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ON EQUIVALENCE OF DISCRETE-DISCRETE AND CONTINUUM-DISCRETE DESIGN SENSITIVITY ANALYSIS†

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Developments in design sensitivity analysis (DSA) method have been made using two fundamentally different approaches as shown in figure 1. In the first approach, a discretized structural finite element model is used to carry out DSA. There are three different methods in the discrete DSA approach: finite difference, semi-analytical, and analytical methods. The finite difference method is a popular one due to its simplicity, but a serious shortcoming of the method is the uncertainty in the choice of a perturbation step size of design variables (ref. 1). In the semi-analytical method, the derivatives of stiffness matrix is computed by finite differences (refs. 2-4) whereas in the analytical method, the derivatives are obtained analytically. For the shape design variable, computation of analytical derivative of stiffness matrix is quite costly (ref. 1). Because of this, the semi-analytical method is a popular choice in discrete shape DSA approach (refs. 3 and 4). However, recently, Barthelemy and Haftka (ref. 5) presented that the semi-analytical method can have serious accuracy problems for shape design variables in structures modeled by beam, plate, truss, frame, and solid elements. They found that accuracy problems occur even for a simple cantilever beam. In the second approach, a continuum model of the structure is used to carry out DSA. For shape design variable, the material derivative concept of continuum mechanics is used to relate variations in structural shape to measures of structural performance (refs. 6-10). Using continuum DSA approach, expressions for shape design sensitivity are obtained in the form of integrals with integrands written in terms of natural physical quantities such as displacements, stresses, strains, and domain shape changes. If exact solutions of the continuum equilibrium equations are used to evaluate these continuum design sensitivity expressions, the method is called continuum-continuum (C-C) method. On the other hand, if the analysis results of the finite element or boundary element methods are used to evaluate these terms, the method is called continuum-discrete (C-D) method. The analytical method of discrete design sensitivity analysis approach will be called discrete-discrete (D-D) method.

METHODS OF DESIGN SENSITIVITY ANALYSIS

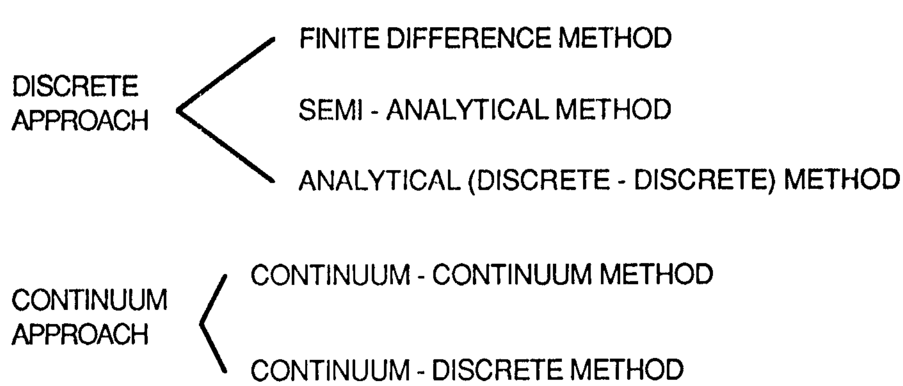


Figure 1

The D-D method starts with the finite element matrix equilibrium equation for linear structural system as shown in figure 2, where K(b) is the reduced global stiffness matrix, z is the reduced displacement vector, F(b) is the external load vector, and b is a design variable vector. Differentiating both sides of the matrix equilibrium equation with respect to b, a matrix equation for the derivative of displacement vector, dz/db, is obtained where the tilde (~) indicates a variable that is to be held constant for the process of partial differentiation. If the derivative dz/db is obtained by solving this equation, the method is called direct differentiation method. If derivatives of a general performance measure are needed, an adjoint variable method can be used (ref. 11). Even though the direct differentiation and adjoint variable methods are different in computational efficiency depending on situations, they are equivalent in accuracy as long as consistent computational procedure is used for both methods. For the D-D method, the derivative of stiffness matrix is obtained analytically, whereas it is obtained by finite differences for the semi-analytical method. The discrete DSA approach is applicable to both sizing and shape design variables. For the shape design case, the design variables are positions of the finite element grid points.

DISCRETE DSA APPROACH (DIRECT DIFFERENTIATION METHOD)

$$K(b)z = F(b)$$

$$K(b)\frac{dz}{db} = -\frac{\partial}{\partial b}(K(b)\tilde{z}) + \frac{\partial F(b)}{\partial b}$$

- **Semi-analytical** and **analytical (D-D)** methods
- For general performance measures, use the **adjoint variable method.**
- Accuracy of the **direct differentiation** and **adjoint variable methods** are equivalent.
- **Discrete approach** is applicable to both **sizing** and **shape design variables.**

Figure 2

For the continuum approach, using the principle of virtual work, the variational equilibrium equation of the structural system can be obtained (ref. 11) as shown in figure 3, where $a_\Omega(\cdot,\cdot)$ denotes energy bilinear form, $\ell_\Omega(\cdot)$ denotes load linear form, Ω is the shape of the structure, z is the displacement, $\bar{z}$ is the kinematically admissible virtual displacement, and Z is the space of kinematically admissible virtual displacements. Note that an approximate finite element matrix equilibrium equation can be obtained by applying the Galerkin method to the variational equilibrium equation for an approximate solution. For shape DSA, taking the material derivative of both sides of the variational equilibrium equation (refs. 10-12), a variational equation for the material derivative $\dot{z}$ of the displacement is obtained where V is the design velocity field. Expressions for $a'_V(z,\bar{z})$ and $\ell'_V(\bar{z})$ can be obtained for various structural components (refs. 10-12). For the C-D method, an approximate finite element matrix equation is used to obtain an approximate solution of the second variational equation for $\dot{z}$. On the other hand, for the C-C method, the analytical solution z of the first variational equation is used in the second variational equation to obtain the analytical solution $\dot{z}$. As in the D-D method, if derivatives of a general performance measure are needed, an adjoint variable method can be used (refs. 10-12). The C-C method provides the exact design sensitivity of the exact model, whereas the C-D method provides an approximate design sensitivity of the exact model. On the other hand, D-D method yields the exact design sensitivity of an approximate finite element model, and both the finite difference and semi-analytical methods yield approximate design sensitivities of an approximate finite element model.

CONTINUUM SHAPE DSA APPROACH (DIRECT DIFFERENTIATION METHOD)

$$a_\Omega(z,\bar{z}) = \ell_\Omega(\bar{z}), \quad \text{for all } \bar{z} \in Z$$

$$a_\Omega(\dot{z},\bar{z}) = \ell'_V(\bar{z}) - a'_V(z,\bar{z}), \quad \text{for all } \bar{z} \in Z$$

- **FEM equation** is an approximate equation of the **variational equation.**
- Use **material derivative** concept of the continuum mechanics for **shape** DSA.
- For general performance measures, use the **adjoint variable** method of DSA.
- **C-C** and **C-D** methods

Figure 3

One question often asked is; "Are the D-D and C-D methods equivalent?" For this question, certain conditions have to be given. First, the same discretization (shape function) used for the finite element analysis method must be used to evaluate the continuum design sensitivity results. Second, exact integrations (instead of numerical integrations) must be carried out for all integrations used for generation of stiffness matrix and evaluation of continuum design sensitivity expressions. The third condition to be met is that the exact solutions (not a numerical solution) of the finite element matrix equation and adjoint equation are used to compare two methods. The fourth condition is that movement of the finite element grid points for shape design change in the D-D method must be consistent with the parameterization method used for the design velocity field of the C-D method. For the sizing design variable, it is shown in reference 11 that the D-D and C-D methods are equivalent under the conditions given in figure 4 using a beam structural component. It has also been argued that the D-D and C-D methods are equivalent for shape design variable under the conditions given in figure 4 (refs. 13 and 14). One point to note is that these four conditions are not easy to satisfy; in many cases, numerical integrations are used and exact solutions of the finite element matrix equations cannot be obtained. In this paper, equivalence study of D-D and C-D methods for shape design variables is carried out under the conditions given in figure 4. To carry out equivalence study of the D-D and C-D method, two simple structural components, a truss and a cantilever beam, are used. The shape DSA results of the D-D and C-D methods derived in the published literature are cited and used here without being derived in this paper.

ARE THE D-D AND C-D METHODS EQUIVALENT?

Equivalence study under the following **conditions:**

1 The **same shape function** used for **FEA** must be used to evaluate the **continuum DSA results.**

2 **Exact integrations** must be used to generate the **stiffness matrix** and evaluate the **continuum DSA results.**

3 **Exact solutions** of the finite element and adjoint **matrix equations** are used to compare two DSA methods.

4 **Movement** of FE **grid points** for the **D-D** method must be **consistent** with the parameterization of the **design velocity field** for the **C-D** method.

Figure 4

In figure 5, the results of equivalence study of the D-D and C-D methods for shape design sensitivity are presented using a simple truss with one end fixed. The truss has a uniform cross-sectional area A and its length is ℓ which is the shape design variable. Three loading cases, a point load p at the tip, a uniformly distributed load f, and a linearly varying load qx/ℓ, are considered as shown in figure 5. For each loading case, linear and quadratic shape functions are used for finite element models. For the linear shape function, two element model is used whereas for the quadratic shape function, one element model is used. For the equivalence study, design sensitivities of the nodal displacements are considered using the adjoint variable method. In figure 5, 'same' denotes that the D-D and C-D methods yield the same result and 'not' denotes that the two methods do not yield the same result. Details of the equivalence study results are given in the following figures.

RESULTS OF EQUIVALENCE STUDY OF D-D AND C-D METHODS FOR TRUSS

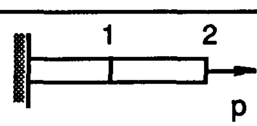
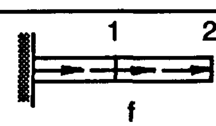
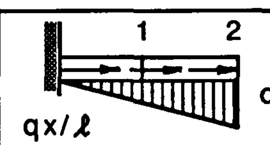
Loading Condition / Design Velocity Field \ Shape Function						
	Linear	Quad	Linear	Quad	Linear	Quad
Linear	Same	Same	Same	Same	Same	Same
Quadratic	Same	Same	Not	Same	Not	Not
Cubic	Same	Same	Not	Not	Not	Not

Figure 5

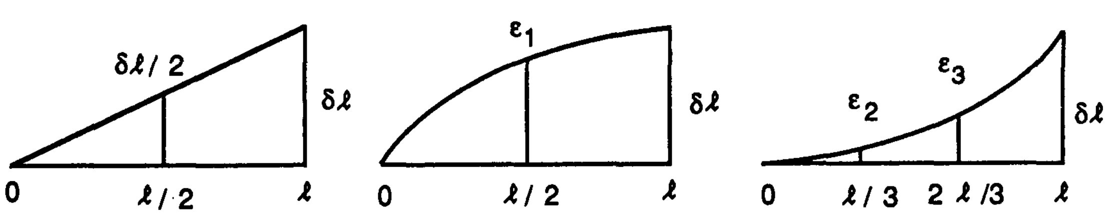
For the design velocity V(x) to be used in the C-D method, three parameterization methods; linear, quadratic, and cubic polynomials are used as shown in figure 6, where

$$\alpha_1 = \frac{4\varepsilon_1 - \delta\ell}{\ell}, \quad \alpha_2 = \frac{2\delta\ell - 4\varepsilon_1}{\ell^2}, \quad \beta_1 = \frac{18\varepsilon_2 - 9\varepsilon_3 + 2\delta\ell}{2\ell}$$

$$\beta_2 = \frac{-45\varepsilon_2 + 36\varepsilon_3 - 9\delta\ell}{2\ell^2}, \quad \beta_3 = \frac{27\varepsilon_2 - 27\varepsilon_3 + 9\delta\ell}{2\ell^3}$$

Note that for all three parameterizations of design velocity, the perturbation of length of the truss is $\delta\ell$ at the tip. Moreover, for the quadratic and cubic design velocities, once ε_i, i=1,2,3, are fixed, then the only one shape design variable is the length ℓ. The movement of the finite element grid points for shape design changes in the D-D method must be consistent with these parameterization methods. For the D-D method, the shape design variables are the positions b_1 and b_2 of the nodal points. If the present design is $b_1=\ell/2$ and $b_2=\ell$, then $V(\ell/2)=\delta b_1=\delta\ell/2$ and $V(\ell)=\delta b_2=\delta\ell$ for the linear velocity, $V(\ell/2)=\delta b_1=\varepsilon_1$ and $V(\ell)=\delta b_2=\delta\ell$ for the quadratic velocity, and $V(\ell/2)=\delta b_1=(9\varepsilon_2+9\varepsilon_3-\delta\ell)/16$ and $V(\ell)=\delta b_2=\delta\ell$ for the cubic velocity.

PARAMETERIZATIONS OF THE DESIGN VELOCITY V(x)



Linear $\quad V_1(x) = x\delta\ell/\ell$

Quadratic $\quad V_2(x) = \alpha_2 x^2 + \alpha_1 x$

Cubic $\quad V_3(x) = \beta_3 x^3 + \beta_2 x^2 + \beta_1 x$

Figure 6

The first case of equivalence study is the truss with the point load p at the tip. For this, the finite element matrix equation, using linear shape function, is given in figure 7 where the stiffness matrix depends on the shape design variables b_i, i=1,2. The finite element matrix equation gives the solutions $z_1 \equiv z(\ell/2)=p\ell/2EA$ and $z_2 \equiv z(\ell)=p\ell/EA$ at the present design $b_1=\ell/2$ and $b_2=\ell$. Thus $z(x)=px/EA$ which is the exact solution of the truss with the point load p at the tip. If the design sensitivities of displacements at two nodal points, z_1 and z_2, are desired, the adjoint equations are given in figure 7 with the adjoint solutions $\lambda^1(x)=x/EA$ for $0 \le x \le \ell/2$, $\lambda^1(x)=\ell/2EA$ for $\ell/2 \le x \le \ell$, and $\lambda^2(x)=x/EA$, respectively. These adjoint solutions are also exact.

FIRST CASE: TRUSS WITH THE POINT LOAD p AT THE TIP

(Linear Shape Function)

$$EA\begin{bmatrix} \dfrac{b_2}{b_1(b_2-b_1)} & \dfrac{1}{b_1-b_2} \\ \dfrac{1}{b_1-b_2} & \dfrac{1}{b_2-b_1} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix} \qquad z(x) = \frac{px}{EA} \text{ (exact)}$$

$$\frac{2EA}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda^1(x) = \begin{Bmatrix} x/EA, & 0 \le x \le \ell/2 \\ \ell/2EA, & \ell/2 \le x \le \ell \end{Bmatrix} \text{ (exact)}$$

$$\frac{2EA}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \lambda^2(x) = \frac{x}{EA} \text{ (exact)}$$

Figure 7

Using the D-D method, design sensitivities for z_1 and z_2 are $z_1'=p\delta\ell/2EA$ and $z_2'=p\delta\ell/EA$, respectively, for the linear velocity as shown in figure 8. On the other hand, if the quadratic velocity is used, then $z_1'=p\varepsilon_1/EA$ and $z_2'=p\delta\ell/EA$. Also for the cubic velocity, the D-D method yields $z_1'=p(9\varepsilon_2+9\varepsilon_3-\delta\ell)/16EA$ and $z_2'=p\delta\ell/EA$. Now, using the C-D method, the design sensitivity expression is obtained as

$$z_i' = \int_0^{\ell} EA\, z_x \lambda_x^i V_x\, dx, \qquad i = 1,2$$

Using the finite element analyses results and the linear velocity in this design sensitivity expression, the C-D method gives $z_1'=p\delta\ell/2EA$ and $z_2'=p\delta\ell/EA$ which are the same as the results of the D-D method. Moreover, the design sensitivity expression yields $z_1'=p\varepsilon_1/EA$ and $z_2'=p\delta\ell/EA$ for the quadratic velocity and $z_1'=p(9\varepsilon_2+9\varepsilon_3-\delta\ell)/16EA$ and $z_2'=p\delta\ell/EA$ for the cubic velocity which are the same as the results of the D-D method. Thus, when the linear shape function is used for finite element model of the truss with the point load p, the D-D and C-D methods are equivalent for all parameterizations of velocity considered as indicated in the second column of figure 5. One point to emphasize in this case is that the original and adjoint responses of finite element models are the exact solutions of the truss with the point load. Note that the design sensitivity $z_2'=p\delta\ell/EA$ is independent of the parameterizations of velocity for the C-D method.

DESIGN SENSITIVITY OF NODAL DISPLACEMENTS (First Case)

- **D-D** and **C-D** methods yield the **same result** for **all parameterizations** of velocity.

Linear Velocity	$z_1'=p\delta\ell/2EA$	$z_2'=p\delta\ell/EA$
Quadratic Velocity	$z_1'=p\varepsilon_1/EA$	$z_2'=p\delta\ell/EA$
Cubic Velocity	$z_1'=p(9\varepsilon_2+9\varepsilon_3-\delta\ell)/16EA$	$z_2'=p\delta\ell/EA$

Figure 8

The second case of study is the truss with the uniformly distributed load f along the truss. For this, using the quadratic shape function, the finite element matrix equation is obtained as given in figure 9. The solutions of the finite element matrix equation are $z_1=z(\ell/2)=3f\ell^2/8EA$ and $z_2=z(\ell)=f\ell^2/2EA$ at the present design $b_1=\ell/2$ and $b_2=\ell$. Thus $z(x)=fx(-x+2\ell)/2EA$ which is the exact solution of the truss with the uniformly distributed load f. If the design sensitivities of z_1 and z_2 are desired, the adjoint equations are given in figure 9 with the adjoint solutions $\lambda^1(x)=(-3x^2/4\ell+5x/4)/EA$ and $\lambda^2(x)=x/EA$, respectively. The adjoint solution $\lambda^2(x)$ is the same as in the linear shape function case which is the exact solution. On the other hand, the adjoint solution $\lambda^1(x)$ is different from the linear shape function case and not exact.

SECOND CASE: TRUSS WITH UNIFORMLY DISTRIBUTED LOAD f

(Quadratic Shape Function)

$$EA\begin{bmatrix} \dfrac{b_2^3}{3b_1^2(b_1-b_2)^2} & -\dfrac{b_2^2}{3b_1(b_1-b_2)^2} \\ -\dfrac{b_2^2}{3b_1(b_1-b_2)^2} & \dfrac{4b_2^2-6b_1b_2+3b_1^2}{3b_2(b_1-b_2)^2} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} \dfrac{f\,b_2^3}{6b_1(b_2-b_1)} \\ \dfrac{f\,b_2(2b_2-3b_1)}{6(b_2-b_1)} \end{bmatrix} \qquad z(x)=\frac{fx(-x+2\ell)}{2EA}\ \text{(exact)}$$

$$\frac{EA}{3\ell}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda^1(x)=\frac{-3x^2/4\ell+5x/4}{EA}\ \text{(approximate)}$$

$$\frac{EA}{3\ell}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \lambda^2(x)=\frac{x}{EA}\ \text{(exact)}$$

Figure 9

Using the D-D method, design sensitivities for z_1 and z_2 are $z'_1=3f\ell\delta\ell/4EA$ and $z'_2=f\ell\delta\ell/EA$, respectively, for the linear velocity as shown in figure 10. On the other hand, if the quadratic velocity is used, then $z'_1=f\ell(\delta\ell+\varepsilon_1)/2EA$ and $z'_2=f\ell\delta\ell/EA$. Also, for the cubic velocity, the D-D method yields $z'_1=f\ell(15\delta\ell+9\varepsilon_2+9\varepsilon_3)/32EA$. Now, using the C-D method, the design sensitivity expression is obtained as

$$z'_i = \int_0^{\ell} (f\lambda^i + EA\, z_x \lambda^i_x)\, V_x\, dx, \quad i = 1,2$$

Using the finite element analyses results and the linear velocity in this expression, the C-D method gives $z'_1=3f\ell\delta\ell/4EA$ and $z'_2=f\ell\delta\ell/EA$ which are the same as the results of the D-D method. Also, using the finite element analyses results and the quadratic velocity in the design sensitivity expression, the C-D method gives $z'_1=f\ell(\delta\ell+\varepsilon_1)/2EA$ and $z'_2=f\ell\delta\ell/EA$. These are the same as the results of the D-D method. However, the design sensitivity expression yields $z'_1=f\ell(42\delta\ell+36\varepsilon_2+9\varepsilon_3)/80EA$ for the cubic velocity which is different from the result of the D-D method. Hence, it can be concluded that the D-D and C-D methods are not equivalent in the second case of study. Notice that the sensitivity results of the D-D method are the same as those of the C-D method up to the linear velocity when the linear shape function is used and up to the quadratic velocity when the quadratic shape function is used. Thus, the second case indicates the D-D and C-D methods might be equivalent under an additional condition that the shape function used in the finite element model is isoparametric with the discretization polynomial of the design velocity. However, this is not true as the results of the next case of study indicate.

DESIGN SENSITIVITY OF NODAL DISPLACEMENTS (Second Case)

	D-D	C-D
Linear Velocity	$z'_1=3f\ell\delta\ell/4EA$ $z'_2=f\ell\delta\ell/EA$	$z'_1=3f\ell\delta\ell/4EA$ $z'_2=f\ell\delta\ell/EA$
Quadratic Velocity	$z'_1=f\ell(\delta\ell+\varepsilon_1)/2EA$ $z'_2=f\ell\delta\ell/EA$	$z'_1=f\ell(\delta\ell+\varepsilon_1)/2EA$ $z'_2=f\ell\delta\ell/EA$
Cubic Velocity	$z'_1=f\ell(15\delta\ell+9\varepsilon_2+9\varepsilon_3)/32EA$	$z'_1=f\ell(42\delta\ell+36\varepsilon_2+9\varepsilon_3)/80EA$

Figure 10

The third case of study is the truss with the linearly varying load qx/ℓ along the truss. Before carrying out design sensitivity computation, dependency of the external load on the shape design has to be defined as shown in figure 11. That is, as the length of the truss changes, the external load will maintain the form of qx/ℓ. For this, using the quadratic shape function, the finite element matrix equation is given in figure 11. The matrix equation gives the solutions $z_1=z(\ell/2)=11q\ell^2/48EA$ and $z_2=z(\ell)=q\ell^2/3EA$ at the present design $b_1=\ell/2$ and $b_2=\ell$. Thus $z(x)=qx(-3x+7\ell)/12EA$ which is not the exact solution of the truss with the linearly varying load. The same adjoint equations that are given in figure 9 are applicable in this case with the solutions $\lambda^1(x)=(-3x^2/4\ell+5x/4)/EA$ and $\lambda^2(x)=x/EA$, respectively. As mentioned before, the adjoint solution $\lambda^1(x)$ is not exact, whereas $\lambda^2(x)$ is exact.

THIRD CASE: TRUSS WITH LINEARLY VARYING LOAD qx/ℓ

(Quadratic Shape Function)

$$EA\begin{bmatrix} \dfrac{b_2^3}{3b_1^2(b_1-b_2)^2} & -\dfrac{b_2^2}{3b_1(b_1-b_2)^2} \\ -\dfrac{b_2^2}{3b_1(b_1-b_2)^2} & \dfrac{4b_2^2-6b_1b_2+3b_1^2}{3b_2(b_1-b_2)^2} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} \dfrac{q\,b_2^3}{12b_1(b_2-b_1)} \\ \dfrac{q\,b_2(3b_2-4b_1)}{12(b_2-b_1)} \end{bmatrix}$$

$$z(x)=\frac{qx(-3x+7\ell)}{12EA} \quad \text{(approximate)}$$

Dependency of the External Load on Shape Design Variable

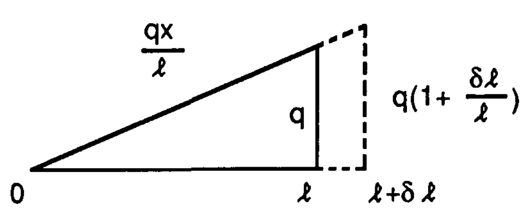


Figure 11

Using the D-D method, design sensitivities for z_1 and z_2 are $z'_1=11q\ell\delta\ell/16EA$ and $z'_2=q\ell\delta\ell/EA$, respectively, for the linear velocity as shown in figure 12. On the other hand, if the quadratic velocity is used, then $z'_1=q\ell(25\delta\ell+16\varepsilon_1)/48EA$. Also for the cubic velocity, the D-D method yields $z'_2=q\ell\delta\ell/EA$. Now, using the C-D method, the design sensitivity expression is obtained as

$$z'_i = \int_0^{\ell} \left[\left(\frac{q}{\ell}\right)\lambda^i V + \left(\frac{qx}{\ell}\right)\lambda^i V_x + EA\, z_x \lambda^i_x V_x \right] dx, \qquad i = 1,2$$

Using the finite element analyses results and the linear velocity in this expression, the C-D method gives $z'_1=11q\ell\delta\ell/16EA$ and $z'_2=q\ell\delta\ell/EA$ which are the same as the results of the D-D method. However, using the finite element analyses results and the quadratic velocity in the design sensitivity expression, the the C-D method gives $z'_1=q\ell(19\delta\ell+292\varepsilon_1)/240EA$, whereas it yields $z'_2=q\ell(249\delta\ell+27\varepsilon_2-27\varepsilon_3)/240EA$ for the cubic velocity. These are not the same as the results of the D-D method. Thus the D-D and C-D methods are not equivalent for the truss with a linearly varying load. Based on the equivalence study of truss problem, the D-D and C-D methods are possibly equivalent only for linear velocity. If this is the case, then both methods will give the exact design sensitivity information of the finite element analysis results that may not be acceptable at all. This is the situation for the fillet problem in reference 15 that the design sensitivity results of the C-D method agrees up to 5 to 6 digits with the finite difference even though the finite element model using constant stress triangular element does not provide accurate analysis result. On the other hand, when automatic regridding methods are employed for shape optimal design (refs. 16 and 17), parameterizations of the design velocity field cannot be limited to be only linear functions.

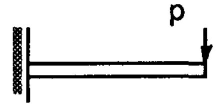
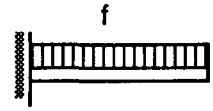
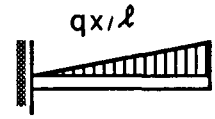
DESIGN SENSITIVITY OF NODAL DISPLACEMENTS (Third Case)

	D-D	C-D
Linear Velocity	$z'_1=11q\ell\delta\ell/16EA$ $z'_2=q\ell\delta\ell/EA$	$z'_1=11q\ell\delta\ell/16EA$ $z'_2=q\ell\delta\ell/EA$
Quadratic Velocity	$z'_1=q\ell(25\delta\ell+16\varepsilon_1)/48EA$	$z'_1=q\ell(19\delta\ell+292\varepsilon_1)/240EA$
Cubic Velocity	$z'_2=q\ell\delta\ell/EA$	$z'_2=q\ell(249\delta\ell+27\varepsilon_2-27\varepsilon_3)/240EA$

Figure 12

The results of analytical equivalence study for a simple cantilever beam with moment of inertia I and length ℓ are given in figure 13. Like the truss problem, three lateral loading cases shown in figure 13 are considered. For all loading cases, Hermite cubic shape functions are used for the finite element model with one element. Also, for the design velocity V(x), the same linear and quadratic parameterizations as in the truss problem are used. In addition to these, Hermitian parameterization of the velocity is used. That is, if the beam is fixed at x=0 and changes its length by $\delta\ell$ at $x=\ell$ and the slope of the velocity is zero at x=0 and θ at $x=\ell$, then the parameterization of the velocity is $V_4(x)$ as shown in figure 13 where γ_3 and γ_2 are given in terms of ℓ, $\delta\ell$, and θ. For the equivalence study, design sensitivity of the tip displacement is considered. The results of equivalence study are summarized in figure 13. As in the truss case, the finite element model for the beam with the point load p at the tip yields the exact solutions of the original and adjoint structures. Hence the D-D and C-D methods give the same design sensitivity results for all parameterizations of velocity as shown in figure 13.

RESULTS OF EQUIVALENCE STUDY OF D-D AND C-D METHODS FOR BEAM

Loading Condition / Design Velocity Field			
Linear	Same	Same	Same
Quadratic	Same	Not	Not
Hermitian	Same	Not	Not

Hermitian Velocity $V_4=\gamma_3 x^3+\gamma_2 x^2$ with $\gamma_3=(\ell\theta-2\delta\ell)/\ell^3$, $\gamma_2=(-\ell\theta+3\delta\ell)/\ell^2$

Figure 13

For the beam with the linearly varying load qx/ℓ along the beam, the finite element matrix equation is given in figure 14 with the solutions $z_1=11q\ell^4/120EI$ and $z_2=q\ell^3/8EI$ at the present design $b=\ell$. Thus $z(x)=q\ell(18\ell x^2-7x^3)/120EI$ which is an approximate solution of the beam with linearly varying load. For the design sensitivity of z_1, the solution of the adjoint equation given in figure 14 can be used. Using the D-D method, design sensitivity for z_1 is $z_1'=11q\ell^3\delta\ell/24EI$ for all parameterizations of velocity. For the C-D method, the design sensitivity expression is

$$z_1' = \int_0^{\ell} \left\{ EI\,[\, 3z_{xx}\lambda_{xx}V_x + (\, z_x\lambda_{xx} + z_{xx}\lambda_x\,)\, V_{xx}\,] + \left(\frac{q}{\ell}\right)\lambda V + \left(\frac{qx}{\ell}\right)\lambda V_x \right\} dx$$

Using the finite element analyses results in this design sensitivity expression, the C-D method yields $z_1'=11q\ell^3\delta\ell/24EI$ for the linear velocity which is the same as the result of the D-D method. However, the design sensitivity expression yields $z_1'=q\ell^3(171\delta\ell-12\varepsilon_1)/360EI$ for the quadratic velocity and $z_1'=q\ell^3(1194\delta\ell-9\ell\theta)/2520EI$ for the Hermitian velocity which are not the same as the results of the D-D method. Thus the D-D method and C-D methods are not equivalent for the beam with linearly distributed load as indicated in figure 13. Based on the equivalence study of the beam problem, the D-D and C-D methods are possibly equivalent only for linear velocity.

DESIGN SENSITIVITY OF NODAL DISPLACEMENT FOR BEAM

(Linearly Varying Load qx/ℓ)

$$\frac{EI}{b^3}\begin{bmatrix} 12 & -6b \\ -6b & 4b^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{7q\,b}{20} \\ -\frac{q\,b^2}{20} \end{bmatrix} \qquad z(x) = \frac{q\ell(18\ell x^2-7x^3)}{120EI} \text{ (approximate)}$$

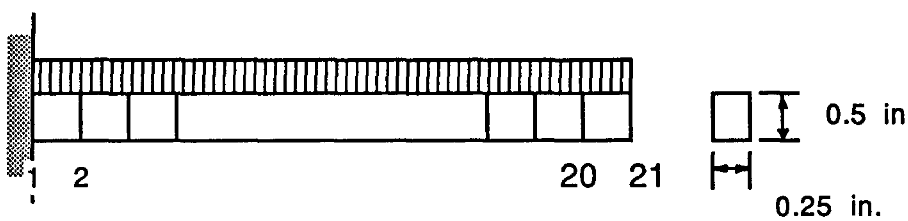
$$\frac{EI}{b^3}\begin{bmatrix} 12 & -6b \\ -6b & 4b^2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda(x) = \frac{x^2(3\ell-x)}{6EI} \text{ (exact)}$$

	D-D	C-D
Linear Velocity	$z_1'=11q\ell^3\delta\ell/24EI$	$z_1'=11q\ell^3\delta\ell/24EI$
Quadratic Velocity	$z_1'=11q\ell^3\delta\ell/24EI$	$z_1'=q\ell^3(171\delta\ell-12\varepsilon_1)/360EI$
Hermitian Velocity	$z_1'=11q\ell^3\delta\ell/24EI$	$z_1'=q\ell^3(1194\delta\ell-9\ell\theta)/2520EI$

Figure 14

Next, a numerical study is carried out for the C-D method using the cantilever beam with the uniformly distributed load to see effect of accuracy of the finite element analysis results on accuracy of the design sensitivity informations obtained. The finite element models with 1, 2, and 20 elements are considered for numerical study. Node numbering for all finite element models starts at the clamped end of the beam and the node number of free end of the beam is (m+1) where m is the number of elements in the model. The beam is 60 in. long and has a uniform rectangular cross-section of 0.5 in. high and 0.25 in. wide. Young's modulus, Poisson's ratio, and uniformly distributed load are $E=30\times10^6$ psi, $\nu=0.3$, and $f=0.5$ ℓb/in., respectively. Finite element analysis is carried out using ANSYS finite element STIF4. Three parameterizations of velocity with 1% perturbation of the length $\ell=60$ in. of the beam are used for numerical study as shown in figure 15. Once the solutions of the original and adjoint structural system are obtained using ANSYS, the continuum design sensitivity expression is numerically integrated using three points Gauss quadrature.

CONTINUUM-DISCRETE METHOD FOR A CANTILEVER BEAM



$\ell=60$ in., $E=30\times10^6$ psi, $\nu=0.3$, $f=0.5$ ℓb/in.

ANSYS STIF4

Parameterizations of Velocity for Numerical Study of the C-D Method

Case	Design Velocity Type	Parameter Values
A	Linear	$\delta\ell=0.6$ in.
B	Quadratic	$\delta\ell=0.6$ in. and $\varepsilon_1=10$ in.
C	Hermitian	$\delta\ell=0.6$ in. and $\theta=-0.3$

Figure 15

To check accuracy of the design sensitivity obtained, the results are compared with the results obtained by finite difference as shown in figures 16 and 17. In these figures, $z(\ell-\delta\ell)$ and $z(\ell+\delta\ell)$ are the displacements of selected nodal points for perturbed backward and forward designs, respectively, $\Delta z=z(\ell+\delta\ell)-z(\ell-\delta\ell)$ is the finite difference, and z' is the difference predicted by the design sensitivity. The ratio of z' and Δz times 100 can be used as a measure of accuracy of the design sensitivity. In figures 16 and 17, for all finite element models, the case A with linear velocity yields excellent agreement between the design sensitivity z' and the finite difference Δz. This confirms with the results of analytic study that the D-D and C-D methods may be equivalent for linear velocity. On the other hand, for one element model, the design sensitivity z' and the finite difference Δz do not agree at all for other parameterizations (cases B and C) of velocity as can be seen in figure 16. For cases B and C, the agreements improve substantially for two elements model.

COMPARISON OF DESIGN SENSITIVITY OF THE C-D METHOD

One Element Model

Case	Node No.	$z(\ell-\delta\ell)$	$z(\ell+\delta\ell)$	Δz	z'	($z'/\Delta z$ x100)%
A	2	0.99593E+00	0.10789E+01	0.41476E-01	0.41471E-01	100.0
B	2	0.99593E+00	0.10789E+01	0.41476E-01	-0.47926E-01	-115.6
C	2	0.99593E+00	0.10789E+01	0.41476E-01	0.64970E-01	156.6

Two Elements Model

Case	Node No.	$z(\ell-\delta\ell)$	$z(\ell+\delta\ell)$	Δz	z'	($z'/\Delta z$ x100)%
A	2	0.35273E+00	0.38210E+00	0.14689E-01	0.14688E-01	100.0
	3	0.99593E+00	0.10789E+01	0.41476E-01	0.41471E-01	100.0
B	2	0.17939E+00	0.59469E+00	0.20765E+00	0.20744E+00	99.9
	3	0.99593E+00	0.10789E+01	0.41476E-01	0.35880E-01	86.5
C	2	0.30947E+00	0.42955E+00	0.60040E-01	0.61126E-01	101.8
	3	0.99593E+00	0.10789E+01	0.41476E-01	0.42939E-01	103.5

Figure 16

On the other hand, for twenty elements model, agreements become excellent as shown in figure 17. This confirms the fact that accurate design sensitivity informations can be obtained as long as accurate finite element analysis results are used for the C-D methods. This fact is not the case for the semi-analytic method, as demonstrated by Barthelemy and Haftka (ref. 5). They found that the design sensitivity error of the semi-analytic method is proportional to the square of the number of elements. This is completely opposite behavior from the C-D method since the design sensitivity error increases very rapidly as the finite element analysis results of the original structure become more accurate. As demonstrated in figures 16 and 17, an essential advantage that may accrue in the C-D method is associated with the ability to identify the effect of numerical error associated with finite element analysis results. That is, if disagreement arises between the design sensitivity of the C-D method and the finite difference, then error has crept into the finite element approximation. If the D-D method is used, in which the structure is discretized and the design variables are imbedded into the stiffness matrix, then any error inherent in the finite element model is consistently parameterized and will never be reported to the user. Therefore, precise design sensitivity coefficients of the matrix model of the structure are obtained without realizing that there may be substantial inherent error in the original model. On the other hand, the C-D method can be used to obtain a warning that approximation error is creeping into the finite element model.

COMPARISON OF DESIGN SENSITIVITY OF THE C-D METHOD (Cont)

Twenty Elements Model

Case	Node No.	$z(\ell-\delta\ell)$	$z(\ell+\delta\ell)$	Δz	z'	$(z'/\Delta z \times 100)\%$
A	2	0.48158E-02	0.52169E-02	0.20055E-03	0.20053E-03	100.0
	6	0.10504E+00	0.11379E+00	0.43744E-02	0.43739E-02	100.0
	11	0.35273E+00	0.38210E+00	0.14689E-01	0.14688E-01	100.0
	16	0.66525E+00	0.72066E+00	0.27704E-01	0.27701E-01	100.0
	21	0.99593E+00	0.10789E+01	0.41476E-01	0.41471E-01	100.0
B	2	0.70800E-03	0.13220E-01	0.62558E-02	0.62566E-02	100.0
	6	0.29727E-01	0.22933E+00	0.99800E-01	0.10127E+00	101.5
	11	0.17939E+00	0.59469E+00	0.20765E+00	0.21023E+00	101.2
	16	0.50771E+00	0.89205E+00	0.19217E+00	0.19269E+00	100.3
	21	0.99593E+00	0.10789E+01	0.41476E-01	0.41467E-01	100.0
C	2	0.47614E-02	0.52749E-02	0.25676E-03	0.25673E-03	100.0
	6	0.95061E-01	0.12480E+00	0.14869E-01	0.14863E-01	100.0
	11	0.30947E+00	0.42955E+00	0.60040E-01	0.60046E-01	100.0
	16	0.60858E+00	0.78134E+00	0.86383E-01	0.86384E-01	100.0
	21	0.99593E+00	0.10789E+01	0.41476E-01	0.41470E-01	100.0

Figure 17

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