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**STRUCTURAL OPTIMIZATION OF FRAMED STRUCTURES  
USING  
GENERALIZED OPTIMALITY CRITERIA**

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## ABSTRACT

This paper presents the application of a generalized optimality criteria to framed structures. The optimality conditions, Lagrangian multipliers, resizing algorithm, and scaling procedures are all represented as a function of the objective and constraint functions along with their respective gradients. The optimization of two plane frames under multiple loading conditions subject to stress, displacement, generalized stiffness, and side constraints is presented. These results are compared to those found by optimizing the frames using a nonlinear mathematical programming technique.

## INTRODUCTION

Weight optimization of large aerospace structures requires the use of efficient optimization methods due to the potentially excessive number of design variables and related constraints. In order for optimization to be seriously used by the preliminary designer, the method must be able to handle multidisciplinary problems (thousands of design variables and their related constraints) and must be efficient (produce designs in hours not days).

In the late sixties and early seventies the optimality criteria approach to structural optimization was developed[1]. At that time, and in subsequent work, the optimality criterion was derived for an individual problem and although very efficient, it was criticized for its lack of generality. Recently, Venkayya[2] has generalized the optimality criteria to apply to any structural optimization problem to which the sensitivities of the objective and constraint functions can be computed. First, this paper will briefly state the optimality conditions. Next, detailed descriptions of the Lagrangian multipliers, scaling formulation along with a redesign procedure for stress, displacement, and generalized stiffness constraints applied to plane frames will be presented. Finally, the optimization results of some plane frames using the generalized optimality criterion will be given along with a comparison to results found by nonlinear mathematical programming.

## PROBLEM STATEMENT

The optimization of a structure for minimum weight can be stated mathematically as:

Minimize the objective function

$$W(A) = \sum_{i=1}^m \rho_i l_i A_i \quad (1)$$

subject to the constraints

$$Z_j(A) \leq \bar{Z}_j \quad j = 1, 2, \dots, n \quad (2)$$

$$A_i^{(L)} \leq A_i \leq A_i^{(U)} \quad (3)$$

where  $W(A)$  is the weight of the structure,  $\rho_i$ ,  $A_i$ , and  $l_i$  are the specific weight, the cross-sectional area, and the length of the  $i$ th element respectively. The  $Z_j(A)$  consist of all  $n$  of the behavioral constraints and  $\bar{Z}_j$  is the given allowable for  $Z_j(A)$ . The summation in equation (1) is over all  $m$  elements in the structure. However, this does not imply that all the elements are required to participate in the design algorithm. In addition to the constraints  $Z_j$ , each design variable ( $A_i$ ) has upper bounds ( $A_i^{(U)}$ ) and lower bounds ( $A_i^{(L)}$ ) referred to as side constraints.

## DESIGN VARIABLES

The design variables chosen for the plane frame are  $A_i$ ,  $I_{iz}$ , and  $S_{iz}$ , where  $A_i$  is the cross-sectional area,  $I_{iz}$  is the moment of inertia about the  $z$ -axis and  $S_{iz}$  is the section modulus about the  $z$ -axis. These

variables are not independent; therefore,  $A_i$  is chosen as the primary variable with  $I_{iz}$  and  $S_{iz}$  expressed as explicit nonlinear functions of  $A_i$  in the form

$$I_{iz} = \alpha_i A_i^{n_i} \quad (4)$$

$$S_{iz} = \gamma_i A_i^{v_i} \quad (5)$$

Depending on the type of cross-section and the assumptions being made  $n_i$  will vary from 1 to 3, and  $v_i$  will vary from 1 to 2. It is important to note that this method is general and any design variable can be chosen such as width of the section, thickness of the flange, or thickness of the web of the section. This section will focus on relations between  $A_i$ ,  $I_{iz}$ , and  $S_{iz}$  for solid rectangular cross-sections and a three spar box section. Three separate cases will be presented for each type of section. Each case will make varying assumptions about the width of the section, depth of the section, thickness of the flange, thickness of the web, and ratios of these quantities.

For the rectangular section in figure 1 having depth  $d$  and width  $b$  the following cases for the relations between  $A_i$ ,  $I_{iz}$ , and  $S_{iz}$  are presented.

Case 1: Assume  $b$  to be constant and  $d$  is allowed to vary.

$$\begin{aligned} I_{iz} &= \alpha_i A_i^{n_i} \quad n_i = 3 \quad \alpha_i = \frac{1}{12b^2} \\ S_{iz} &= \gamma_i A_i^{v_i} \quad v_i = 2 \quad \gamma_i = \frac{1}{6b} \end{aligned} \quad (6)$$

Case 2: Assume the ratio  $b/d$  is equal to some constant  $C_o$  and let  $b$  and  $d$  vary.

$$\begin{aligned} I_{iz} &= \alpha_i A_i^{n_i} \quad n_i = 2 \quad \alpha_i = \frac{1}{12C_o} \\ S_{iz} &= \gamma_i A_i^{v_i} \quad v_i = 3/2 \quad \gamma_i = \frac{1}{6\sqrt{C_o}} \end{aligned} \quad (7)$$

Case 3:  $d$  is assumed to be constant while  $b$  is allowed to vary.

$$\begin{aligned} I_{iz} &= \alpha_i A_i^{n_i} \quad n_i = 1 \quad \alpha_i = \frac{d^2}{12} \\ S_{iz} &= \gamma_i A_i^{v_i} \quad v_i = 1 \quad \gamma_i = \frac{d}{6} \end{aligned} \quad (8)$$

Next, consider the three spar box section in figure 2. The area  $A_i$ , moment of inertia  $I_{iz}$ , and the section modulus can be expressed as:

$$A_i = bd \left\{ 2 \left( 1 + \frac{t_w}{b} \right) \frac{t_f}{d} + 3 \left( 1 - \frac{t_f}{d} \right) \frac{t_w}{b} \right\}$$

or

$$A_i = bdC_1 \quad (9)$$

$$I_{iz} = \frac{bd^3}{12} \left\{ \left( 1 + \frac{t_w}{b} \right) \left( 1 + \frac{t_f}{d} \right)^3 - \left( 1 - \frac{2t_w}{b} \right) \left( 1 - \frac{t_f}{d} \right)^3 \right\}$$

or

$$I_{iz} = \frac{bd^3}{12} C_2 \quad (10)$$

$$S_{iz} = \frac{bd^2}{6} \left\{ \left( \frac{1}{1 + t_f/d} \right) \left\{ \left( 1 + \frac{t_w}{b} \right) \left( 1 + \frac{t_f}{d} \right)^3 + \right. \right.$$

$$-\left(1 - \frac{2t_w}{b}\right)\left(1 - \frac{t_f}{d}\right)^3 \}$$

or

$$S_{iz} = \frac{bd^2}{6} C_2 C_3 \quad (11)$$

Following these definitions the three cases can now be presented. Case1: The  $t_w$ ,  $b$ , and  $t_f/d$  are all held constant.

$$\begin{aligned} A_i &= bdC_1 \\ I_{iz} &= \alpha_i A_i^{n_i} \quad n_i = 3 \quad \alpha_i = \frac{C_2}{12b^2 C_1^3} \\ S_{iz} &= \gamma_i A_i^{v_i} \quad v_i = 2 \quad \gamma_i = \frac{C_2}{6C_1^2 C_3 b} \end{aligned} \quad (12)$$

Case 2:  $b/d$  a constant  $C_4$ ,  $t_w/d$ , and  $t_f/d$  a constant.

$$\begin{aligned} A_i &= bdC_1 \\ I_{iz} &= \alpha_i A_i^{n_i} \quad n_i = 2 \quad \alpha_i = \frac{C_2}{12C_1^2 C_4} \\ S_{iz} &= \gamma_i A_i^{v_i} \quad v_i = 3/2 \quad \gamma_i = \frac{C_2 C_4}{6C_3 (C_1 C_4)^{3/2}} \end{aligned} \quad (13)$$

Case 3:  $b$  allowed to vary with  $d$ ,  $t_w/b$ , and  $t_f/d$  held constant.

$$\begin{aligned} A_i &= bdC_1 \\ I_{iz} &= \alpha_i A_i^{n_i} \quad n_i = 2 \quad \alpha_i = \frac{d^2 C_2}{12C_1} \\ S_{iz} &= \gamma_i A_i^{v_i} \quad v_i = 3/2 \quad \gamma_i = \frac{dC_2}{6C_1 C_3} \end{aligned} \quad (14)$$

These relations for the three spar box can easily be extended to a  $n$  spar box cross-section. Also, the I-section can be show to be a special case of the spar box with  $n = 1$ .

### CONDITIONS OF OPTIMALITY

The optimization problem defined by equation (1) can be restated in Lagrangian form as

$$\begin{aligned} L(\underline{A}, \underline{\lambda}) &= W(\underline{A}) - \sum_{j=1}^p \lambda_j (Z_j - \bar{Z}_j) \\ p &= \text{number of active constraints} \end{aligned} \quad (15)$$

$L$  is the Lagrangian function, and  $\lambda_j$  are the Lagrangian multipliers corresponding to the active constraints. A constraint will be defined as active if  $Z_j = \bar{Z}_j$ . Minimization of the Lagrangian function with respect to the design variable  $A_i$  gives

$$\frac{\partial L}{\partial A_i} = \frac{\partial W}{\partial A_i} - \sum_{j=1}^p \lambda_j \frac{\partial Z_j}{\partial A_i} = 0 \quad i = 1, 2, \dots, m \quad (16)$$

Equation (16) can be rewritten as

$$\sum_{j=1}^p e_{ij} \lambda_j = 1 \quad i = 1, 2, \dots, m \quad (17)$$

where  $e_{ij}$  is the ratio of constraint to objective function sensitivities and is given by

$$e_{ij} = \frac{\partial Z_j / \partial A_i}{\partial W / \partial A_i} \quad (18)$$

or in matrix form

$$\epsilon \lambda = \underline{1} \quad (19)$$

where  $\epsilon$  is an  $m \times p$  matrix,  $\lambda$  is an  $p \times 1$  matrix and  $\underline{1}$  is a  $m \times 1$  matrix.

### LAGRANGIAN MULTIPLIERS

The Lagrangian formulation introduces more unknowns in addition to the  $m$  design variables. These additional unknowns are the Lagrangian multipliers and there are as many Lagrangian multipliers as active constraints  $p$ . Thus, it is necessary to solve for  $m + p$  unknowns. This section discusses some methods of solution for the Lagrangian multipliers.

Premultiplying equation (19) by  $\epsilon^t w$  yields

$$\epsilon^t w \epsilon \lambda = \epsilon^t w \underline{1} = \overline{Z} \quad (20)$$

where the weighting matrix  $w$  is an  $m \times m$  positive definite diagonal matrix. Here the diagonal elements of the  $w$  matrix are taken to be the individual weights of each element ( $w_{ii} = \rho_i l_i A_i$ ). Equation (20) can now be stated as

$$H \lambda = \overline{Z} \quad (21)$$

Although the  $H$  matrix is non-singular, it is an implicit function of the final design variables. Thus, equation (21) represents a non-linear set of equations. Thus, some approximate methods are used instead of trying to actually solve for the  $\lambda$ s by some iterative scheme. The first approximate method considered consists of using the information at the current design and solving the equation (21) for the Lagrangian multipliers by inverting the  $H$  matrix.

$$\lambda = H^{-1} \overline{Z} \quad (22)$$

One of the drawbacks of this method is that there is no guarantee that all the Lagrangian multipliers will have the appropriate sign. This causes problems when attempting to resize the design variables. The second method considered was originally developed in 1973 by Venkayya and coworkers[3]. This method assumes that only one constraint is active at any one time. For a multi-constraint problem the  $\lambda$ 's simply become weighting parameters. For a single constraint equation (21) reduces to

$$\lambda = \frac{W}{\overline{Z}} \quad (23)$$

In the case of multiple constraints

$$\lambda_j = \frac{W}{\overline{Z}_j} \quad (24)$$

where  $\overline{Z}_j$  is found to be

$$\overline{Z}_j = \sum_{i=1}^m e_{ij} w_{ii} \quad (25)$$

Calculating the  $\lambda$ 's with this approximate method is computationally very efficient compared to inverting the  $H$  matrix. This is because  $w_{ii}$  are the weights of the individual elements and  $\overline{Z}_j$  is a simple function of the given constraint. When finding the  $\lambda$ s by this approximate method it is always assured that the  $\lambda$ s will have the necessary sign.

## RESIZING ALGORITHM

Using the optimality criterion described in equation (17) an iterative resizing algorithm can be found by multiplying equation (17) by  $A_i^\zeta$  and solving for  $A_i$

$$A_i^{r+1} = A_i^r \left[ \sum_{j=1}^p e_{ij} \lambda_j \right]^{1/\zeta} \quad (26)$$

where  $\zeta$  is defined as a step size parameter and  $r$  indicates the  $r$ th iteration. For most problems  $\zeta = 2$  is chosen and gives a good rate of convergence.

## SCALING PROCEDURE

Once a new design has been generated by the resizing algorithm the constraint surface must be found. This is done by the use of a scaling procedure. Let  $\bar{A}$  be the current design vector with the new design found by  $\bar{A} = \Lambda A$  where  $\Lambda$  is the scale factor. If  $dA$  is the difference between two designs, it can be written as

$$dA = \bar{A} - A = (\Lambda - 1)A \quad (27)$$

If the response of the structure is  $R$ , then performing a first order Taylor Series expansion on  $R$  about the current design point  $A$  yields

$$\bar{R} = R + \sum_{i=1}^m \frac{\partial R}{\partial A_i} dA_i \quad (28)$$

Substituting equation (27) for  $dA$ , dividing both sides of the equation by the response  $R$ , and realizing that  $\bar{R} - R = dR$  yields

$$\frac{dR}{R} = (\Lambda - 1) \frac{\sum_{i=1}^m \frac{\partial R}{\partial A_i} A_i}{R} \quad (29)$$

In equation (29) the term  $\left[ \sum_{i=1}^m \frac{\partial R}{\partial A_i} A_i \right] / R$  is either  $\leq 0$  or  $\geq 0$  depending on the type of constraint being considered. For this work only stress and displacement constraints are being investigated and for these cases  $\left[ \sum_{i=1}^m \frac{\partial R}{\partial A_i} A_i \right] / R$  is always  $\leq 0$ . Now defining  $\mu$  as

$$\mu = - \frac{\sum_{i=1}^m \frac{\partial R}{\partial A_i} A_i}{R} \quad (30)$$

equation (29) can be expressed as

$$\frac{dR}{R} = (1 - \Lambda)\mu \quad (31)$$

Solving equation (31) for the inverse of the scale factor  $\Lambda$  and defining  $b = \frac{1}{\mu} \frac{dR}{R}$  gives

$$\frac{1}{\Lambda} = \frac{1}{1 - b} \quad b \ll 1 \quad (32)$$

By performing a binomial expansion and ignoring higher order terms equation (32) becomes

$$\frac{1}{\Lambda} = 1 + b \quad (33)$$

Rearranging equation (33), adding 1 to both sides, and defining  $\beta$  as the target response ratio equation (33) becomes

$$\beta = \frac{\mu}{\Lambda} = \mu + 1 \quad (34)$$

where  $\beta$  is

$$\beta = \frac{\text{New Response}}{\text{Initial Response}} = \frac{R + dR}{R} \quad (35)$$

Finally, solving equation (34) for the scale factor gives

$$\Lambda = \frac{\mu}{\beta + \mu - 1} \quad (36)$$

In the case of truss or membrane structures  $\mu = 1$  and  $\Lambda$  reduces to  $1/\beta$  which is the exact scale factor for stress and displacement constraints.

### SPECIALIZATION TO BENDING ELEMENTS

In the following sections the  $e_{ij}$ ,  $\lambda_j$ , and  $\Lambda$  for bending elements subject to stress, displacement, and generalized stiffness constraints will be discussed.

#### DISPLACEMENT CONSTRAINTS

To find the  $e_{ij}$  for displacement constraints the gradient of the constraint with respect to the design variable ( $\partial Z_j / \partial A_i$ ) is required. There are several methods for finding these gradients (finite difference, direct differentiation, virtual load method), but for this work the virtual load method was incorporated. The virtual load method consists of expressing the active constraint  $Z_j$  in terms of a virtual load vector  $F_j^t$  and the global displacements  $u$ . Thus, the displacement constraint  $Z_j = u_j$  can be written as

$$Z_j = u_j = F_j^t u \quad (37)$$

where  $F_j^t$  is the virtual load vector in which  $F_j = 1$  for  $i = j$  and  $F_j = 0$  if  $i \neq j$ . The  $e_{ij}$  can be found by first partitioning the element stiffness matrix  $K_i$  into axial  $K_{Ai}$  and bending  $K_{Bi}$  components. Next, substitute the relation  $\alpha_i A_i^n$  for  $I_{iz}$  and note that  $\partial W / \partial A_i = \rho_i l_i$ . Then  $e_{ij}$  for a displacement constraint becomes

$$e_{ij} = \frac{f_j^t (K_{Ai} + n_i K_{Bi}) u}{\rho_i l_i A_i} \quad (38)$$

where  $f_j^t$  is the virtual displacement vector corresponding to the virtual load vector  $F_j^t$  and is obtained from the relation

$$F_j = K f_j^t \quad (39)$$

Once  $e_{ij}$  is known the Lagrangian multiplier  $\lambda_j$  for the  $j$ th active constraint can be found by using equation (23). The resulting  $\lambda$ 's are

$$\lambda_j = - \frac{W}{Z_j (\mu_{Aj} + \mu_{Bj})} \quad (40)$$

where the parameters  $\mu_{Aj}$  and  $\mu_{Bj}$  are

$$\mu_{Aj} = \frac{\sum_{i=1}^m f_j^t K_{Ai} u}{F_j^t u} \quad (41)$$

$$\mu_{Bj} = \frac{\sum_{i=1}^m n_i f_j^t K_{Bi} u}{F_j^t u} \quad (42)$$

For the scaling factor of equation (36) the parameter  $\mu$  can be broken up into axial and bending parts  $\mu_{Aj}$ ,  $\mu_{Bj}$ . Therefore,

$$\mu = \mu_{Aj} + \mu_{Bj} \quad (43)$$

and the scale factor for displacement constraints becomes

$$\Lambda = \frac{\mu_{A_j} + \mu_{B_j}}{\beta + \mu_{A_j} + \mu_{B_j} - 1} \quad (44)$$

$1/\beta$  which is the scale factor for membrane structures originally found by Venkayya in 1971[1]. By inspection of  $\mu_{A_j}$  and  $\mu_{B_j}$  and recalling the limits on  $n_i$  for bending elements, it follows that

$$1 \leq \mu_{A_j} + \mu_{B_j} \leq 3 \quad (45)$$

### STRESS CONSTRAINTS

For bending elements the stress in the  $j$ th member is expressed in terms of its bending and axial components as;  $\sigma_j = \sigma_{jA} + \sigma_{jB}$ . The gradients of the stress constraints  $(\partial Z_j / \partial A_i)$  are found by the adjoint variable method which is a generalization of the virtual load method. That is, the constraints are recast in terms of a virtual load vector  $F_j^t$  and the global displacement vector  $u$ . The stress in a given member was written by Venkayya [2] as

$$\sigma_j = T_j^t Q_j \quad (46)$$

where the vector  $T_j^t$  is defined as

$$T_j^t = \begin{bmatrix} \frac{SGN}{A_j} & 0 & \frac{SGN}{S_j} & 0 & 0 & 0 \end{bmatrix} \quad \text{end A} \quad (47)$$

$$T_j^t = \begin{bmatrix} 0 & 0 & 0 & \frac{SGN}{A_j} & 0 & \frac{SGN}{S_j} \end{bmatrix} \quad \text{end B} \quad (48)$$

where  $SGN$  is the sign on the entries of the element force vector  $Q_j$  and  $S_j$  is the section modulus defined as  $S_j = \gamma_j A_j^{v_j}$ . The element force matrix can be expressed in terms of the global displacements  $u$ , the local element stiffness  $k_j$ , and a transformation matrix  $a_j$  as

$$Q_j = k_j a_j u \quad (49)$$

Now the stress  $\sigma_j$  can be written in terms of a virtual load vector  $F_j^t$  and the global displacements as

$$\sigma_j = F_j^t u \quad (50)$$

where the virtual load vector  $F_j^t = T_j^t k_j a_j$ . Unlike displacement constraints the derivative of the virtual load vector with respect to the design variable is not equal to zero ( $\partial F_j^t / \partial A_i \neq 0$ ). This fact causes somewhat more complicated expressions for  $e_{ij}$ ,  $\lambda_j$  and  $\Lambda$ . For stress constraints  $e_{ij}$  becomes

$$e_{ij} = \frac{\delta_{ij} [T_j^t \bar{Q}_j - \bar{T}_j^t Q_j] - f_j^t [K_{A_i} + n_i K_{B_i}] u}{\rho_i l_i A_i} \quad (51)$$

where  $\delta_{ij}$  is the Kronecker delta and  $\bar{T}_j^t$  and  $\bar{Q}_j$  are defined as

$$\bar{T}_j^t = -\frac{\partial T_j^t}{\partial A_j} A_j \quad (52)$$

$$\bar{Q}_j = (k_{A_j} + n_j k_{B_j}) a_j u_j \quad (53)$$

Now, the Lagrangian multiplier for the  $j$ th constraint can be found as

$$\lambda_j = -\frac{W}{Z_j (\mu_{A_j} + \mu_{B_j} - \mu_j)} \quad (54)$$



where  $\mu_{Aj}, \mu_{Bj}$  are the same as those for the displacement constraints.  $\mu_j$  is a new term introduced due to the fact that  $\partial F_j^t / \partial A_i \neq 0$  for stress constraints and it is found to be

$$\mu_j = \frac{T_j^t \bar{Q}_j - \bar{T}_j^t Q_j}{F_j^t u} \quad (55)$$

Finally, following the derivation of equation (36) the scale factor for bending elements subject to stress constraints can be found to be

$$\Lambda = \frac{\mu_{Aj} + \mu_{Bj} - \mu_j}{\beta + \mu_{Aj} + \mu_{Bj} - \mu_j - 1} \quad (56)$$

### GENERALIZED STIFFNESS CONSTRAINT

If  $P$  is the generalized forces and  $u$  is the corresponding generalized displacements then the generalized stiffness constraint can be written as

$$Z_j(A) = \frac{1}{2} P_i^t u_i \quad i = 1, 2, \dots \text{load cases} \quad (57)$$

The  $e_{ij}$ ,  $\mu_{Aj}$ ,  $\mu_{Bj}$ , and  $\lambda_j$  can be found to be

$$\begin{aligned} e_{ij} &= -\frac{\frac{1}{2} u^t [K_{Ai} + n_i K_{Bi}] u}{\rho_i l_i A_i} \\ \mu_{Aj} &= \sum_{i=1}^m \frac{u^t K_{Ai} u}{P_j^t u} \\ \mu_{Bj} &= \sum_{i=1}^m \frac{n_i u^t K_{Bi} u}{P_j^t u} \\ \lambda_j &= -\frac{W}{\bar{Z}_j (\mu_{Aj} + \mu_{Bj})} \end{aligned} \quad (58)$$

It is worthwhile to note that the generalized stiffness constraint does not require the use of the virtual load and displacement vector. This is because the information needed for the gradient of the constraint is already available and no new computations are needed.

### MEETING THE CONDITIONS OF OPTIMALITY

The conditions of optimality state that the product between  $e_{ij} \lambda_j$  summed over all active constraints should be equal to unity at the optimum design. The  $e_{ij}$  are the ratios of the constraint gradients to objective gradients. These gradients are taken with respect to each active design variable  $A_i$ . A design variable is considered active if it satisfies the following criteria:

1. The variable is chosen to participate in the design iteration.
2. The variable is within the given allowable limits.
3. The sensitivity of the  $j$ th active constraint with respect to the design variable  $A_i$  is negative ( $\frac{\partial Z_j}{\partial A_i} < 0$ ). It is important to note that this third criteria is constraint dependent.

If the design variable does not satisfy the above criteria it is considered passive and the conditions of optimality will not be satisfied for that particular design variable. That is  $\sum e_{ij} \lambda_j$  may not be equal to unity at the optimum design. If the variable does not satisfy the first two criteria then the variable is simply eliminated from the design for that particular iteration. However, if the variable passes the first two criteria there is some question on how to handle the third criteria since it is constraint dependent. It is easy to see that  $A_i$  may be passive for one constraint ( $\frac{\partial Z_j}{\partial A_i} > 0$ ) but active for another constraint ( $\frac{\partial Z_{j+1}}{\partial A_i} < 0$ ). This raises the question of how does one deal with the  $e_{ij}$  when a member is passive relative to a particular constraint? In this work if  $\partial Z_j / \partial A_i > 0$  then  $e_{ij}$  for that particular constraint and design variable was set to zero.

## MODIFYING THE RESIZING ALGORITHM

When using the resizing algorithm equation (26) as it appears (that is taking the sum of all the  $e_{ij}\lambda_j$ ) it was found that this tends to over constrain the problem. The converged optimum was well above the known optimum. This was particularly true for multiple loading conditions. Here, instead of using the entire sum for resizing the maximum value of  $e_{ij}\lambda_j$  for the particular design variable was chosen. This can be interpreted as each variable being resized based on the constraint that is most critical for that particular element. This method was found to work well and allowed the algorithm to converge to the known optimum.

### REDUCING THE ERROR IN THE SCALE FACTORS

Due to the Taylor Series and binomial approximations the scaling factors in equations (40,56) are only valid within certain limits. This is especially true when the structure is primarily in bending. This is because equations (40,56) do not reduce to the exact bending scale factor  $(1/\beta)^{1/n}$  if axial contributions are ignored. It is desired that the limits on  $\beta$  extend indefinitely without allowing the error in the scale factor or the response to exceed 5%. If this can be accomplished, then no additional detailed analyses are required to scale the design to the constraint surface. Venkayya[3] achieved this by writing an interaction formula in the non-dimensional parameter space  $\mu^s$ . Since the limits on  $\mu_{Aj}$ ,  $\mu_{Bj}$  and  $\Lambda$  are known this is easily accomplish. The  $\mu_{Aj}$  and  $\mu_{Bj}$  indicate what portion of each scale factor ( $\Lambda_{axial}$ ,  $\Lambda_{bending}$ ) must be used to generate the scale factor for the combined axial bending case. A linear interpolation was used and the error on the scale factor and response was found to be  $< 2\%$  regardless of the  $\beta$ . The scale factor can be represented by a linear interaction formula as

$$\Lambda = \frac{\mu_{Aj}}{\bar{\mu}_{Aj}} \left( \frac{1}{\beta} \right) + \frac{\mu_{Bj}}{\bar{\mu}_{Bj}} \left( \frac{1}{\beta} \right)^{\frac{1}{n}} \quad (59)$$

where  $\mu_{Aj}$ ,  $\mu_{Bj}$  are the non-dimensional parameters found in equations (41,42), and  $\bar{\mu}_{Aj} = 1$  and  $\bar{\mu}_{Bj} = n$ . One could also fit a higher order polynomial between the two ranges of  $\Lambda$  to totally eliminate the error, but an error  $\leq 2\%$  is generally sufficient. If each element in the structure has a different  $n$  then equation (59) can be written as

$$\Lambda = \frac{\mu_{Aj}}{\bar{\mu}_{Aj}} \left( \frac{1}{\beta} \right) + \frac{\mu_{Bj}}{\bar{\mu}_{Bj}} \left( \frac{1}{\beta} \right)^{\frac{1}{n}} \quad (60)$$

where now  $\bar{\mu}_{Bj} = \bar{n}$  and  $\bar{n} = \frac{\mu_{Bj}}{\bar{\mu}_{Bj}}$  with

$$\bar{\mu}_{Bj} = \frac{\sum_{i=1}^m f_j^t K_{Bi} u}{F_j^t u} \quad (61)$$

## SAMPLE PROBLEMS AND RESULTS

On the basis of the preceding derivations, a computer program written in FORTRAN 77 was developed. The frames used in these examples are steel structures with a specific weight of .283 lb/cubic inch and a modulus of elasticity of 29ksi. The type of section used is I-sections and the values for  $\alpha_i$ ,  $n_i$ ,  $\gamma_i$  and  $v_i$  are .2072, 3.0, .393, 2.0 respectively for all members. All problems were solved on a VAX 8600. For these problems the resizing was based on the generalized stiffness and displacement constraints where the scaling was done with respect to the stress and displacement constraints.

The first example optimized is a ten-story symmetric frame show in figure 3 that was reported by Tabak and Wright[4]. In this work the distributed loads used in reference [4] were replaced by concentrated load (figure 4), thus creating new nodes at the mid span of each floor. By doing this the thirty member plane frame reported in reference [4] became a forty member structure. This frame was optimized with stress constraints of 22ksi on each element and displacement constraints of two inches in the horizontal direction for all the nodes. Figure 5 shows that the math programming method converged [5],[6],[7] in seven iterations to a final weight of 35,051 pounds in 37.28 cpu seconds while the optimality criteria converged to 36,421

pounds in four iterations with a cpu time of 4.33 seconds [8]. The optimality criteria final weight is slightly higher (4%) but the cpu time is significantly less (over eight times less) than that of math programming.

The final example optimized is the 313 member frame in figure 6. This frame is subject to five loading conditions (figure 7), along with stress and displacement constraints. The displacement constraints are 4.0 inches in the vertical and 12.0 in the horizontal direction at all nodes. The limit for the stresses in each element was 29 ksi. In figure 8 it can be seen that math programming converges to a final weight of 120,419 pounds in fourteen iterations with a cpu time of 58 minutes. Where the optimality criteria converged to a final weight of 125,166 pounds in twenty-five iterations using approximately 8 minutes of cpu time. Again, the optimality criteria converges to a slightly higher weight (4% higher). The optimality criteria took a significantly large number of iterations to converge compared to math programming. This poor convergence is partly due to constraint switching. Even with the large number of iterations the cpu time for the optimality criteria algorithm is much lower than that for math programming.

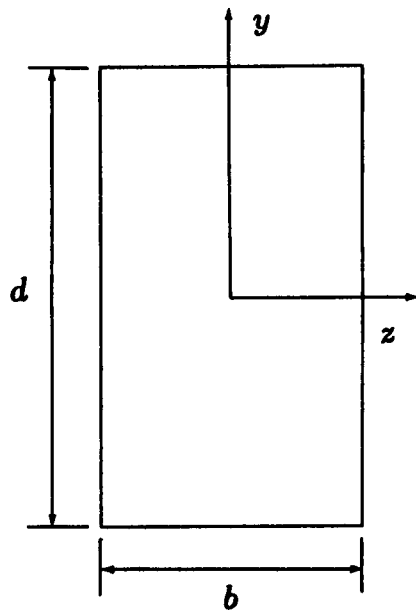
## CONCLUSIONS

The generalized optimality criteria presented in this paper can be applied to any structural optimization problem and related constraints provided that the constraints and their respective gradients are available. The math programming method finds a new design by adding and subtracting gradient information to the current design. Searching from point to point can be a very long and costly procedure. On the other hand, in the optimality criteria a redesign is computed by multiplying (not adding) gradient information to the current design thus, sweeping the design space instead of performing a point search. The optimality criteria is also fairly independent of the the number of design variables, thus allowing literally thousands of independent design variables. This is not the case with math programming where an upper limit on the number of independent design variables is between three to four hundred. When this limit is exceeded computer time becomes excessive and convergence is uncertain. For these problems the math programming method although computationally heavy gave a smooth rate of convergence and overall very good results.

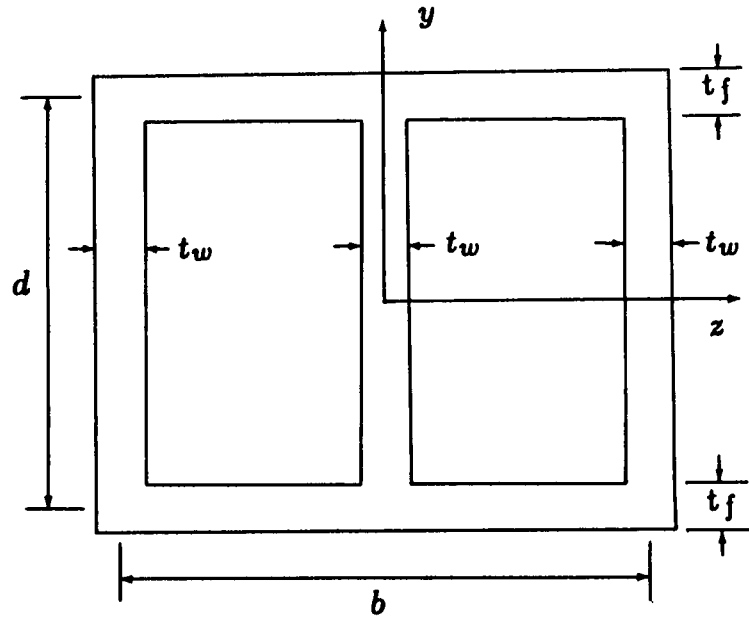
There are some disadvantages to using the optimality criteria which are evident in the sample problems. The optimality criteria, with the current implementation, converges to a 2% to 4% higher weight than that found by the mathematical programming method. Also, for the 313 member frame the optimality criteria method took a very large number of iterations to converge to the optimum. This is very undesirable since detailed analyses for large problems can become extremely expensive.

## REFERENCES

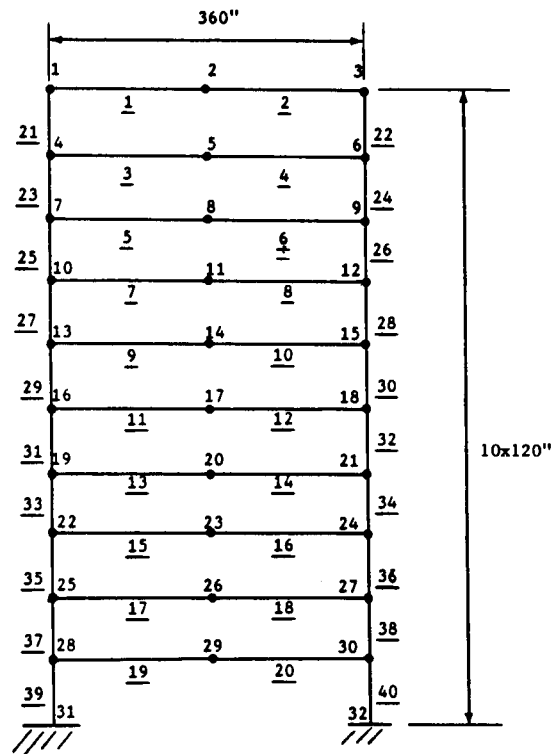
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Rectangular Section  
Figure 1



Three Spar Box Section  
Figure 2



40 Member Frame  
Figure 3

Node	x-Load(lbs)	Load case 1 y-Load(lbs)	Moment(inch · lbs)
1,3		-6000.	
2,		-24000.	
4,6,7,9,12		-12000.	
13,15,16,18,19		-12000.	
21,22,24,25,27		-12000.	
28,30		-12000.	
5,8,11,14,17,20		-48000.	
23,26,29		-48000.	

Node	x-Load(kips)	Load case 2 y-Load(kips)	Moment(inch · lbs)
1,3		-4500.	
2,		-18000.	
4,6,7,9,12		-9000.	
13,15,16,18,19		-9000.	
21,22,24,25,27		-9000.	
28,30		-9000.	
5,8,11,14,17,20		-36000.	
23,26,29		-36000.	

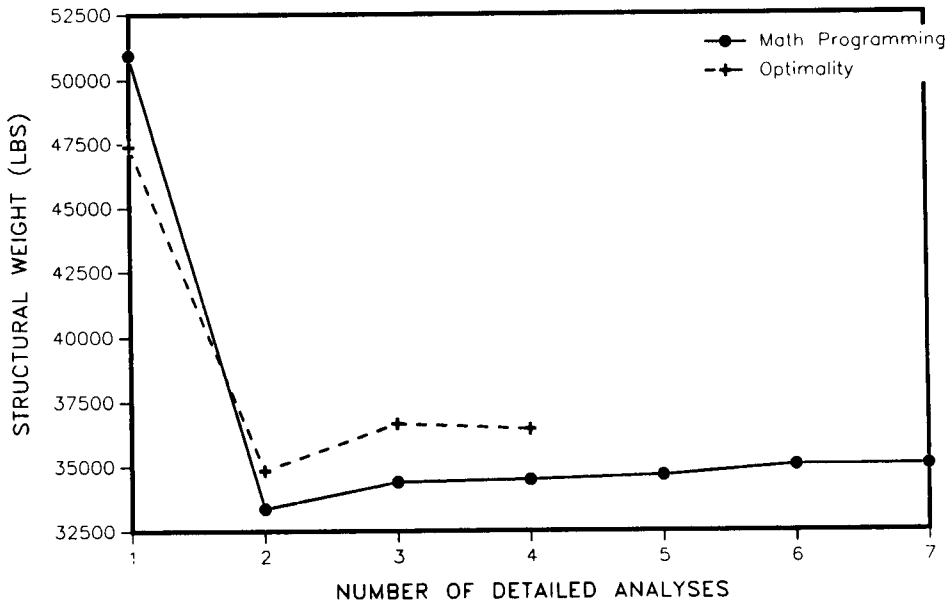
1	3180.
4	6230.
7	6080.
10	5920.
13	5740.
16	5540.
19	5300.
22	5000.
25	4610.
28	4010.

Node	x-Load(kips)	Load case 3 y-Load(kips)	Moment(inch · lbs)
1,3		-4500.	
2,		-18000.	
4,6,7,9,12		-9000.	
13,15,16,18,19		-9000.	
21,22,24,25,27		-9000.	
28,30		-9000.	
5,8,11,14,17,20		-36000.	
23,26,29		-36000.	

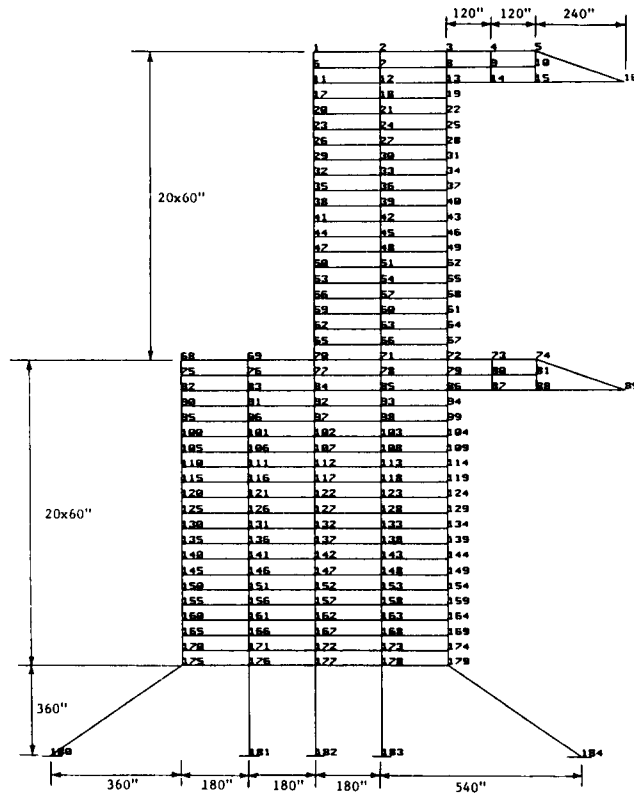
1	-3180.
4	-6230.
7	-6080.
10	-5920.
13	-5740.
16	-5540.
19	-5300.
22	-5000.
25	-4610.
28	-4010.

**Loading Information For 40 Member Frame  
Figure 4**

### 40 ELEMENT STRUCTURE



**Weight Iteration History**  
**Figure 5**



**313 Member Frame**  
**Figure 6**

		Load case 1		
Node	x-Load(kips)		y-Load(kips)	Moment(inch · kip)
15			-26.	
16			-30.	
88			-18.	
89			-20.	

		Load case 2		
Node	x-Load(kips)		y-Load(kips)	Moment(inch · kip)
1	2.0			
6 thru 65 by 5	4.0			
68,75,82	4.0			
90 thru 170 by 5	4.0			

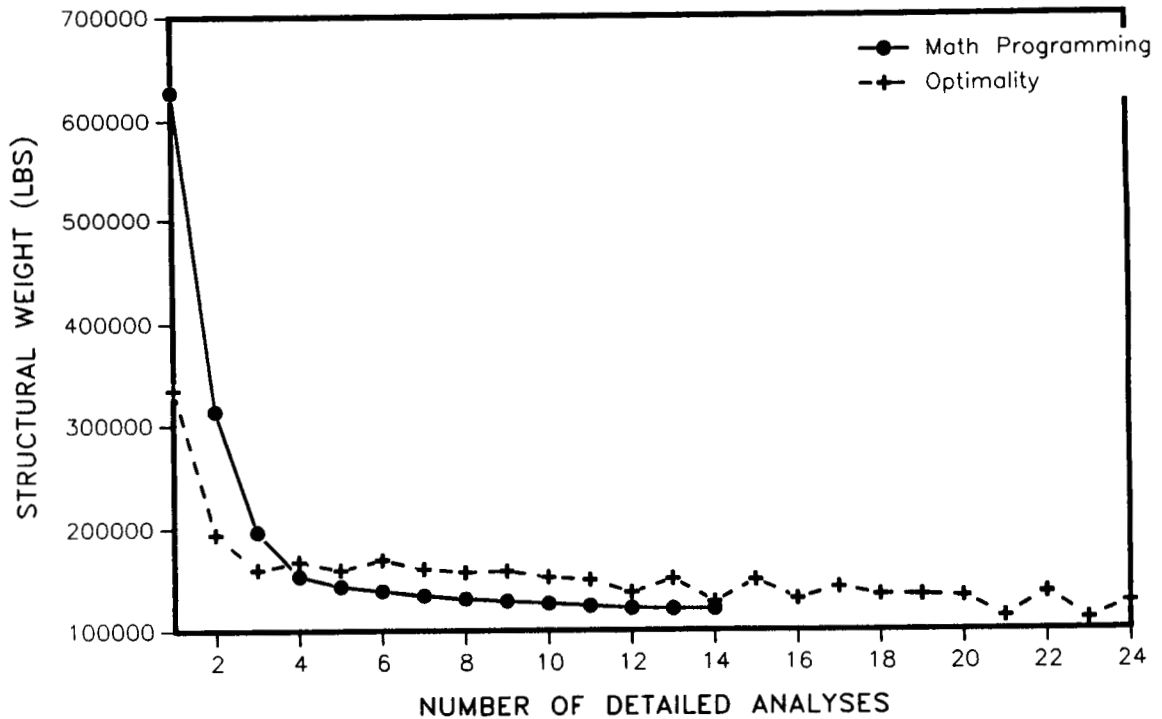
		Load case 3		
Node	x-Load(kips)		y-Load(kips)	Moment(inch · kip)
5	-2.0			
16,19	-4.0			
22 thru 67 by 5	-4.0			
74,89	-4.0			
94 thru 174 by 5	-4.0			

Load case 4 = Load case 1 + Load case 2

Load case 5 = Load case 1 + Load case 3

**Loading Information For 313 Member Frame  
Figure 7**

**313 ELEMENT STRUCTURE**



**Weight Iteration History  
Figure 8**