Geometric Foundations of the Theory of Feedback Equivalence

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1. INTRODUCTION

For the past ten years, a group of researchers--mathematicians and theoretical engineers, centered at, and partially supported by, the Flight Control group at NASA-AMES--have attempted to push beyond what was done in the 1960's for linear control theory, and develop effective methods for controlling systems whose dynamical equations are fundamentally *nonlinear*. Our applied focus has been the practical problems encountered in designing aircraft and helicopters, but our methodology--based as it is on fundamental mathematical principles--is adaptable to robotic systems.

Conversely, we hope that use of the new ideas under development in the computer science and AI community will help us use computer technology in a more effective way to handle types of control problems--particularly of a "discrete event" nature--that have been difficult to include in a differential-equations based methodology.

Taking a historical view of progress in engineering and engineering-related mathematics, the situation becomes clarified. The breakthroughs of the 1960's in control theory were closely linked to the development of computers, which could solve differential equations very efficiently. Mathematically, assumptions on *linearity* worked well because of the nature of the engineering problems that needed to be solved, especially in the Apollo Program, where the space craft could be treated satisfactorily as point particles, or at worst as rigid bodies. In the 1970's we attempted to adapt the mathematical techniques developed in the 1960's to the more difficult problems of control of aircraft and helicopters in circumstances where the assumptions of linearity of the dynamics could no longer be realistically justified. Recently, there has been a change in computer technology--such as LISP logic-based symbolic computation and greatly increased potentialities for parallelism--that has not yet been fully integrated into the main body of control theory. Further, computer science has achieved greater maturity and substance, and I believe that there are great scientific and technological possibilities in combining the talents and insights in the two communities. What control theory has to offer is a mature, mathematically based overview of a certain class of engineering problems, based on concepts of *differential equations* and *dynamics*, while the youthful vigor of the computer science discipline is generating a lot of energy, but exhibiting the need (in my opinion, at least) for more scientific and mathematical direction.

For the past two years, I have been trying--with George Meyer's advice and support on the engineering questions--to push in two directions. First, to understand how the control techniques of feedback linearization--developed as a useful control algorithm by Hunt, Meyer and Su at NASA-AMES [1,2]--can be integrated into the mainstream of differential geometry and extended in the direction of understanding the relation between *global* and *local* feedback linearization. Second, I have tried to familiarize myself with the LISP and logic-based computer technology and algorithms, and help in the job of introducing it into control theory. Since the first part of this program is further along--a major mathematical paper is now completed [3] and awaits publication--I will describe some of the ideas it contains here, and leave my ideas about developing relations between computer science and control theory to another occasion.

2. A VIEW OF FEEDBACK CONTROL IN THE CONTEXT OF DIFFERENTIAL EQUATIONS, DIFFERENTIAL GEOMETRY, AND LIE THEORY

A feedback control system can be taken as an underdetermined system of ordinary differential equations of the following general form:

$$f\left(x, \frac{dx}{dt}, u\right) = 0 \qquad (2.1)$$

$$x \in R^n; \quad u \in R^m$$

$$f \text{ is a map: } R^{2n+m} \to R^p$$

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"x" is a vector of R^n describing components of the system (aircraft, helicopter, spacecraft, robot,...) that are fixed in value, such as velocities, positions, angular or linear momenta, etc. "u" are the control variables, which we must choose in some way to achieve a prescribed goal.

Feedback control can be described as follows. A *feedback map* or *law* is a map

$$x \to F(x) = u \qquad (2.2)$$

$$O \to R^m$$

from an open subset O of R^n to the control space R^m. A *trajectory* of the feedback control law (2.2) is a curve

$$t \to x(t) \qquad (2.3)$$

in R^n that satisfies the following ordinary differential equation:

$$f\left(x(t), \frac{dx}{dt}, F(x(t))\right) = 0 \qquad (2.4)$$

In engineering practice, we will want to choose the feedback law (2.2), so that the family of trajectories defined by (2.3) and (2.4) will have certain stability, robustness, and design properties. (For example, for the latter one might want the trajectory (2.3) to start off at time $t = 0$ at a point x_0 and end up exactly or approximately at a point x_1 at $t = t_1$.) *Stabilization* is the property that is best understood mathematically, hence I will use it as a touchstone here.

Much of the work in control theory of the 1960's--which was very successful on both the mathematical and practical fronts--was oriented toward *linear* control systems, i.e., those of the form:

$$\frac{dx}{dt} - Ax - Bu = 0 \qquad (2.5)$$

where A and B are constant matrices of appropriate size. Here, it is natural to require that the feedback (2.1) preserve this linearity. This can be accomplished by specifying that the feedback map (2.2) be of the following form:

$$u = Kx \qquad (2.6)$$

where K is an $m \times n$ real matrix. The trajectory equations (2.2) are then of the following form:

$$\frac{dx}{dt} = (A + BK) \qquad (2.7)$$

One may then require that these trajectories have a prescribed degree of stability. Because (2.7) is a system of differential equations that can be handled with well-known mathematical tecnniques, we know that this behavior can be specified by imposing conditions on the eigenvalues of the $n \times n$ matrix

$$A + BK \qquad (2.8)$$

In turn, this "pole-placement" problem can be handled with well-known mathematical techniques (matrix Riccati equations or Kronecker pencil theory) that were applied in the 1960's, but that of course go back many years in the mathematical literature.

It is especially interesting that the useful sufficient conditions for stabilization of (2.5) via linear feedback (2.6), i.e., "pole-placement" in the engineering jargon, involve *controllability* of the control system (2.5)and is a mathematical concept that is--as I showed many years ago [4]--essentially *differential-geometric* and *Lie-theoretic* in nature. Thus, it should be no surprise that the problem of stabilization and feedback control of a more general nonlinear system of type (2.1) also involves differential geometry and Lie theory.

Indeed, the work of Hunt, Meyer, and Su [1,2] (preceeded by work of Krener [5], Brockett [6], Sommer [7], Jakubzyk and Respondek [8]) demonstrate this in a decisive way. Their work only dealt with feedback control of a certain class of systems (the *feedback linearizable* ones, with the functions f(, ,) occurring in (2.1) satisfying certain conditions) if the trajectory stayed within a small neighborhood R^n, whose size could not be specified in advance. This posed the question of finding conditions for *global* feedback equivalence. There has been important partial work on this problem by Boothby, Dayawansa, and Elliot [9-11] using the tools of differential topology and foliation theory. In my paper [3] I have begun to develop ways of applying the Ehresmann-Haefliger [12] theory of pseudogroup cohomology to this problem, but there is a long way to go before the results that are useful in practical situations will come forth.

The mathematical heart of the methods I have developed in [3] is the theory of vector field systems (or distributions) on a manifold and their equivalence. I will now sketch some of this basic differential-geometric theory, then return to the control situation.

3. VECTOR FIELD SYSTEMS AND FEEDBACK EQUIVALENCE

I will now use the formalism "calculus on manifolds," particularly the theory of vector fields (i.e., first-order linear partial differential operators) and the Jacobi-Lie bracket [,] (i.e., commutator) of such vector fields. See Isidori's book [13] for an engineer's introduction to these concepts.

Let Z be a manifold, with $\underline{V}(Z)$ the space of vector fields. In terms of coordinates (z^i) for Z, $1 \leq i,j \leq N = \dim Z$, a $V \in \underline{V}(Z)$ is a differential operator of the following form:

$$V = A^i(z) \frac{\partial}{\partial z^i} \tag{3.1}$$

(summation convention in force)

If

$$V' = B^i \frac{\partial}{\partial z^i} \tag{3.2}$$

then

$$[V,V'] = \left(A^i \frac{\partial (B^j)}{\partial z^i} - B^i \frac{\partial A^j}{\partial z^i} \right) \frac{\partial}{\partial z^j} \tag{3.3}$$

Let $\underline{F}(Z)$ be the ring of C^∞, real-valued functions on Z. $\underline{V}(Z)$ is a *module* over $\underline{F}(Z)$, since vector fields can be multiplied by functions:

$$(f,V) \to fA^i \frac{\partial}{\partial z^i} \tag{3.4}$$

Definition. A *vector field system* $\underline{W}$ on Z is a subspace of $\underline{V}(Z)$ satisfying the following condition:

$$fV \in \underline{W} \quad \text{for} \quad V \in \underline{W}$$

$$V_1 + V_2 \in \underline{W} \quad \text{for} \quad V_1, V_2 \in \underline{W}$$

i.e., $\underline{W}$ is a *submodule* of $\underline{V}(Z)$.

Let $\underline{W}$ be such a vector field system. For $z \in Z$, set

$$\underline{W}(z) = \{V(z): V \in \underline{W}\} \tag{3.5}$$

$W(z)$ is a linear subspace of the space of tangent vectors at z. Its dimension is called the *rank* of $\underline{W}$ at z. $\underline{W}$ is said to be *nonsingular* if the rank is constant as z ranges over Z. In this paper we will assume that all vector field systems considered have constant rank, unless specified otherwise. The concept defined next will play a basic role in this work.

Definition. Let $\underline{W} \subset \underline{V}$ be a vector field system. Set

$$C(\underline{W}): \{V \in \underline{W}: [V,\underline{W}] \subset \underline{W}\} \tag{3.6}$$

$C(\underline{W})$ is called the *Cauchy Characteristic system* associated with $\underline{W}$.

Theorem 3.1. $C(\underline{W})$ is another vector field system on Z with the following properties:

$$C(\underline{W}) \subset \underline{W} \tag{3.7}$$

$$[C(\underline{W}), C(\underline{W})] \subset C(\underline{W}) \tag{3.8}$$

i.e., $C(\underline{W})$ is *Frobenius integrable* as a vector field system

$$[C(\underline{W}),\underline{W}] \subset \underline{W} \tag{3.9}$$

Proof. Follows from (3.6).

Definition. A curve $t \to z(t)$ in Z is called an *orbit curve* of the vector field system $\underline{W}$ if the following condition is satisfied:

There is a vector field V

$$V = A^i \frac{\partial}{\partial z^i}$$

in $\underline{W}$ such that $t \to z(t)$ is an orbit curve of V, i.e., if

$$\frac{dz}{dt} = V(z(t)) \qquad (3.10)$$

or, in coordinate terms:

$$\frac{dz}{dt} = A(z(t)) \qquad (3.11)$$

In this way, a vector field system defines a family of curves on Z. It is this geometric property that is the key to the usefulness of vector field systems in control theory. As we have seen, control systems are also defined by families of curves, namely solutions of the control equations:

$$f\left(x(t), \frac{dz}{dt}, u(t)\right) = 0 \qquad (3.12)$$

$$x \in R^n, \quad u \in R^m$$

Set:

$$Z = R^n \times R^m$$

$$z = (x,u)$$

We can then define a vector field system $\underline{W}$ on Z as the smallest submodule of $\underline{V}(Z)$ whose orbit curves are solutions of the control equation (3.12).

Let X and X' be manifolds. Let $\underline{W}$ and $\underline{W}'$ be nonsingular vector field systems on X and X', respectively. Let

$$\alpha: X \to X'$$

be a diffeomorphism.

Definition. α is called an *equivalence* from the vector field system $\underline{W}$ to the vector field system $\underline{W}'$ if the following condition is satisfied:

$$\alpha_*(\underline{W}(x)) = \underline{W}'(\alpha(x)) \qquad (3.13)$$

for all $x \in X$

i.e., if α maps an orbit curve of $\underline{W}$ into an orbit curve of $\underline{W}'$. Our problem is to describe numbers attached to vector field systems that are *invariant* under equivalence. We shall cite (without proofs) some of the theorems from [3] that do provide such invariants.

Theorem 3.2. Let $\alpha: X \to X'$ be a diffeomorphism from X to X' that is an equivalence of vector field system $\underline{W}$ to vector field system $\underline{W}'$. Let $C(\underline{W})$ and $C(\underline{W}')$ be the Cauchy characteristic systems of $\underline{W}$ and $\underline{W}'$, respectively. Then, the following condition is satisfied:

$$\alpha_*(C(\underline{W})) = C(\underline{W}') \qquad (3.14)$$

i.e., α is an equivalence between the Cauchy characteristic systems of the given vector field systems.

Definition. For the vector field system $\underline{W}$, set:

$$\underline{W}^1 = \underline{W} + [\underline{W},\underline{W}] \qquad (3.15)$$

It is called a *derived system of* $\underline{W}$.

Theorem 3.3. Let α be an isomorphism from $\underline{W}$ to $\underline{W}'$. Then, it is an isomorphism of the derived system $\underline{W}^1$ to $\underline{W}'^1$.

We can now iterate. Set

$$(\underline{W}^1)^1 = \underline{W}^2, \ldots \tag{3.16}$$

to define the *successive derived systems* of the given vector field system $\underline{W}$, denoted as $\underline{W}^1, \underline{W}^2, \ldots$ We obtain an increasing filtration of submodules of the module of all vector fields on X:

$$\underline{W} \subset \underline{W}^1 \subset \underline{W}^2 \subset \ldots \tag{3.17}$$

Theorem 3.4. We have:

$$C(\underline{W}) \subset C(\underline{W}^1) \subset C(\underline{W}^2) \subset \ldots \tag{3.18}$$

In words, this says that the Cauchy characteristics of the derived systems also form an ascending, filtered sequence of submodules of the module of all vector fields on X.

We assume that all the modules (3.17) and (3.18) are of constant rank. Set

$$\begin{aligned} r &= \text{rank } \underline{W} \\ c &= \text{rank } C(\underline{W}) \\ r_1 &= \text{rank } \underline{W}^1 \\ c_1 &= \text{rank } C(\underline{W}^2) \end{aligned} \tag{3.19}$$

and so on.

Theorem 3.5. The sequence of integers

$$\begin{aligned} r &\le r_1 \le r_2 \le \cdots \\ c &\le c_1 \le c_2 \le \cdots \end{aligned} \tag{3.20}$$

attached to the vector field system $\underline{W}$ are numerical *equivalence invariants*.

Let us now apply these results to control systems in state space form.

4. FEEDBACK INVARIANTS FOR CONTROL SYSTEMS IN STATE SPACE FORM

Let us now specialize the feedback control system to consider those of the following *state space* form:

$$\frac{dx}{dt} = f(x,u) \tag{4.1}$$

$$y \in R^n, \quad u \in R^m \ .$$

Theorem 4.1. Let

$$\begin{aligned} Z &= R^n \times R^m \\ &= \{(x,u) : x \in R^n,\ u \in R^m\} \end{aligned}$$

Let $\underline{W}$ be the vector field system Z generated by the components of the following vector-valued vector fields on Z:

$$\underline{W} = \left\{ f(x,u)\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right\} \tag{4.2}$$

Then, the orbit curves on $\underline{W}$ are precisely the curves $t \to (x(t),u(t))$ that satisfy the control equation (4.1).

Theorem 4.2. Let $dx/dt = f(x,u)$, and $dz/dt = h(z,v)$ be two feedback control systems with the same number of states and controls. Let

$$\underline{W} = \left\{ f(x,u)\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right\}$$

and

$$\underline{W}' = \left\{ h(y,v)\frac{\partial}{\partial y}, \frac{\partial}{\partial v} \right\}$$

be the vector field systems assigned to these control systems. Let

$$T: R^{n+m} \to R^{n+m} \tag{4.3}$$

be a C^∞ map of the following form:

$$T(x,u) \to (y,v) \tag{4.4}$$

with

$$\begin{aligned} y &= \alpha(x) \\ v &= \beta(x,u) \end{aligned} \tag{4.5}$$

Then T maps the control system $\{dx/dt = f(x,u)\}$ into the control system $\{dy/dt = h(y,v)\}$, in the sense that it maps solution curves of the first system of ordinary differential equations into solution curves of the second, if and only if T is an equivalence from the vector field system $\underline{W}$ to the vector field system $\underline{W}'$. In particular, the integers $r, r_1, \ldots; c, c_1, \ldots$ assigned by (3.3) to $\underline{W}$ are invariant under feedback equivalence. In the case of a linear control system, these integers can be computed in terms of the *controllability indices*.

Let us now consider the vector field systems associated with a linear, scalar input, control system, i.e., one of the following form:

$$\frac{dx}{dt} = Ax + bu \tag{4.6}$$

$$x \in R^n, \quad u \in R, \quad b \in R^n$$

Associate with that system the following pairs of vector fields on R^n:

$$\begin{aligned} V &= Ax\frac{\partial}{\partial x} \\ V_0 &= b\frac{\partial}{\partial x} \end{aligned} \tag{4.7}$$

Let $\underline{W}$ be the vector field systems on R^n spanned by these two vector fields and $\partial/\partial u$. Set:

$$\begin{aligned} V_i &= Ad^i(V)(V_0) \\ &= [V, V_{i-1}] \\ &\text{for } i \geq 0 \end{aligned} \tag{4.8}$$

Theorem 4.3. The following commutation relations hold among these vector fields on R^n:

$$\begin{aligned} [V, V_i] &= V_{i+1}, \qquad \text{for } i = 0,1,2,\ldots \\ [V_i, V_j] &= 0, \qquad \text{for } i,j = 0,1.\ldots \end{aligned} \tag{4.9}$$

The derived system $\underline{W}^j$ is the vector field system generated by

$$\left\{ \frac{\partial}{\partial u}, V, V_i;\ i = i,\ldots,j \right\} \tag{4.10}$$

Theorem 4.4. If the system (4.6) is controllable, then

$$C(\underline{W}_{j+1}) = \left\{ \frac{\partial}{\partial u}, V_0, V_1, \ldots, V_j \right\} \tag{4.11}$$

As I show in [3], Theorem 4.4 is the geometric heart of the sufficient conditions that Hunt, Meyer, and Su [1,2] provided in their work on feedback linearization, namely:

Theorem 4.5. Let V_0, V_1 be vector fields on R^n generating a single input, controllable control system of the following form

$$\frac{dx}{dt} = V(x) + uV_0(x) \tag{4.12}$$

Suppose the following condition is satisfied:

The vector field systems

$$\{V_0,\ V_1 = [V,V_0], \ldots,\ V_j = [V,V_{j-1}]\} \tag{4.13}$$

are Frobenius integrable for all j .

Then, the system (4.12) is *locally* feedback equivalent to a chosen system.

In the Hunt-Meyer-Su work, the transformation T, which establishes the feedback equivalence of (4.12) with a linear system, is obtained as a solution of a system of first order, partial differential equations, and we can only prove existence of such linearizing transformations *locally*. A basic question is:

How to piece together such local feedback equivalences to find a *global* one?

The answer can be described in terms of cohomology theory [12]. Indeed, this is a typical problem of *global* differential geometry:

Find conditions for the existence (and computational feasibility!) of a *global* solution of a system of partial differential equations when the conditions for existence of local solutions are satisfied.

What complicates the analysis of the conditions for existence of a global solution is that the cohomology theory one must use involves an algebraic object--the groupoid of feedback automorphisms of the linear control systems--that is *infinite dimensional*, so that standard topological techniques are not very helpful. It is interesting to note that elementary particle physicists at the frontiers--in the so-called *string theory*--are involved with mathematical monstrocities that are very similar to these! Work on this question is in progress.

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