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PART I: VELOCITY SELECTION

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Analytic theory for the determination of velocity and stability of bubbles in a Hele-Shaw cell

Part I: Velocity selection

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Abstract

An asymptotic theory is presented for the determination of velocity and linear stability of a steady symmetric bubble in a Hele-Shaw cell for small surface tension. In the first part, the bubble velocity U relative to the fluid velocity at infinity is determined for small surface tension T by determining transcendentally small correction to the asymptotic series solution. It is found that for any relative bubble velocity U in the interval $(U_c, 2)$, solutions exist at a countably infinite set of values of T (which has zero as its limit point) corresponding to the different branches of bubble solutions. U_c decreases monotonically from 2 to 1 as the bubble area increases from 0 to ∞. However, for a bubble of arbitrarily given size, as $T \rightarrow 0$, solution exists on any given branch with relative bubble velocity U satisfying the relation $2 - U = c\,T^{2/3}$, where c depends on the branch but is independent of the bubble area. The analytical evidence further suggests that there are no solutions for $U > 2$. These results are in agreement with earlier analytical results for a finger.

In Part II, an analytic theory is presented for the determination of the linear stability of the bubble in the limit of zero surface tension. Only the solution branch corresponding to the largest possible U for given surface tension is found to be stable, while all the others are unstable, in accordance with earlier numerical results.

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1. Introduction

The problem of a single finger of less viscous fluid steadily moving through a more viscous fluid has received considerable attention in recent literature. Bensimon et al (1986), Homsy (1987), and Saffman (1986) summarize the state of affairs as of 1986. The exact solutions of Saffman & Taylor (1958) and Taylor & Saffman (1959) form a degenerate set of solutions for zero surface tension. For specified geometry of the cell and pressure gradient far ahead of the finger, it was found that both the finger width relative to the channel width and the distance of the finger tip from the channel centerline are arbitrary in contradiction to their experiments which resulted in stable symmetric fingers with relative width close to a half for relatively large capillary number. For a steady finger assumed to be symmetric about the channel centerline, Mclean & Saffman's (1980) numerical calculation showed that surface tension determines solution with fixed finger width λ ; subsequent numerical work of Romero (1982) and Vanden-Broeck (1983) suggested that a discrete infinity of such steady solutions is possible. However, for a finger assumed to be symmetric about the channel centerline, Mclean & Saffman (1980) were able to calculate a uniformly valid perturbation expansion for small surface tension in the physical domain of interest without any restriction on the finger width, suggesting that steady solutions exist for any λ . The discrepancy between numerics and perturbation has now been resolved. The analytical work of of Combescot et al (1986), Hong and Langer (1986), and Shraiman (1986) confirm Kessler & Levine's (1985) suggestion based on numerical evidence that for fingers assumed to be symmetric, transcendentally small terms in surface tension neglected in the original Mclean-Saffman (1980) analysis determine a discrete infinity of solutions with the relative finger width tending to one half as surface tension tends to zero. In the context of a generally nonsymmetric finger, analytical evidence by Tanveer (1987b) suggests that only symmetric fingers are possible are possible for nonzero surface tension.

Taylor & Saffman (1959) found that for a bubble of given area assumed symmetric about the channel centerline, the bubble velocity remains arbitrary. Subsequently, Tanveer

(1987a) and Kadanoff (private comm.) independently found exact zero surface tension solutions for non-symmetric bubbles. It was initially reported (Tanveer, 1987a) that there was a discrepancy between the two works as to the number of independent parameters in the problem. However, since then, a relation between the four parameters in the Kadanoff representation has been found, and there is now no conflict between the two independent works. In the general case, for specified bubble area, both the bubble velocity and the distance of the bubble centroid from the channel centerline remains arbitrary for zero surface tension. Numerical calculations (Tanveer, 1986, 1987a) suggest that for nonzero surface tension, a discrete set of symmetric bubble solutions is possible, each characterized by a different velocity. It was not clear from numerical calculation if this discrete set was finite or countably infinite. For small symmetric bubble, explicit asymptotic expression for bubble velocity in terms of a bubble size parameter for fixed surface tension was found by Tanveer(1986) for one branch of solution. This expression was not uniformly valid when surface tension tends to zero. Further, an issue not settled at the time was whether other branches of solutions exist for small bubble size.

The analytical determination of steady bubble velocity for small surface tension for arbitrary sized bubble has only been made recently by Combescot & Dombre (1988) while a first draft of this paper containing most of the results of the first part of this paper was completed. Their analysis confirms our earlier numerical results (Tanveer, 1986, 1987a) and further shows that like the finger, there is a discrete infinity of branches. They also conclude that no solutions are possible when $U > 2$. They also find that some solution branches disappear when the bubble size is made small, a result that was not investigated in the original Tanveer draft. Here, in the first part of this paper, we present an analytic theory for the steady state selection for bubbles that are assumed symmetric about the channel centerline. We find that there exists solution for arbitrary U in the interval $(U_c, 2)$ at a countably infinite values of surface tension which has zero as a limit point. These correspond to a countably infinite branches of bubble solutions. It is found that

U_c depends on the bubble size and decreases monotonically from 2 to 1 as the bubble area is increased from 0 to ∞. However, on any of these branches, as surface tension tends to zero, we find that U tends to 2 with $(2-U) \sim cT^{2/3}$ to the leading order, where c depends on the branch but is independent of the bubble area. At a given value of T and bubble area, only those solution branches exist for which $cT^{2/3} < (2-U_c)$. Since U_c tends to 2 as the bubble area is reduced, a given branch of solution at a particular value of T ceases to exist when the area is so small that the corresponding U_c does not satisfy this condition. In that sense, we agree with Combescot & Dombre (1988) statement that solution branches disappear for small bubble area. However, for any given area for any given branch characterized by a specific c, if surface tension T is small enough, solution exists and so in that sense all the solutions branches exists even for small bubbles though at a much smaller range of values of surface tension. Aside from this discrepancy, our results are in agreement with Combescot & Dombre's conclusion. We also point out that the asymptotic power law dependence of $2-U$ on T for small T is not uniformly valid as the bubble area tends to zero. In the limit, when the bubble area tends to 0 (area is is then proportional to α^2 as defined later) such that the ratio of bubble area to the square of surface tension is small, the analytic expression given by Tanveer (1986) [Eqn. 38] holds.

The method presented here is rather different from Combescot & Dombre and is more in accordance with the Tanveer (1987b) analysis for a non-symmetric finger. Though limited only to symmetric bubbles, our method is easily generalizable to the more general time dependent problem (linear stability for example, as shown in the second part of the paper) or to the steady problem with boundary conditions incorporating the transverse curvature and the thin film effects as will be seen in an upcoming paper.

In the second part of this paper, we extend the approach developed in the first part to address the question of linear stability of a steady bubble for small non-zero surface tension where the usual numerical approaches break down because of the inherent ill-posedness of the stability operator. Since the equations for the bubble and the finger are quite similar, Tanveer & Saffman (1987) conjectured that the bubble on the Mclean-Saffman branch like the finger (Kessler & Levine, 1985, 1986, Tanveer, 1987c) would remain stable in the limit of zero surface tension and that all the other branches would be unstable. Extrapolation of their numerical results to zero surface tension is consistent with this conjecture though there is room for doubt since the modal decay rates changed very substantially at the smallest value of surface tension for which reliable numerical results could be obtained. We confirm that conjecture analytically in the second part of this paper, though only for symmetric disturbances. Though the final result is not unexpected and the theory is not all that different from the theory developed earlier for the finger (Tanveer, 1987c), it is expected to provide a framework for future study into the non-linear interactions of the modes which is thought to be responsible for the experimentally observed instability (Bensimon, 1986). The idealization of a semi-infinite finger introduces complications in a possible nonlinear analysis owing to the geometric singularity at the tail and the problem of specifying boundary conditions appropriate for the class of disturbances that convect to infinity or changes the width of the finger. The previous analytic theory (Tanveer, 1987c) ruled out these classes of disturbances by assuming that the finger was asymptotically parallel to the channel and that the finger width was time invariant. The study of bubble stability may be expected to be an important link towards understanding phenomena such as dendritic instability (Couder et al, 1986) or perhaps also the mechanism of complicated fractal like patterns observed experimentally or the interfacial motion in a Hele-Shaw cell (Maxworthy, 1987). In this respect, it may be pointed out that we found discrete modes affecting the sides and the back of a bubble which have no analog for the finger.

2. Analytical continuation of equations to the unphysical plane

Our starting point will be Tanveer (1986). For an overview of the entire bubble problem and recent progress in this area, the reader is referred to Tanveer (1988). We nondimensionalize all lengths and velocities by a and V, which are respectively the half width of the cell and the displaced fluid velocity at ∞ . The Tanveer (1986) paper shows that under some simplifying assumptions on the flow conditions at the bubble boundary, the flow velocity and the shape of a steady symmetric bubble with a smooth boundary are all describable in terms of a function f , analytic on and inside the unit semi-circle in the ς plane such that it satisfies boundary conditions given by equations (7) and (8) of that paper. For convenience of analysis, we rescale f and h of that paper by constants such that equations (7) and (8) now become

$$Im\ f = 0. \tag{1}$$

on the real diameter and on $\varsigma = e^{i\nu}$, ν real between 0 and π ,

$$Re\ f = -\frac{\gamma}{|f' + h|}\left\{1 + Re\ \varsigma\frac{d}{d\varsigma}\ell n(f' + h)\right\} \tag{2}$$

where primes denote derivative with respect to ς . γ is a dimensionless surface tension parameter defined by

$$\gamma = \frac{\pi^2 U}{4\alpha^2}\ \frac{b^2\ T}{12\mu}\frac{1}{[U(1+\alpha^2) - 2\alpha^2]^2} \tag{3}$$

where b , T , and μ denote the narrow gap width, surface tension and the viscosity of the more viscous fluid, the viscosity of the less viscous fluid being neglected. The effect of finite viscosity ratio of the two fluids has recently been studied (Tanveer & Saffman, 1988) and the results show only qualitative changes in the stability properties. The function $h(\varsigma)$ in (2) is defined as

$$h(\varsigma) = \frac{(1 - p^2\varsigma^2)}{(\varsigma^2 - \alpha^2)(1 - \varsigma^2\ \alpha^2)} \tag{4}$$

where

$$p^2 = \frac{U(1+\alpha^2) - 2}{U(1+\alpha^2) - 2\alpha^2}. \tag{5}$$

Here the symbol α (the same notation as Tanveer (1986)) is a parameter between 0 and 1 and will be treated as given along with γ . Small bubbles correspond to small values of α . For large bubbles, α approaches one. The determination of relative bubble velocity

U for specified bubble area and surface tension for given cell geometry, fluid viscosity and flow conditions at infinity is equivalent to determining the parameter p for specified α and γ.

It is to be noted (Tanveer, 1986) that the analyticity of f on the circular arc $|\varsigma| = 1$ is equivalent to requiring that the bubble boundary be smooth. In particular, the analyticity of f at $\varsigma = \mp 1$ corresponds to a smooth tip and a back. It follows that for a smooth bubble tip and back, equation (1) must also be valid in some open intervals on the real ς axis containing ∓ 1.

For $\gamma = 0$, i.e., zero surface tension, it is clear from (1) and (2) that $f = 0$ is the solution for any α, and this corresponds to the Taylor-Saffman solution for which p^2 is arbitrary in the interval $(-1, 1)$, corresponding to U in the open interval $(1, \infty)$. Since we are interested in the perturbing effect of small γ on the Taylor-Saffman solution, we will assume that p^2 is still in (-1, 1) for small but nonzero γ. Tanveer (1986) constructed a formal perturbation expansion of the form

$$f = \sum_{n=1}^{\infty} \gamma^n f_n \tag{6}$$

and showed that such an expansion is consistent to every order in γ without any further constraint on parameters α and p. Every term f_n is found to be analytic on the unit semi-circular boundary including ± 1. Thus, on the real ς axis there exists an open interval around each of $\varsigma = 1$ and -1 where f_n has vanishing imaginary part in view of (1). However, the ability to construct a consistent perturbation expansion need not imply that a solution f satisfying all the conditions exists for nonzero γ. This has been illustrated earlier (Tanveer, 1986) in the context of a small bubble. In order that a solution does exist, transcendentally small terms in γ in the asymptotic expansion neglected in (6), must also be analytic in $|\varsigma| \leq 1$ along with terms of algebraic order. For arbitrary p, this condition on analyticity is generally violated by these transcendentally small terms at $\varsigma = -1$, and this means that the bubble tip is generally not smooth. Analytical evidence based on explicit calculation of the leading order transcendentally small correction to (6) suggests that it is possible to satisfy the smooth tip condition only for a discrete set of p with $p^2 < \alpha^2$, i.e., for $U < 2$.

In order to calculate the transcendentally small terms in surface tension, it is necessary to analytically continue the equations outside the unit circle to points where the perturbation expansion (6) is nonuniform. The sources of non-uniformity in the unphysical region contribute to transcendentally small terms in surface tension in the physical region. This idea in the context of a nonlinear problem is originally due to Kruskal & Segur (1986). To find the sources of non-uniformity of the perturbation expansion (6), we now analytically continue the leading order perturbation term f_1 outside the unit circle across the arc of the unit semi-circle.

On substituting (6) into (1) and (2), it is clear that f_1 satisfies (1) on the real diameter of the unit semi-circle and that on the arc of the unit semi-circle, $\varsigma = e^{i\nu}$ for ν real in $[0, \pi]$,

$$Re\ f_1 = -\frac{1}{|h|}\ Re\ [1 + \varsigma\ \frac{h'}{h}\ ]. \tag{7}$$

In view of (1), Schwarz reflection principle holds, and we find that (7) must hold on the entire circle $|\varsigma| = 1$. From Poisson's integral formula relating a harmonic function and its conjugate in the interior of the unit circle to its boundary value, one finds that for $|\varsigma| < 1$,

$$f_1(\varsigma) = \frac{1}{2\pi} \int_0^{2\pi} d\nu'\ (\frac{\varsigma + \varsigma'}{\varsigma' - \varsigma})\ Re\ f_1\ (\varsigma') \tag{8}$$

where $\varsigma' = e^{i\nu'}$. Substituting for h from (4) into (7) and realizing that on the circular boundary the complex conjugate of ς' is $1/\varsigma'$, one finds that for $|\varsigma| < 1$,

$$f_1(\varsigma) = \frac{1}{2\pi\ i} \oint_{|\varsigma'|=1} \frac{d\varsigma'}{\varsigma'} (\frac{\varsigma + \varsigma'}{\varsigma' - \varsigma})\ K(\varsigma')\{1 + \frac{p^2 \varsigma'^2}{1 - p^2 \varsigma'^2} + \frac{p^2}{\varsigma'^2 - p^2}\} \equiv I_1(\varsigma) \tag{9}$$

where the contour of integration is in the counterclockwise sense and

$$K(\varsigma) = \frac{(\varsigma^2 - \alpha^2)\ (1 - \alpha^2 \varsigma^2)}{\varsigma (1 - p^2 \varsigma^2)^{1/2}\ (\varsigma^2 - p^2)^{1/2}} \tag{10}$$

By the standard technique of analytic continuation through contour deformation in the ς' plane, one finds that for $|\varsigma| > 1$

$$f_1(\varsigma) = I_1(\varsigma) + 2\,[1 + \frac{p^2\varsigma^2}{1 - p^2\varsigma^2} + \frac{p^2}{\varsigma^2 - p^2}]\, K(\varsigma) \tag{11}$$

Clearly f_1 is singular at $\varsigma = \pm 1/p$ and it is easy to see that in the neighborhoods of these points the perturbation expansion (6) will be non-uniform. It is therefore necessary to construct an inner expansion by rescaling dependent and independent variables near these points and match it to the outer expansion. The terms of the inner expansion that do not match with the outer expansion are transcendentally small in the outer region that includes the physical domain and it is our objective to calculate these transcendental terms to the leading order. For that purpose, we analytically continue equations (1) and (2) to determine f for $|\varsigma| > 1$.

First, we define $g(\varsigma)$ to be an analytic function of ς inside the unit semi-circle with the boundary conditions

$$Im\ g = 0. \tag{12}$$

on the real axis and on the arc of the unit semi-circle, we require

$$Re\ g = |h + f'|\ Re\ f \tag{13}$$

If f is analytic on the boundary of the unit-semi circle, it is clear the g will be analytic as well, since $f' + h$ is nonzero and analytic on the boundary of the unit circle (see Tanveer, 1986). Further from the reflection principle, it follows that g is analytic in and on the entire unit circle if f is. Equation (2) can be written as

$$Re\ [1 + \varsigma\frac{(f'' + h')}{(f' + h)} + \frac{g}{\gamma}] = 0 \tag{14}$$

The term within the square parentheses in (14) is an analytic function in $|\varsigma| \leq 1$ except for simple pole singularities at $\varsigma = \pm\alpha$. By adding a function that is purely imaginary on $\varsigma = e^{i\nu}$ and subtracts off the simple poles at $\varsigma = \pm\alpha$, we find that (14) is equivalent to

$$Re\,[1 + \varsigma\frac{(f'' + h')}{(f' + h)} + \frac{g}{\gamma} + \frac{2\varsigma^2}{\varsigma^2 - \alpha^2} - \frac{2}{1 - \alpha^2\varsigma^2}] = 0. \tag{15}$$

Since the expression within the square parentheses in (15) is analytic within the unit circle whose imaginary part is easily seen to vanish on the real diameter because of (1) and (12), it follows that for $|\varsigma| \le 1$

$$1 + \varsigma\frac{(f'' + h')}{(f' + h)} + \frac{g}{\gamma} + \frac{2\varsigma^2}{\varsigma^2 - \alpha^2} - \frac{2}{1 - \alpha^2\varsigma^2} = 0. \tag{16}$$

For $|\varsigma| < 1$, it follows from Poisson's integral formula (8) applied to the analytic function g satisfying (12) and (13) and using the definition of h from (4) that

$$g(\varsigma) = \frac{1}{4\pi i}\oint_{|\varsigma'|=1} d\varsigma' \left\{ \frac{(l_1(\varsigma')\, l_2(\varsigma'))^{1/2}}{\varsigma'(\varsigma'^2 - \alpha^2)(1 - \varsigma'^2\alpha^2)} \right\} \cdot$$
$$\{f(\varsigma') + f(1/\varsigma')\} \left\{ \frac{\varsigma + \varsigma'}{\varsigma' - \varsigma} \right\} \equiv I(f, \varsigma) \tag{17}$$

where

$$l_1(\varsigma) = [\varsigma^2(\varsigma^2 - p^2) + (1 - \alpha^2\varsigma^2)(\varsigma^2 - \alpha^2) f'(1/\varsigma)] \tag{18}$$

$$l_2(\varsigma) = [(1 - p^2\varsigma^2) + (\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2) f'(\varsigma)]. \tag{19}$$

The analytic continuation of g for $|\varsigma| > 1$ is given by

$$g(\varsigma) = I(f, \varsigma) + \frac{\{l_1(\varsigma)\, l_2(\varsigma)\}^{1/2}}{(\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)} (f(\varsigma) + f(1/\varsigma)). \tag{20}$$

For $|\varsigma| > 1$, equation (16) is valid where g is related to f through (20). On expressing g in terms of f, we find that (16) is a non-linear integro-differential equation for f, which appears to be too formidable to be of any practical use. However, we note that the transcendentally small terms in the physical domain of interest are not transcendentally small everywhere in $|\varsigma| > 1$, and, therefore the transcendental correction to the asymptotic expansion (6) is significant in some regions. However, the integrand in I involves f and f' on the unit circle where any correction to the perturbation expansion (6) is assumed to be transcendentally small. Therefore it is legitimate to substitute (6) for f in the integral I. Again, for the same reason, the term $f(1/\varsigma)$ can be replaced by the perturbation expansion (6) with transcendental error since $1/\varsigma$ is inside the circle when ς is outside. To the leading order we have

$$f(1/\varsigma) \sim \gamma f_1(1/\varsigma) \tag{21}$$

and

$$\gamma^{-1} I(f,\varsigma) \sim I_2(f_1,\varsigma) \tag{22}$$

where

$$I_2(f_1,\varsigma) \equiv \frac{1}{4\pi i} \oint_{|\varsigma'|=1} d\varsigma' \left\{ \frac{(\varsigma'^2 - p^2)^{1/2}(1-p^2\varsigma'^2)^{1/2}}{(1-\varsigma'^2\alpha^2)(\varsigma'^2-\alpha^2)} \right\} . \\ \{f_1(\varsigma') + f_1(1/\varsigma')\} \left\{ \frac{\varsigma+\varsigma'}{\varsigma'-\varsigma} \right\} \tag{23}$$

It is clear from observation of the right hand side of (23) and comparison with Poisson's integral formula (8) that for $|\varsigma| \leq 1$, $I_2(f_1,\varsigma) = G_1(\varsigma)$ where $G_1(\varsigma)$ is defined by the boundary conditions

$$Im\, G_1 = 0 \tag{24}$$

on the real axis and on the arc of the semi-circle $\varsigma = e^{i\nu}$

$$Re\, G_1 = |h|\, Re\, f_1 = -\, Re[1+\varsigma h'/h]. \tag{25}$$

The latter equality in (25) follows from (7). From (24) and (25), using the definition of h from (4), it follows that for any ς

$$G_1(\varsigma) = 1 + \frac{2p^2\varsigma^2}{1-p^2\varsigma^2}. \tag{26}$$

Again from (23), we can easily prove the property that

$$I_2(f_1,\varsigma) = -I_2(f_1,\varsigma^{-1}).$$

From this and (26), it follows that for $|\varsigma| > 1$,

$$I_2(f_1,\varsigma) = -1 - \frac{2p^2}{\varsigma^2 - p^2}. \tag{27}$$

On substituting for I and $f(1/\varsigma)$ as indicated in (21) and (22), (16) becomes a non-linear 2nd order ODE for the function $f(\varsigma)$ outside the unit circle:

$$\frac{\varsigma \frac{f''+h'}{f'+h} + \frac{1}{\gamma} \frac{(l_1 l_2)^{1/2}}{(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)} f = -\frac{(l_1 l_2)^{1/2}}{(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)} f_1(}{1\varsigma) - \frac{2\varsigma^2}{\varsigma^2-\alpha^2} + \frac{2}{1-\varsigma^2\alpha^2} + \frac{2p^2}{\varsigma^2-p^2}} \tag{28}$$

One can attempt to directly calculate numerical solutions to (28). This method of solution would not be any easier than solving equations (1) and (2) directly as done earlier (Tanveer, 1986). Also, the ill posedness for very small surface tension as in the previous solution method (Tanveer, 1986) will not removed if one were to directly calculate numerical solutions to (28) because f changes drastically over small distances in parts of the complex ς plane, as our analytical calculations reveal.

3. Leading order WKB transcendental correction for small surface tension

By the standard procedure of dominant balancing (see Bender & Orszag, e.g.), one finds that the leading order behavior of solution to (28) for small γ is given by

$$f(\varsigma) \sim \frac{2\gamma(\varsigma^2-\alpha^2)\,(1-\alpha^2\varsigma^2)}{(1-p^2\varsigma^2)^{1/2}(\varsigma^2-p^2)^{1/2}\varsigma}\left[1 + \frac{p^2}{\varsigma^2-p^2} + \frac{p^2\varsigma^2}{1-p^2\varsigma^2}\right] - \gamma f_1(1/\varsigma) \tag{29}$$

and this is exactly $\gamma\, f_1$ as given by (11), since $f_1(1/\varsigma) = -I_1(1/\varsigma)$. If we keep on using dominant balancing arguments to extract higher and higher order terms in γ from (28), we recover the perturbation expansion in (6). However, in order that (29) is the dominant behavior of the solution to (28), we must ensure that any transcendental terms in γ that can be part of the solution to (28) is indeed small and not large in regions of the complex ς plane adjoining the physical region $|\varsigma| \leq 1$. To find the leading order transcendental correction to (6), we first linearize (28) about (29). The homogeneous part of the linearized equation is found to be

$$f'' - \frac{H'}{H} f' + \frac{L}{\gamma} f = 0 \tag{30}$$

where

$$H = \frac{\left(1 - p^2 \varsigma^2\right)^{3/2}}{\left(\varsigma^2 - p^2\right)^{1/2} (\varsigma^2 - \alpha^2)(1 - \alpha^2 \varsigma^2)} \tag{31}$$

and

$$L = \frac{\left(1 - p^2 \varsigma^2\right)^{3/2} \left(\varsigma^2 - p^2\right)^{1/2}}{\left(\varsigma^2 - \alpha^2\right)^2 \left(1 - \alpha^2 \varsigma^2\right)^2}. \tag{32}$$

At this point it is appropriate to point out an error in an earlier work (Tanveer, 1987b) for the finger in linearizing the non-linear equation equivalent to (28). It was found that the coefficient of f' in linear equation equivalent to (30) is h'/h where h in that case is defined slightly differently from here. The actual coefficient of f' should have been H'/H where H is defined in an analogous manner relative to h. Fortunately, this error has no bearing on the final result.

Returning to (30), the leading order WKB solutions to (30) for small γ, denoted by g_1 and g_2 are given by

$$g_1(\varsigma) = H^{1/2} L^{-1/4} e^{\gamma^{-1/2} P} \tag{33}$$

$$g_2(\varsigma) = H^{1/2} L^{-1/4} e^{-\gamma^{-1/2} P} \tag{34}$$

where

$$P(\varsigma) = \int_{1/p}^{\varsigma} i L^{1/2}(\varsigma')\, d\varsigma' = i \int_{1/p}^{\varsigma} d\varsigma \frac{\left(1 - p^2 \varsigma^2\right)^{3/4} \left(\varsigma^2 - p^2\right)^{1/4}}{(\varsigma^2 - \alpha^2)(1 - \alpha^2 \varsigma^2)}. \tag{35}$$

g_1 and g_2 are transcendentally large or small in γ depending on whether $Re\ P$ is greater or smaller than zero. The lines $Re\ P = constant$ emanating from each of the critical points $\varsigma = \pm\frac{1}{p}$ will be called the Stokes lines and divide the upper half ς plane into several sectors. The leading order transcendental correction away from the immediate neighborhoods of the turning points (will be more precise later) $\varsigma = \pm\frac{1}{p}$ (where the WKB solutions (33) and (34) are invalid) must be constant multiples of g_1 and g_2 in each of the sectors. On analytical continuation of f across the Stokes lines, the linear combination of g_1 and g_2 necessary to describe the leading order transcendental correction changes. This question is addressed in more detail for the finger problem (Tanveer, 1987b). The connection between the set of constant multiples of g_1 and g_2 in each of the different sectors (the so-called connection formulas) can be found by examination of the leading order inner solutions near the turning points $\varsigma = \pm\frac{1}{p}$. The details of the inner outer matching in each of the different sectors differ according to the Stokes lines picture in the complex upper half ς plane, and these are different for the cases $\alpha^2 > p^2 > 0$, $p^2 > \alpha^2$ and $p^2 < 0$. From definitions of p^2 in (5), these cases correspond to $\frac{2}{1+\alpha^2} < U < 2$, $U > 2$ and $1 < U < \frac{2}{1+\alpha^2}$ respectively. We consider each of the three different cases in the next three sections. At this point, it is appropriate to point out that the WKB solutions are also invalid near $\varsigma = \pm p$, but the Stokes lines emanating from these points do not affect our analysis since it is easily seen that the local solutions constructed near these points can be made to match with the same linear combination of g_1 and g_2 even when we cross these lines. Thus we ignore the Stokes lines emanating from $\varsigma = \pm p$.

4. The case of $\alpha^2 > p^2 > 0$

In this case, the Stokes lines of constant $Re\ P$ emanating from $\varsigma = \pm\frac{1}{p}$ are shown in Figure 1. This picture was obtained by numerical integration in (35). The Stokes lines divide the upper half ς plane into different sectors. It is necessary to have the leading order asymptotic behavior for small γ given by (29) to be valid in each of the sectors I, II, and III since they extend to the inside of the physical region $|\varsigma| < 1$ in the upper half ς plane. On inclusion of the leading order transcendentally small correction in γ (which is beyond all algebraic order) in the description of the asymptotic behavior, in sector I of Fig. 1,

$$f \sim \gamma f_1 + C_1\, g_1 + C_2\, g_2. \tag{36}$$

It is easily seen from (35) that in the upper half plane $Re\ P$ has a maximum value of $\gamma^{1/2}\beta$ in sector I at any point on the real axis in the open interval $(\alpha, 1/\alpha)$, where

$$\beta = \frac{\pi}{2\alpha}\,\frac{\left(\alpha^2 - p^2\right)^{3/4}\left(1 - \alpha^2 p^2\right)^{1/4}}{\left(1 - \alpha^4\right)\,\gamma^{1/2}}. \tag{37}$$

Thus, in order that the term $C_1 g_1$ be transcendentally small compared to γf_1 , it is necessary that

$$|C_1|e^{\beta} = O(1). \tag{38}$$

On the other hand g_2 is continuously decreasing in size as we move away from $\varsigma = 1/p$ in sector I. However, near $Re\ P = 0$, the decay rate is arbitrarily small. Thus in order that $C_2 g_2$ be transcendentally small everywhere in sector I, it is necessary that

$$|C_2| = O(1). \tag{39}$$

In order that (36) be real on the real axis in the some open interval containing $\varsigma = 1$, which it must be for the bubble back to be smooth, we must have

$$C_2 = C_1^* e^{2\beta}. \tag{40}$$

Actually, the condition of a smooth bubble back implies more than (40). Even the higher order transcendental corrections not accounted for in (36) must be real on the real ς axis for some interval containing $\varsigma = 1$.

In sector II of Fig. 1,

$$f \sim \gamma f_1 + C_3\, g_1 + C_4\, g_2. \tag{41}$$

It is easy to see that $Re\ P$ is negative in this sector and has a maximum and minimum value of 0 and $Re\ P_m$ where

$$P_m \equiv P(-1/p) = 2^{1/2}(-1+i) \int_{\frac{1}{p}}^{\infty} d\varsigma \frac{(p^2\varsigma^2 - 1)^{3/4} (\varsigma^2 - p^2)^{1/4}}{(\varsigma^2 - \alpha^2)(\alpha^2\varsigma^2 - 1)} \tag{42}$$

has negative real part. Thus in order that $C_4 g_2$ remains transcendentally small compared to γf_1 , it is necessary that

$$|\, C_4\, e^{-\gamma^{-1/2} P_m} | = O(1). \tag{43}$$

Again, since $Re\ P$ in this sector can be arbitrarily close to 0 as the Stokes line $Re\ P = 0$ is approached, it is necessary that

$$|C_3| = O(1). \tag{44}$$

as well.

In sector III of Fig. 1,

$$f \sim \gamma f_1 + C_5 g_1. \tag{45}$$

There cannot be any g_2 term in (45) because it grows without bounds in this sector as $\varsigma = -\alpha$ is approached. Since $Re\ P$ can be arbitrarily close to $Re\ P_m$, the maximum value of $Re\ P$ in this sector, in order that the transcendental term in (45) be small, it is necessary that

$$|C_5\, e^{\gamma^{-1/2} P_m}| = O(1). \tag{46}$$

The condition that the bubble tip be smooth implies that f is real on the real axis for some open interval containing $\varsigma = -1$. Note that γf_1 and all other terms of the perturbation expansion (6), satisfy this condition automatically.

From the definition of g_1 in (33) and noting from (35) that $P(\varsigma) - P(-1/p)$ has a constant imaginary part β on the real axis in the interval $(-1/\alpha, -\alpha)$, it follows that we must satisfy the condition

$$Arg\ \left[C_5\, e^{\gamma^{-1/2}\, P_m}\, e^{-i3\pi/4}\, e^{i\beta}\right] = n\,\pi, \tag{47}$$

where n is any integer. Again, the condition of the smooth bubble tip implies more than (47). Even the higher order transcendental correction neglected in (45) should be real on the real ς axis in some open interval containing -1. We now need to determine the unknown coefficients C_1 through C_5 subject to conditions (40) and (47) to complete the determination of leading order transcendentally small terms in the physical domain. Note that equations (40) and (47) may be considered as two real equations to determine one complex constant C_5, since each of the pairs $(C_3,\ C_4)$ and $(C_1,\ C_2)$ are functions of C_5 because equations (36), (41), and (45) describe the asymptotic behavior of the same analytic function f continued across the Stokes lines. To determine the relation between $(C_1,\ C_2)$, $(C_3,\ C_4)$ and C_5, i.e. to determine the so-called connection formulas, it is clear from Fig. 1 that one needs to investigate the analytic function f in the neighborhood of the critical (or so-called turning) points $\varsigma = \pm 1/p$ where the asymptotic behavior (36), (41), and (45) are not valid uniformly as $\gamma \to 0$. The form of the equations determining the leading order behavior near each of these points depends on the size of α and p. Again within the case considered in this section, there are four possible subcases:

(a) $\alpha^2 - p^2 = O(1)$ with α , $p = O(1)$,

(b)$\alpha^2 - p^2 << 1$ with α , $p = O(1)$

(c) $\alpha << 1$

(d) $\alpha = O(1)$, $p << 1$.

4a. Subcase: $\alpha^2 - p^2 = O(1)$; α , $p = O(1)$

From the definition of p^2 in (5), it follows that this subcase corresponds to U in the interval $(\frac{2}{1+\alpha^2}, 2)$ with U not too close to the end points of this interval. The condition on α implies that the bubble size is not too small.

For analysis in the immediate neighborhood of $\varsigma = 1/p$, we introduce local change of variable

$$(1 - p\varsigma) = e^{-i2\pi/7} r^{-2/7} \mu_2 \tag{48}$$

$$f = -\frac{2 r^{-4/7} p^3 e^{-i4\pi/7}}{(1 - \alpha^2 p^2)(\alpha^2 - p^2)} H_2(\mu_2) \tag{49}$$

where

$$r = \frac{2^{3/2} p^5 (1-p^4)^{1/2}}{\gamma (\alpha^2 - p^2)^2 (1-\alpha^2 p^2)^2}. \tag{50}$$

Given the assumptions on this subcase, r is large for small γ. For large r, we find from (28) that the leading order equation for $\mu_2 = O(1)$ is given by

$$H_2'' - (\mu_2 - H_2')^{3/2} H_2 = 1. \tag{51}$$

For large μ_2, it is easy to see that

$$H_2 \sim -\frac{1}{\mu_2{}^{3/2}}, \tag{52}$$

and this matches with $f = \gamma f_1$ as given in (11) when $(1 - p\varsigma) \to 0$. The leading order transcendental term can be found by linearizing (51) about (52). The homogeneous part of this linear equation is:

$$\frac{d^2\,H_t}{d\mu_2{}^2} - \frac{3}{2\mu_2}\frac{d\,H_t}{d\mu_2} - \mu_2{}^{3/2}\,H_t \;=\; 0. \tag{53}$$

By WKB method, one finds that the leading order asymptotic behavior of two independent solution to (53) for large μ_2 is given by

$$\mu_2{}^{3/8}\,e^{\pm 4\,\mu_2{}^{7/4}/7}. \tag{54}$$

These must be the leading order transcendental corrections to (52). As $\mu_2 \to \infty$ with $Arg\ \mu_2$ in $(-2\pi/7,\ 2\pi/7)$, the leading order behavior (on inclusion of transcendental behavior) is:

$$H_2 \;\sim\; -\frac{1}{\mu_2{}^{3/2}} \;+\; \gamma_{22}\,\mu_2{}^{3/8}\,e^{-4\,\mu_2{}^{7/4}/7} \;+\; \gamma_{21}\,\mu_2{}^{3/8}\,e^{4\,\mu_2{}^{7/4}/7}, \tag{55}$$

and this matches to (36) in sector I of Fig. 1 if

$$\frac{C_1}{\gamma_{21}} \;=\; \frac{C_2}{\gamma_{22}} \;=\; -2^{5/8}\,\frac{p^{9/4}\,(1-p^4)^{3/8}}{(1-\alpha^2p^2)(\alpha^2-p^2)}\,e^{-i\,13\,\pi/28}\,r^{-13/28}. \tag{56}$$

As $\mu_2 \;\to\; \infty$ with $Arg\ \mu_2$ in $(-6\pi/7,\ -2\pi/7)$, the solution to the leading order is

$$H_2 \;\sim\; -\frac{1}{\mu_2{}^{3/2}} \;+\; \gamma_{24}\,\mu_2{}^{3/8}\,e^{-4\,\mu_2{}^{7/4}/7} \;+\; \gamma_{23}\,\mu_2{}^{3/8}\,e^{4\,\mu_2{}^{7/4}/7} \tag{57}$$

and this matches to (41) if

$$\frac{C_3}{\gamma_{23}} \;=\; \frac{C_4}{\gamma_{24}} \;=\; -2^{5/8}\,\frac{p^{9/4}\,(1-p^4)^{3/8}}{(1-\alpha^2p^2)(\alpha^2-p^2)}\,e^{-i\,13\,\pi/28}\,r^{-13/28}. \tag{58}$$

From (40) and (56), we find

$$\gamma_{22} \;=\; e^{i13\pi/14}\,e^{2\,\beta}\,\gamma_{21}^{*}. \tag{59}$$

In the immediate neighborhood of $\varsigma \;=\; -1/p$, we introduce new variables

$$(1+p\varsigma) \;=\; r^{-2/7}\,\mu_1 \tag{60}$$

$$f = \frac{2\, r^{-4/7}\, p^3}{(1 - \alpha^2 p^2)(\alpha^2 - p^2)} H_1(\mu_1) \tag{61}$$

where r is as defined in (50). Then to leading order in $r^{-2/7}$, the local equation derived from (28) is:

$$H_1'' - (\mu_1 - H_1')^{3/2} H_1 = 1. \tag{62}$$

For large μ_1, with $Arg\, \mu_1$ in $(2\pi/7,\, 6\pi/7)$, the leading order behavior is

$$H_1 \sim -\frac{1}{\mu_1{}^{3/2}} + \gamma_{11}\, \mu_1{}^{3/8}\, e^{4\, \mu_1{}^{7/4}/7} + \gamma_{12}\, \mu_1{}^{3/8}\, e^{-4\, \mu_1{}^{7/4}/7}, \tag{63}$$

and this matches with (45) in sector III provided

$$\frac{C_5 e^{\gamma^{-1/2} P_m}}{\gamma_{11}} = 2^{5/8} \frac{p^{9/4}\,(1-p^4)^{3/8}}{(1-\alpha^2 p^2)(\alpha^2 - p^2)}\, e^{i\, 3\, \pi/4}\, r^{-13/28} \tag{64}$$

$$\gamma_{12} = 0. \tag{65}$$

For large μ_1 with $Arg\, \mu_1$ in $(6\pi/7,\, \pi]$, we have

$$H_1 \sim -\frac{1}{\mu_1{}^{3/2}} + \gamma_{13}\, \mu_1{}^{3/8}\, e^{4\, \mu_1{}^{7/4}/7} + \gamma_{14}\, \mu_1{}^{3/8}\, e^{-4\, \mu_1{}^{7/4}/7}, \tag{66}$$

and this matches to (41) in sector II, provided

$$\frac{C_3\, e^{\gamma^{-1/2} P_m}}{\gamma_{13}} = \frac{C_4\, e^{-\gamma^{-1/2} P_m}}{\gamma_{14}} = 2^{5/8} \frac{p^{9/4}\,(1-p^4)^{3/8}}{(1-\alpha^2 p^2)(\alpha^2 - p^2)}\, e^{i\, 3\, \pi/4}\, r^{-13/28}. \tag{67}$$

Note that the matching for large μ_1 for $Arg\, \mu_1$ in the interval $(-2\pi/7,\, 2\pi/7)$ is not relevant as far as finding transcendentally small terms in the physical region since the corresponding sector to the immediate right of $\varsigma = 1/p$ on the real axis (see Fig. 1) does not extend all the way to the physical region on or inside the unit semi-circle since $Re\, P_m > Re\, P(-1)$ as found on numerical experimentation over a range of values of α and p.

From (58) and (67) that

$$\gamma_{13} = -\gamma_{23}\, e^{-i 17\pi/14}\, e^{\gamma^{-1/2}\, P_m} \tag{68}$$

$$\gamma_{14} = -\gamma_{24}\, e^{-i17\pi/14}\, e^{-\gamma^{-1/2}\, P_m}. \tag{69}$$

Equations (51) and (62) each have to be solved subject to the conditions (59), (65), (68), and (69). This will determine unique solutions to each of (51) and (62) as we now argue. Clearly in order that (63) and (66) be the asymptotic behavior of the same function $H_1(\mu_1)$ analytically continued past a Stokes line γ_{13} and γ_{14} must each be a function of γ_{11} and γ_{12} that is determined solely from (62) which is free of any parameters. Thus (65), (68), and (69) determine each of γ_{23} and γ_{24} in terms of γ_{11}. Again, in order that (55) and (57) be the asymptotic behavior of the same analytic function $H_2(\mu_2)$, γ_{21}, and γ_{22} are each function of γ_{23} and γ_{24} that is determined solely by equation (51) which is again devoid of any parameters. Thus each of γ_{21} and γ_{22} are determinable functions of γ_{11}. The condition (59) then determines γ_{11} and hence all the other γ's. The precise calculation of γ_{11} involves finding appropriate solution of (62). We note that as far as the leading order asymptotics, the conditions for determining unique solutions to each of the equations (51) and (62) become decoupled. From (43) and (58), it is clear that γ_{24} is transcendentally small compared to γ_{23}. Again from the definition of β in (37), it is clear that $e^{-2\beta}$ is transcendentally small for this subcase, and so from (59), γ_{21} is transcendentally smaller than γ_{22}. Therefore, with negligible error in calculating γ_{22} and γ_{23}, we can require that (51) have a solution that does not grow at $\mu_2 \to \infty$ with $Arg\ \mu_2$ in the interval $(-5\pi/7,\ 2\pi/7)$. This determines a unique solution to (51) and therefore the values of γ_{22}, γ_{23}. These must be pure numbers since equation (51) contains no parameters. As far as equation (62), we note from (44) and (67) that γ_{13} is small compared to γ_{14}, therefore, we neglect γ_{13}. Together with condition (65), this means that we are interested in a solution to (62) that does not grow for large μ_1 for $Arg\ \mu_1$ in the interval $(2\pi/7,\ \pi]$. This determines the solution uniquely together with the coefficients γ_{11}, γ_{14} which are pure numbers independent of any parameters. This problem was encountered earlier in the context of a finger (Tanveer, 1987b) where explicit calculation of these coefficients was made for some linearized form of (62). For the full nonlinear equation (51), numerical calculations were done by Dorsey & Martin (1987) for an equivalent problem, and their numerical calculations translated to our problem gives us $Arg\ \gamma_{11} = 1.455$. Once such unique solutions to (51) and (62) are found, we impose the remaining condition smooth tip condition (47) where C_5 is now known in terms of γ_{11} from the relation (64). We get the selection rule for this subcase

$$\beta + Arg\ \gamma_{11} = n\,\pi. \tag{70}$$

Since by definition β is positive, it follows that the integer n on the right of (70) must be positive since it holds in the limit of large β . Also from the definition of β , the above relation determines p and hence U for given α and γ . Since γ must be small for the theory to hold, n must be a very large positive integer since β is large under the conditions of this subcase.

From the conditions of this subcase, it means that if U is any number in $(\frac{2}{(1+\alpha^2)}, 2)$ that is not too close to the end points of the interval, there is a solution corresponding to that U at countably infinite values of the surface tension parameter γ determined by (70). Again for fixed integer n , as $\gamma \rightarrow 0$, (70) could not possibly hold since from definition of β in (37) $\alpha - p$ will then be small. Similar relations were found for the finger (Tanveer, 1987b). Note that in this problem, the selection of velocity arose exclusively from consideration of the tip condition, and the condition on the rear of of the bubble is automatically satisfied to the leading order of transcendental correction. It is for this reason that the selection rule for the bubble and the fingers turns out to be similar.

4b. Analysis for $\alpha - p << 1$ with $\alpha = O(1)$

For this subcase, using the definition of p, we note that $(2-U)$ is being assumed small but positive and the bubble size not too small. Note that for this subcase, introduction of change of variables as in (48), (49), (60), and (61) in section 4a does not reduce (28) to (51) and (62) to the leading order. It is clear some other change in variable is needed.

Near $\varsigma = -1/p$, where the WKBJ solutions (33) and (34) are invalid, it is appropriate in this subcase to introduce new variables

$$x_1 = (1 + \alpha \varsigma)/\epsilon \tag{71}$$

$$\tilde{D}(x_1) = \frac{(1-\alpha^4)}{\alpha\epsilon} f \tag{72}$$

$$\beta' = \frac{\epsilon^{3/2} \, 2^{-1/2} \, \alpha}{\gamma(1-\alpha^4)^{3/2}} \tag{73}$$

where

$$\epsilon \equiv (\alpha - p)/\alpha \tag{74}$$

is small when α is of order unity. Then to leading order in ϵ, for $x_1 = O(1)$, equation (28) becomes:

$$\tilde{D}'' + \frac{1 - \beta' \tilde{M} \tilde{D}}{x_1} \tilde{D}' - \beta' \frac{(1+x_1)}{x_1{}^2} \tilde{M} \tilde{D} = -\frac{1}{x_1} \tag{75}$$

where

$$\tilde{M} = (1 + x_1 + x_1 \tilde{D}')^{1/2}. \tag{76}$$

For large x_1, from dominant balance argument, the asymptotic behavior will be

$$\tilde{D} \sim \frac{1}{\beta' (1+x_1)^{1/2}}, \tag{77}$$

and this matches with (29) to the leading order when $|1 + \alpha\varsigma| << 1$ but $|(1 + \alpha\varsigma)/\epsilon| >> 1$. To ensure that (77) is indeed the leading order asymptotic behavior we must ensure that any transcendental correction to (77) is small and not large. To find transcendental terms that are small, we linearize equation (75) about the leading order algebraic behavior (77) to get

$$D''_H - \frac{1}{2(1+x_1)} D'_H - \beta' (1+x_1)^{-1/2} D_H = 0. \tag{78}$$

The leading order behavior for large x_1 of the two independent solutions to (78) is denoted by $\tilde{D}_{H1}$ and $\tilde{D}_{H2}$ where

$$\tilde{D}_{H1} = (1+x_1)^{3/8} \, e^{-\frac{4}{3}\beta'^{1/2}(1+x_1)^{3/4}} \tag{79}$$

$$\tilde{D}_{H2} = (1+x_1)^{3/8} e^{\frac{4}{3}\beta'^{1/2}(1+x_1)^{3/4}}. \tag{80}$$

On inclusion of the leading order transcendentally small term for large x_1 with corresponding ς in sector III of Fig. 1, we get

$$\tilde{D} \sim \frac{1}{\beta'(1+x_1)^{1/2}} + \delta_{11} \tilde{D}_{H1}, \tag{81}$$

and this matches to (45) when $(1 + \alpha \varsigma)$ is small but $(1+\alpha\varsigma)/\epsilon$ large provided

$$\frac{C_5 e^{\gamma^{-1/2} P_m}}{\delta_{11}} = \frac{\alpha^{\frac{1}{4}} \epsilon^{5/8}}{(1-\alpha^4)^{5/8}} 2^{-3/8} e^{i3\pi/4} e^{\beta'^{1/2} R} \tag{82}$$

where

$$R = \int_{-1}^{\infty} dx_1 \left[\frac{(1+x_1)^{3/4}}{x_1} - \frac{1}{(1+x_1)^{1/4}} \right] - i\,\pi. \tag{83}$$

For large x_1 with corresponding ς in sector II of Fig. 1,

$$\tilde{D} \sim \frac{1}{\beta'(1+x_1)^{1/2}} + \delta_{12} \tilde{D}_{H1} + \delta_{13} \tilde{D}_{H2}, \tag{84}$$

and this matches to (41) in sector II provided

$$\frac{C_3}{\delta_{12}} e^{-\beta'^{1/2} R} e^{P_m \gamma^{-1/2}} = \frac{C_4}{\delta_{13}} e^{\beta'^{1/2} R} e^{-P_m \gamma^{-1/2}} = \frac{\alpha^{\frac{1}{4}} \epsilon^{5/8}}{(1-\alpha^4)^{5/8}} 2^{-3/8} e^{i3\pi/4}. \tag{85}$$

It is clear that there must be a relation between each of δ_{12} and δ_{13} with δ_{11} since (81) and (84) describe the same asymptotic behavior of the same analytic function $\tilde{D}(x_1)$. Also, it is clear that these relations can only involve the parameter β' that occurs in (75).

Now we move to the neighborhood of $\varsigma = 1/\alpha$ by introducing new variable

$$x_2 = (1 - \alpha \varsigma)/\epsilon \tag{86}$$

$$D_2(x_2) = \frac{(1-\alpha^4)}{\alpha\epsilon} f. \tag{87}$$

Then to leading order in ϵ, we get

$$D_2'' + \frac{1 - \beta' M_2 D_2}{x_2} D_2' + \beta' \frac{(1+x_2)}{x_2{}^2} M_2 D_2 = \frac{1}{x_2} \tag{88}$$

where

$$M_2 = (1 + x_2 - x_2\ D_2')^{1/2}. \tag{89}$$

For large x_2 , from dominant balance arguments

$$D_2 \sim \frac{1}{\beta'\ (1+x_2)^{1/2}} \tag{90}$$

and this matches with (29) when $|1-\alpha\varsigma| << 1$ but $|(1-\alpha\varsigma)/\epsilon| >> 1$. Equation (90) is only valid if transcendental corrections for large x_2 are small and not large. To find transcendental terms that are small, we linearize equation (88) about the leading order algebraic behavior and as before with equation (75), we find that the two independent WKB solutions for large x_2 to the homogeneous part of the linearized equation. Including the leading transcendental correction, the behavior of D_2 for large x_2 with corresponding ς in sector I of Fig. 1

$$D_2 \sim \frac{1}{\beta'(1+x_2)^{1/2}} + \delta_{21}\ D_{H1} + \delta_{22}\ D_{H2} \tag{91}$$

where

$$D_{H1} = (1+x_2)^{3/8}\ e^{i\ \frac{4}{3}\beta'^{1/2}(1+x_2)^{3/4}} \tag{92}$$

$$D_{H2} = (1+x_2)^{3/8}\ e^{-i\ \frac{4}{3}\beta'^{1/2}(1+x_2)^{3/4}}. \tag{93}$$

The solution (91) matches with (36) in sector I, provided

$$\frac{C_1\ e^{-i\ \beta'^{1/2}\ R}}{\delta_{21}} = = \frac{C_2\ e^{i\ \beta'^{1/2}\ R}}{\delta_{22}} = \frac{\alpha^{\frac{1}{4}}\epsilon^{5/8}}{(1-\alpha^4)^{5/8}}\ 2^{-3/8}. \tag{94}$$

The condition (40) translates into the requirement that

$$\delta_{21} = \delta_{22}^*. \tag{95}$$

For large x_2 with corresponding ς in sector II (Fig. 1),

$$D_2 \sim \frac{1}{\beta'(1+x_2)^{1/2}} + \delta_{23}\ D_{H1} + \delta_{24}\ D_{H2}. \tag{96}$$

Note that δ_{23} , δ_{24} must be related to δ_{21} and δ_{22} in order that (91) and (96) are the asymptotic behavior of the same analytic function in different sectors. This relations can only involve the parameter β' that is present in (88). The solution (96) matches with (41) provided

$$\frac{C_3\ e^{-i\ \beta'^{1/2}\ R}}{\delta_{23}} = = \frac{C_4\ e^{i\ \beta'^{1/2}\ R}}{\delta_{24}} = \frac{\alpha^{\frac{1}{4}}\epsilon^{5/8}}{(1-\alpha^4)^{5/8}}\ 2^{-3/8}. \tag{97}$$

From (85) and (97), we get

$$\delta_{23} = \delta_{12}e^{\beta'^{1/2}(1-i)R}\ e^{-\gamma^{-1/2}P_m}\ e^{i3\pi/4} \tag{98}$$

$$\delta_{24} = \delta_{13}e^{-\beta'^{1/2}(1-i)R}\ e^{\gamma^{-1/2}P_m}\ e^{i3\pi/4}. \tag{99}$$

Thus (75) and (88) have to be solved subject to the conditions (95), (98), and (99). As stated previously each of δ_{12} and δ_{13} are determined in terms of δ_{11}

From (98) and (99) δ_{23} , δ_{24} are determined in terms of δ_{11} . Again δ_{21} and δ_{22} are determined in terms of δ_{23} and δ_{24} since (91) and (96) describe the same analytic function $D_2(x_2)$. Thus δ_{21} and δ_{22} are determined in terms of δ_{11} . Equation (95) then determines δ_{11} and therefore all the other coefficients. Fortunately, finding the solutions to (75) and (88) that satisfy the above set of relations is, to the leading order, quite simple since conditions determining unique solutions to (75) and (88) decouple. Equations (43) and (97) imply that δ_{24} is negligible complared to δ_{23} since $e^{\gamma^{-1/2}P_m}$ is transcendentally small compared to $e^{-i\ R\ \beta'^{1/2}}$ from the given conditions of this subcase. Thus a unique solution to (88) is found by requiring that the solutions to (88) contain no large terms as x_2 tends to ∞ with corresponding ς in sector II of Fig. 1 and requiring that the solution be real on the real positive x_2 axis as implied by the condition of a smooth bubble back.

From Fig. 1, it is clear that if $x_2 \to \infty$ with $Arg\ x_2 = -\pi$,i.e., $Arg\ (1-\alpha\varsigma) = -\pi$ then we are assured of the corresponding ς being in sector II. Thus to solve (88) numerically, one can consider the two point boundary value problem for open contour which starts at $x_2 = \infty\ e^{i\pi}$ and ends at $x_2 = \infty\ e^{-i\pi}$ as it goes around $x_2 = 0$.

We would impose zero conditions at the end of the open contour. That D_2 should be 0 at $\infty\, e^{i\pi}$ follows from the following consideration. The condition that f be real on the real ς axis for some open interval containing $\varsigma = 1$. is equivalent to requiring that D_2 be real on the positive real axis.

From Schwarz reflection principle this is equivalent to $D_2 = 0$ at $x_2+1 = \infty\, e^{i\pi}$ since $D_2 = 0$ when $x_2 + 1 = \infty\, e^{-i\pi}$. However, it turns out that for the determination of the leading order transcendentally small term, this two point boundary problem need not actually be solved so long as a solution exists in principle. We will assume that such a solution indeed exists and is unique as can be expected from leading order asymptotics for large β' which gives results similar to the last section for problem (51).

From (43) and (85), we also obtain the condition that δ_{12} is transcendentally small compared to δ_{13} and hence we can set δ_{12} to zero. Together with (81), this means that we want that solution to (75) which goes to zero at $x_1 \to \infty$ with corresponding ς in sectors II or III in Fig.1. Note that under the conditions of this subcase the limit $x_1 \to \infty$ on the real axis corresponds to ς in sector III and not IV (See Fig. 1). The sector containing $\varsigma = -1/\alpha$ shrinks to zero size when $p \to \alpha$. ¿From Fig. 1, it is clear that we want solution to (75) with $\tilde{D} \to 0$ as $x_1 \to \infty$ for $Arg\ x_1$, i.e. $Arg\ (1 + \alpha\varsigma)$ in the interval $[0,\ \pi]$. This determines a unique solution to (75) as shall be argued shortly. The remaining requirement of smoothness of the bubble tip then determines β' . We note that the problem near the tip completely decouples from the back of the bubble once again for this subcase.

We now relate the problem as posed above for finding an appropriate solution to (75) to an equation that arose in the context of the finger (Tanveer, 1987b). This can be seen by introducing a change of variable:

$$D = \frac{1}{2}\tilde{D} \tag{100}$$

$$\xi = -i\, x_1^{1/2} \tag{101}$$

This transforms (75) into

$$D'' + \frac{1 - i\tilde{\beta}' \, M \, D}{\xi} D' + i \, \tilde{\beta}' \, \frac{(\xi - 1) \, (\xi + 1)}{\xi^2} M \, D = 2 \tag{102}$$

where

$$M = [(\xi - 1)(\xi + 1) \; - \; \xi \, D']^{1/2}$$

where $\tilde{\beta}' = 4\beta'$. Equation (102) is the same as equation (127) of the Tanveer (1987b) paper in the case when the finger is assumed symmetric (the parameter $\tilde{\beta}'$ is called δ in that paper). The requirement on $\tilde{D}$ transforms into the requirement that the solution D to equation (102) tends to zero as $\xi \rightarrow \infty$ $Arg \, \xi$ in the interval $[-\pi/2, \, 0]$. The condition of smooth tip implies that D is purely real on the negative imaginary ξ axis. From reflection principle and the symmetry of equation (102), it follows that an equivalent condition is that we find solution D to (102) which contain no transcendentally growing term for $Arg \, \xi$ in the interval $[-\pi, \, 0]$. This problem was solved in numerically in section VIII of the Tanveer (1987b) paper and it was found that the two smallest value of $\tilde{\beta}'$ which satisfies the above condition are $\tilde{\beta}' = 1.47$ and 10.1 corresponding to the Mclean-Saffman and the Romero-Vanden-Broeck branch of bubble solutions. Dorsey & Martin independently carried out an equivalent numerical calculation using a different formulation for the finger problem. They calculate the corresponding $\tilde{\beta}'$ to higher precision and for two other branches. For large $\tilde{\beta}'$ an asymptotic analysis was carried out by Tanveer (1987b) for the finger, as was done earlier by Combescot et al (1986) using a different formulation. From one of the results of the Tanveer (1987b) paper, we find that in our terminology,

$${\tilde{\beta}'}^{1/2} \; = \; (2n - \frac{2}{\pi} \, \gamma_{11}) \tag{103}$$

where n is a positive integer and γ_{11} is the constant in (63) found by solving (62) with conditions that solution does not grow for large μ_1 with $Arg \, \mu_1$ in $(2\pi/7, \, \pi]$. As mentioned previously, the numerical value of $Arg \, \gamma_{11}$ is 1.455. Going back to the definition of $\tilde{\beta}'$, β , p , and γ , we find that for this subcase $\beta' = (2-U)^{3/2} 2^{3/4} \pi^{-1} \mathcal{B}^{-1/2}$ where $\mathcal{B} = \frac{b^2 T}{12\mu}$. Thus $\tilde{\beta}'$ is completely independent of the bubble size parameter α . Since $\tilde{\beta}'$ is a pure number, it follows that

$$2 - U = k \mathcal{B}^{2/3} \tag{104}$$

where k is a pure number independent of the bubble size. This result is in agreement with earlier numerical results (Tanveer, 1987a). Further, we will see in the next section that (104) holds even for small bubbles, i.e. for $\alpha << 1$ provided $\frac{\alpha}{\gamma} >> 1$. For large n , i.e. for branches with large $\tilde{\beta}'$ so that (103) holds, we have

$$k = 2^{1/3} \pi^{4/3} \left(n - 0.4630\right)^{4/3} \tag{105}$$

4c. Subcase $\alpha << 1$

The restriction in this subcase corresponds to a bubble that is very small compared to the width $2a$ of the of the Hele-Shaw cell. However, in order for the Hele-Shaw approximation to be valid, it is necessary to have the bubble much bigger than the gap width of the cell. Given the restriction posed for all of case 4, we must have $p << 1$ as well, implying that $\alpha - p << 1$ and so $2 - U$ is small.

In this case, the inner neigborhood where the WKB solutions (33) and (34) become invalid are close to ∞ in the ς plane. It is therefore appropriate to introduce the new independent and dependent variables defined by

$$\chi = \frac{1}{\alpha \varsigma} \tag{106}$$

and

$$F(\chi) = \alpha^{-1} [f(\varsigma) + \gamma f_1(0)]. \tag{107}$$

To the leading order in α , equation (28) becomes

$$F'' + \frac{2\chi}{\chi^2 - 1} F' + \frac{mF}{(\chi^2 - 1)^2} \left[\chi^2 - k^2 - (\chi^2 - 1) F'\right]^{3/2} = \frac{2\chi}{\chi^2 - 1} \tag{108}$$

where

$$m = \frac{\alpha}{\gamma} \tag{109}$$

$$k = \frac{p}{\alpha} \tag{110}$$

For large χ

$$F \sim \frac{2}{m} \tag{111}$$

and this matches with f given by (29) for $|\varsigma| >> 1 \quad \alpha|\varsigma| << 1$ provided transcendental correction to the behavior given by (111) is small and not large. Linearizing (108) about (111), one finds that the homogeneous part of the linearized equation, to the leading order, for large χ is of the form

$$F_H'' - \frac{1}{\chi} F_H' + \frac{m}{\chi} F_H = 0. \tag{112}$$

and so the form of the transcendental correction to (111) must be to the leading order linear combinations of two independent solutions to (112) which is asymptotically given by F_{H_1} and F_{H_2} where

$$F_{H_1} = \chi^{3/4} e^{-2im\chi^{1/2}} \tag{113}$$

$$F_{H_2} = \chi^{3/4} e^{2im\chi^{1/2}}. \tag{114}$$

In sector I in Fig. 1, for the limit $|\varsigma| >> 1$, $\alpha|\varsigma| << 1$, the behavior given by (36) must match with the asymptotic behavior of F given by

$$F \sim \frac{2}{m} + b_1 \chi^{3/4} e^{-2i\chi^{1/2}} + b_2 \chi^{3/4} e^{2i\chi^{1/2}} \tag{115}$$

and the matching condition is

$$\frac{C_1 e^{-im^{1/2}\bar{R}}}{b_1} = \frac{C_2 e^{im^{1/2}\bar{R}}}{b_2} = \alpha^{1/4} \tag{116}$$

where

$$\bar{R} = \int_k^\infty d\chi \left[\frac{(\chi^2 - k^2)^{3/4}}{(\chi^2 - 1)} - \frac{1}{\chi^{1/2}}\right] + \frac{\pi}{2} i(1 - k^2)^{3/4} - 2k^{1/2}. \tag{117}$$

The condition (40) is equivalent to

$$b_1 = b_2^*. \tag{118}$$

For large χ with corresponding ς in sector II in Fig. 1,

$$F \sim \frac{2}{m} + b_3\, \chi^{3/4}\, e^{-2i\chi^{1/2}} + b_4\, \chi^{3/4}\, e^{2i\chi^{1/2}}. \tag{119}$$

Matching with (41) provides us with the relation that

$$\frac{C_3 e^{-im^{1/2}\bar{R}}}{b_3} = \frac{C_4 e^{im^{1/2}\bar{R}}}{b_4} = \alpha^{1/4}. \tag{120}$$

For large χ in sector III,

$$F \sim \frac{2}{m} + b_5\, \chi^{3/4}\, e^{-2i\chi^{1/2}} + b_6\, \chi^{3/4}\, e^{2i\chi^{1/2}}. \tag{121}$$

Matching with (45) gives

$$\frac{C_5 e^{-im^{1/2}\bar{R}}}{b_5} = \alpha^{1/4} \tag{122}$$

and

$$b_6 = 0. \tag{123}$$

Now clearly each of the pairs $(b_1,\ b_2)$ and $(b_3,\ b_4)$ is determined in terms of b_5 since (115), (119), and (121) are the asymptotic behavior of the same analytic function F analytically continued across Stokes lines. Condition (118) therefore closes the system of unknowns and determines b_5 and hence all the other b s. The condition of smooth tip (47) implies

$$Arg\ b_1 = (n + 3/4)\ \pi. \tag{124}$$

The above equation determines k for given m , i.e., bubble velocity for given bubble size parameter α and surface tension parameter γ . We note that b_1 is not a pure number and depends on both m and k . Thus when m is of order unity or smaller, it is better to determine k for given m directly by solving the two point boundary value problem posed by solving (108) on a complex contour that starts from $\chi = \infty\, e^{-i\pi}$, goes around $\chi = +1$, and moves to the upper half χ plane before going to $\xi = \infty\, e^{i\pi}$ with F tending to $\frac{2}{m}$ at the two endpoints of such a contour. Note that (118) ensures that F is purely real on the χ axis right of $\chi = 1$. Thus from reflection principle it follows that we can impose that $F = \frac{2}{m}$ at $\infty\, e^{i\pi}$ in the upper half χ plane. By requiring that such a solution be real on just one point on the real χ axis left of -1, we can effectively impose the smooth bubble tip condition (124) and this will determine k for given m . We have not carried out this numerical evaluation, though it appears straightforward in view of a similar kind of two point boundary problem that was solved for the finger problem (Tanveer, 1987b).

It may be mentioned that in Tanveer (1986), we found an expression [Eqn. 38] for bubble velocity in terms of α in the limit of small α for any value of surface tension and the expression was found not to be uniformly valid for all values of surface tension. This previous analysis when applied to the case of small surface tension corresponds to the subcase being considered here in the limit of small m . This means that both α and γ are small such that their ratio is also small. It is interesting to point out that in this limit only one solution was found (Tanveer, 1986).

For m large compared to 1, provided $(1-k^2)^{3/4} m^{1/2} >> 1$, it is possible to find the expressions for k analytically as we now show. When this condition on $1-k^2$ is not valid, as for the first few solution branches, it is necessary to carry out a set of scalings near $\chi = \pm k$ similar to what was done for case 4b. We do not present the details in that case since the end result is exactly the same as in case 4b.

We now return to the case when m is large and $(1-k^2)^{3/4} m^{1/2}$ large as well. In the neighborhood of $\chi = k$, we introduce the local independent and dependent variables ξ_2 and G_2 determined by

$$\xi_2 = \frac{(2k)^{3/7}}{(1-k^2)^{4/7}} m^{2/7} (\chi - k) \tag{125}$$

$$G_2(\xi_2) = m^{4/7} \frac{(2k)^{-1/7}}{(1-k^2)^{1/7}} F. \tag{126}$$

Then to the leading order in $m^{-2/7}$,

$$\frac{d^2 G_2}{d\xi_2^2} + G_2 \left[\xi_2 + \frac{dG_2}{d\xi_2}\right]^{3/2} = -1. \tag{127}$$

For large ξ_2 , $G_2 \sim -\xi_2^{-3/2}$ and linearizing (127) about this, we find the approximate WKB solutions for large ξ_2 for the two independent solutions to the homogeneous part of the linear equation. These are the leading order transcendental corrections. For large ξ_2 with $Arg\ \xi_2$ in $(-4\pi/7,\ 0)$,

$$G_2 \sim -\xi_2^{-3/2} + B_1 \xi_2^{3/8}\, e^{i4\xi_2^{7/4}/4} + B_2 \xi_2^{3/8}\, e^{-i4\xi_2^{7/4}/4}, \tag{128}$$

and this matches with (36) for large ξ_2 in sector I provided

$$\frac{B_1}{C_1} = \frac{B_2}{C_2} = (1-k^2)^{1/14}(2k)^{1/14} m^{13/28} \alpha^{-1/4}. \tag{129}$$

The condition (40) translates into

$$B_2 = B_1^*\, e^{m^{1/2}\pi(1-k^2)^{3/4}}. \tag{130}$$

For large ξ_2 for $Arg\ \xi_2$ in $[-\pi,\ -4\pi/7)$,

$$G_2 \sim -\xi_2^{-3/2} + B_3 \xi_2^{3/8}\, e^{i4\xi_2^{7/4}/4} + B_4 \xi_2^{3/8}\, e^{-i4\xi_2^{7/4}/4} \tag{131}$$

and this matches to (41) provided

$$\frac{B_3}{C_3} = \frac{B_4}{C_4} = (1-k^2)^{1/14}(2k)^{1/14} m^{13/28} \alpha^{-1/4}. \tag{132}$$

Similarly in the neighborhood of $\chi = -k$ we introduce local independent and dependent variables ξ_1 and G_1 by the relations

$$\xi_1 = e^{i\pi} \frac{(2k)^{3/7}}{(1-k^2)^{4/7}} m^{2/7}(\chi + k) \tag{133}$$

$$G_1(\xi_1) = m^{4/7} \frac{(2k)^{-1/7}}{(1-k^2)^{1/7}} F(\chi(\xi_1)). \tag{134}$$

Then the nonlinear equation (108) to leading order in m becomes

$$\frac{d^2G_1}{d\xi_1^2} - G_1 \left[\xi_1 - \frac{dG_1}{d\xi_1}\right]^{3/2} = 1. \tag{135}$$

Again for large ξ_1 dominant balance argument produces $G_1 \sim -\xi_1^{-3/2}$. Linearizing (135) about this and considering the homogeneous part of the linearized equation, we try WKB solutions for large ξ_1 to get the asymptotic behavior for two independent solutions to homogeneous 2nd order linear ODE. For large ξ_1 with $Arg\ \xi_1$ in $(6\pi/7,\ \pi]$,

$$G_1 \sim -\xi_1^{-3/2} + B_5\xi_1^{3/8}\, e^{4\xi_1^{7/4}/4} + B_6\xi_1^{3/8}\, e^{-4\xi_1^{7/4}/4} \tag{136}$$

and this matches with (41) in sector II provided

$$\frac{B_5 e^{-\gamma^{-1/2}\,P_m}}{C_3} = \frac{B_4 e^{\gamma^{1/2}\,P_m}}{C_4} = (1-k^2)^{1/14}(2k)^{1/14}m^{13/28}\alpha^{-1/4}e^{-i3\pi/4}. \tag{137}$$

For large ξ_1 with $Arg\ \xi_1$ in $(2\pi/7,\ 6\pi/7)$, we get

$$G_1 \sim -\xi_1^{3/2} + B_7\xi_1^{3/8}\, e^{i4\xi_1^{7/4}/4}, \tag{138}$$

and this matches with (45) in sector III provided

$$\frac{B_7 e^{-\gamma^{-1/2}\,P_m}}{C_5} = (1-k^2)^{1/14}(2k)^{1/14}m^{13/28}\alpha^{-1/4}e^{-i3\pi/4}. \tag{139}$$

From (132) and (137) we obtain

$$e^{-\gamma^{-1/2}P_m}\, e^{i3\pi/4}\, B_5 = B_3 \tag{140}$$

$$e^{\gamma^{-1/2}P_m}\, e^{i3\pi/4}\, B_6 = B_4. \tag{141}$$

Within the order of approximation made here in this subcase

$$\gamma^{-1/2}\, P_m = (-1+i)2^{1/2}\, m^{1/2} \int_0^k \frac{d\chi}{(1-\chi^2)}\, (k^2-\chi^2)^{3/4}. \tag{142}$$

So for large m , the condition (44) that C_3 is of order unity at most, implies that B_5 is negligible in view of (137). Similarly condition (43) implies that B_4 is negligible in view of (132). Thus the determination of appropriate solutions to equations (127) and (135) becomes decoupled. We need not analyze the actual solutions to (127). The tip equation (135) can now be solved subject to the requirement that for large ξ_1 there be no transcendentally large terms in ξ_1 for $Arg\ \xi_1$ in the interval $(2\pi/7,\ \pi]$, which is exactly the same problem as for (62) in section 3a. So $B_7 = \gamma_{11}$. Condition (47) becomes Then condition (47) becomes

$$m^{1/2}\,(1-k^2)^{3/4}\,\pi/2 = n\,\pi \quad - Arg\ \gamma_{11} \tag{144}$$

where $Arg\ \gamma_{11} = 1.455$, as mentioned earlier.

From the definition of m , k in (109) and (110), of p and γ in (5) and (3), it is easily seen that (143) implies that if α is small but γ even smaller so that their ratio is large then the bubble velocity U obeys the same relation (105) as before for α and p of order unity with $\alpha - p$ not small. Thus the range of validity of (104) in the B variable shrinks to zero as the size of the bubble goes to zero as noted earlier from numerical evidence (Tanveer, 1987a).

4d. Subcase: p small but $\alpha = O(1)$

In this case it is appropriate to transform both the dependent and independent variables by

$$\xi = \frac{1}{p\varsigma} \tag{144}$$

$$f = \frac{p^3}{\alpha^2}\,D(\xi) \tag{145}$$

then to the leading order in p , equation (10) is equivalent to

$$D'' + \tilde{\beta}\,(\xi^2 - 1 + D')^{1/2} DD' + \tilde{\beta}(\xi^2-1)(\xi^2 - 1 + D')^{1/2} D = -2\xi \tag{146}$$

where

$$\tilde{\beta} = \frac{p^5}{\alpha^4\gamma}. \tag{147}$$

For large ξ , as $\xi \to \infty$, from dominant balancing

$$D \sim -\frac{2}{\tilde{\beta}\xi^2}. \tag{148}$$

By linearizing (146) about this equation for large ξ , the homogeneous part of the linearized equation is

$$D_H'' - \frac{3}{\xi}D_H' + \tilde{\beta}\,(\xi^3 - \frac{3}{2}\xi)D_H = 0. \tag{149}$$

The asymptotic behavior of two independent solutions to this equation is given by

$$D_1 = \xi^{3/4}\, e^{i\frac{2}{5}\xi^{5/2}\tilde{\beta}^{1/2}}\, e^{-i\frac{3}{2}\xi^{1/2}\tilde{\beta}^{1/2}} \tag{150}$$

$$D_2 = \xi^{3/4}\, e^{-i\frac{2}{5}\xi^{5/2}\tilde{\beta}^{1/2}}\, e^{i\frac{3}{2}\xi^{1/2}\tilde{\beta}^{1/2}}. \tag{151}$$

Therefore including the leading order asymptotic correction for large ξ with corresponding ς in sector I of Fig. 1, we have

$$D \sim -\frac{2}{\tilde{\beta}\xi^2} + \kappa_1\, D_1 + \kappa_2\, D_2 \tag{152}$$

and this matches to (36) in sector I, provided

$$\frac{C_1 e^{i\tilde{R}\tilde{\beta}^{1/2}}}{\kappa_1} = \frac{C_2 e^{-i\tilde{R}\tilde{\beta}^{1/2}}}{\kappa_2} = \frac{p^{9/4}}{\alpha^2} \tag{153}$$

where

$$\tilde{R} = \int_1^\infty d\xi \left[(\xi^2-1)^{3/4} - \xi^{3/2} + \frac{3}{4}\xi^{-1/2}\right] + \frac{11}{10}. \tag{154}$$

The condition of smooth bubble back (40) implies

$$\kappa_2 = \kappa_1^*\, e^{\frac{\pi\alpha^{1/2}}{\gamma^{1/2}(1-\alpha^4)}}. \tag{155}$$

Under the conditions of this subcase, it is clear from (155) that κ_1 is transcendentally small compared to κ_2 . Now for large ξ with corresponding ς in sector II of Fig. 1,

$$D \sim -\frac{2}{\tilde{\beta}\xi^2} + \kappa_3\, D_1 + \kappa_4\, D_2 \tag{156}$$

and this matches with (41) provided

$$\frac{C_3 e^{i\tilde{R}\tilde{\beta}^{1/2}}}{\kappa_3} = \frac{C_4 e^{-i\tilde{R}\tilde{\beta}^{1/2}}}{\kappa_4} = \frac{p^{9/4}}{\alpha^2}. \tag{157}$$

Now for large ξ with corresponding ς is sector III in Fig. 1,

$$D \sim -\frac{2}{\tilde{\beta}\xi^2} + \kappa_5 D_1 + \kappa_6 D_2 \tag{158}$$

and this matches with (45) provided

$$\frac{C_5 e^{i\tilde{R}\tilde{\beta}^{1/2}}}{\kappa_5} = \frac{p^{9/4}}{\alpha^2} \tag{159}$$

$$\kappa_6 = 0. \tag{160}$$

Since κ_1 is small as argued earlier, by setting it to 0 along with condition (160), we are guaranteed a unique solution to (146). This can be implemented numerically similar to other problems. Note κ_5 is only a function of $\tilde{\beta}$ The smooth tip condition becomes

$$Arg\ \kappa_5 - \tilde{\beta}^{1/2}\left\{\frac{11}{10} + \int_1^\infty d\xi \left[(\xi^2-1)^{3/4} - \xi^{3/2} + 3\xi^{-1/2}/4\right]\right.$$
$$\left. +2^{1/2}\tilde{\beta}^{1/2} \int_0^1 d\xi (1-\xi^2)^{3/4}\right\} + \frac{\pi\alpha^{1/2}}{2(1-\alpha^4)\gamma^{1/2}} = (n+3/4)\pi. \tag{161}$$

It is clear from the above condition that for fixed integer n , as γ tends to zero, we cannot satisfy the condition (161) since under the conditions of this subcase $\tilde{\beta}$ term is much smaller than the term involving α . On the other hand, for fixed value of p , there are solutions at a countable infinite set of values of γ given by relation (161). These solutions are very sensitive to changes in γ and will be difficult to observe in a numerical calculation.

5. Case of $\alpha - p < 0$

In this case, the Stokes lines are shown in Fig. 2. The asymptotic solution in sector III away from the immediate neighborhood of the critical point is given by

$$f = \gamma f_1 + C_3 g_1. \tag{162}$$

There can be no g_2 term once again as it tends to ∞ at $\varsigma = -\alpha$. The condition of a smooth tip is now

$$Arg\left[C_3\, e^{-i3\pi/4}\, e^{\gamma^{-1/2}P_m}\right] = n\pi \tag{163}$$

where n is again an integer and

$$P_m = 2^{1/2}(-1+i)\int_{\frac{1}{p}}^{\infty} d\varsigma \frac{(p^2\varsigma^2-1)^{3/4}\,(\varsigma^2-p^2)^{1/4}}{(\varsigma^2-\alpha^2)\,(\alpha^2\varsigma^2-1)}. \tag{164}$$

It is not difficult to show that $\frac{(p^2\varsigma^2-1)^{3/4}(\varsigma^2-p^2)^{1/4}}{(\varsigma^2-\alpha^2)}$ is an increasing function of ς on the real axis in the interval $(1/p,\ \infty)$. This implies

$$\int_{1/p}^{\infty} d\varsigma\ \frac{(p^2\varsigma^2-1)^{3/4}(\varsigma^2-p^2)^{1/4}}{(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)} < (\frac{p^2}{\alpha^2}-1)^{3/4}(\frac{1}{\alpha^2}-p^2)^{3/4}(\frac{1}{\alpha^2}-\alpha^2)^{-1} \text{ times}$$

$$\int_{1/p}^{\infty} d\varsigma \frac{1}{1-\alpha^2\varsigma^2} = -\frac{1}{2\alpha}\, ln\, \left(\frac{p+\alpha}{p-\alpha}\right)\frac{(p^2-\alpha^2)^{3/4}(1-\alpha^2p^2)^{1/4}}{(1-\alpha^4)}. \tag{165}$$

The right hand side of (165) is clearly negative and so $Re\ P_m$ is negative as for case 4.

In sector II

$$f\ =\ \gamma\, f_1\ +\ C_1\, g_1\ +\ C_2\, g_2. \tag{166}$$

The condition that (166) is real on the real axis in some open interval containing $\varsigma = 1$ is equivalent to

$$C_1\ =\ C_2^*. \tag{167}$$

Note in this case, the behavior of f in sector I is irrelevant as far as finding transcendentally small terms in the physical domain since sector I does not extend to the immediate neighborhood of $|\varsigma| = 1$.

To see how conditions (163) and (167) determine each of the unknown coefficients C_3 , C_1 , and C_2 it is necessary to analyze the neighborhood of $\varsigma = -1/p$ to find the connection coefficients between the coefficients C_1 , C_2 , to C_3 before we can impose the conditions (163) and (167). Once again the leading order nonlinear equations in this neighborhood depends on the size of α and p . We have three distinct cases

(a) $p - \alpha = O(1)$ with $p = O(1)$,

(b) $p - \alpha << 1$ with $\alpha = O(1)$ and

(c) $p - \alpha << 1$ with $\alpha << 1$.

5a. Subcase $p - \alpha = O(1)$ with $p = O(1)$

Let us consider the immediate neighborhood of the critical point $\varsigma = -1/p$. We introduce the same transformation (60) and (61) where r is again given by (50). Here in taking fractional power of r , only the positive root will be referred to. Once again equation (62) holds. For $Arg\, \mu_1$ in $(-2\pi/7,\ 2\pi/7)$, we like the asymptotic behavior for large μ_1 to be given by

$$H_1 \sim -\frac{1}{{\mu_1}^{3/2}} + \gamma_{11}\, {\mu_1}^{3/8}\, e^{-4\, {\mu_1}^{7/4}/7} \tag{168}$$

since this matches with (162) in sector 1 of Fig. 2 provided

$$\frac{C_3 e^{\gamma^{-1/2} P_m}}{\gamma_{11}} = 2^{5/8}\, \frac{p^{9/4}\, (1-p^4)^{3/8}}{(1-\alpha^2 p^2)(\alpha^2 - p^2)}\, e^{i\, 3\, \pi/4}\, r^{-13/28}. \tag{169}$$

For large μ_1 with $Arg\, \mu_1$ in $(2\pi/7,\ 6\pi/7)$ we get

$$H_1 \sim -\frac{1}{{\mu_1}^{3/2}} + \gamma_{12}\, {\mu_1}^{3/8}\, e^{-4\, {\mu_1}^{7/4}/7} + \gamma_{13}\, {\mu_1}^{3/8}\, e^{4\, {\mu_1}^{7/4}/7} \tag{170}$$

and this matches with (166) in sector II provided

$$\frac{C_1\, e^{\gamma^{-1/2} P_m}}{\gamma_{12}} = \frac{C_2\, e^{-\gamma^{-1/2} P_m}}{\gamma_{13}} = 2^{5/8}\, \frac{p^{9/4}\, (1-p^4)^{3/8}}{(1-\alpha^2 p^2)(\alpha^2 - p^2)}\, e^{i\, 3\, \pi/4}\, r^{-13/28}. \tag{171}$$

From (166) and (171) it follows that

$$\gamma_{13}^{*} = -ie^{-2\gamma^{-1/2}\, Re P_m}\, \gamma_{12}. \tag{172}$$

Thus γ_{12} is negligible compared to γ_{13} and therefore can be neglected to the leading order. One then requires that the solution to the nonlinear equation (62) have no transcendentally large correction for $Arg\, \mu_1$ in the interval $(0,\ 6\pi/7)$. This gives a unique solution, and one obtains $Im\, \gamma_{11} = 0.875$ as obtained numerically. The smooth tip condition (163), which is now equivalent to γ_{11} being real can therefore never be satisfied implying no solution exists in this case.

5b. Subcase $p - \alpha << 1$ with $p = O(1)$

Define new independent and dependent variable

$$x_1 = \frac{(1+\alpha\varsigma)}{\epsilon} \tag{173}$$

$$D_1(x_1) = \frac{1-\alpha^4}{\alpha\epsilon} f(\varsigma(x_1)) \tag{174}$$

and we obtain

$$D_1'' + \frac{1}{x_1}\left(1 - \beta_1(x_1 - 1 + x_1 D_1')^{1/2} D_1\right) D_1' - \beta_1 \frac{(x_1-1)}{x_1^2}[x_1 - 1 + x_1 D_1']^{1/2} D_1 = -\frac{1}{x_1} \tag{175}$$

where

$$M_1 = (x_1 - 1 + x_1 D')^{1/2} \tag{176}$$

and

$$\beta_1 = \frac{\epsilon^{3/2} 2^{-1/2} \alpha}{\gamma (1-\alpha^4)^{3/2}}. \tag{177}$$

For large x_1, from dominant balancing argument

$$D_1 \sim \beta_1{}^{-1}(x_1 - 1)^{-1/2}. \tag{178}$$

Linearizing (175) and finding WKB solutions for large x_1, we find the behavior of the leading order transcendental correction to (178). For large x_1 with corresponding ς in sector III of Fig. 2,

$$D_1 \sim \beta_1{}^{-1}(x_1 - 1)^{-1/2} + \delta_1 (x_1 - 1)^{3/8} e^{-\frac{4}{3}\beta_1^{1/2}(x_1-1)^{3/4}} \tag{179}$$

and this matches to (162) when

$$\frac{C_3 e^{\gamma^{-1/2} P_m} e^{\beta_1{}^{1/2} \tilde{R}_1}}{\delta_1} = \frac{\alpha^{1/4}\epsilon^{5/8}}{(1-\alpha^4)^{5/8}} 2^{-3/8} e^{i3\pi/4} \tag{180}$$

where

$$\tilde{R}_1 = \int_1^\infty dx_1 \left[\frac{(x_1 - 1)^{3/4}}{x_1} - \frac{1}{(x_1 - 1)^{1/4}} \right]. \tag{181}$$

For large x_1 with corresponding ς in sector II of Fig. 2,

$$D_1 \sim \beta_1{}^{-1}(x_1 - 1)^{-1/2} + \delta_2 (x_1 - 1)^{3/8} e^{-\frac{4}{3}\beta_1^{-1/2}(x_1-1)^{3/4}} + \delta_3\, (x_1 - 1)^{3/8}\, e^{-\frac{4}{3}\beta_1^{-1/2}(x_1-1)^{3/4}} \tag{182}$$

Matching with (166) gives the condition that

$$\frac{C_3 e^{\gamma^{-1/2} P_m} e^{\beta_1{}^{1/2}\tilde{R}_1}}{\delta_2} = \frac{C_4 e^{-\gamma^{-1/2} P_m} e^{-\beta_1{}^{1/2}\tilde{R}_1}}{\delta_3} = \frac{\alpha^{1/4}\epsilon^{5/8}}{(1 - \alpha^4)^{5/8}}\, 2^{-3/8}\, e^{i3\pi/4}. \tag{183}$$

The condition (167) for a smooth bubble back implies

$$\delta_3^* = -i\, e^{-2\gamma^{-1/2} Re P_m}\, e^{-2\beta_1{}^{1/2}\tilde{R}_1}\, \delta_2 \tag{184}$$

and the smooth tip condition (163) implies

$$Arg\, \delta_1 = 0. \tag{185}$$

It is clear from the estimate (165) that for small γ ,

$$-\gamma^{-1/2}\, Re\, P_m >> \beta_1{}^{1/2}$$

for this subcase, so condition (184) implies that δ_2 is negligible compared to δ_3 . Thus one can solve equation (175) subject to the condition that D_1 contains no transcendentally large terms for for large x_1 in sectors II and III. This was done numerically and no values of β_1 ensured the condition (185) implying no solution exists for this case either.

5c. Subcase $\alpha - p << 1$, with $\alpha << 1$

We introduce the same change in variable as in (106) and (107). Equation (108) is again valid, except that k now exceeds 1. Equation (112) through (114) is also valid. Now for large χ with corresponding ς in sector III of Fig.2,

$$F \sim \frac{2}{m} + b_1\, \chi^{3/4}\, e^{-2i\chi^{1/2}} \tag{186}$$

and this will match with (162) provided

$$e^{\gamma^{-1/2}P_m}C_3e^{-im^{1/2}\tilde{R}_1}b_1{}^{-1} = \alpha^{1/4} \tag{187}$$

where $\tilde{R}_1$ is now given by

$$\tilde{R}_1 = \int_{-k}^{-\infty} d\chi \left[\frac{(\chi^2-k^2)^{3/4}}{(\chi^2-1)} - \frac{1}{\chi^{1/2}}\right] + 2ik^{1/2}. \tag{188}$$

For large χ with corresponding ς in sector II

$$F \sim \frac{2}{m} + b_3\,\chi^{3/4}\,e^{-2i\chi^{1/2}} + b_4\,\chi^{3/4}\,e^{2i\chi^{1/2}}, \tag{189}$$

and this matches with (166) provided

$$e^{\gamma^{-1/2}P_m}C_1e^{-im^{1/2}\tilde{R}_1}b_3{}^{-1} = e^{-\gamma^{-1/2}P_m}C_2e^{im^{1/2}\tilde{R}_1}b_4{}^{-1} = \alpha^{1/4}. \tag{190}$$

The condition of the smooth back (167) implies that

$$b_3 = b_4^*, \tag{191}$$

and the condition of smooth tip (163) implies

$$Arg\ b_1 = 3\pi/4. \tag{192}$$

On numerical experimentation continuously changing k over a wide range for given value of m no such solutions could be found. For large m, an asymptotic analysis similar to the one in section 4c was done, and it was found that no solutions exist in that limit.

6. Case of p^2 negative

When p is imaginary, we write $p = -iq$, where q is positive, without any loss of generality. The Stokes lines in this case are sketched in Fig. 3. $P(\varsigma)$ defined in equation (35) is now given by

$$P(\varsigma) = i\int_{i/q}^{\varsigma} \frac{(1+q^2\varsigma^2)^{3/4}(\varsigma^2+q^2)^{1/4}}{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)}d\varsigma \tag{193}$$

where branch cuts and branches are chosen so that argument of $(1 \pm i q \varsigma)$ is in the interval $(-\pi, \pi)$ and argument of $\varsigma \pm i q$ is in the interval $(-\pi/2, 3\pi/2)$. In sector I of Fig.3, the leading order asymptotic behavior (on inclusion of the leading order transcendental behavior as well) is given by

$$f \sim \gamma f_1 + C_1 g_1 + C_2 g_2, \tag{194}$$

and this is real on the real axis on some interval containing +1 provided

$$C_1^* = e^{-2 \gamma^{-1/2} Re P(1)} C_2. \tag{195}$$

In sector II of Fig.3, the leading order asymptotic behavior is given by

$$f \sim \gamma f_1 + C_3 g_1. \tag{196}$$

There can be no g_2 term as it grows without bounds at $\varsigma = -\alpha$ inside the physical region. In sector III of Fig.3,

$$f \sim \gamma f_1 + C_4 g_1 + C_5 g_2. \tag{197}$$

The condition of a smooth tip implies that f is real on some open interval containing $\varsigma = -1$. Two cases are possible depending on whether $Re\, P(-1) < 0$ or ≥ 0. In the first case, the region outside the unit circle in the immediate vicinity of $\varsigma = -1$ is part of sector II and therefore the asymptotic behavior of f in sector III is irrelevant as far as enforcing the tip condition. This is the picture shown in Fig. 3. The smooth tip condition is

$$Arg \left[C_3 \, e^{\gamma^{-1/2} P(-1)} e^{-i3\pi/4} \right] = n \, \pi \tag{198}$$

where n is some integer. However, if $Re\, P(-1) \geq 0$, then the region outside the unit circle in the immediate vicinity of $\varsigma = -1$ is part of sector III and the smooth tip condition implies that

$$Arg \left[C_4 \, e^{\gamma^{-1/2} P(-1)} e^{-i3\pi/4} \right] = n \, \pi \tag{199}$$

$$Arg\left[C_5\, e^{-\gamma^{-1/2}P(-1)} e^{-i3\pi/4}\right] = n'\,\pi \tag{200}$$

where n and n' are integers not necessarily equal. It is clear both equations (199) and (200) could not possibly hold if these conditions are independent since for given α and γ only one parameter p is left to be determined. This indicates that if p is such that $Re\ P(-1) \geq 0$ then solutions do not exist. We will confirm this later.

It is clear that the coefficients C_1 , C_2 , C_3 , C_4 , and C_5 can only be determined by exploring the neighborhood of $\varsigma = i/q$ where the WKB behavior is not valid since that will provide the connection coefficients. Note that the neighborhood of $\varsigma = -i/q$ is not convenient since with the choice of branch cuts in defining $P(\varsigma)$ there is no way to move from sector I to II in this neighborhood. Again, the form of the local equation near $\varsigma = i/q$ depends on the relative size of q and α .

6a. Subcase, q order unity :

We introduce the change in variable

$$(1 + iq\varsigma) = e^{-i\pi/7}\, r^{-2/7}\, \mu_1 \tag{201}$$

$$f = -\frac{2\, i\, r^{-4/7}\, q^3\, e^{-i\,2\,\pi/7}}{(1 + \alpha^2 q^2)\,(\alpha^2 + q^2)}\, H_1\,(\mu_1) \tag{202}$$

where

$$r = \frac{2^{3/2}\, q^5\, \left(1 - q^4\right)^{1/2}}{\gamma\, \left(\alpha^2 + q^2\right)^2\, \left(1 + \alpha^2 q^2\right)^2}; \tag{203}$$

the resulting equation is

$$\frac{d^2 H_1}{d{\mu_1}^2} - \left(\mu_1 - \frac{dH_1}{d\mu_1}\right)^{3/2} H_1 = 1 \tag{204}$$

The asymptotic behavior for large μ_1 for argument μ_1 in $(-2\pi/7,\ 2\pi/7)$ is given by

$$H_1 \sim -\frac{1}{{\mu_1}^{3/2}} + \gamma_3\, {\mu_1}^{3/8}\, e^{-4\,{\mu_1}^{7/4}/7} \tag{205}$$

and this matches with (196) in sector II of Fig. 3 provided

$$\frac{C_3}{\gamma_3} = 2^{5/8}\, \frac{q^{9/4}\, \left(1 - q^4\right)^{3/8}}{(1 + \alpha^2 q^2)(\alpha^2 + q^2)}\, e^{-i\,5\,\pi/14}\, r^{-13/28}. \tag{206}$$

For argument of μ_1 in the interval $(2\,\pi/7,\ 6\,\pi/7)$, for large μ_1 , we find

$$H_1 \sim -\frac{1}{\mu_1{}^{3/2}} + \gamma_1\,\mu_1{}^{3/8}\,e^{-4\,\mu_1{}^{7/4}/7} + \gamma_2\,\mu_1{}^{3/8}\,e^{4\,\mu_1{}^{7/4}/7}, \tag{207}$$

and this matches with (194) in sector I in Fig.3 provided

$$\frac{C_1}{\gamma_1} = \frac{C_2}{\gamma_2} = 2^{5/8}\,\frac{q^{9/4}\,(1-q^4)^{3/8}}{(1+\alpha^2q^2)(\alpha^2+q^2)}\,e^{-i\,5\,\pi/14}\,r^{-13/28}. \tag{208}$$

Now condition (195) implies that

$$\gamma_1^* = e^{-i5\pi/7}\,e^{-2\,\gamma^{-1/2}\,Re\,P(1)}\,\gamma_2. \tag{209}$$

For large μ_1 with $Arg\,\mu_1$ in the interval $(-6\pi/7,\ -2\pi/7)$, we require that

$$H_1 \sim -\frac{1}{\mu_1{}^{3/2}} + \gamma_4\,\mu_1{}^{3/8}\,e^{-4\,\mu_1{}^{7/4}/7} + \gamma_5\,\mu_1{}^{3/8}\,e^{4\,\mu_1{}^{7/4}/7} \tag{210}$$

and this matches with (197) provided

$$\frac{C_5}{\gamma_5} = \frac{C_4}{\gamma_4} = 2^{5/8}\,\frac{q^{9/4}\,(1-q^4)^{3/8}}{(1+\alpha^2q^2)(\alpha^2+q^2)}\,e^{-i\,5\,\pi/14}\,r^{-13/28}. \tag{211}$$

If $Re\,P(-1) < 0$, the condition of smooth tip (198) translates into the requirement that

$$-31\,\pi/28 + \gamma^{-1/2}\,Im\,P(-1) + Arg\,\gamma_3 = n\,\pi. \tag{212}$$

On the other hand, if $Re\,P(-1) \geq 0$, the condition of smooth tip (199) and (200) translates into the requirements that

$$-31\,\pi/28 + \gamma^{-1/2}\,Im\,P(-1) + Arg\,\gamma_4 = n\,\pi \tag{213}$$

$$-31\,\pi/28 - \gamma^{-1/2}\,Im\,P(-1) + Arg\,\gamma_5 = n'\,\pi \tag{214}$$

where n and n' are not necessarily equal integers. It is easy to see from (193) that

$$Re\,P(-1) = Im\,P(1) = -\!\!\!\!\!\!\int_1^\infty \frac{(1+q^2\varsigma^2)^{3/4}(\varsigma^2+q^2)^{1/4}}{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)}d\varsigma \tag{215}$$

and

$$Im\,P(-1) = Re\,P(1) = -\int_{1/q}^{\infty} \frac{\left(-1+q^2y^2\right)^{3/4}\left(y^2-q^2\right)^{1/4}}{(1+\alpha^2y^2)(y^2+\alpha^2)}dy + \frac{\pi\left(\alpha^2+q^2\right)^{3/4}}{2\alpha(1-\alpha^4)}\left(1+\alpha^2q^2\right)^{1/4} \tag{216}$$

Now

$$\int_{1/q}^{\infty} \frac{\left(-1+q^2y^2\right)^{3/4}\left(y^2-q^2\right)^{1/4}}{(1+\alpha^2y^2)(y^2+\alpha^2)}dy \;<\; \int_{1/q}^{\infty} \frac{\left(q^2y^2\right)^{3/4}\left(y^2\right)^{1/4}}{(1+\alpha^2y^2)y^2}dy$$
$$= \frac{q^{3/2}}{\alpha}\left(\frac{\pi}{2} - tan^{-1}\,\frac{\alpha}{q}\right) \;<\; \frac{q^{3/4}\pi}{2\alpha} \tag{217}$$

Thus

$$Re\;P(1) \;>\; \frac{\pi}{2\alpha}\;\left[\left(\alpha^2+q^2\right)^{3/4}\left(1+\alpha^2q^2\right)^{1/4} \;-\; q^{3/4}\right] \;>\; 0 \tag{218}$$

It is clear from (209) and (218) that γ_1 is transcendentally small compared to γ_2 and will therefore be neglected. This means that we are interested in a solution to (204) such that the solution decays for large μ_1 with $Arg\,\mu_1$ in the interval $[0,\ 6\pi/7)$. This determines a unique solution to (204) and each of γ constants determined. Clearly these must be order one constants as (204) does not involve any parameters. Thus when $Re\ P(-1)\ <\ 0$, for given value of q it is possible to satisfy the tip condition (212) only at isolated values of γ . However if n is held fixed and $\gamma\ \rightarrow\ 0$, then equation (212) could not possibly be satisfied since $\gamma^{-1/2}\ Im\ P(-1)$ is very much larger than any other term of the equation. When $Re\ P(-1)\ \geq\ 0$, equations (213) and (214) are too restrictive since they impose two independent conditions on parameter q and could not possibly be satisfied except perhaps at isolated values of γ or α . Numerically, it is seen that $Re\ P(-1)$ increases with q . Thus $Re\ P(-1)\ =\ 0$ condition determines critical values of q and hence U , q_c , and U_c beyond which there are no solutions. Table 1 lists the q_c and corresponding U_c for various values of α . The value of the corresponding bubble area in the limit of zero surface tension J determined from Equation (21) of the Tanveer (1986) paper is also listed. As can be seen from the table, in the limit of α tending to zero, $q_c\ \rightarrow\ 0$, implying $U_c\ \rightarrow\ 2$. Now, it was pointed out earlier (Taylor & Saffman, 1959, Tanveer, 1986) in the limit of zero bubble area, $U\ =\ 2$ corresponds to a circular bubble when

$\gamma = 0$. Since $U_c \rightarrow 2$ as $\alpha \rightarrow 0$, together, with the results in section 4c, we conclude that every branch of solution in this limit is circular. It is interesting to note that when the effect of the cell walls is totally neglected, only one branch of solution was found for which the bubble is circular with $U = 2$ (Tanveer, 1986). This means that as soon as any amount of cell wall effect is introduced, we get multiple solutions all tending to $U = 2$ Taylor-Saffman bubble in the limit of zero surface tension.

Case 6b: $q << 1$ **, but** $\alpha = O(1)$

We introduce independent and dependent variables

$$\xi = \frac{1}{q\zeta} \tag{219}$$

$$F(\xi) = \frac{\alpha^2}{q^3} f. \tag{220}$$

Then to the leading order in q , equation (28) becomes

$$F'' + \tilde{\beta}_1 \left(\xi^2 + 1 + F'\right)^{3/2} F = -2\xi \tag{221}$$

where

$$\tilde{\beta}_1 = \frac{q^5}{\alpha^4 \gamma}. \tag{222}$$

The asymptotic behavior for large ξ , with corresponding ζ in sector I of Fig. 3 is

$$F \sim -\frac{2}{\tilde{\beta}_1 \xi^2} + d_1 \xi^{3/4} \, e^{i\tilde{\beta}_1^{1/2}\left[\frac{2}{5}\xi^{5/4} + \frac{3}{2}\xi^{1/2}\right]} + d_2 \xi^{3/4} \, e^{-i\tilde{\beta}_1^{1/2}\left[\frac{2}{5}\xi^{5/4} + \frac{3}{2}\xi^{1/2}\right]} \tag{223}$$

and this matches with (194) in sector I provided

$$\frac{C_1}{d_1} e^{i\tilde{\beta}_1^{1/2}\tilde{J}} = \frac{C_2}{d_2} e^{-i\tilde{\beta}_1^{1/2}\tilde{J}} = \frac{q^{9/4}}{\alpha^2} \tag{224}$$

where

$$\tilde{J} = 2^{-1/2}(-1+i) \int_1^\infty dy \left[\left(y^2 - 1\right)^{3/4} - y^{3/2} + \frac{3}{4} y^{-1/2} \right] + \frac{11}{10} 2^{-1/2}(-1+i). \tag{225}$$

The condition (195) for a smooth bubble back implies that

$$e^{-2\tilde{\beta}_1^{1/2} Im \, \tilde{J}} e^{-2\gamma^{-1/2} Re \, P(1)} \, d_2 = d_3^*. \tag{226}$$

The asymptotic behavior of F for large ξ with corresponding ς in sector II in Fig. 3 is given by

$$F \sim -\frac{2}{\tilde{\beta}_1 \xi^2} + d_3 \xi^{3/4} \, e^{i\tilde{\beta}_1^{1/2}[\frac{4}{5}\xi^{5/4}+\frac{3}{2}\xi^{1/2}]}, \tag{227}$$

and this matches with (196) in sector II provided

$$\frac{C_3}{d_3} \, e^{i\tilde{\beta}_1^{1/2}\tilde{J}} = \frac{q^{9/4}}{\alpha^2}. \tag{228}$$

In this subcase, from numerical experimentation, it was found that $Re\ P(-1)$ was always negative (This is confirmed by the results in Table 1.) and therefore the corresponding condition for smooth tip (198) translates into

$$Arg\ d_3 - \tilde{\beta}_1^{1/2} Re\ \tilde{J} + \gamma^{-1/2}\ Im\ P(-1) = (n + \frac{31}{28})\pi \tag{229}$$

From estimate (217), it is clear that d_1 is transcendentally small compared to d_2 . Thus one can find solution to (221) subject to the condition that there be no transcendentally large term in ξ for corresponding ς in sectors I and II of Fig. 3. This will determine a unique solution and therefore d_3 . This d_3 can only be a function of $\tilde{\beta}_1$ since this is the only parameter appearing in (221). From the selection condition (229), it is clear that since $\gamma^{-1/2}\ Im\ P(-1)$ much larger than terms involving $\tilde{\beta}_1$, the selection criteria could not be satisfied as $\gamma \rightarrow 0$ if we hold integer n to be a constant. On the other hand, there are solutions for any given q and hence $\tilde{\beta}_1$ at isolated but countably infinite values of γ which have 0 as a limit point.

6c. Case of both q and $\alpha << 1$

Here we if $q > q_c$, i.e., $Re\ P(-1) \geq 0$, from simple counting conditions, it follows that there will be no solution as discussed in section 6c.

From the slope of q_c vs α curve (see table 1), it follows that if both q and α are small with the ratio q/α fixed, then $Re\ P(-1) < 0$. We will assume this is the case in what follows.

In this case, we introduce the variable

$$\chi = \frac{1}{\alpha\varsigma} \tag{230}$$

$$F(\chi) = \alpha^{-1} \left(f + \gamma f_1(0)\right). \tag{231}$$

Then to the leading order in α (28) reduces to

$$F'' + \frac{2\chi}{\chi^2 - 1}F' + \frac{mF}{(\chi^2 - 1)^2}\left[\chi^2 + k^2 - (\chi^2 - 1)F'\right]^{3/2} = \frac{2\chi}{\chi^2 - 1} \tag{232}$$

where

$$m = \frac{\alpha}{\gamma} \tag{233}$$

and

$$k = \frac{q}{\alpha}. \tag{234}$$

For large χ , with corresponding ς in sector II, we get

$$F \sim \frac{2}{m} + d_3\, \chi^{3/4}\, e^{-2im^{1/2}\chi^{1/2}} \tag{235}$$

and this matches with (196) provided

$$\frac{C_3}{d_3} e^{-m^{1/2}2^{-1/2}(1+i)\tilde{J}_1} = \alpha^{-1/4} \tag{236}$$

where

$$\tilde{J}_1 = -2k^{1/2} + \int_k^\infty dy \left[\frac{(y^2 - k^2)^{3/4}}{(y^2 + 1)} - y^{-1/2}\right]. \tag{237}$$

For large χ with corresponding ς in sector I,

$$F \sim \frac{2}{m} + d_1\, \chi^{3/4}\, e^{-2im^{1/2}\chi^{1/2}} + d_2\, \chi^{3/4}\, e^{2im^{1/2}\chi^{1/2}} \tag{238}$$

and this matches with (194) in sector I provided

$$\frac{C_1}{d_1} e^{-m^{1/2}2^{-1/2}(1+i)\tilde{J}_1} = \frac{C_2}{d_2} e^{m^{1/2}2^{-1/2}(1+i)\tilde{J}_1} = \alpha^{-1/4}. \tag{239}$$

The condition of the smooth bubble back (195) is equivalent to

$$d_1^* = e^{-2^{1/2} m^{1/2} \tilde{J}_1} e^{-2\gamma^{-1/2} Re\ P(1)} d_2. \tag{240}$$

The condition of a smooth tip (198) becomes

$$m^{1/2} \tilde{J}_1 + \gamma^{-1/2} Im\ P(-1) = (n + \frac{3}{4})\pi. \tag{241}$$

As before, in the subcase (4c), rather than using condition (241) directly, it is more convenient to solve the two point boundary problem involving the ODE (232) on a complex contour in the χ plane that starts at $\infty e^{-i\pi}$ and goes around $\chi = 1$ before moving over to the upper half χ plane and ending at $\chi = \infty e^{i\pi}$. From the condition of a smooth bubble back, Schwarz reflection principle applies and one can require that the endpoint boundary conditions on the complex contour be $F = \frac{2}{m}$. Once, the two point boundary value problem is solved for each k , we change k to ensure that such a solution is real at some point on the negative real χ axis. This condition will automatically ensure that the bubble tip is smooth. We have not carried out this piece of numerical calculation. Again, as in section (4c), we can do asymptotics for large m , and we find that the results of the section 6a applies.

Conclusion

We have considered the problem of selection of bubble velocity in a Hele-Shaw cell. Previous analytical results for the case of a small bubble (Tanveer, 1986) have been extended to include bubbles of arbitrary size.

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Figure and Table Captions

Fig 1: Stokes lines in the ς plane for $\alpha^2 > p^2 > 0$.

Fig. 2: Stokes lines in the ς plane for $p^2 > \alpha^2$.

Fig. 3: Stokes lines in the ς plane for $p^2 = -q^2 < 0$.

Table 1: Critical bubble velocity U_c for different bubble area J . Corresponding values of parameters $p \equiv iq_c$ and $\alpha \equiv \alpha_c$ are also given.

Table 1

α	J	q_c	U_c
0.01	$1.27e-04$	0.062	1.992
0.02	$5.09e-04$	0.093	1.982
0.03	$1.15e-03$	0.117	1.971
0.04	$2.03e-03$	0.138	1.959
0.05	$3.18e-03$	0.156	1.948
0.06	$4.57e-03$	0.172	1.936
0.07	$6.23e-03$	0.186	1.924
0.08	$8.13e-03$	0.199	1.912
0.09	$1.03e-02$	0.211	1.900
0.10	$1.27e-02$	0.222	1.888
0.20	$5.01e-02$	0.300	1.771
0.30	0.110	0.341	1.661
0.40	0.189	0.358	1.560
0.50	0.282	0.356	1.465
0.60	0.381	0.335	1.376
0.70	0.476	0.286	1.291
0.80	0.550	0.168	1.208

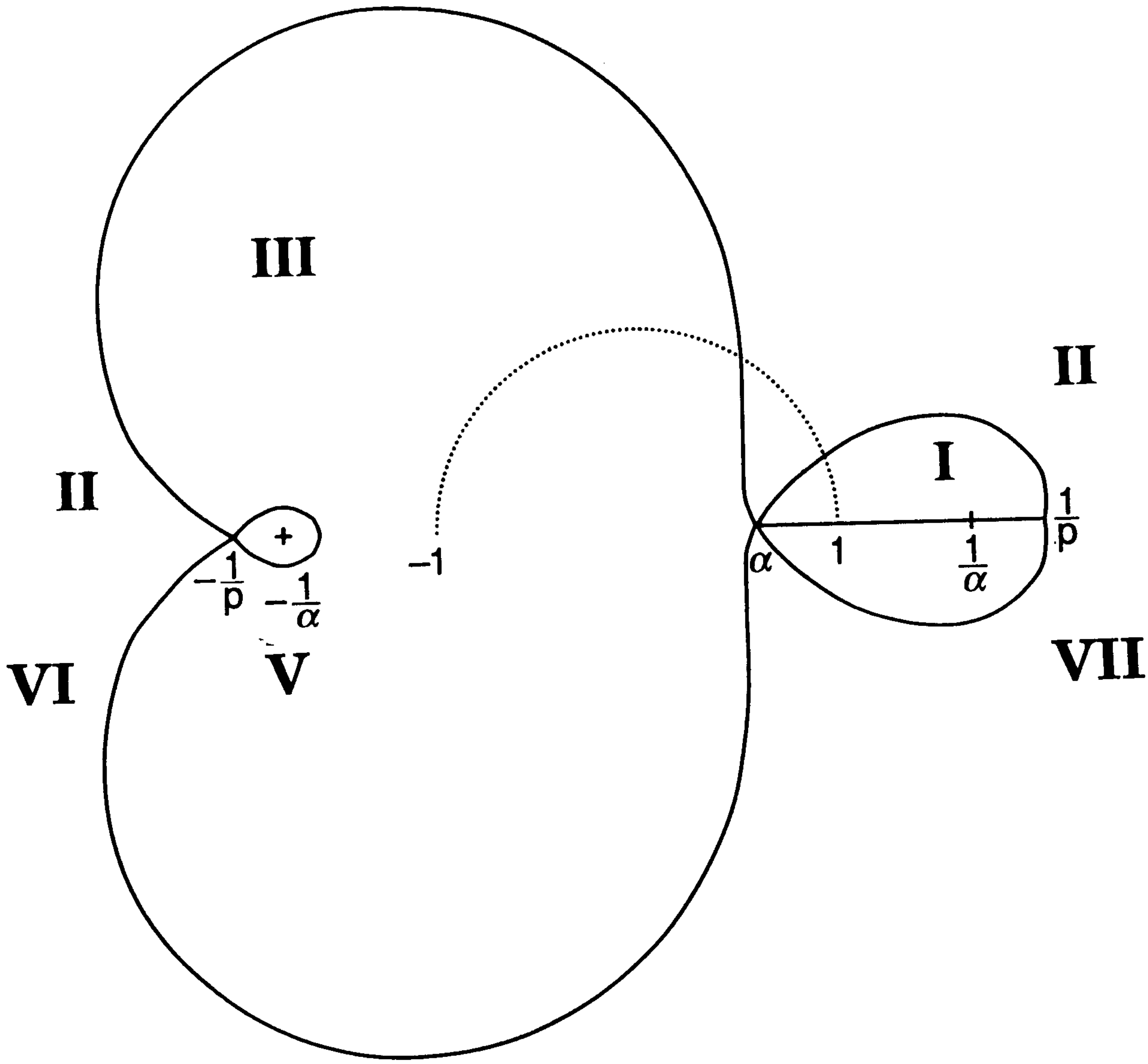

Figure 1 Stokes Lines for $a^2 > p^2 > 0$ in the ς-Plane.

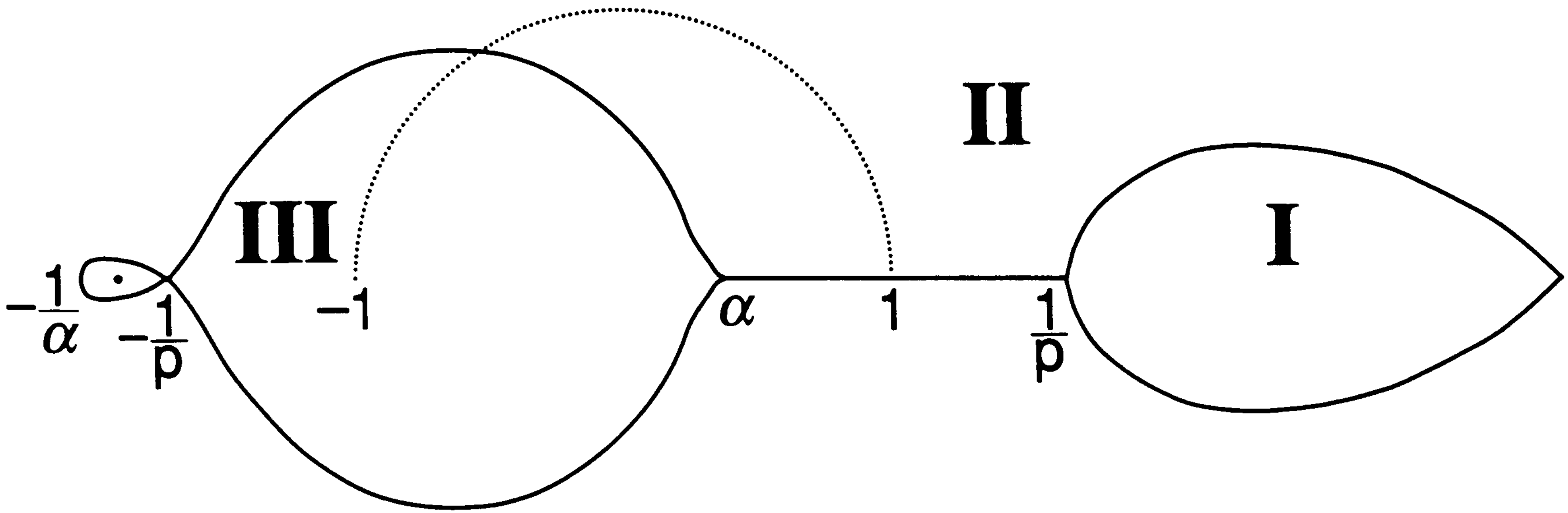

Figure 2 Stokes Lines for $a^2 < p^2$ in the ς-Plane.

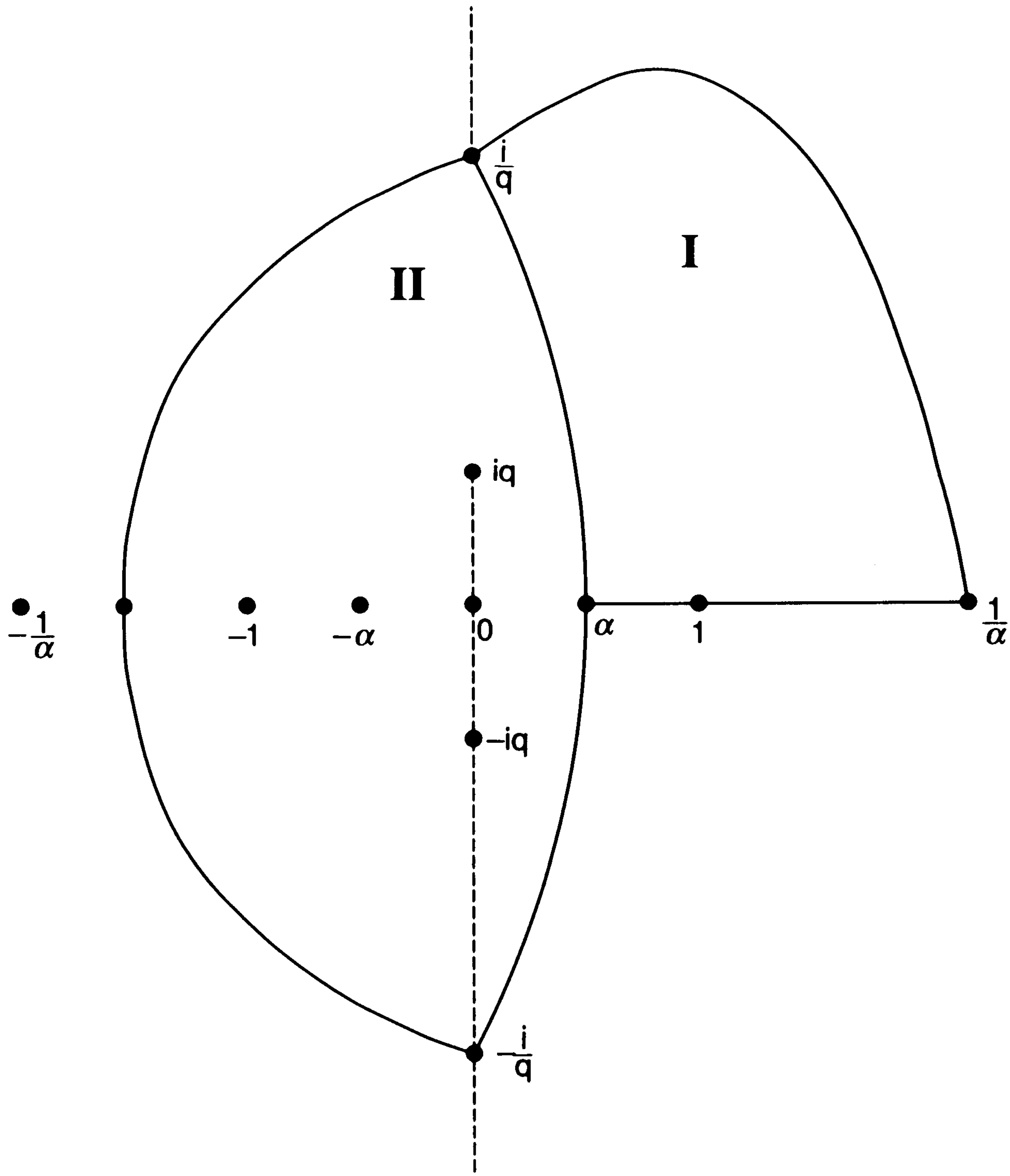

Figure 3 Stokes Lines for $p^2 \equiv -q^2 < 0$ in the ς-Plane.

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16. Abstract

An asymptotic theory is presented for the determination of velocity and linear stability of a steady symmetric bubble in a Hele-Shaw cell for small surface tension. In the first part, the bubble velocity U relative to the fluid velocity at infinity is determined for small surface tension T by determining the leading order transcendentally small correction to the asymptotic series solution. It is found that for any relative bubble velocity U in the interval $(U_c, 2)$, solutions exist at a countably infinite set of values of T (which has zero as its limit point) corresponding to the different branches of bubble solutions. U_c decreases monotonically from 2 to 1 as the bubble area increases from 0 to ∞. However, for a bubble of arbitrarily given size, as $T \rightarrow 0$, solution exists on any given branch with relative bubble velocity U satisfying the relation $2 - U = c\, T^{2/3}$, where c depends on the branch but is independent of the bubble area. The analytical evidence further suggests that there are no solutions for $U > 2$. These results are in agreement with earlier analytical results for a finger.

In the second part of this paper, an analytic theory is presented for the determination of the linear stability of the bubble in the limit of zero surface tension. Only the solution branch corresponding to the largest possible U for given surface tension is found to be stable, while all the others are unstable, in accordance with earlier numerical results.

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