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Sampling inhomogeneous turbulent fields

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1. Background

Consider a real random process $U(y)$ whose statistical properties vary inhomogeneously in the y-direction on an interval I. The Karhunen-Loeve expansion of the process is

$$U(y) = \sum_{m=1}^{\infty} a_m \phi_m(y), \tag{1}$$

where the expansion coefficients are

$$a_m = \int_I u(y)\phi_m(y)\, dy \tag{2}$$

and the expansion functions ϕ_m are eigen-solutions of

$$\int_I R(y,y')\phi_m(y')dy' = \lambda_m \phi_m(y). \tag{3}$$

In (3), $R(y,y')$ is the two-point correlation of of the process $U(y)$. The eigenfunctions are orthogonal on the interval I, that is

$$\int_I \phi_m(y)\phi_m(y')dy = \delta_{mn}. \tag{4}$$

Also, the expansion coefficients are related to the eigenvalues (λ) by

$$< a_m a_n >= \lambda_m \delta_{mn}, \tag{5}$$

thus they are statistically orthogonal.

The KL functions provide the "best" orthonormal set of functions in the sense that the expansion converges faster than expansions in terms of any other orthonormal set, as measured in the L_2 norm. These eigenfunctions reduce to the trigonometric functions if $U(y)$ is statistically homogeneous, so the primary significance of the KL expansion lies in its ability to efficiently represent statistically inhomogeneous processes.

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Shannon's Sampling Theorem provides a method for representing a homogeneous (i.e. stationary) band-limited continuous random process in terms of its samples at a finite number of points whose spacing is determined by the Nyquist criterion. In this case, the best representation of the process is an expansion in terms of a finite sum of trigonometric functions, and the samples must be taken periodically at a rate exceeding two times the maximum frequency in the signal spectrum. The question arises as to whether a similar result pertains to inhomogeneous random process and how such a result might be related to the KL expansion.

An inhomogeneous process that is band limited in the generalized sense that it can be represented by a KL expansion with M modes with zero mean square error, can also be represented by a sum of the form

$$U(y) = \sum_{\alpha=1}^{M} A_\alpha(y) U(y_\alpha), \tag{6}$$

where $y_\alpha, (\alpha = 1, \ldots, M)$ are a set of M sample points, and the interpolating functions are solutions of

$$\sum_{\alpha=1}^{M} A_\alpha(y) \phi_m(y_\alpha) = \phi_m(y), \qquad m = 1, \ldots, M \tag{7}$$

By analogy to Shannon's sampling theorem, we are particularly interested in the zeros of ϕ (*i.e.* $\phi(y_\alpha) = 0$) as sample points.

2. Computations for turbulent channel flow

The relevance of the foregoing to numerical computation of turbulence was explored by examining the spectrum of eigenvalues and the eigenfunctions of the one-dimensional process $U(y)$ in the turbulent channel flow data base. Inspection of the eigenvalue spectrum for the case of KL expansion over the wall layer domain $0 \le y^+ \le 40$ shows that the first eigenmode contains 73% of the total energy, and the first two eigenmodes contain 92% of the total (Moin and Moser 1989). Clearly the eigenfunction spectrum is rapidly decreasing, and there is some order M beyond which it is not necessary to include additional terms. The selection of M is somewhat arbitrary, but by $M = 10$, we find that the energy of the tenth mode relative to that of the first mode is 0.6×10^{-4}, so $M = 10$ is a reasonable cutoff.

Figure 1 shows $\phi_1(y)$ and $\phi_{10}(y)$ for KL expansion on $0 \le y^+ \le 40$. It should be possible to reconstruct $U(y)$ from its samples taken at the zeros of ϕ_{10}. There are nine zeroes (the zero at the wall is not counted because $U(y) = 0$ is a boundary condition). For comparison, the Chebychev polynomial grid points used in the direct numerical simulation are indicated by the squares. There are twenty-nine of these points. It is interesting to note that the spacings of KL sampling points are nearly equal (see Figure 1*b*. It appears that for sampling in the direction of flow inhomogeneity in boundary layers one does not have to use numerous points in the vicinity of the walls. This is because the KL eigenfunctions carry with them much

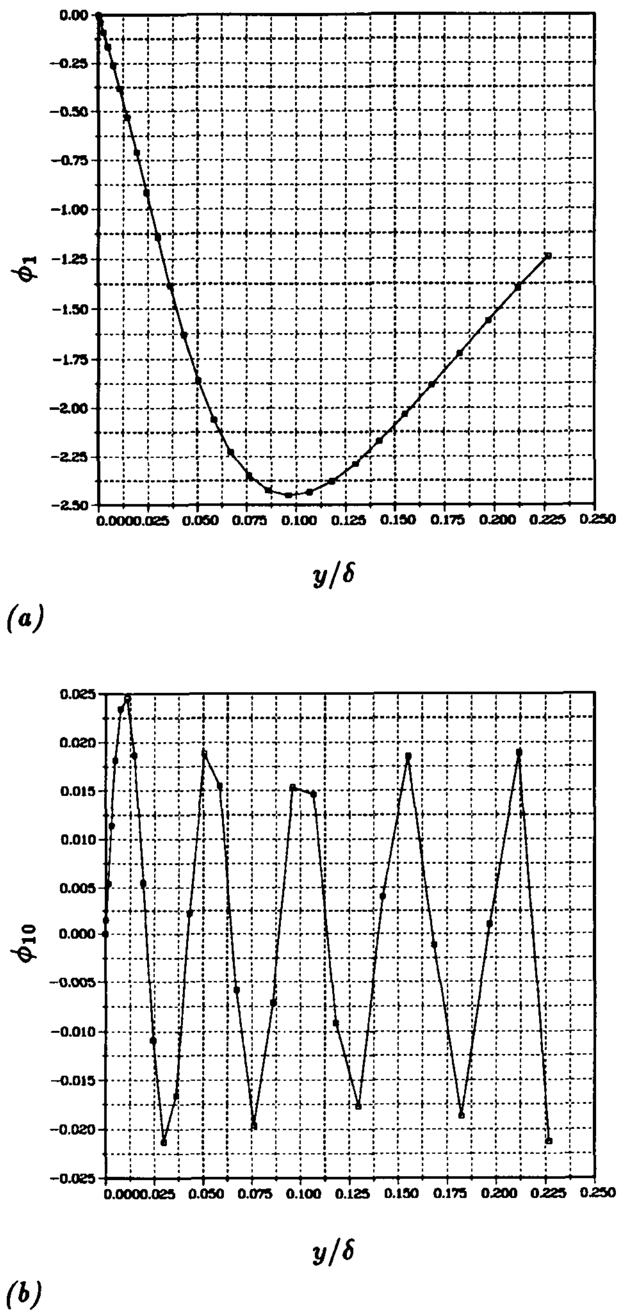

FIGURE 1. Eigenfunctions of the Karhunen-Loeve expansion for u over the interval $0 \le y^+ \le 40$. The sample points y_α are at the intersection of the solid lines and the horizontal line through zero. The open symbols are chebychev points. (a) $\phi_1(y)$, (b)$\phi_{10}(y)$.

of the rapidly varying parts of $U(y)$. Note, for example, the rapid variation of $\phi_1(y)$ close to the wall.

Computations for KL expansion on the full domain $0 \leq y^+ \leq 180$ reveals some interesting consequences of varying the domain (see Moin & Moser 1989). The eigenvalue spectrum is wider than that of the wall layer expansion. The bandwidth required to reach 0.27×10^{-4} of the energy in the first mode is $M = 30$, compared to $M = 11$ for the wall layers. However, the extent of the domain is over four times as large as the wall layer, so many of the additional modes are actually slowly varying functions needed to represent the longer fluctuations that occur in the wider layer.

The first and thirtieth eigenfunctions for the full channel width are shown in Figure 2. There are 64 Chebychev points on this interval (not counting $y = 0$) and 29 KL points

In contrast to the wall layer the spacing of sample points for expansion over the full layer increases by more than a factor of 2.5 from the wall to the centerline. This is presumably due to milder variation of the eigenfunctions in the vicinity of the walls. The spacing of the sample points may be related to the variation of turbulent length scale as one moves away from the wall.

3. Conclusions

The reconstruction of an inhomogeneous random process from a finite number of discrete samples can be performed in terms of the Karhunen- Loeve expansion for that process. The n^{th} eigenfunction has $n - 1$ zero crossings which are the sampling points for the inhomogeneous process. The rapid variation of the KL eigenfunctions makes it unnecessary to have a high density of sampling (or grid points) near the wall. However, this result should not be construed as to indicate that with spectral simulations significantly fewer grid points are required with the KL expansion as compared to other orthogonal expansions. Moin & Moser (1989) have shown that the the advantage of the KL expansion over Chebychev expansion rapidly diminishes when high percentage (say 90%) energy recovery is demanded.

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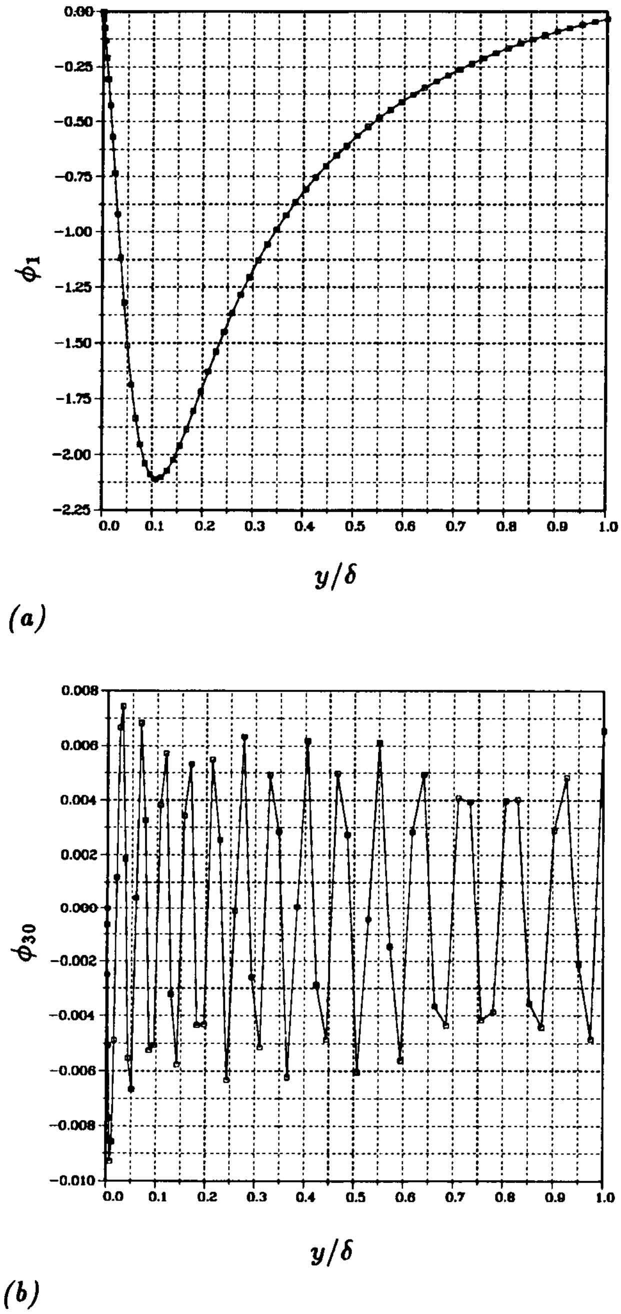

FIGURE 2. Eigenfunctions of the Karhunen-Loeve expansion for U over the interval $0 \leq y/\delta \leq 1$. The sample points y_α are at the intersetion of the solid lines and the horizontal line through zero. The open symbols are Chebychev points. (a) $\phi_1(y)$, (b)$\phi_{30}(y)$.