1^3\,\kappa^2}\left[\, K(\kappa) \,-\, E(\kappa)\,\right] \tag{54}$$

and

$$F_{00,2} \;=\; \frac{-4x_2}{a_1^3\,\kappa^2}\left[\, \frac{E(\kappa)}{\kappa'^2} \,-\, K(\kappa)\,\right] \tag{55}$$

Equations (54) and (55) agree with those obtained by Kassir and Sih [13]. Thus the displacements u_1 and u_2 at any point on the crack face are

$$\begin{aligned} u_1(x_1,x_2,0) &= -(1-2\nu)\,\frac{Sx_1}{2\mu}\,\frac{a_2^2}{a_1^2 - a_2^2}\left[\frac{K(\kappa)}{E(\kappa)} - 1\right] \\ u_2(x_1,x_2,0) &= -(1-2\nu)\,\frac{Sx_2}{2\mu}\,\frac{a_2^2}{a_1^2 - a_2^2}\left[\frac{1}{\kappa'^2} - \frac{K(\kappa)}{E(\kappa)}\right] \end{aligned} \tag{56}$$

Because of symmetry about the x_1 and x_2 axes, $u_2(x_1,0,0) \equiv u_1(0,x_2,0) \equiv 0$. Eqs. (56) identically satisfy this requirement. The maximum u_1 and u_2 displacements occur at the end of major and minor axes, respectively, and are given by Eq. (56) with $x_1 = a_1$ and $x_2 = a_2$.

Penny-shaped Crack

For a penny-shaped crack, $a_1 = a_2 = a$, $E(\kappa) = \pi/2$, and Eq. (51) reduces to

$$u_3(x_1,x_2,0) \;=\; \frac{(1-\nu)}{\mu}\,\frac{2Sa}{\pi}\left[1 - (x_1^2/a^2) - (x_2^2/a^2)\right]^{\frac{1}{2}} \tag{57}$$

and the maximum opening displacement is

$$u_3(0,0,0) = \frac{(1-\nu)}{\mu}\frac{2Sa}{\pi} \tag{58}$$

Similarly, the displacements $u_1(x_1, x_2, 0)$ and $u_2(x_1, x_2, 0)$ can be obtained from Eq. (56), using the limit as $\kappa \to 0$ (see Appendix II) as

$$\begin{aligned} u_1(x_1, x_2, 0) &= -(1-2\nu)\frac{Sx_1}{4\mu} \\ u_2(x_1, x_2, 0) &= -(1-2\nu)\frac{Sx_2}{4\mu} \end{aligned} \tag{59}$$

Use of the Crack-face Displacement Equations

As previously mentioned, the crack-face displacement equations, Eqs. (30) and (44), can be directly used in the FEAM to calculate crack-face displacements for various three-dimensional crack configurations. To illustrate the use of these equations consider plate containing a semi-elliptical surface crack with $(a_1/a_2) = 0.6$. The plate is subjected to remote tensile loading as shown in Fig. 2. The crack configuration was analyzed by the FEAM and the three-dimensional finite element method with singularity elements using the models and methods described in referece 19. In the FEAM, the maximum degree of the polynomial (M) was equal to 5. The maximum crack opening displacement at $(x_1 = x_2 = x_3 = 0)$ obtained by both methods is compared in Table 2 for three crack sizes (defined by crack depth-to-plate thickness ratio) of 0.2, 0.5, and 0.8. The results from the two methods agreed well. For a deep crack, the differences between the two displacements is only about 3 percent.

Concluding Remarks

The evaluation of the crack-face displacements of an embedded elliptic crack in an infinite solid subjected to arbitrary tractions is addressed. The arbitrary tractions are assumed to be applied to the crack faces and are assumed to be expressed in a polynomial form in terms of the Cartesian coordinates that describe the ellipse.

The exact solution obtained by Vijayakumar and Atluri, and Nishioka and Atluri was extended to obtain closed form expressions for the crack-face displacements. The crack-face displacements for special cases of uniform normal tractions agree identically with those obtained by Green and Sneddon, and Kassir and Sih.

The evaluation of the crack-face displacements for embedded, surface, and corner cracked solids in finite bodies require the use of numerical methods such as finite element, finite-element alternating and boundary-element alternating methods. The displacement expressions obtained in this paper were used with the finite-element alternating method (FEAM). An example of a semi-elliptic surface crack in a finite solid subjected to remote tensile loading was analyzed with the finite-element alternating method and finite-element method with singularity elements. The maximum crack opening dispalcements were calculated with both methods. In the FEAM, the crack-face displacements equations derived in this paper were used. The maximum opening displacements from both methods agreed well for both shallow and deep cracks.

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Appendix I

This appendix shows that the second partial derivatives, $F_{kl,3\gamma}$, do not have singularities on the $x_3 = 0$ plane, inside the elliptic crack ($\xi_3 = 0$).

The second partial derivatives $F_{kl,3\gamma}$ are [7,8,11,15]

$$F_{kl,3\gamma} \;=\; \int_{\xi_3}^{\infty} \frac{\partial^{k_1+l_1+m_1}}{\partial x_1^{k_1}\,\partial x_2^{l_1}\,\partial x_3^{m_1}}\,\omega^{k+l+1}\,\frac{ds}{\sqrt{Q(s)}} \;+\; F^0_{kl,3\gamma} \tag{I.1}$$

where

$$F^0_{kl,3\gamma} \;=\; (k+l+1)!\,\frac{x_1^{k_1}\,x_2^{l_1}\,x_3^{m_1}}{(\xi_3-\xi_1)\,(\xi_3-\xi_2)}\left[(\frac{-2}{a_1^2+\xi_3})^{k_1}\,(\frac{-2}{a_2^2+\xi_3})^{l_1}\,(\frac{-2}{\xi_3})^{m_1}\,\sqrt{Q(\xi_3)}\right] \tag{I.2}$$

with

$$k_1 \;=\; k\,+\,\delta_{1\gamma};\;\; l_1 \;=\; l\,+\,\delta_{2\gamma};\;\; m_1 \;=\; 1\,+\,\delta_{3\gamma} \tag{I.3}$$

and $\gamma = 1, 2, 3$. These derivatives are to be evaluated at $x_3 = \xi_3 = 0$.

The first term on the right hand side of Eq. (I.1) is given by Eq. (20) with k_1, l_1, and m_1 defined in Eq. (I.3). Using Eqs. (25) through (29), this term can be easily shown to be finite at $x_3 = \xi_3 = 0$.

Now consider the second term in Eq. (I.1). At $x_3 = \xi_3 = 0$, this term can be written as

$$F^0_{kl,3\gamma} \;=\; (k+l+1)!\,\frac{x_1^{k_1}\,x_2^{l_1}\,x_3^{m_1}}{\xi_1\,\xi_2}(\frac{-2}{a_1^2})^{k_1}\,(\frac{-2}{a_2^2})^{l_1}\,(\frac{-2}{\xi_3})^{m_1}\,\sqrt{\xi_3\,a_1^2\,a_2^2}\;\;\Big|_{x_3,\xi_3=0} \tag{I.4}$$

Using Eq. (3) and rearranging, one obtains

$$F^0_{kl,3\gamma} \;=\; (k+l+1)!\,\frac{x_1^{k_1}\,x_2^{l_1}}{a_1\,a_2}(\frac{-2}{a_1^2})^{k_1}\,(\frac{-2}{a_2^2})^{l_1}\,(-2)^{m_1}\,\frac{\sqrt{\xi_3}}{x_3}\,(\frac{x_3}{\xi_3})^{m_1-1}\,\Big|_{x_3,\xi_3=0} \tag{I.5}$$

Using Eq. (4), Eq. (I.5) can be written as

$$F^0_{kl,3\gamma} \;=\; (k+l+1)!\,\frac{x_1^{k_1}\,x_2^{l_1}}{a_1\,a_2}(\frac{-2}{a_1^2})^{k_1}\,(\frac{-2}{a_2^2})^{l_1}\,(-2)^{m_1}\,\cdot$$

$$\frac{1}{[1\,-\,(x_1/a_1)^2\,-\,(x_2/a_2)^2]^{\frac{1}{2}}}(\frac{x_3}{\xi_3})^{m_1-1}\;\;\Big|_{x_3,\xi_3=0} \tag{I.6}$$

For the terms $F^0_{kl,31}$ and $F^0_{kl,32}$, $m_1 = 1$ and hence $(x_3/\xi_3)^{m_1-1} = 1$. Thus, no singularity exists in the right side of Eq. (I.6) and the term is finite.

For the term $F^0_{kl,33}$, $m_1 = 2$ and therefore Eq. (I.5) can be written as

$$F^0_{kl,33} = (k+l+1)! \frac{x_1^k x_2^l}{a_1 a_2} (\frac{-2}{a_1^2})^k (\frac{-2}{a_2^2})^l \frac{4}{\sqrt{\xi_3}} |_{x_3,\xi_3=0} \qquad (I.7)$$

This term has a singularity at $\xi_3 = 0$. However, as shown in reference 15, when Eq. (I.7) is combined with the first term in Eq. (I.1), the terms involving $1/\sqrt{\xi_3}$ cancel each other. Alternatively, because F_{kl} is harmonic,

$$F_{kl,33} = -F_{kl,11} - F_{kl,22} \qquad (I.8)$$

The derivatives $F_{kl,11}$ and $F_{kl,22}$ can be easily shown to be finite at $x_3 = \xi_3 = 0$. Thus the derivative $F_{kl,33}$ also remains finite at $x_3, \xi_3 = 0$.

Appendix II

The appendix derives the recurrence relationship of Eq. (23) and the integral in Eq. (39). This appendix also presents the identities of the Jacobian elliptic functions used elsewhere in the paper.

Recurrence Relation Of Eq. (23)

The integral $L_{p,q-r,r}$ is

$$L_{p,q-r,r} = \int_0^u (sn^{2p}\, t)\,(nd^{2q-2r}\, t)\,(nc^{2r}\, t)\; dt \tag{II.1}$$

Integration by parts was used to obtain the recurrence relationship as

$$\begin{aligned}
L_{p,q-r,r} &= \int_0^u (sn^{2p-1}\, t)\,(nd^{2q-2}\, t)\,(sn\, t\;\; nc^{2r}\, t\;\, nd^{2-2r}\, t)\; dt \\
&= \int_0^u (sn^{2p-1}\, t)\,(nd^{2q-2}\, t)\; d\left[\frac{1}{\kappa'^2\,(2r-1)}\,(\frac{dn\, t}{cn\, t})^{2r-1}\right] \\
&= \frac{1}{\kappa'^2\,(2r-1)}\Bigg[\Big\{(sn^{2p-1}\, t)\,(nc^{2r-1}\, t)\,(nd^{2q-2r-1}\, t)\;\Big\}_0^u \\
&\quad - (2p-1)\int_0^u (sn^{2p-2}\, t)\,(nd^{2q-2r-2}\, t)\,(nc^{2r-2}\, t)\; dt \\
&\quad - \kappa^2\,(2q-2)\int_0^u (sn^{2p}\, t)\,(nd^{2q-2r}\, t)\,(nc^{2r-2}\, t)\; dt\Bigg]
\end{aligned} \tag{II.2}$$

Therefore

$$\begin{aligned}
L_{p,q-r,r} = \frac{1}{(2r-1)\,\kappa'^2}\Big[&(sn^{2p-1}\, u)\,(nc^{2r-1}\, u)\,(nd^{2q-2r-1}\, u) \\
&- (2p-1)\,L_{p-1,q-r-1,r-1} - \kappa^2\,(2q-2)\,L_{p,q-r,r-1}\Big]
\end{aligned} \tag{II.3}$$

Derivation Of Eq. (39)

Using one of the Jacobian identities the integral in Eq. (39) can be written as

$$\int_0^u sn^{2p}\, t\; nd^{2q}\, t\; dt$$

$$= \int_0^u \left(\frac{1 - dn^2\, t}{\kappa^2}\right)^p nd^{2q}\, t\; dt$$

$$= \frac{1}{\kappa^{2p}} \sum_{j=0}^{p} (-i)^j \frac{p\,(p-1)\,(p-2)\cdots(p-j+1)}{j!} \int_0^u dn^{2j}\, t\; nd^{2q}\, t\; dt$$

$$= \frac{1}{\kappa^{2p}} \sum_{j=0}^{p} \frac{(-i)^j\, p!}{(p-j)!\, j!} \int_0^u nd^{2q-2j}\, t\; dt$$

$$= \frac{1}{\kappa^{2p}} \sum_{j=0}^{p} \frac{(-i)^j\, p!}{(p-j)!\, j!}\, I_{2(q-j)}$$

In the above derivations and elsewhere in the paper the following identities of the Jacobian elliptic functions [18] were used.

$$sn^2\, u + cn^2\, u = 1 \quad ; \quad \kappa^2\, sn^2\, u + dn^2\, u = 1$$

$$dn^2\, u - \kappa^2\, cn^2\, u = \kappa'^2 \quad ; \quad \kappa'^2\, sn^2\, u + cn^2\, u = dn^2\, u$$

$$cd\, u = \frac{cn\, u}{dn\, u} \quad ; \quad cs\, u = \frac{cn\, u}{sn\, u} \quad ; \quad dc\, u = \frac{dn\, u}{cn\, u}$$

$$ds\, u = \frac{dn\, u}{sn\, u} \quad ; \quad nc\, u = \frac{1}{cn\, u} \quad ; \quad nd\, u = \frac{1}{dn\, u}$$

$$sd\, u = \frac{sn\, u}{dn\, u} \quad ; ns\, u = \frac{1}{sn\, u}$$

$$tn\, u = sc\, u = \frac{sn\, u}{cn\, u}$$

$$\kappa^2 = \frac{(a_1^2 - a_2^2)}{a_1^2} \quad ; \quad \kappa'^2 = \frac{a_2^2}{a_1^2}$$

$$d(sn\,u) = cn\,u\;dn\,u \quad ; \quad d(nd\,u) = \kappa^2\,sd\,u\;cd\,u$$

$$K(\kappa) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2\,sin^2\phi}}$$

$$E(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2\,sin^2\phi}\;d\phi$$

$$sn\,K = 1 \quad ; \quad cn\,K = 0 \quad ; \quad dn\,K = \kappa'$$

$$\lim_{\kappa \to 0} \frac{K(\kappa) - E(\kappa)}{\kappa^2} = \pi/4$$

$$\lim_{\kappa \to 0} \frac{E(\kappa) - \kappa'^2\,K(\kappa)}{\kappa^2} = \pi/4$$

Table 1 Values of $F_{kl,3}$ at the center of the elliptic crack $x_1 = x_2 = x_3 = 0$ for various values of k and l, $(0 \leq k, l \leq 5)$.

k	l	$a_1 a_2 F_{kl,3}$
0	0	-4
1	0	0
0	1	0
2	0	$32/a_1^2$
1	1	0
0	2	$32/a_2^2$
3	0	0
2	1	0
1	2	0
0	3	0
4	0	$-1536/a_1^4$
3	1	0
2	2	$-512/(a_1^2 a_2^2)$
1	3	0
0	4	$-1536/a_2^4$
5	0	0
4	1	0
3	2	0
2	3	0
1	4	0
0	5	0

Table 2 Comparison of the maximum total crack opening displacement for a surface cracked plate by FEAM and finite element method with singularity elements. $(a_1/a_2 = 0.6)$

Crack Depth-to-Plate Thickness Ratio	$\frac{\mu\, u_3(0,0,0)\, E(\kappa)}{(1-\nu)\, S\, a_2}$	
	FEAM	FEM [19]
0.2	2.677	2.699
0.5	3.132	3.076
0.8	3.954	3.839

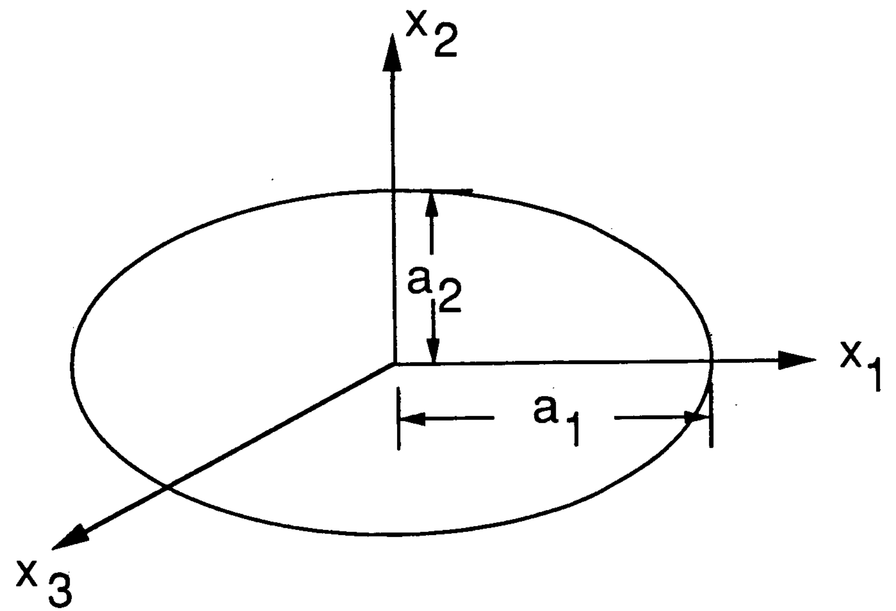

Figure 1 - Embedded elliptic crack in an infinite solid.

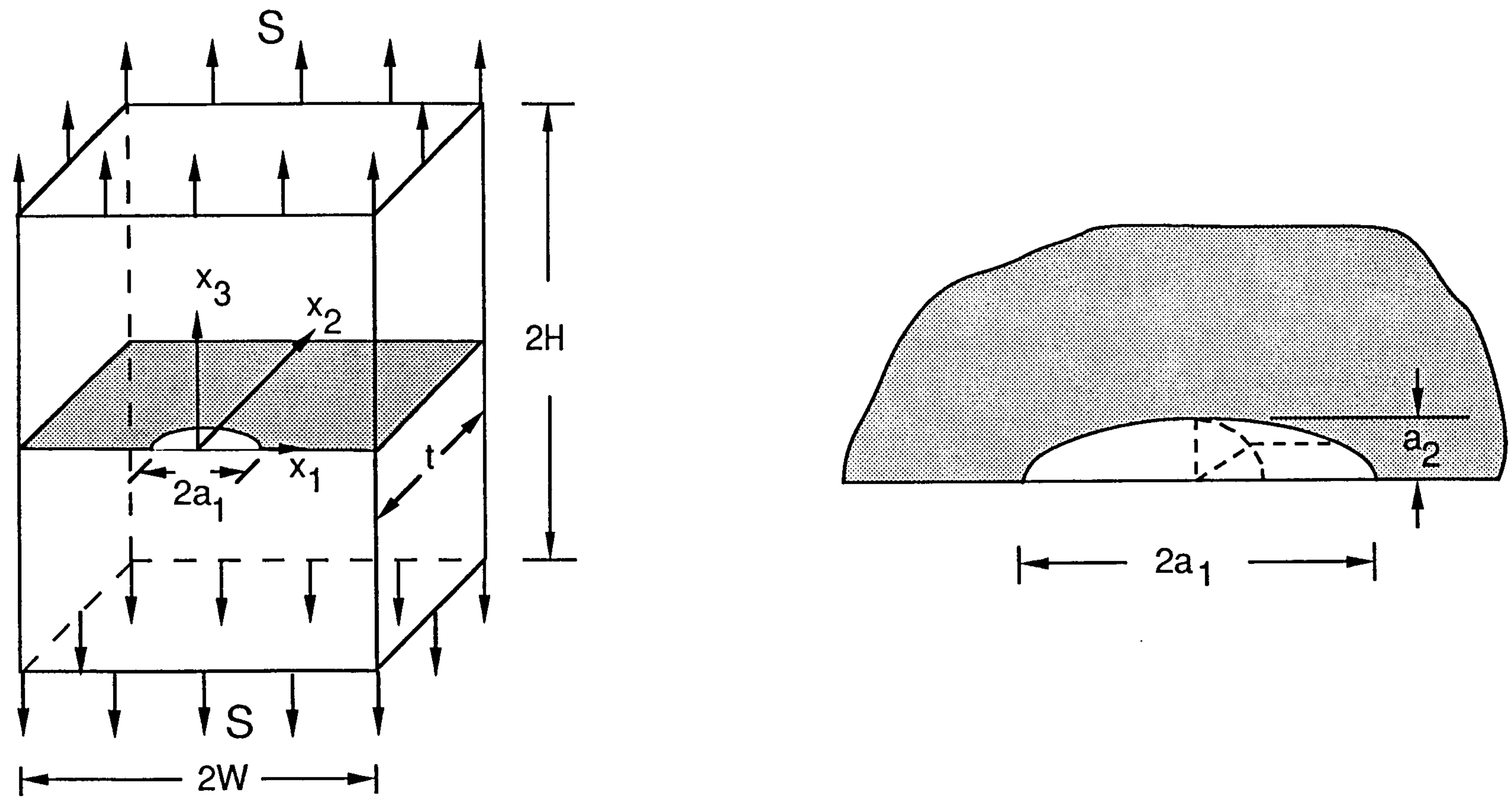

Figure 2. Surface crack in a finite plate.
($W > 30a_1$; $H/W = 4$; $0.2 < a_2/t < 0.8$)

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16. Abstract

Analytical expressions for the crack-face displacements of an embedded elliptic crack in infinite solid subjected to arbitrary tractions are obtained. The tractions on the crack faces are assumed to be expressed in a polynomial form. These displacements expressions complete the exact solution of Vijayakumar and Atluri, and Nishioka and Atluri. For the special case of an embedded crack in an infinite solid subjected to uniform pressure loading, the present displacements agree with those by Green and Sneddon. The displacement equations derived in this paper were used with the finite-element alternating method (FEAM) for the analysis of a semi-elliptic surface crack in a finite solid subjected to remote tensile loading. The maximum opening displacements obtained with FEAM are compared to those with the finite-element method with singularity elements. The maximum crack opening displacements by the two methods showed good agreement.

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