A New Class of Random Processes With Application to Helicopter Noise

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ABSTRACT

This paper introduces a new class of random processes $X(t)$ whose autocorrelation $R_x(t_1,t_2)$ satisfies a relation of the type

$$R_x(t_1, t_2) = \sum_{|j|=1}^{N} a_j R_x(t_1 + \tau_j, t_2 + \tau_j)$$

for all t_1 and t_2 in some interval of the t-axis. Such random processes are denoted as linearly correlated. This class is shown to include the familiar stationary and periodically correlated processes as well as many other, both harmonizable and non-harmonizable, nonstationary processes. When a process is linearly correlated for all t and harmonizable, its two-dimensional power spectral density $S_x(\omega_1,\omega_2)$ is shown to take a particularly simple form, being non-zero only on lines such that $\omega_1-\omega_2=\pm r_k$ where the r_k's are (not necessarily equally spaced) roots of a characteristic function. The relationship of such processes to the class of stationary processes is examined. In addition, the application of such processes in the analysis of typical helicopter noise signals is described.

INTRODUCTION

The concept of dividing random processes into classes having particular desirable properties has long been employed. For example, the classes of stationary[1], locally stationary[2], periodically correlated[3,4,5], and harmonizable[6] processes readily come to mind.* In this paper, a new class of random processes, called linearly correlated, is introduced which includes many of the aforementioned processes as well as other interesting processes which fall into none of the above classes. The properties of these new processes and their relation to the other classes of processes are examined. In addition, an application to helicopter noise, which cannot generally be rigorously analyzed as either stationary or periodically correlated, is discussed.

DEFINITION AND EXAMPLES

Consider the class of random processes X(t) which satisfy the relation

$$R_x(t_1, t_2) = \sum_{|j|=1}^{N} a_j R_x(t_1 + \tau_j, t_2 + \tau_j) \tag{1}$$

for all t_1 and t_2 in some interval of the time axis. Here $R_x(t_1,t_2)$ is the autocorrelation of the random process X(t), the a_j's and τ_j's are real constants, and N is a finite positive integer. Such processes will be denoted as linearly correlated.

As examples of this class of processes, consider:

1. Stationary Random Processes

For a stationary random process, $R_x(t_1, t_2) = R_x(t_1+t, t_2+t)$ for all t. In this case, the τ_j's may be taken to be arbitrary and Eq. (1) becomes

$$R_x(t_1, t_2) = \sum_{|j|=1}^{N} a_j R_x(t_1 + \tau_j, t_2 + \tau_j)$$

$$= R_x(t_1, t_2) \sum_{|j|=1}^{N} a_j$$

Thus, as long as $\sum_{|j|=1}^{N} a_j = 1$, stationary random processes are in the above class.

2. Periodically Correlated Random Processes

For a periodically correlated random process, $R_x(t_1,t_2)=R_x(t_1+np,t_2+np)$ for all n where p is the period of the process. In this case, choosing $\tau_j=jp$, Eq. (1) becomes

$$R_x(t_1, t_2) = \sum_{|j|=1}^{N} a_j R_x(t_1 + jp, t_2 + jp)$$

$$= R_x(t_1, t_2) \sum_{|j|=1}^{N} a_j$$

Thus, again with the requirement $\sum_{|j|=1}^{N} a_j = 1$, periodically correlated random processes are seen to be members of the above class.

3. The Wiener Process

For the Wiener process, $R_x(t_1,t_2)=\sigma^2 \min(t_1,t_2)$ for $t\geq 0$, where σ is constant. Thus, assuming $\tau_j \geq 0$ for all j, Eq. (1) becomes

$$R_x(t_1, t_2) = \sigma^2 \min(t_1, t_2)$$

$$= \sum_{|j|=1}^{N} a_j R_x(t_1 + \tau_j, t_2 + \tau_j) = \sum_{|j|=1}^{N} a_j \sigma^2 \min(t_1 + \tau_j, t_2 + \tau_j)$$

$$= \sum_{|j|=1}^{N} a_j \sigma^2 [\min(t_1, t_2) + \tau_j]$$

$$= \sigma^2 \min(t_1, t_2) \sum_{|j|=1}^{N} a_j + \sigma^2 \sum_{|j|=1}^{N} a_j \tau_j$$

For this process, the dual requirements $\sum_{|j|=1}^{N} a_j = 1$ and $\sum_{|j|=1}^{N} a_j \tau_j = 0$ must be met, along with the condition $\tau_j \geq 0$. As an example, suppose $\tau_1=1$, $\tau_2=2$, $a_1=2$, $a_2=-1$ with all others a_j's=0. Then

$$R_x(t_1, t_2) = 2R_x(t_1 + 1, t_2 + 1) - R_x(t_1 + 2, t_2 + 2)$$

Thus, the Wiener process is linearly correlated for $t\geq 0$.

4. Processes of the Form $X(t)=\alpha^t Y(t)$

Here α is a real constant and Y(t) is stationary. This is an example of a locally stationary random process[2]. For such processes,

$$R_x(t_1, t_2) = \alpha^{t_1 + t_2} R_Y(t_1, t_2)$$

Thus, for arbitrary τ_j, Eq. (1) becomes

$$R_x(t_1, t_2) = \alpha^{t_1 + t_2} R_Y(t_1, t_2)$$

$$= \sum_{|j|=1}^{N} a_j \alpha^{t_1 + t_2 + 2\tau_j} R_Y(t_1 + \tau_j, t_2 + \tau_j)$$

$$= \alpha^{t_1 + t_2} R_y(t_1, t_2) \sum_{|j|=1}^{N} a_j \alpha^{2\tau_j}$$

and such processes are seen to be linearly correlated if $\sum_{|j|=1}^{N} a_j \alpha^{2\tau_j} = 1$. For example, with $\tau_1=1$ and $\tau_2=2$, $a_1=3\alpha^{-2}$ and $a_2=-2\alpha^{-4}$ and all other a_j's=0,

$$R_x(t_1, t_2) = 3\alpha^{-2} R_x(t_1 + 1, t_2 + 1) - 2\alpha^{-4} R_x(t_1 + 2, t_2 + 2)$$

5. Processes of the Form $X(t)=t^k Y(t)$

Here k is a positive integer and Y(t) is stationary. For such processes, $R_X(t_1,t_2)=t_1^k t_2^k R_Y(t_1,t_2)$. Thus, Eq. (1) becomes

$$R_X(t_1, t_2) = t_1^k t_2^k R_Y(t_1, t_2)$$

$$= \sum_{|j|=1}^{N} a_j (t_1 + \tau_j)^k (t_2 + \tau_j)^k R_Y(t_1 + \tau_j, t_2 + \tau_j)$$

$$= R_Y(t_1, t_2) \sum_{|j|=1}^{N} a_j \sum_{m=0}^{k} \binom{k}{m} t_1^{k-m} \tau_j^m \sum_{n=0}^{k} \binom{k}{n} t_2^{k-n} \tau_j^n$$

$$= t_1^k t_2^k R_Y(t_1, t_2) \sum_{m=0}^{k} \sum_{n=0}^{k} \binom{k}{m} \binom{k}{n} t_1^{-m} t_2^{-n} \sum_{|j|=1}^{N} a_j \tau_j^{m+n}$$

Thus, if

$$\sum_{|j|=1}^{N} a_j \tau_l^{m+n} = \begin{cases} 1 & m = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

then such processes are linearly correlated. For example, suppose k=1, $\tau_1=1$, $\tau_2=2$, $\tau_3=3$ and $a_1=3$, $a_2=-3$, and $a_3=1$ with all other a_j's=0. Then

$$R_X(t_1, t_2) = 3R_X(t_1 + 1, t_2 + 1) - 3R_X(t_1 + 2, t_2 + 2)$$
$$+ R_X(t_1 + 3, t_2 + 3)$$

6. Processes where Y(t) is Periodically Correlated

The previous two examples may be extended to the case where Y(t) is periodically correlated by taking $\tau_j=jp$ where p is the period of the process.

HARMONIZABILITY

A random process X(t) is said to be harmonizable if the double Fourier transform of its autocorrelation

$$S_X(\omega_1, \omega_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \, R_X(t_1, t_2) \, e^{i(\omega_1 t_1 - \omega_2 t_2)} \tag{2}$$

exists. The function $S_X(\omega_1,\omega_2)$ is called the two-dimensional power spectral density of the random process X(t).

Although not all linearly correlated random processes are harmonizable, as can be seen from the previous examples, one of the useful properties of linearly correlated random processes is that, if they are harmonizable, their two dimensional power spectral density takes a particularly simple form. If the random process X(t) is harmonizable and linearly correlated for all t, this can be seen by using Eq. (2) in Eq. (1),

$$S_x(\omega_1,\omega_2) = \sum_{|j|=1}^{N} a_j \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, R_x(t_1+\tau_j, t_2+\tau_j)\, e^{i(\omega_1 t_1 - \omega_2 t_2)}$$

$$= \sum_{|j|=1}^{N} a_j e^{-i(\omega_1-\omega_2)\tau_j} \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, R_x(t_1+\tau_j, t_2+\tau_j)\, e^{i[\omega_1(t_1+\tau_j) - \omega_2(t_2+\tau_j)]}$$

$$= S_x(\omega_1,\omega_2) \sum_{|j|=1}^{N} a_j e^{-i(\omega_1-\omega_2)\tau_j}$$

This equation can be satisfied if and only if $S_x(\omega_1,\omega_2)=0$ everywhere except where

$$\sum_{|j|=1}^{N} a_j e^{-i(\omega_1-\omega_2)\tau_j} = 1$$

or, letting $\omega=\omega_1-\omega_2$, at the real roots of the equation

$$f(\omega) = \sum_{|j|=1}^{N} a_j e^{-i\omega\tau_j} - 1 = 0 \qquad (3)$$

These roots, $\omega=r_k$, imply that $\omega_1-\omega_2=r_k$ and thus the two-dimensional spectra can have nonzero values only on lines $\omega_1-\omega_2=r_k$ parallel to the line $\omega_1=\omega_2$. One of these roots must be $\omega=0$ since $S_x(\omega_1,\omega_1) = E\left[|X(\omega_1)|^2\right]$ which cannot be zero for all ω_1 unless X(t)=0. Here

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{-i\omega t} dt$$

is the (generalized) Fourier transform of X(t).

Thus,

$$f(o) = \sum_{|j|=1}^{N} a_j - 1 = 0$$

and

$$\sum_{|j|=1}^{N} a_j = 1$$

is seen to be a necessary condition for harmonizability. Further, the roots of Eq. (3) occur symmetrically since if $\omega = r_k$ is a root,

$$f(-r_k) = f^*(r_k) = 0$$

This leads to a two-dimensional spectra with support as shown on Figure 1. Processes with this type of spectra have been called "almost periodically correlated" by Hurd(7).

In order to develop examples of such spectra, it is useful to introduce the change of variables

$$\bar{t} = \frac{t_1 + t_2}{2} \qquad \tau = t_2 - t_1$$

into Eq. (2). Then,

$$S_x(\omega_1, \omega_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau\, R_x(\bar{t}, \tau)\, e^{-i(\omega_1 - \omega_2)\bar{t}}\, e^{\left(\frac{\omega_1 + \omega_2}{2}\right)\tau} \tag{4}$$

1. Stationary Random Processes

All stationary random processes are harmonizable and $R_x(\bar{t}, \tau) = R_x(\tau)$. In this case, the double integral separates yielding

$$S_x(\omega_1, \omega_2) = S_x\left(\frac{\omega_1 + \omega_2}{2}\right)\delta(\omega_1 - \omega_2)$$

where $\delta(\cdot)$ is the Dirac delta function and

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)\, e^{i\omega\tau} d\tau$$

is the one-dimensional power spectral density of the random process X(t). Thus, the two-dimensional power spectra density will have values only along the line $\omega_1=\omega_2$ where Eq. (3) is known to have a root.

2. Periodically Correlated Random Processes

Continuous periodically correlated random processes are not necessarily harmonizable[8]. However, since $R_x(\bar{t}, \tau) = R_x(\bar{t} + np, \tau)$ for all n, the autocorrelation is periodic in $\bar{t}$ and may be written

$$R_x(\bar{t}, \tau) = \sum_{n=-\infty}^{\infty} R_n(\tau) e^{i\frac{2n\pi}{p}\bar{t}} \tag{5}$$

Thus, for those periodically correlated processes which are harmonizable, utilizing this relation in Eq. (4),

$$S_x(\omega_1, \omega_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\bar{t} \int_{-\infty}^{\infty} d\tau \sum_{n=-\infty}^{\infty} R_n(\tau) e^{i\frac{2n\pi}{p}\bar{t}} e^{-i(\omega_1 - \omega_2)\bar{t}} e^{\left(\frac{\omega_1 + \omega_2}{2}\right)\tau}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \sum_{-\infty}^{\infty} R_n(\tau)\, \delta\left(\omega_1 - \omega_2 - \frac{2n\pi}{p}\right) e^{i\left(\frac{\omega_1 + \omega_2}{2}\right)\tau}$$

$$= \sum_{n=-\infty}^{\infty} S_n\left(\frac{\omega_1 + \omega_2}{2}\right) \delta\left(\omega_1 - \omega_2 - \frac{2n\pi}{p}\right)$$

where

$$S_n(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_n(\tau)\, e^{i\omega\tau} d\tau$$

is the Fourier transform of the nth coefficient in the Fourier series for the autocorrelation function given by Eq. (5). Thus, the two-dimensional spectra will have values only along the lines $\omega_1-\omega_2=2n\pi/p$, n=0, ±1,±2,. . .

Note that for the case of a periodically correlated random process, one may take $a_1=1$ and $\tau_1=p$ with all other a_j's=0 in Eq. (1). Then, becomes Eq. (3) becomes

$$e^{-i\omega p} = 1$$

with roots

$$\omega = \frac{2n\pi}{p} \quad n=0,\pm1,\pm2,\ldots$$

Thus, Eq. (3) will have roots at those points where the two-dimensional spectrum is nonzero.

As an example of this type of spectrum, consider binary noise, B(t). This is a simple random process in which the time axis is divided into intervals of length p and, for $np<t\leq(n+1)p$, $P[B(t)=1]=P[B(t)=-1]=1/2$. If t_1 and t_2 are not in the same interval, $B(t_1)$ and $B(t_2)$ are independent random variables. Then, it can be seen that

$$R_B(t_1,t_2) = \begin{cases} 1 & t_1 \text{ and } t_2 \text{ in same int.} \\ 0 & t_1 \text{ and } t_2 \text{ in different int.} \end{cases}$$

as shown in Figure 2. The two dimensional power spectrum of this process may be readily obtained by carrying out the integral of Eqn. (4). It is

$$S_B(\omega_1,\omega_2) = \frac{p}{2\pi} e^{-i\left(\frac{\omega_1-\omega_2}{2}\right)p} \left(\frac{\sin\frac{\omega_1}{2}p}{\frac{\omega_1 p}{2}}\right)\left(\frac{\sin\frac{\omega_2 p}{2}}{\frac{\omega_2}{2}p}\right) \sum_{j=-\infty}^{\infty} \delta\left(\omega_1-\omega_2-\frac{2\pi j}{p}\right)$$

Note that this spectrum is real.

The amplitude (i.e. coefficients of the delta functions) of this relationship is shown in Figure 3 for p=2.

3. Sum of Two Independent Periodically Correlated Random Processes with Incommensurate Periods

Let

$$S(t) = X(t) + Y(t)$$

be the sum of two independent, mean zero, periodically correlated random processes X(t) and Y(t) with periods p_1 and p_2 respectively. Further, suppose that the periods are incommensurate. Then

$$R_s(\bar{t}, \tau) = R_X(\bar{t}, \tau) + R_Y(\bar{t}, \tau) \tag{6}$$

Such a random process is not itself periodically correlated. This can be seen by supposing the contrary, that S(t) is periodically correlated with period T. Then, using Eqn. (5), Eqn (6) becomes

$$R_s(\bar{t}, \tau) = \sum_{n=-\infty}^{\infty} R_{sn}(\tau)\, e^{\frac{i2n\pi\bar{t}}{T}} = \sum_{n=-\infty}^{\infty} R_{xn}(\tau) e^{\frac{i2n\pi\bar{t}}{p_1}} + \sum_{n=-\infty}^{\infty} R_{yn}(\tau) e^{\frac{i2n\pi\bar{t}}{p_2}}$$

$$= \sum_{n=-\infty}^{\infty} R_{xn}(\tau) e^{\frac{i2n\pi}{p_1}(\bar{t}+T)} + \sum_{n=-\infty}^{\infty} R_{yn}(\tau)\, e^{\frac{i2n\pi}{p_2}(\bar{t}+T)}$$

since, by definition, $R_s(\bar{t}, \tau) = R_s(\bar{t} + T, \tau)$. Thus,

$$\sum_{n=-\infty}^{\infty} R_{xn}(\tau)\left[1 - e^{\frac{i2n\pi}{p_1}T}\right] e^{\frac{i2n\pi}{p_1}\bar{t}} + \sum_{n=-\infty}^{\infty} R_{yn}(\tau)\left[1 - e^{\frac{i2n\pi}{p_2}T}\right] e^{\frac{i2n\pi}{p_2}\bar{t}} = 0$$

Fourier transforming this equation with respect to $\bar{t}$ yields

$$\sum_{n=-\infty}^{\infty} R_{xn}(\tau)\left[1 - e^{\frac{i2n\pi}{p_1}T}\right] \delta\left(\omega - \frac{2n\pi}{p_1}\right) + \sum_{n=-\infty}^{\infty} R_{yn}(\tau)\left[1 - e^{\frac{i2n\pi}{p_2}T}\right] \delta\left(\omega - \frac{2n\pi}{p_2}\right) = 0$$

This relationship can be true if and only if T/p_1=integer=N_1 and T/p_2=integer=N_2. However, this implies that $p_1/p_2 = N_2/N_1$ which violates the assumption that the periods are incommensurate. Thus S(t) is not periodically correlated.

Since the operator of Eq. (2) is linear, assuming that X(t) and Y(t) are harmonizable implies that S(t) is harmonizable with two-dimensional spectrum

$$S_s(\omega_1, \omega_2) = S_X(\omega_1, \omega_2) + S_Y(\omega_1, \omega_2)$$

which is of the form shown in Figure 1 for a linearly correlated random process. Further, it can be proved that S(t) is, in fact, a linearly correlated process: Let $a_1 = a_2 = 1$

and a_3=-1 with $\tau_1=p_1$, $\tau_2=p_2$, $\tau_3=p_1+p_2$ and all other a_j's=0. In terms of the transform variables $\bar{t}$ and τ, Eqn. (1) becomes

$$R_x(\bar{t},\tau) = \sum_{j=1}^{N} a_j R_x\left(\bar{t}+\tau_j, \tau\right) \tag{7}$$

Thus, for the present example,

$$R_s(\bar{t},\tau) = R_s\left(\bar{t}+p_1,\tau\right) + R_s\left(\bar{t}+p_2,\tau\right) - R_s\left(\bar{t}+p_1+p_2,\tau\right)$$

since

$$R_s\left(\bar{t}+p_1,\tau\right) + R_s\left(\bar{t}+p_2,\tau\right) - R_s\left(\bar{t}+p_1+p_2,\tau\right)$$

$$= R_x\left(\bar{t}+p_1,\tau\right) + R_Y\left(\bar{t}+p_1,\tau\right) + R_x\left(\bar{t}+p_2,\tau\right) + R_Y\left(\bar{t}+p_2,\tau\right)$$

$$- R_x\left(\bar{t}+p_1+p_2,\tau\right) - R_y\left(\bar{t}+p_1+p_2,\tau\right) = R_x(\bar{t},\tau) + R_y\left(\bar{t}+p_1,\tau\right) + R_x\left(\bar{t}+p_2,\tau\right)$$

$$+ R_y(\bar{t},\tau) - R_x\left(\bar{t}+p_2,\tau\right) - R_y\left(\bar{t}+p_1,\tau\right)$$

$$= R_x(\bar{t},\tau) + R_y(\bar{t},\tau) = R_s(\bar{t},\tau)$$

upon use of Eqns (5) and (6). Thus, the sum of two harmonizable, independent, periodically correlated processes with incommensurate periods is not only harmonizable (note that $\sum_{j=1}^{N} a_j = 1$) but also linearly correlated. This example is of interest in the analysis of helicopter noise where the sound reaching an observer at a location fixed relative to the helicopter consists of two periodically correlated, mean zero random noise processes generated by the main and tail rotors respectively. Ordinarily, the periods of these two signals are incommensurate. This application is illustrated in Figure 4.

RELATIONSHIP WITH STATIONARY PROCESSES

The class of stationary random processes is familiar and well understood, particularly as to the interpretation of its spectra. Thus, it is of interest to attempt to relate this new class of linearly correlated processes with the familiar stationary class. However, since all stationary processes are harmonizable, a close relation could only be anticipated for the harmonizable subset of linearly correlated processes. This attempt is lent credence by the powerful theorem of Miamee and Salehi [9] who have shown that any harmonizable random process is the projection of a stationary process in a larger space.

Perhaps the place to start is with periodically correlated processes which are known [8] to be closely related to stationary processes. If X (t) is a periodically correlated random process with period p, then

$$E[X(t)] = \sum_{n=-\infty}^{\infty} A_n e^{\frac{i2n\pi}{p}t}$$

and

$$R_x(\bar{t}, \tau) = \sum_{n=-\infty}^{\infty} R_n(\tau) e^{\frac{i2n\pi}{p}\bar{t}}$$

i.e. its mean is a periodic function of t and its auto correlation is a periodic function of $\bar{t} = \frac{t_1 + t_2}{2}$.

Now, define

$$Y(t) = X(t - \theta)$$

where θ is a random variable which is independent of $X(t)$ and uniformly distributed over the internal (o, p).

Then

$$E[Y(t)] = E[X(t-\theta)] = E_\theta[E_x[X(t-\theta)|\theta]]$$

$$= \sum_{n=-\infty}^{\infty} A_n e^{\frac{i2n\pi}{p}t} \frac{1}{p}\int_0^p e^{-\frac{i2n\pi}{p}\theta} d\theta = A_o$$

since

$$\frac{1}{p}\int_0^p e^{-\frac{i2n\pi}{p}\theta} d\theta = \delta_{no}$$

where δ_{nm} is the Kronecker delta function. Thus, the mean value of $Y(t)$ is constant. In fact, it can be seen that

$$E[Y(t)] = \frac{1}{p}\int_0^p E[X(t)]\,dt = A_o$$

is just the average value of the mean of $X(t)$ over the period. Further,

$$R_y(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[X(t_1-\theta)X(t_2-\theta)]$$

$$= E_\theta[E_x[X(t_1-\theta)X(t_2-\theta)|\theta]]$$

$$= \sum_{n=-\infty}^{\infty} R_n(\tau) e^{\frac{i2n\pi}{p}\bar{t}} \frac{1}{p}\int_0^p e^{-\frac{i2n\pi}{p}\theta} d\theta = R_o(\tau)$$

Thus, the autocorrelation of $Y(t)$ depends only upon the time difference $\tau = t_2 - t_1$, and the portion of the autocorrelation $R_x(\bar{t}, \tau)$ which is independent of $\bar{t}$ is seen to be the autocorrelation of $Y(t)$. Again, note that

$$R_y(\tau) = \frac{1}{p}\int_0^p R_x(\bar{t}, \tau)\,d\bar{t} = R_o(\tau)$$

is just the average value of $R_x(\bar{t}, \tau)$ over a period.

Since the mean value of $Y(t)$ is constant and its autocorrelation is only a function of the time difference, $Y(t)$ satisfies the definition of a weakly stationary random process. Thus, a periodically correlated random process becomes stationary under the action of a random phase shift uniformly distributed over its period. Further, the power spectral density of the stationary process is

$$S_y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_y(\tau)\, e^{-i\omega\tau} d\tau$$
$$= \text{mass of } S_x(\omega_1, \omega_2) / _{\omega_1 = \omega_2 = \omega}$$

i.e. just the mass of the two dimensional spectrum of $X(t)$ which lies along the line $\omega_1 = \omega_2$.

A similar approach can be taken with the sum of two independent periodically correlated random processes, when they can be summed with independent random phases, since the sum of two independent weakly stationary random processes is a weakly stationary random process. However, in the important application to helicopter noise signals, the relative phase of the two signals is fixed by mechanical gearing within the helicopter drive train. Thus, a different approach must be employed.

Suppose

$$S(t) = X(t) + Y(t)$$

is the sum of the two independent, mean zero, periodically correlated random processes with incommensurate periods p_1 and p_2 as discussed previously. Recall that $S(t)$ is not periodically correlated, but is linearly correlated. Now, let

$$\theta = \theta_1 + \theta_2$$

where θ_1 and θ_2 are independent random variables uniformly distributed over the intervals (o, p_1) and (o, p_2) respectively. Assume, without loss of generality, that $p_1 \le p_2$. Then, it can easily be shown that θ is a nonuniformly distributed random variable having density function

$$f_\theta(\theta) = \begin{cases} \dfrac{\theta}{p_1 p_2} & 0 \le \theta \le p_1 \\ \dfrac{1}{p_2} & p_1 \le \theta \le p_2 \\ \dfrac{p_1 + p_2 - \theta}{p_1 p_2} & p_2 \le \theta \le p_1 + p_2 \end{cases}$$

as shown in Figure 5. Define

$$Z(t) = S(t - \theta)$$

as the linearly correlated random process after undergoing two random phase shifts. Then,

$$E[Z(t)] = E[S(t-\theta)] = E_\theta\left[E_s[S(t-\theta)\,|\,\theta]\right] = o$$

and

$$E[Z(t_1)Z(t_2)] = E[S(t_1 - \theta)S(t_2 - \theta)]$$

$$= E_\theta\left[E_s\left[S(t_1-\theta)S(t_2-\theta)\,|\theta\right]\right] = E_\theta\left[R_s(\bar{t}-\theta,\tau)\right]$$

$$= E_\theta\left[R_x(\bar{t}-\theta,\tau)\right] + E_\theta\left[R_y(\bar{t}-\theta,\tau)\right]$$

$$= \sum_{n=-\infty}^{\infty} R_{xn}(\tau)\, e^{\frac{i2n\pi}{p_1}\bar{t}}\, E_\theta\left[e^{\frac{i2n\pi}{p_1}\theta}\right]$$

$$+ \sum_{n=-\infty}^{\infty} R_{yn}(\tau)\, e^{\frac{i2n\pi}{p_2}\bar{t}}\, E_\theta\left[e^{\frac{-i2n\pi}{p_2}\theta}\right]$$

$$= R_{xo}(\tau) + R_{yo}(\tau) = R_{So}(\tau)$$

since

$$E_\theta\left[e^{\frac{-i2n\pi}{p_1}\theta}\right] = E_\theta\left[e^{\frac{-i2n\pi}{p_2}\theta}\right] = \delta_{no}$$

Thus, the mean of Z(t) is zero and the autocorrelation of Z(t) is seen to depend only upon the time difference τ which implies that Z(t) is a weakly stationary random process as well. This result indicates that the random signals which typically arise in helicopter noise measurements can be rendered stationary by the application of two random phase shifts. A similar approach can apparently be utilized with any harmonizable linearly correlated random process [10].

DISCUSSION AND CONCLUSIONS

This paper has introduced a new class of random processes, denoted as linearly correlated, and has examined the relationship of this class to other classes of random processes, in particular stationary, periodically correlated and harmonizable. The relations are illustrated by the Venn diagram shown on Figure 6. Stationary random processes, which may be viewed as periodically correlated with arbitrary period, are a subset of the periodically correlated random processes which are, in turn, a subset of the linearly correlated random processes. The class of harmonizable processes, which includes stationary processes, contains only some of the periodically and linearly correlated processes, although it contains others which are in neither class.

For those processes which are both linearly correlated for all t and harmonizable, it was shown that the two-dimensional power spectrum $S_x(\omega_1, \omega_2)$ takes a particularly simple form, being non-zero only on lines such that $\omega_1 - \omega_2 = \pm r_k$ where the r_k' s are (not necessarily equally spaced) roots of a characteristic function. It was also noted and demonstrated by example that such processes may be transformed into stationary random processes by the introduction of a suitably chosen phase shift θ, i.e. if X(t) is an arbitrary harmonizable linearly correlated random process, then a θ can be found such that

$$Z(t)=X(t-\theta)$$

is a weakly stationary random process.

What implication does this result have for the analysis of such processes? As a concrete example, suppose one has a typical helicopter noise measurement where the main rotor and tail rotor have incommensurate periods. The signal is not periodic in any sense and thus cannot be analyzed as periodic. However, the above result seems to indicate that one can either view the measurement as a sample function from a highly nonstationary linearly correlated random process, in which case further analysis and its interpretation are more difficult, or one can view the measurement as a sample function from a stationary random process, which was obtained from the nonstationary process by an appropriate phase shift, in which case the further analysis and interpretation is straightforward. How is this strange result possible and what does it mean?

The answer lies in recalling the objective of Fourier analysis which is to represent a given time history by a collection of sinusoids. If one is content to answer the typical question as to the amplitude of the various sinusoidal components required, then analysis as a stationary random process is sufficient and will yield a representation such that

$$S_x(\omega_1, \omega_2) = E\left[X^*(\omega_1)X(\omega_2)\right] = 0 \qquad \omega_1 \neq \omega_2$$

that is, there will be no linear relationship (correlation) between the Fourier components at different frequencies. Thus, all phase relationships in the signal will have been lost in the representation. However, such linearly correlated signals do exhibit linear phase relationships between the different frequency components which account for the off-diagonal components in $S_x(\omega_1, \omega_2)$. Thus, if one is engaged in analysis (prediction, filtering, detection, etc.) where phase relationships are important, then analysis as a nonstationary linearly correlated process provides much more information about the signal and will allow a more optimum analysis.

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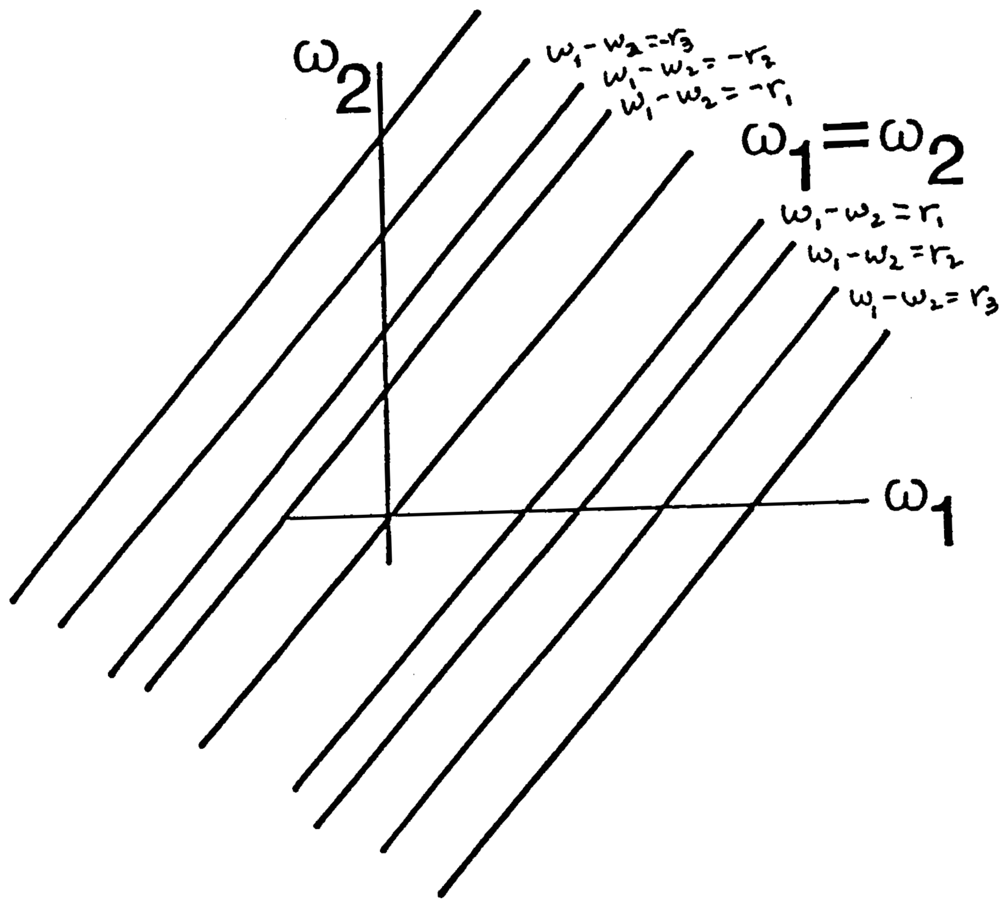
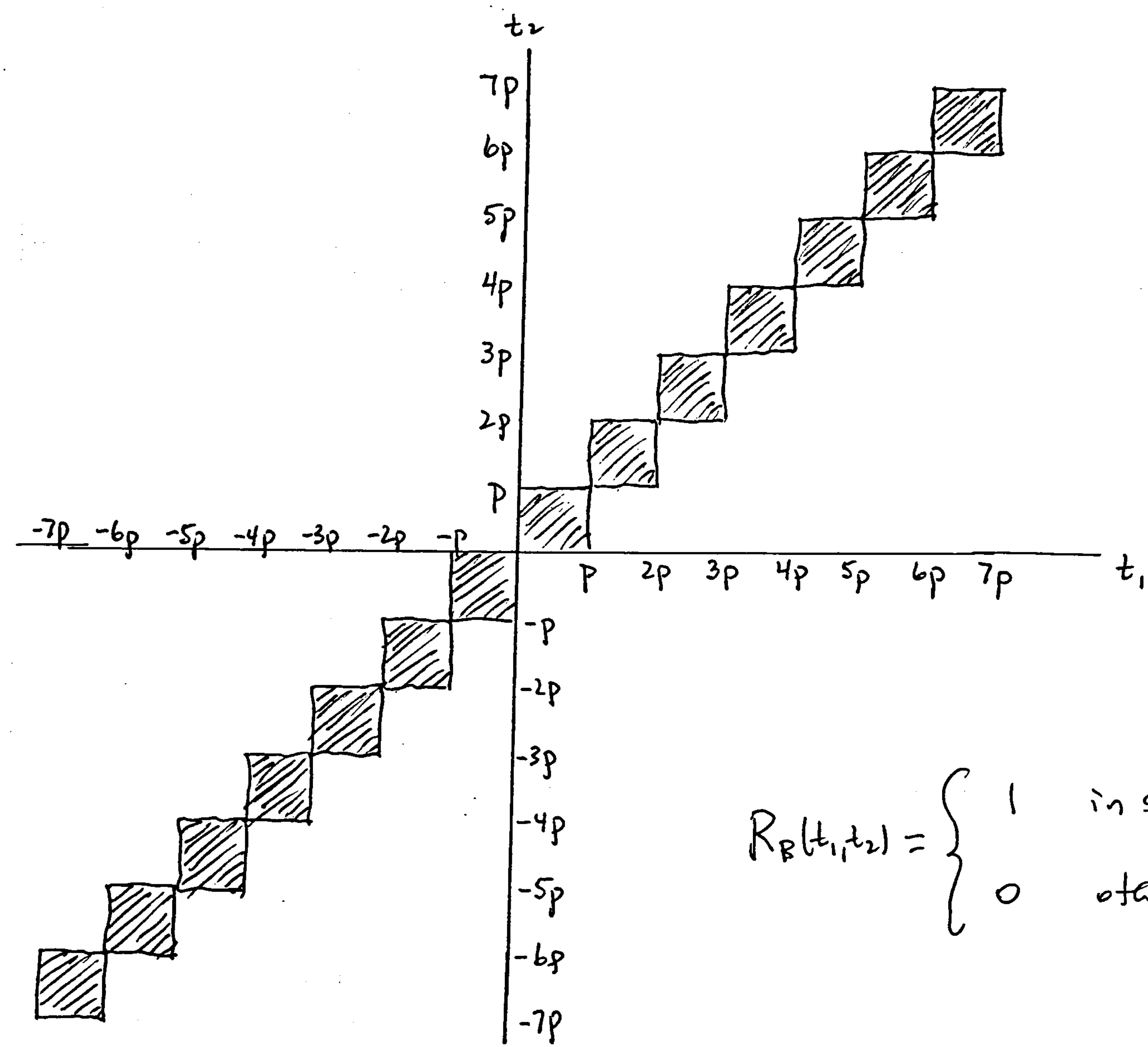


Figure 1: Support of 2-D Power Spectral Density



$$R_B(t_1,t_2) = \begin{cases} 1 & \text{in shaded areas} \\ 0 & \text{otherwise} \end{cases}$$

Figure 2: Autocorrelation of Binary Noise

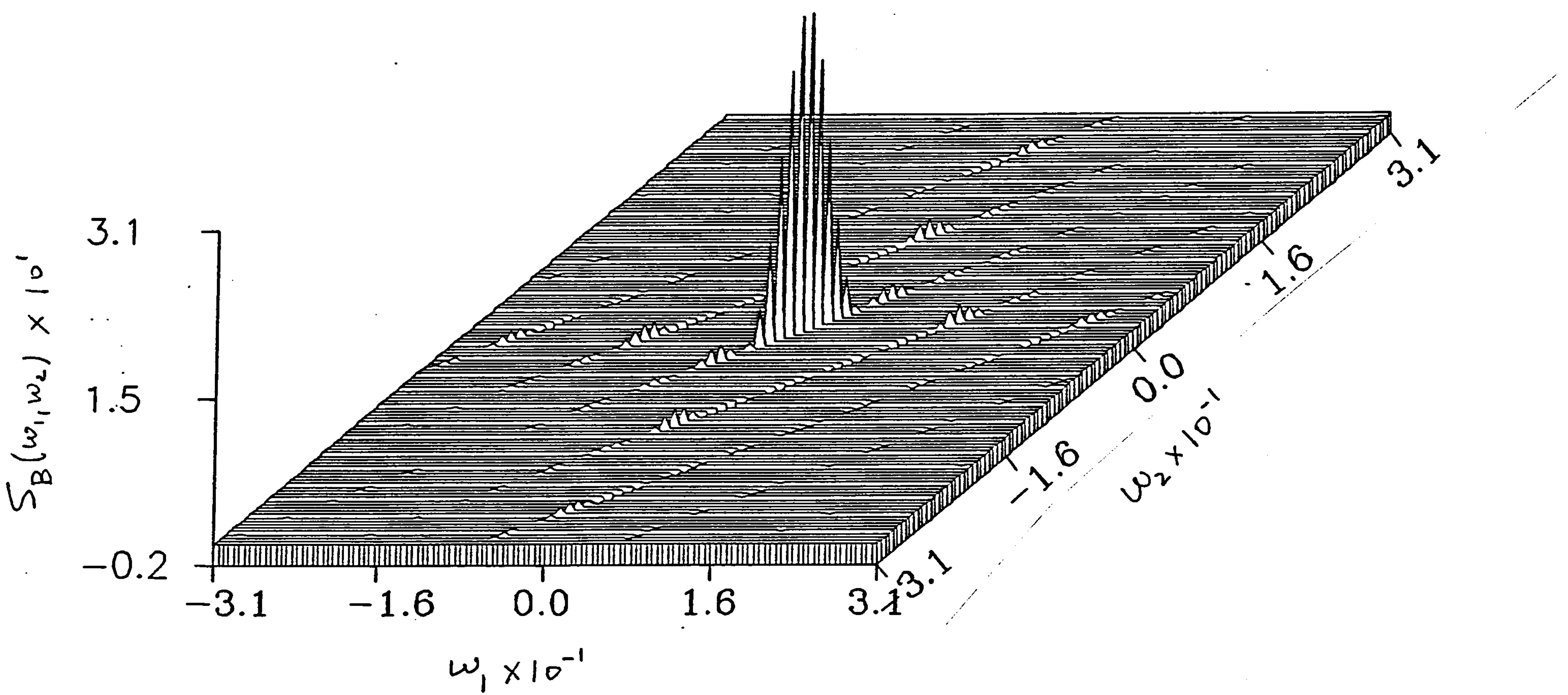


Figure 3: Two Dimensional Spectrum of Binary Noise

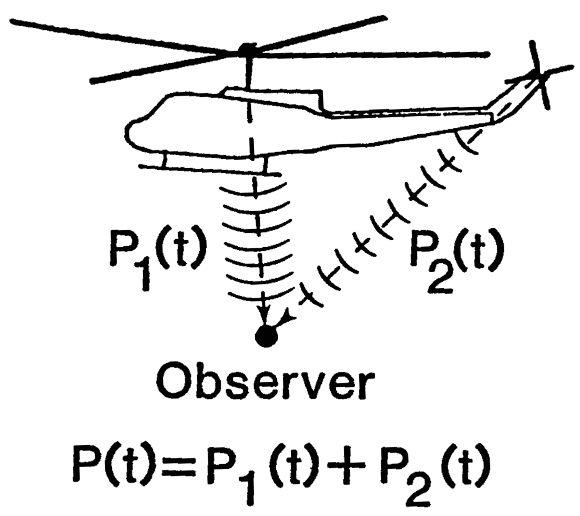


Figure 4: HELICOPTER NOISE FIELD

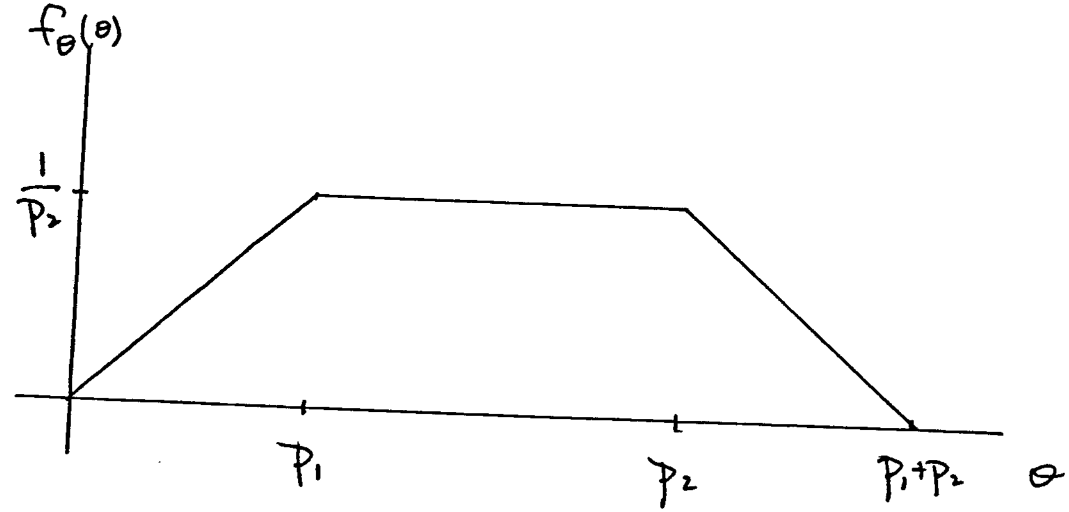


Figure 5: Density Function of Required Shift

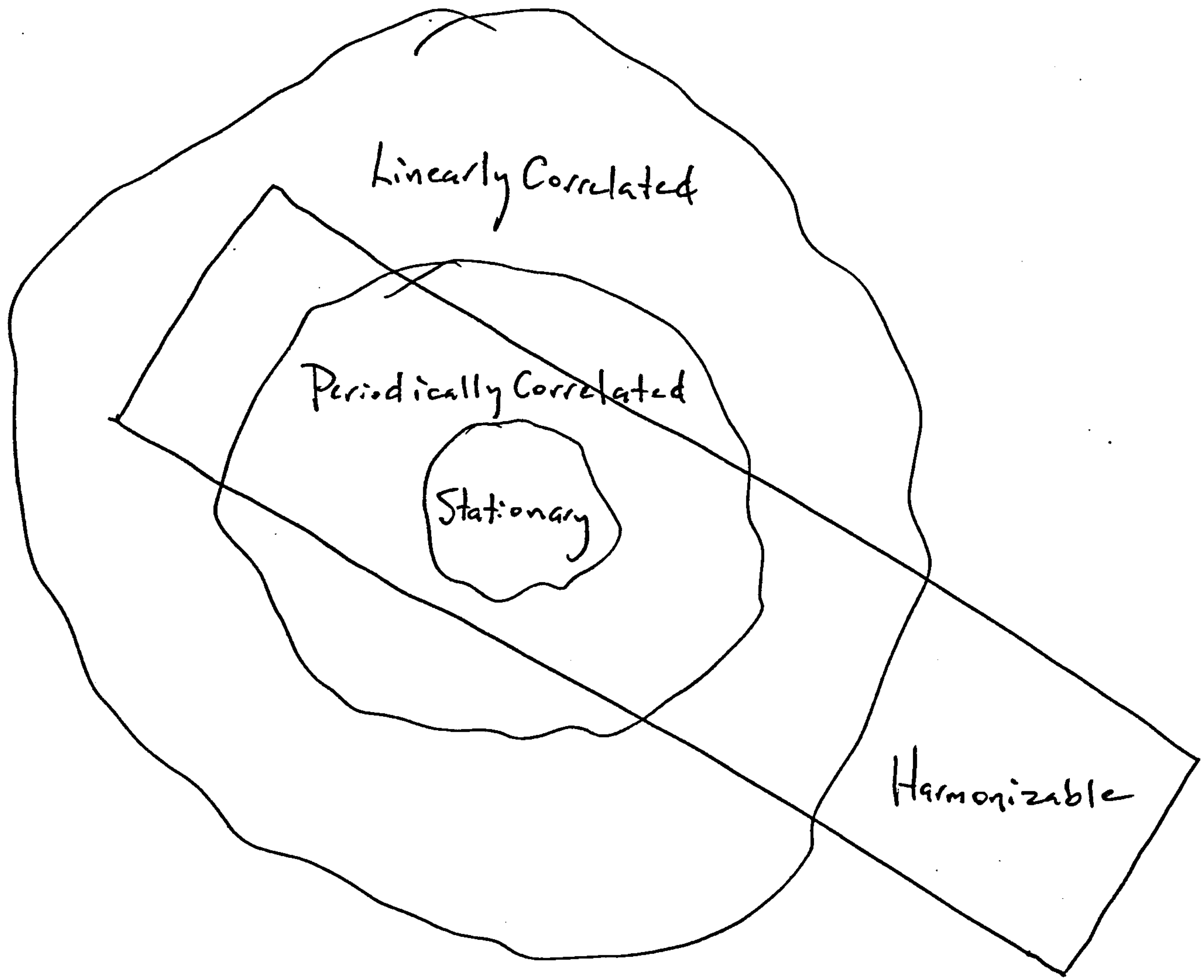


Figure 6: Venn Diagram of Class Relations