TREATMENT OF BODY FORCES IN BOUNDARY ELEMENT DESIGN SENSITIVITY ANALYSIS*

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INTRODUCTION

The inclusion of body forces has received a good deal of attention in boundary element research. The consideration of such forces is essential in the design of high performance components such as fan and turbine disks in a gas turbine engine. Due to their critical performance requirements, optimal shapes are often desired for these components. The boundary element method (BEM) offers the possibility of being an efficient method for such iterative analysis as shape optimization.

A survey of efforts in the area of sensitivity analysis in BEM was given by Mota, Soares and Choi [1]. The shape sensitivity using a finite-difference formulation was given by Wu [2] and using the implicit-differentiation formulation by Barone and Yang [3], Saigal et al. [4-6], and Rice and Mukherjee.* Mukherjee and Chandra [7] presented a BEM sensitivity formulation for materially nonlinear problems. The treatment of body forces for sensitivity analysis has not received much attention.

In this paper, the implicit-differentiation of the boundary integral equations [8] is performed to obtain the sensitivity equations. The body forces are accounted for by either the particular integrals [9,10] for uniform body forces or by a surface integration [11] for non-uniform body forces. The corresponding sensitivity equations for both these cases are presented. The validity of present formulations is established through a close agreement with exact analytical results.

*Rice, J.S. and Mukherjee, S., "Design Sensitivity Coefficients for Axisymmetric Elasticity Problems by Boundary Element Methods", private communication.

BOUNDARY ELEMENT ANALYSIS EQUATIONS

Including the effect of temperature variation ϕ, the stress tensor σ_{ij} is given in equation (1), and the equation of equilibrium is given in equation (2). Starting with a weak statement of equation (2) and using the divergence theorem twice the integral equation (3) is obtained. Assuming steady state condition, using the divergence theorem, and applying Green's second function leads to equation (4) where the thermal effects have been reduced to a boundary integral form.

$$\sigma_{ij} = \frac{E}{(1+\nu)} e_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \delta_{ik} e_{kk} - \frac{E}{(1-2\nu)} \delta_{ij\alpha\phi} \tag{1}$$

$$\sigma_{ij,j} + F_i = 0 \tag{2}$$

$$u_i(p) = -\int_\Gamma T_{ij}(p,q) u_j(q) d\Gamma + \int_\Gamma U_{ij}(p,q) t_j(q) d\Gamma + \int_\Omega U_{ij}(p,q) F_i(q) d\Omega - \frac{\alpha E}{(1-2\nu)} \int_\Omega U_{ij}(p,q) \phi \, d\Omega \tag{3}$$

$$u_i(p) = -\int_\Gamma T_{ij}(p,q) u_j(q) d\Gamma + \int_\Gamma U_{ij}(p,q) t_j(q) d\Gamma + \frac{\alpha(1+\nu)}{8\pi(1-\nu)} \int_\Gamma \left\{ \frac{\phi}{R} \left(n_i - R_{,i} \frac{\partial R}{\partial n} \right) - R_{,i} \frac{\partial \phi}{\partial n} \right\} d\Gamma + \int_\Omega U_{ij}(p,q) F_i(q) d\Omega \tag{4}$$

E, ν, and α are the modulus of elasticity, Poisson's ratio, and coefficient of thermal expansion, respectively. T_{ij} and U_{ij} are the fundamental (Kelvin) solutions for traction and displacement, respectively. p and q are the load point and the field point, respectively, and R is the distance between these points. u_i, t_i , and F_i are the components of displacement, traction and body force, respectively.

GRAVITATIONAL AND CENTRIFUGAL FORCE SENSITIVITY

If the total displacement is written as a sum of a complementary and a particular integral component as in equation (5), then in the absence of temperature variation, the last two terms in equation (4) drop out giving equation (6). Discretizing the boundary using boundary elements with displacements and tractions interpolated as shown in equation (7), we get the matrix relationship (8).Substituting from equation (5) into equation (8), we obtained a relationship in equation (9) including the effect of particular solutions due to body forces. Particular solutions $\{u^p\}$ and $\{t^p\}$ were given by Banerjee and co-workers [9-10] at SUNY - Buffalo. Implicit differentiation of equation (9) with respect to the design variable X_L results in the sensitivity equation (10). The contribution of the body forces is included in the vector $\{f^p\}$ given in equation (11).

$$u_i = u_i^c + u_i^p \tag{5}$$

$$u_i^c(p) = -\int_\Gamma T_{ij}(p,q)u_j^c(q)d\Gamma + \int_\Gamma U_{ij}(p,q)t_j^c(q)d\Gamma \tag{6}$$

Discretizing equation (6) using interpolation functions for displacements and tractions

$$u = [H]\{u\} \ ; \ t = [H]\{t\} \tag{7}$$

The matrix form of equation (6) is obtained as

$$[F]\{u^c\} = [G]\{t^c\} \tag{8}$$

Substituting equation (5) in equation (8)

$$[F]\{u\} = [G]\{t\} + [F]\{u^p\} - [G]\{t^p\} \tag{9}$$

Differentiating with respect to the design variable, X_L

$$[F]\{u\}_{,L} = [G]_{,L}\{t\} + [G]\{t\}_{,L} - [F]_{,L}\{u\} + \{f^p\} \tag{10}$$

Where

$$[f^p\} = [F]\{u^p\}_{,L} + [F]_{,L}\{u^p\} - [G]\{t^p\}_{,L} - [G]_{,L}\{t^p\} \tag{11}$$

The superscripts c and p refer to the complementary and particular soultions, respectively. [H] is a matrix of interpolation functions. $(\ ),_L$ denotes the derivative of ( ) with respect to the design variable X_L.

THERMOELASTIC SENSITIVITY

For the case of temperature variation ϕ and temperature gradient $\phi,_n$, the term with volume integral in equation (4) drops out. Then using the interpolation given in equation (7), we get the matrix relationship given in the equation (12). Implicit -differentiation of equation (12) leads to an equation similar to equation (10) but with a different definition for vector $\{f^p\}$. This relationship is given in equations(13) and (14). The matrix [V] involves thermoelastic kernels which include elliptic integrals of the first and the second kind. The present sensitivity analysis requires derivatives of these elliptic integrals which are easily determined through chain rule of differentiation.

$$[F]\{u\} = [G]\{t\} + [V]\{T\} \tag{12}$$

$$[F]\{u\},_L = [G\},_L\{t\} + [G]\{t\},_L - [F],_L\{u\} + \{f^p\} \tag{13}$$

$$\{f^p\} = [V],_L\{T\} + [V]\{T\},_L \tag{14}$$

{T} is the vector of nodal temperatures.

SEMI-ANALYTICAL SENSITIVITY FORMULATION

The sensitivities can now be obtained using equations (10) and (11) for centrifugal and gravitational body forces, and using equations (13) and (14) for thermal body forces. We, however, need to determine sensitivity matrices such as $[F]_{,L}$ and $[G]_{,L}$; and sensitivity vectors such as $\{u^p\}_{,L}$ and $\{t^p\}_{,L}$. In the semi-analytical approach, the design variable X_L is first perturbed by an amount ΔX_L. The system matrices $[F(X_L+\Delta X_L)]$, $[G(X_L+\Delta X_L)]$, etc., are generated based on the new geometry. The sensitivities are then simply obtained using forward-difference relationships shown in equations (15) and (16). It is noted that the sensitivity results will depend on the perturbation step size ΔX_L.However, this step will result in substantial simplification of the implementation of the sensitivity algorithm.

$$[F]_{,L} = \frac{[F(X_L+\Delta X_L)] - [F(X_L)]}{\Delta X_L}\text{, etc} \tag{15}$$

$$\{u^p\}_{,L} = \frac{[u^p(X_L+\Delta X_L)] - [u^p(X_L)]}{\Delta X_L}\text{, etc} \tag{16}$$

FULL ANALYTICAL SENSITIVITY FORMULATION

For the full-analytical approach, the sensitivity matrices and vectors are directly calculated from their analytical expressions given in equations (17) and (18). These expressions, however, need the sensitivities of geometry quantities such as $x_{,L}$, $y_{,L}$, $n_{,L}$, etc. The initial geometry is first used for solution of vectors $\{u\}$ and $\{t\}$ in equations (8) or (12). This geometry is then changed through a perturbation ΔX_L of the design variable. Only the geometry sensitivities are then calculated using forward-difference approximation. These geometry sensitivities are needed for evaluating terms in equations (17) and (18).

$$[F]_{,L} = \sum_{j=1}^{N} \int_0^1 \{ [t^*]_{,L}^T [H] J + [t^*]^T [H] J_{,L} \} d\xi$$

$$[G]_{,L} = \sum_{j=1}^{N} \int_0^1 \{ [u^*]_{,L}^T [H] J + [u^*]^T [H] J_{,L} \} d\xi \tag{17}$$

The superscripts T and * refer to the transpose of the matrix and the fundamental solutions, respectively.

3-D Centrifugal Loading Particular Integral Sensitivities:

$$u1_{,L}=(2c_1(xx_{,L}+yy_{,L})+2c_2zz_{,L})x+(c_1(xx+yy)+c_2zz)x_{,L}$$
$$u2_{,L}=(2c_1(xx_{,L}+yy_{,L})+2c_2zz_{,L})y+(c_1(xx+yy)+c_2zz)y_{,L}$$
$$u3_{,L}=2c_3(xx_{,L}+yy_{,L})z+c_3(xx+yy)z_{,L}$$

$$c_1= -\rho\omega^2(5\lambda+4\mu)/4(\lambda+u)$$
$$c_2= -\rho\omega^2\mu/8(\lambda+2\mu)/(\lambda+\mu)$$
$$c_3= \rho\omega^2/8/(\lambda+2\mu) \tag{18}$$

λ, μ are the Lame's constants. x_L denotes the derivative of coordinate x with respect to the design variable X_L and y_L and z_L have similar definitions.The corresponding traction sensitivities can be found using the constitutive relation.

SINGULAR TERMS IN SYSTEM SENSITIVITY MATRIX

For determining terms in $[F]_{,L}$ an extension of rigid body technique used for singular terms in [F] is used. This extension is based on the fact that the sensitivities corresponding to rigid body displacements and tractions are zero leading to a row-sum type property for $[F]_{,L}$. Thus from equation (10), the singular terms for 3-D can be obtained as given in equation (21). For 2-D, these terms are similarly obtained. For axisymmetric case , a rigid body motion in Z - direction and an inflation mode in the radial direction are used.

$$\{u_{rigid}\}_{x_1} = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \{u_{rigid}\}_{x_2} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{Bmatrix} \tag{19}$$

$$\{t\}_{rigid} = \{0\} \tag{20}$$

$$\sum_{j=1,3,\cdots}^{n-1} F_{ij,L} = 0 \; ; \quad \sum_{j=2,4,\cdots}^{n} F_{ij,L} = 0 \; ; \quad \text{for each } i \tag{21}$$

STRESS SENSITIVITY RECOVERY

The solution of sensitivity equations yields the boundary traction sensitivities only. The stress sensitivities at other locations can be obtained directly using differentiated elasticity equations describing stress-Strain relationships. These relations are given in equation (22) for the axisymmetric case.

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix} = \begin{bmatrix} n_r^2 & n_z^2 & -2n_rn_z \\ n_z^2 & n_r^2 & 2n_rn_z \\ n_rn_z & -n_rn_z & (n_r^2-n_z^2) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{,L}$$

n_r, n_z are the components of the outward normal in the r and z directions, respectively.

$$\sigma_{22,L} = \frac{\nu}{(1-\nu)}\sigma_{11,L} + \frac{E}{(1-\nu^2)}(e_{22,L}+\nu e_{\theta\theta,L})$$

$$e_{22,L} = -\frac{J_{,L}}{J^2}u_{2,\xi} + \frac{1}{J}u_{2,\xi L}$$

$$e_{\theta\theta,L} = \frac{1}{r^2}(u_{r,L}\,r - u_r r_{,L}) \tag{22}$$

$u_{2,\xi L}$ denotes the mixed derivative with respect to the dimensionless coordinates ξ and the design variable X_L.

NUMERICAL EXAMPLES

The above formulations were applied to a series of selected examples to determine the design sensitivities for displacements and tractions. These examples include: (a) a rotating circular disk of constant thickness with a central hole shown in Figure 1 analyzed using two-dimensional elements, (b) a rotating circular disk with hyperbolic varying thickness and with a central hole shown in Figure 2 analyzed using axisymmetric elements, (c) a hollow cylinder under plane strain shown in Figure 3 subjected to pressure and temperature change and analyzed using axisymmetric elements, (d) a pressurized hollow cylinder under temperature variation shown in Figure 4 and analyzed using axisymmetric elements, (e) a solid circular bar (Figure 5) under self-weight analyzed using three-dimensional elements, and (f) a rotating circular disk (Figure 6) analyzed using three-dimensional elements. For examples without temperature variation the material data used were $E=30\times10^7$ psi, $\nu=0.3$; and for examples with temperature variation the material data used were $E=1$ psi, $\nu=0.3$, and $\alpha=0.02\ /^{0}F$. The results obtained from the present formulations were compared with the exact solutions to check these formulations. For exact sensitivities the elasticity solutions were first expressed in terms of the design variable and then differentiated with respect to this design variable. A good comparison of the present results was seen from the results presented in the following pages.

Design Sensitivity Analysis of a Rotating Circular Disk.

	Sensitivity			
	Radial Displacement x (10^{-3})		Traction (x 10^{3})	
Location*	Analytical	This Study	Analytical	This Study
A	6.7678	6.7677	4.8061	4.7625
B	6.5929	6.5929	8.7301	8.6993
C	2.8329	2.8330	3.6905	3.6923
D	2.6684	2.6685	3.8066	3.8103
E	6.7678	6.7678		
F	6.7678	6.7676		

* Location coordinates (r, z) in inches are: A(4.333, 0.), B(5.061, 0.), C(17.061, 0.), D(19.0, 0.), E(3.864, 1.035), F(3.967, 0.522)

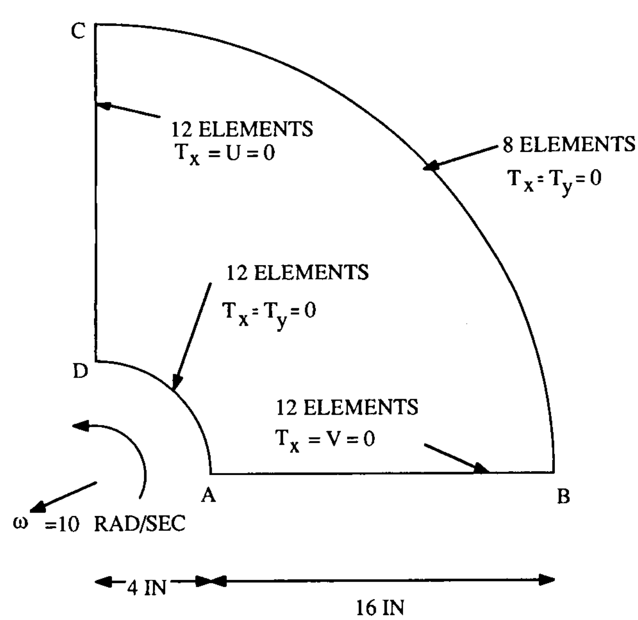

FIGURE 1

Design Sensitivity Analysis of a Rotating Disk with Hyperbolic Varying Thickness

RADIUS (inch)	SENSITIVITY				
	DISPLACEMENT X 10^3			HOOP STR. X 10^{-3}	
	Exact	Mesh A	Mesh B	Exact	Mesh B
4.00	2.05931	1.9987	1.9986	5.07485	3.9470
4.25	2.05819	--	1.9887	5.66207	5.4810
4.50	2.04760	1.9725	1.9725	5.99760	5.6433
4.75	2.02989	--	1.9522	6.16066	5.8450
5.00	2.00685	1.9296	1.9294	6.20577	5.8609
5.25	1.97984	--	1.9046	6.17095	5.8355
5.50	1.94995	1.8789	1.8785	6.08307	5.7574
5.75	1.91803	--	1.8512	5.96129	5.6542
6.00	1.88478	1.8236	1.8229	5.81938	5.5318
6.25	1.85078	--	1.7940	5.66731	5.4024
6.50	1.81653	1.7653	1.7645	5.51231	5.2668
6.75	1.78246	--	1.7348	5.35968	5.1372
7.00	1.74893	1.7060	1.7052	5.21325	5.0076
7.25	1.71628	--	1.6759	5.07583	4.8915
7.50	1.68481	1.6482	1.6474	4.94945	4.7787
7.75	1.65480	--	1.6198	4.83557	4.6831
8.00	1.62650	1.5945	1.5937	4.73521	4.5934
8.25	1.60016	--	1.5692	4.64908	4.5222
8.50	1.57602	1.5474	1.5466	4.57767	4.4589
8.75	1.55428	--	1.5261	4.52126	4.4141
9.00	1.53517	1.5089	1.5081	4.48003	4.3782
9.25	1.51889	--	1.4928	4.45403	4.3596
9.50	1.50565	1.4811	1.4803	4.44327	4.3511
9.75	1.49564	--	1.4710	4.44769	4.3540
10.00	1.48906	1.4660	1.4652	4.46718	4.4120

Note: Exact solution is for the assumption of plane stress.
Mesh A: 15 element model; Mesh B: 30 element model

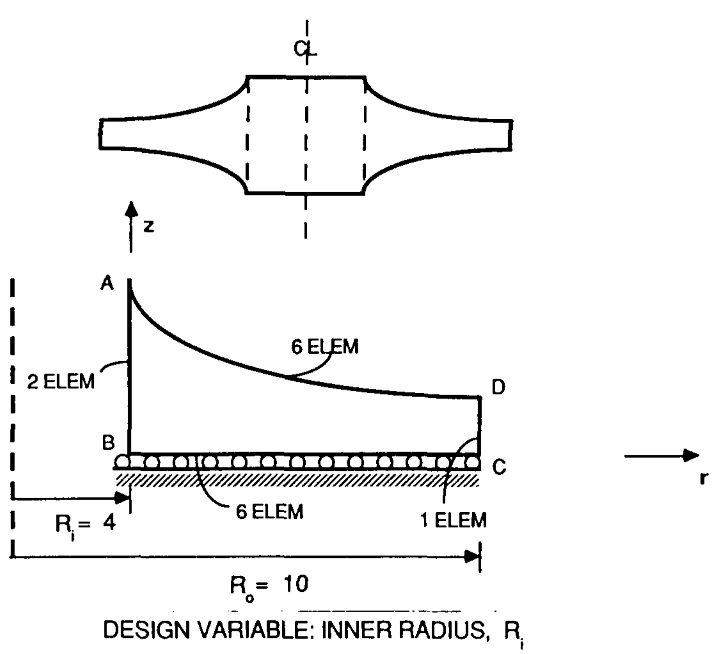

FIGURE 2

Design Sensitivity Analysis of a Plane Strain Hollow Cylinder Under Pressure and Temperature Variation

RADIUS (inch)	SENSITIVITY					
	RADIAL DISPLACEMENT		RADIAL STRESS		CIRCUMFERENTIAL STRESS	
	Exact	This Study	Exact	This Study	Exact	This Study
3.0	3.6305	3.6315	0.000	0.000	0.5955	0.5956
3.5	3.6694	3.6712	-0.1058	-0.1074	0.6995	0.6993
4.0	3.6168	3.6174	-0.1181	-0.1186	0.7112	0.7110
4.5	3.5302	3.5309	-0.0971	-0.0975	0.6903	0.6902
5.0	3.4341	3.4350	-0.0653	-0.0654	0.6591	0.6594
5.5	3.3396	3.3402	-0.0318	-0.0318	0.6264	0.6265
6.0	3.2513	3.2523	0.000	0.000	0.5955	0.5956

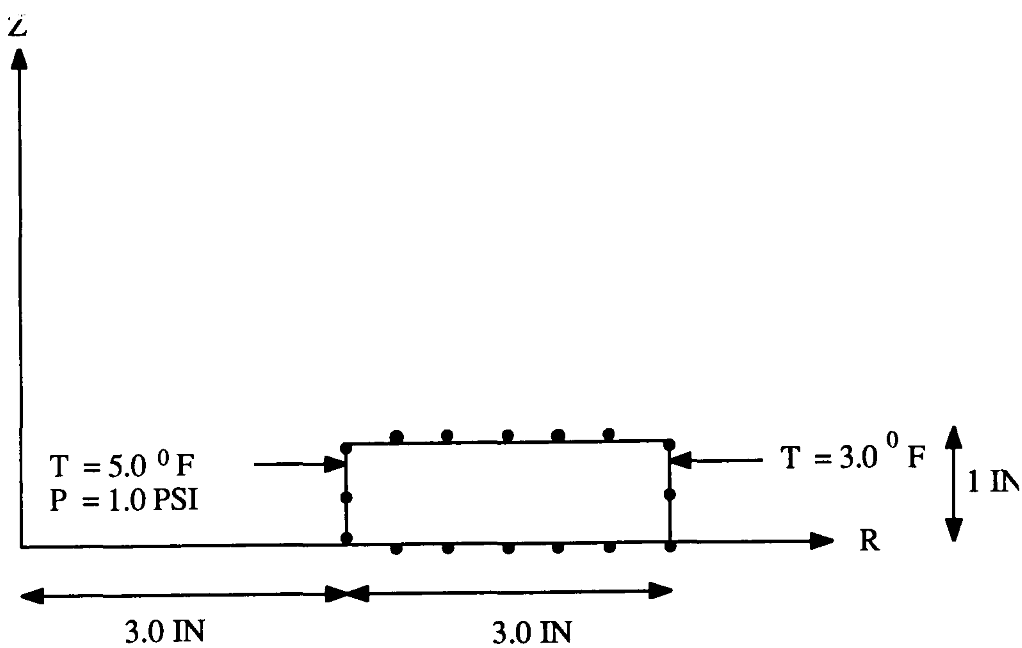

FIGURE 3

Pressurized Hollow Sphere Under Temperature Variation

RADIUS (inch)	SENSITIVITY					
	RADIAL DISPLACEMENT		RADIAL STRESS		CIRCUMFERENTIAL STRESS	
	Exact	This Study	Exact	This Study	Exact	This Study
1.00	1.5126	1.4992	0.000	0.0003	1.500	1.495
1.125	1.758	1.7448	-0.754	-0.618	1.857	1.886
1.25	1.8563	1.8436	-0.901	-0.927	1.923	1.910
1.375	1.8985	1.887	-0.815	-0.791	1.879	1.876
1.50	1.9247	1.9144	-0.653	-0.642	1.801	1.796
1.625	1.9526	1.9433	-0.471	-0.466	1.715	1.709
1.75	1.9896	1.9813	-0.297	-0.290	1.634	1.630
1.875	2.0384	2.0311	-0.139	-0.128	1.562	1.561
2.00	2.0996	2.0952	0.000	0.0003	1.500	1.496

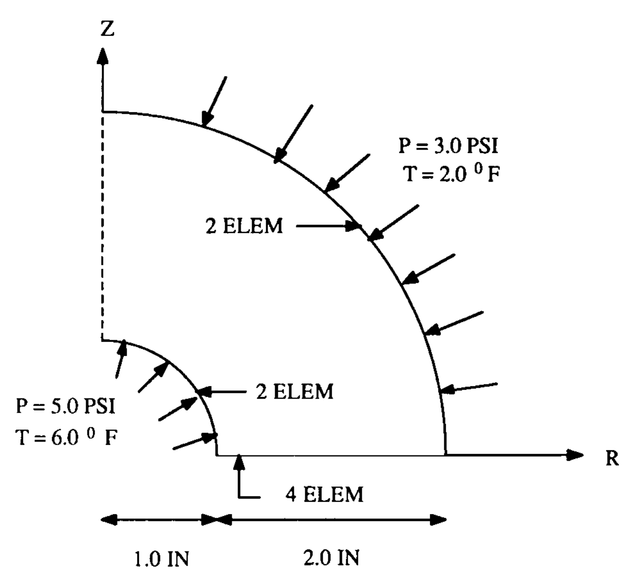

FIGURE 4

Solid Circular Bar under Self-weight

Location	Sensitivity of displacements in Z direction (10^3)			Sensitivity of Stress in Z direction		
x =3.536 y =3.536	Exact	Full-Analytical	Semi-Analytical	Exact	Full-Analytical	Semi-Analytical
A(z=0.0)	0.0	0.0	0.0	2318.4	2320.4	2351.6
B(z=5.0)	6.7620	6.7613	6.7614	1738.8	1747.6	1758.3
C(z=10.0)	11.592	11.589	11.589	1159.2	1188.7	1189.8
D(z=15.0)	14.490	14.488	14.488	579.6	587.6	589.4
E(z=20.0)	15.456	15.448	15.448	0.0	0.0	0.0

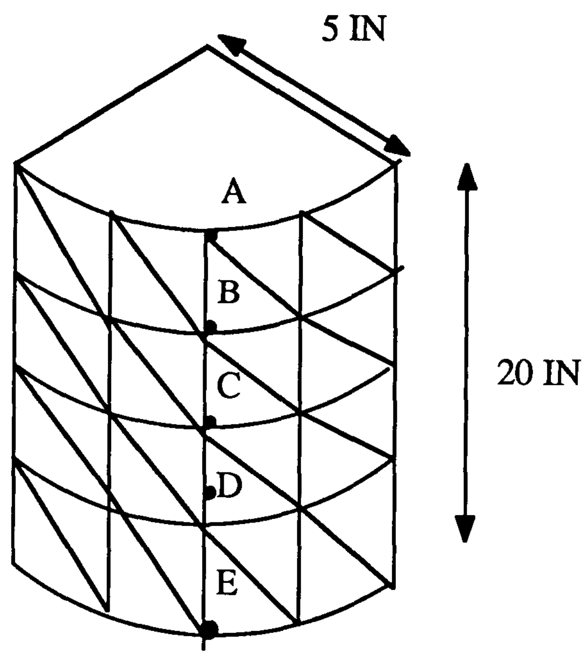

FIGURE 5

Three Dimensional Rotating Circular Disk

Location y=0.0 z=1.5	Displacement Sensitivity $u_{r,L}$ (10^{-3})			Radial Stress Sensitivity (10^4)		
	Exact	Full-Analytical	Semi-Analytical	Exact	Full-Analytical	Semi-Analytical
x= 4.0	0.6768	0.6775	0.6735	0.0840	0.1907	0.2023
x= 5.0	0.6613	0.6610	0.6571	0.8564	0.8413	0.8511
x= 6.0	0.6216	0.6212	0.6179	0.9650	0.9674	0.9644
x= 7.0	0.5768	0.5765	0.5737	0.9018	0.8966	0.8945
x= 8.0	0.5329	0.5326	0.5302	0.8004	0.8061	0.8032
x=16.0	0.2981	0.2978	0.2969	0.3740	0.3814	0.3800
x=17.0	0.2840	0.2838	0.2829	0.3691	0.3649	0.3635
x=18.0	0.2736	0.2733	0.2725	0.3715	0.3782	0.3770
x=19.0	0.2668	0.2667	0.2660	0.3807	0.3752	0.3740
x=20.0	0.2640	0.2639	0.2631	0.3960	0.4083	0.4083

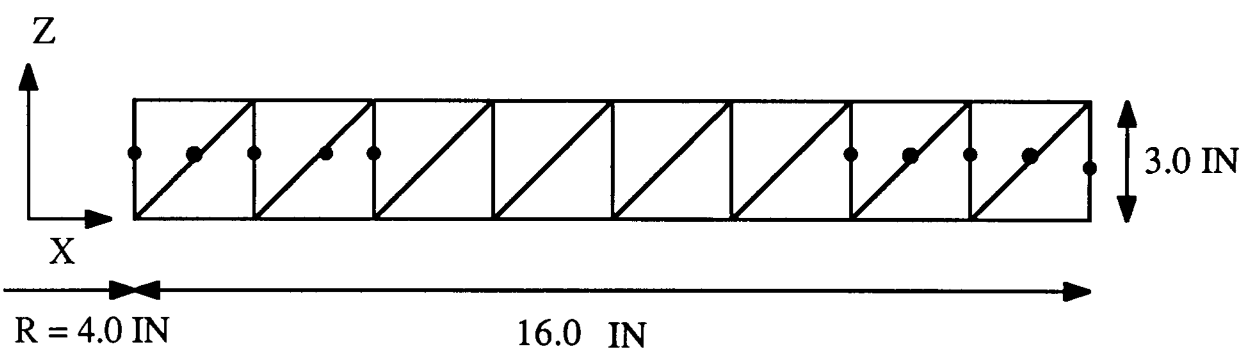

FIGURE 6

CONCLUSIONS

The treatment of body forces of the centrifugal, gravitational, and thermal types in the implicit-differentiation formulation for the design sensitivity analysis of two-dimensional, axisymmetric, and three-dimensional problems is presented. The particular integral sensitivity expressions for the gravitational and centrifugal type body forces are developed. The thermoelastic sensitivity kernels are given for the thermal type body forces. A semi-analytical and a full-analytical approach for determining the sensitivity system matrices are used. A wide range of problems are solved for design sensitivities due to body forces and the results are validated through comparisons with exact analytical solutions.

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