



# Fermi National Accelerator Laboratory

FERMILAB-Pub-89/122-A  
May 17, 1989

IN-90-CR

## Gravitational Radiation from Extreme Kerr Black Hole

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### ABSTRACT

We analytically investigate gravitational radiation induced by a test particle falling into an extreme Kerr black hole. Assuming the radiation is dominated by the infinite sequence of quasi-normal modes which has the limiting frequency  $m/(2M)$ , where  $m$  is an azimuthal eigenvalue and  $M$  is the mass of the black hole, we find the radiated energy diverges logarithmically in time. Then we evaluate the back reaction to the black hole by appealing to the energy and angular momentum conservation laws. We find the radiation has a tendency to increase the ratio of the angular momentum to mass of the black hole, which is completely different from non-extreme case, while the contribution of the test particle is to decrease it.

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N89-26794

Unclas  
0217291

G3/90

(NASA-CR-185350) GRAVITATIONAL RADIATION  
FROM EXTREME KERR BLACK HOLE (FERMI  
National Accelerator Lab.) 20 p CSCL 03B



## 1. Introduction

Coalescence of binary neutron stars or black holes has been considered as one of the strongest sources of the gravitational radiation. As for the total energy radiated in the process of the coalescence, several analyses have been done. Gilden and Shapiro<sup>[1]</sup> calculated the head-on collision of two neutron stars by Newtonian hydrodynamics code and obtained the typical efficiency ( $\Delta E/M$ ) of 0.1% for colliding  $1.4M_{\odot}$  neutron stars. As for fully general relativistic calculations, Smarr<sup>[2]</sup> performed simulations of the head-on collisions of two black holes and obtained the efficiency of  $\lesssim 0.1\%$ . Stark and Piran<sup>[3]</sup> calculated the formation process of rotating black holes and obtained the efficiency of  $\lesssim 0.1\%$ . One of the important results in these calculations is the wave pattern is characterized by a quasi-normal mode (QNM) of the black hole except for the very early stage. Especially the wave pattern by Stark and Piran<sup>[3]</sup> is very similar to that from a perturbation calculation of a rotating dust ring of mass  $\mu$  falling into a Kerr black hole of mass  $M$  ( $\gg \mu$ ).<sup>[4]</sup> The collapsing dust ring mimics the flattening in the equatorial plane of the collapsing polytrope. One also finds very good agreement in the amplitudes if one scales  $\mu$  up to the reduced mass of the system which is no longer infinitesimal. These experiences suggest us that the perturbation calculations are a very good guide to fully nonlinear general relativistic calculations.<sup>[5]</sup>

One may think that the above fully nonlinear simulations show the weakness of the sources of gravitational waves. However all these calculations are restricted to axially symmetric systems. Perturbation calculations for the efficiency of the emission of gravitational waves from a test particle falling into a Schwarzschild or Kerr black hole for various cases<sup>[4]</sup> show that in general the efficiency for axially symmetric systems is much smaller than that for non-axially symmetric systems due to the phase cancellation effects.<sup>[6]</sup> The perturbation calculations for Kerr cases<sup>[4,7,8]</sup> show that the wave pattern is also dominated by a QNM of the black hole which depends on the azimuthal eigenvalue  $m$ . The energy from a particle

with orbital angular momentum falling into a Kerr black hole increases with the increase in the Kerr parameter  $a$ . This is due to the fact that the imaginary part of the fundamental QNM for  $m = 2$  goes to zero in the limit of  $a = M$  where  $M$  is the mass of the black hole. The small imaginary part means the large damping time, hence much energy. However what will happen for an extreme Kerr black hole ( $a = M$ ) case was not clear in the previous studies<sup>[4,7,8]</sup> since they were only for  $a \leq 0.99M$ . In this paper we shall study analytically the gravitational waves emitted from a particle falling into an extreme Kerr black hole as a first step to know the maximum efficiency of the gravitational radiation.

One may think that the formation of an extreme Kerr black hole is of only academic interest. But it is not so. Observationally there are 10 binary pulsars at present. Eight of them have small eccentricity and small mass companion ( $\lesssim 0.4M_{\odot}$ ). Three of them have large eccentricity and large mass companion ( $\sim 1.4M_{\odot}$ ). Especially PSR1913+16 is believed to consist of two neutron stars of mass  $1.445M_{\odot}$  and  $1.384M_{\odot}$ .<sup>[9]</sup> Due to the emission of gravitational waves, two neutron stars in PSR1913+16 will coalesce in  $\sim 10^8$  yrs. The recently discovered binary millisecond pulsar PSR0021-72A<sup>[10]</sup> will coalesce in much shorter time  $\sim 10^6$  yrs. Under the assumption of a steady state, it is estimated that coalescence of binary neutron stars will occur  $\sim 10$  events/year up to a distance of 100 Mpc. Therefore they can be relevant sources of gravitational waves. Now the total mass of such a binary like PSR1913+16 is much larger than the maximum mass of the neutron star. Therefore a natural final destiny of such a coalescence is the formation of a black hole. Moreover the total angular momentum of the system when the coalescence begins is larger than that for the extreme Kerr black hole. So the final black hole can be an extreme Kerr black hole.

This paper is organized as follows. In §2, we review the property of QNMs of the extreme Kerr black hole. In §3, the wave form of the gravitational radiation induced by a test particle falling into a black hole is analytically estimated. The effect of back reaction is also considered. §5 is devoted to conclusions and astrophysical implications of the results.

## 2. QNMs of the extreme Kerr black hole

There exist infinite number of QNMs which accumulate onto the critical frequency  $\omega_c = am/(2Mr_+) = m/(2M)$  for  $a = M$ . This occurs for modes with  $|m| = l$  and it has been shown both analytically by Detweiler<sup>[11]</sup> and numerically by Leaver.<sup>[12]</sup> Following Detweiler, we first briefly review the derivation of this fact, rewrite the result in the form more convenient for us, and then discuss the properties of the QNMs.

The QNMs correspond to complex zeros of  $Z_{in}(\omega)$ , the amplitude of ingoing radiation at infinity of the mode which is purely ingoing at the horizon, called the "in" mode. Its form at  $a \sim M$  has been discussed in Appendix A of Teukolsky and Press.<sup>[13]</sup> Taking the limit  $a \rightarrow M$ , one finds that

$$Z_{in} = \alpha(-\delta, \hat{\omega}) e^{i\pi(s+3/2)} (2\hat{\omega})^{-2s-1} (-8\hat{\omega}\tau)^{s+1/2-i(2\hat{\omega}+\delta)} \left[ r e^{i\theta} - (-8\hat{\omega}\tau)^{2i\delta} \right], \quad (2.1)$$

where

$$\begin{aligned} \hat{\omega} &= \omega M, \\ \tau &= \hat{\omega} - \frac{m}{2}, \\ \alpha(-\delta, \hat{\omega}) &= \frac{\Gamma(-2i\delta)\Gamma(1-2i\delta)}{\Gamma(s+1/2-i(2\hat{\omega}+\delta))\Gamma(-s+1/2-i(2\hat{\omega}+\delta))}, \\ r &= \left| \frac{\alpha(\delta, \hat{\omega})}{\alpha(-\delta, \hat{\omega})} \right| = \frac{\cosh \pi(2\hat{\omega} - \delta)}{\cosh \pi(2\hat{\omega} + \delta)}, \\ \theta &= \arg \left[ \frac{-\alpha(\delta, \hat{\omega})}{\alpha(-\delta, \hat{\omega})} \right], \end{aligned} \quad (2.2)$$

$$\delta^2 = m^2 - \left(s + \frac{1}{2}\right)^2 - {}_s\lambda,$$

with  $s$  being the spin of the wave ( $s = \pm 2$  for gravitational waves) and  ${}_s\lambda$ , the eigenvalue of the spin-weighted spheroidal harmonic which depends on  $\hat{\omega}$ ,  $l$  and

$m$  (for notational simplicity, we omit the angular indices  $(l, m)$  unless there is a chance of confusion). For our purpose, we shall take  $s = -2$  and suppress the spin indices in the following. In addition, it is known that  $\delta$  is real for the case of our interest (*i.e.*, for all modes with  $|s| = 2$  and  $|m| = l \geq 2$ , where  $m > 0$  for  $\text{Re}(\omega) > 0$  and  $m < 0$  for  $\text{Re}(\omega) < 0$ ).<sup>[13,14]</sup> Then it can be taken as a positive number without loss of generality.

For later convenience, we translate the above formula for  $Z_{in}$  into the amplitude  $A_{in}$  of the corresponding "in" mode solution for the equation derived by Sasaki and Nakamura,<sup>[15]</sup> which has a nice property of being regular at both infinity and horizon. The derivation of the relation between  $Z_{in}$  and  $A_{in}$  is given in the Appendix. One finds

$$A_{in} = \sqrt{8} \frac{\tau^2}{\hat{\omega}^2} Z_{in}. \quad (2.3)$$

Since the infinite sequence of QNMs, which is given by solving for zeros of the square bracket term in Eq.(2.1), appears for  $\hat{\omega}$  very close to  $\hat{\omega}_c = m/2$  (*i.e.*, for  $\tau \sim 0$ ), one may replace  $\hat{\omega}$  by  $\hat{\omega}_c$  in the rest of the expression (2.1) or (2.2). Letting  $-\hat{\omega}\tau \equiv z$ , one obtains after a straightforward algebra that

$$A_{in}(z; \hat{\omega}_c) = e^{i\phi_0} \frac{\cosh \pi(2\hat{\omega}_c + \delta)}{i\hat{\omega}_c \sinh 2\pi\delta} z^{1/2 - i(2\hat{\omega}_c + \delta)} \left[ r e^{i\theta_0} - z^{2i\delta} \right], \quad (2.4)$$

where

$$\begin{aligned} \phi_0 &= \arg[\alpha(-\delta, \hat{\omega}_c)] - (2\hat{\omega}_c + \delta) \ln 8, \\ \theta_0 &= \theta - 2\delta \ln 8. \end{aligned} \quad (2.5)$$

Then it is readily seen that the zeros of  $A_{in}$  are given by

$$z = z_n = e^{\frac{\theta_0 - 2n\pi}{2\delta}} e^{i\varphi_c}, \quad (2.6)$$

for any (large) positive integers of  $n$ , where the phase  $\varphi_c$  is given by

$$\begin{aligned} \varphi_c &= -\frac{1}{2\delta} \ln r \\ &= \frac{1}{2\delta} \ln \left[ \frac{\cosh \pi(2\hat{\omega}_c + \delta)}{\cosh \pi(2\hat{\omega}_c - \delta)} \right]. \end{aligned} \quad (2.7)$$

In terms of  $\hat{\omega}$ , these QNM frequencies are

$$\hat{\omega}_n = \hat{\omega}_c + \tau_n = \hat{\omega}_c - \frac{1}{\hat{\omega}_c} e^{\frac{\theta_0 - 2n\pi}{2\delta}} e^{i\varphi_c}. \quad (2.8)$$

To see the properties of  $\hat{\omega}_n$  in more details, let us rewrite the above result by noting the expression (2.7) for the phase  $\varphi_c$  and the fact that  $\hat{\omega}_c = m/2$ . We then obtain

$$\hat{\omega}_n = \frac{m}{2} - \frac{2}{m} e^{\frac{\theta_0 - 2n\pi}{2\delta}} \cos |\varphi_c| - i \frac{2}{|m|} e^{\frac{\theta_0 - 2n\pi}{2\delta}} \sin |\varphi_c|, \quad (2.9)$$

where

$$|\varphi_c| = \frac{1}{2\delta} \ln \left[ \frac{\cosh \pi(|m| + \delta)}{\cosh \pi(|m| - \delta)} \right] < \pi.$$

Thus the imaginary part is always negative, irrespective of the sign of  $m$ . Note also that the QNMs of Eq.(2.9) have the symmetry  $\tilde{\omega}_{n;l,-m} = -\hat{\omega}_{n;l,m}$ . In any case, this implies that an extreme Kerr black hole is stable against the excitation of a finite number of the QNMs belonging to the infinite sequence given above. However, since the imaginary part tends to zero as  $n$  approaches infinity, it is not obvious that the excitation of an infinite number of the QNMs would not lead to some kind of instability. In fact, we shall see in the next section that the radiation induced by a test particle diverges logarithmically in general.

### 3. Gravitational radiation induced by a test particle

In this section, we investigate analytically the wave form of gravitational radiation induced by a test particle of energy  $\varepsilon$  falling into an extreme Kerr black hole of mass  $M$ . We assume  $\varepsilon \ll M$ .

The basic equation we consider is the radial equation derived in Ref.15 which takes the form,

$$\left[ \frac{d^2}{dr^{*2}} - \mathcal{F} \frac{d}{dr^*} - \mathcal{U} \right] X = \varepsilon \mathcal{S}, \quad (3.1)$$

where the explicit forms of  $\mathcal{F}$  and  $\mathcal{U}$ , and the relation of the source term  $\mathcal{S}$  to that of the original Teukolsky equation are given in Ref.15. Here we only mention

that  $\mathcal{F}$  and  $\mathcal{U}$  vanish sufficiently rapidly for both  $r^* = \pm\infty$ , and so does  $\mathcal{S}$  for any trajectory of a test particle.

Let  $X_{in}$  and  $X_{out}$  be the homogeneous solutions to Eq.(3.1) (i.e., with  $\mathcal{S} = 0$ ) whose boundary conditions are given by

$$\begin{aligned} X_{in} &\rightarrow \begin{cases} e^{-ikr^*} & \text{for } r^* \rightarrow -\infty; \\ A_{in} e^{-i\omega r^*} + A_{out} e^{i\omega r^*} & \text{for } r^* \rightarrow +\infty, \end{cases} \\ X_{out} &\rightarrow \begin{cases} B_{in} e^{-ikr^*} + B_{out} e^{ikr^*} & \text{for } r^* \rightarrow -\infty; \\ e^{i\omega r^*} & \text{for } r^* \rightarrow +\infty, \end{cases} \end{aligned} \quad (3.2)$$

where  $k = \omega - m\omega_+$  and  $\omega_+$  is the angular velocity of the horizon. Then the solution of Eq.(3.1) can be expressed as

$$X(r^*) = \frac{\varepsilon}{W} \left[ X_{out}(r^*) \int_{-\infty}^{r^*} X_{in} \mathcal{S} dr^{*'} + X_{in}(r^*) \int_{r^*}^{\infty} X_{out} \mathcal{S} dr^{*'} \right], \quad (3.3)$$

where  $W$  is the Wronskian of  $X_{in}$  and  $X_{out}$ ;

$$W(\omega) = X_{in} \frac{dX_{out}}{dr^*} - X_{out} \frac{dX_{in}}{dr^*} = 2i\omega A_{in}(\omega). \quad (3.4)$$

Since we are interested in the radiation emitted to infinity, we only need the form of  $X$  at  $r^* \rightarrow +\infty$ , which becomes

$$X \rightarrow \frac{\varepsilon G(\omega)}{2i\omega A_{in}(\omega)} e^{i\omega r^*}; \quad G(\omega) = \int_{-\infty}^{\infty} X_{in} \mathcal{S} dr^{*'}. \quad (3.5)$$

Let  $h_+$  and  $h_\times$  be amplitudes of the usual plus and cross modes, respectively, of gravitational waves at infinity (i.e., the  $1/r$  part of the metric perturbation)

and let  $h = h_+ + ih_-$ . Then  $h$  is given in terms of  $G$  and  $A_{in}$  as<sup>[18]</sup>

$$h = \frac{8\varepsilon}{r} \sum_{l,m} \int_{-\infty}^{\infty} d\omega \frac{G_{l,m}(\omega) e^{-i\omega u}}{2i\omega c_0(\omega) A_{in;l,m}(\omega)} \frac{S_{l,m}^{a\omega}(\theta)}{\sqrt{2\pi}} e^{im\varphi}, \quad (3.6)$$

where  $S_{l,m}^{a\omega}$  is the  $s = -2$  spin-weighted spheroidal function,  $u = t - r^*$  is the retarded time,  $c_0 = \lambda_{l,m}^{a\omega}(\lambda_{l,m}^{a\omega} + 2) - 12(a^2\omega^2 - m a \omega) - 12iM\omega$  and the mode indices  $(l, m)$  are recovered to avoid ambiguity.

Let us now evaluate the above integral formula for  $h$ . Although it can only be done numerically in general, the late time behavior of  $h$  can be approximately evaluated if one notes the fact that it is dominated by QNMs which are excited by a test particle. Furthermore, the dominant contribution comes from those QNMs which are least damped. In the present case, they are the ones we obtained in the previous section.

Assuming that the integrand vanishes at large  $|\omega|$  on the lower half complex plane of  $\omega$  (which should be true for large positive  $u$ ), the integral in Eq.(3.6) is given approximately by the sum of residues of the integrand at poles of  $1/A_{in}(\omega)$  which are the QNM frequencies;

$$h \sim -\frac{\varepsilon}{r} \sum_{l,m=\pm l} \sum_n \text{Res} \left( \frac{1}{A_{in}(\omega)}; \omega_n \right) \frac{8\pi G_{l,m}(\omega_n) S_{l,m}^{\dot{\omega}_n}}{c_0(\omega_n) \omega_n \sqrt{2\pi}} e^{-i\omega_n u} e^{im\varphi}. \quad (3.7)$$

The residues  $\text{Res}(1/A_{in}(\omega); \omega_n)$  can be calculated from Eq.(2.4). Inserting the result into Eq.(3.7) and approximating all  $\omega_n$  by  $\omega_c (= \dot{\omega}_c/M)$  except  $z_n$  in the residues and the exponent of  $e^{-i\omega_n u}$ , we obtain

$$\begin{aligned} h \sim & -\frac{\varepsilon}{r} \sum_{l,m=\pm l} e^{-i\phi_0} e^{i\varphi_c(1/2+i(2\dot{\omega}_c-\delta))} \frac{\sinh 2\pi\delta}{\delta \cosh \pi(2\dot{\omega}_c + \delta)} \\ & \times \frac{4\pi G_{l,m}(\omega_c) S_{l,m}^{\dot{\omega}_c}}{c_0(\omega_c) \dot{\omega}_c \sqrt{2\pi}} e^{-i\omega_c u + im\varphi} \\ & \times \sum_n \exp \left[ -\frac{2n\pi - \theta_0}{2\delta} \left( \frac{1}{2} + i(2\dot{\omega}_c - \delta) \right) - \beta u \exp \left( -\frac{2n\pi - \theta_0}{2\delta} \right) \right], \end{aligned} \quad (3.8)$$



where

$$\beta = \frac{e^{i\varphi_c}}{i\hat{\omega}_c M}.$$

Note that the real part of  $\beta$  is positive definite (see Eq.(2.9)).

Now we evaluate the sum over  $n$  in the above equation. Let  $N$  be a large positive integer and consider the sum over  $n \geq N$ . An inspection of Eq.(3.8) shows that this is equivalent to replacing  $\theta_0$  by  $\theta_0 - 2N\pi$  and summing over all  $n \geq 0$ . Furthermore, since the important contribution comes from large  $n$ , the sum over  $n$  can be replaced by an integral;

$$\begin{aligned} & \sum_{n=0}^{\infty} \exp \left[ -\frac{n\pi}{\delta} \left( \frac{1}{2} + i(2\hat{\omega}_c - \delta) \right) - e^{-\frac{n\pi}{\delta}} e^{\frac{\theta_0 - 2N\pi}{2\delta}} \beta u \right] \\ & \rightarrow \frac{\delta}{\pi} \int_0^{\infty} dx \exp [-\kappa x - \nu u e^{-x}] \equiv \frac{\delta}{\pi} F(\kappa, \nu; u), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \kappa &= \frac{1}{2} + i(2\hat{\omega}_c - \delta), \\ \nu &= \beta e^{\frac{\theta_0 - 2N\pi}{\delta}}. \end{aligned}$$

The evaluation of the function  $F(\kappa, \nu; u)$  is given in Appendix B. It is expressed in terms of the incomplete gamma function as

$$F(\kappa, \nu; u) = (\nu u)^{-\kappa} \gamma(\kappa, \nu u), \quad (3.10)$$

where note that  $\gamma(\kappa, \infty) = \Gamma(\kappa)$ . Using Eq.(3.10), we finally arrive at the wave form at large  $u$ ;

$$\begin{aligned} h &\sim -\frac{\varepsilon}{r} \sum_{l,m=\pm l} e^{-i\phi_0} \frac{\sinh 2\pi\delta}{\cosh \pi(2\hat{\omega}_c + \delta)} \gamma(\kappa, \nu u) \\ &\times \left( i\hat{\omega}_c \frac{M}{u} \right)^{\frac{1}{2} + i(2\hat{\omega}_c - \delta)} \frac{4G_{l,m}(\omega_c)}{c_0(\omega_c)\hat{\omega}_c} \frac{S_{l,m}^{\hat{\omega}_c}}{\sqrt{2\pi}} e^{-i\omega_c u + im\varphi}, \end{aligned} \quad (3.11)$$

where the phase of  $i\hat{\omega}_c$  should be taken as  $\pi/2$  if  $\hat{\omega}_c > 0$  and  $-\pi/2$  if  $\hat{\omega}_c < 0$ .

The most interesting feature of the wave form is that  $|h| \propto u^{-1/2}$  for  $u \rightarrow \infty$ . This implies that the integrated flux of gravitational radiation diverges logarithmically for  $u \rightarrow \infty$ . Thus the extreme Kerr black hole seems marginally unstable as suggested by Detweiler.<sup>[11]</sup> However, the above analysis neglects the presence of a test particle and the effect of radiation reaction. Although there is no established method for computing the radiation reaction, one may estimate the effect by appealing to the energy and angular momentum conservation laws.

Let  $(M, M^2)$  be the mass and angular momentum, respectively, of the initial (extreme Kerr) black hole,  $(\varepsilon, j)$  be those of a test particle, and  $(E_r, J_r)$  be those of gravitational radiation emitted to infinity. Then the final mass and angular momentum of the black hole,  $(M_f, J_f)$  are given by

$$\begin{aligned} M_f &= M + \varepsilon - E_r; & \frac{E_r}{M} &= O\left[\left(\frac{\varepsilon}{M}\right)^2\right], \quad \frac{\varepsilon}{M} \ll 1, \\ J_f &= M^2 + j - J_r; & \frac{J_r}{M^2} &= O\left[\left(\frac{\varepsilon}{M}\right)^2\right], \quad \frac{j}{M^2} = O\left(\frac{\varepsilon}{M}\right), \end{aligned} \quad (3.12)$$

where it is assumed that there is no incoming radiation from infinity. Inserting the above into the relation  $J_f = M_f^2 q_f$ , solving for  $\Delta q (\equiv q_f - 1)$ , and retaining the terms up to  $O[(\varepsilon/M)^2]$  we obtain

$$\Delta q = -\frac{2M\varepsilon - j}{(M + \varepsilon)^2} - \frac{\varepsilon^2}{M^2} + \frac{2ME_r - J_r}{M^2} + O\left[\left(\frac{\varepsilon}{M}\right)^3\right]. \quad (3.13)$$

It is known that any particle which can fall into the extreme Kerr black hole must have  $\varepsilon \geq j/(2M)$ .<sup>[16]</sup> Hence if  $\varepsilon - j/(2M) = O(\varepsilon)$ , the first term dominates over the rest and one has  $\Delta q < 0$ . However, if  $\varepsilon - j/(2M) \ll \varepsilon$ , all the terms may contribute equally to  $\Delta q$ . To consider such a case, let us parametrize  $j$  as

$$j = 2M\varepsilon \left(1 - b\frac{\varepsilon}{M}\right), \quad (3.14)$$

where  $b$  is a positive constant of order unity. Then

$$\Delta q = -(1 + 2b)\frac{\varepsilon^2}{M^2} + \frac{2ME_r - J_r}{M^2}. \quad (3.15)$$

It is apparent that the contribution of the test particle is still negative. On the other hand, it is not at all clear if the second term is also negative definite, or if it is always smaller in magnitude than the first term. To investigate this problem, note that  $E_r$  and  $J_r$  themselves cannot be evaluated individually, since they are logarithmically divergent. Instead, we evaluate the energy flux  $dE_r/du$  and the angular momentum flux  $dJ_r/du$ . We have

$$\frac{dq_r}{du} = \frac{1}{M^2} \left( 2M \frac{dE_r}{du} - \frac{dJ_r}{du} \right), \quad (3.16)$$

where  $dq_r/du$  means the part of  $dq/du$  which is due to radiation.

The formulas for the radiated energy flux and angular momentum flux are

$$\begin{aligned} \frac{dE_r}{du} &= \frac{1}{16\pi} \int_{r \rightarrow \infty} |\dot{h}|^2 r^2 d\Omega, \\ \frac{dJ_r}{du} &= -\frac{1}{16\pi} \int_{r \rightarrow \infty} \text{Re} \left( \dot{h} \bar{h}' \right) r^2 d\Omega, \end{aligned} \quad (3.17)$$

where  $\dot{h} = \partial h / \partial u$  and  $h' = \partial h / \partial \varphi$ . Though the precise evaluation of  $dE_r/du$  and  $dJ_r/du$  can be done only if the motion of a test particle is specified and only by a careful numerical analysis, a qualitative estimate can be done by using the result obtained in §3, assuming that the QNMs give the dominant contribution to radiation. From Eq.(3.11), for each  $m = \pm l$  component, we find

$$\begin{aligned} \dot{h} &= -i\omega_c h \left( 1 + \frac{\gamma(\kappa + 1, \nu u)}{i\omega_c u \gamma(\kappa, \nu u)} \right), \\ h' &= imh, \end{aligned} \quad (3.18)$$

where we have suppressed the indices  $(l, m)$  for  $h$ . Inserting the above into Eqs.(3.17), we obtain

$$\begin{aligned} \left( \frac{dE_r}{du} \right)_{l,m} &= \frac{\omega_c^2}{16\pi} \int_{r \rightarrow \infty} |h|^2 r^2 d\Omega \left( 1 + 2 \frac{2\hat{\omega}_c - \delta}{\hat{\omega}_c} \frac{M}{u} \right), \\ \left( \frac{dJ_r}{du} \right)_{l,m} &= \frac{m\omega_c}{16\pi} \int_{r \rightarrow \infty} |h|^2 r^2 d\Omega \left( 1 + \frac{2\hat{\omega}_c - \delta}{\hat{\omega}_c} \frac{M}{u} \right), \end{aligned} \quad (3.19)$$

where only the terms that are important at large  $u$  have been retained. Then noting that  $\omega_c = \hat{\omega}_c/M = \omega/(2M)$ , Eq.(3.16) gives

$$\left(\frac{dq_r}{du}\right)_{l,m=\pm l} = \frac{m(m-\delta)}{16\pi M^2 u} \int_{r \rightarrow \infty} |h|^2 r^2 d\Omega. \quad (3.20)$$

Now let us evaluate the sign of  $dq_r/du$ . To do so, it should be reminded that the Teukolsky equation has the symmetry such that

$$\bar{Q}(\omega, l, m) = Q(-\omega, l, -m), \quad (3.21)$$

for any physical amplitude. In particular, this implies

$$|G_{l,m}(\omega_c)|^2 = |G_{l,-m}(-\omega_c)|^2,$$

where  $G_{l,m}$  is the amplitude appearing in Eq.(3.11). Hence  $|h_{l,m}|^2$  and  $|h_{l,-m}|^2$  are not independent and one has to sum the both terms if one wants to know the sign of  $(dq_r/du)_l$ . From Eq.(3.11), we find

$$\int d\Omega r^2 |h|_{l,\pm l}^2 = \frac{16 \sinh^2(2\pi\delta) e^{-\pi(l\mp\delta)}}{l \cosh \pi(l \mp \delta) \cosh^2 \pi(l \pm \delta)} \left| \frac{\varepsilon G_{l,l}}{c_0} \right|^2 \left( \frac{M}{u} \right), \quad (3.22)$$

where the upper and lower signs correspond to  $m = l$  and  $m = -l$ , respectively. Hence  $(dq_r/du)_l$  is given by

$$\begin{aligned} \left(\frac{dq_r}{du}\right)_l &\equiv \left(\frac{dq_r}{du}\right)_{l,l} + \left(\frac{dq_r}{du}\right)_{l,-l} \\ &= \frac{M}{u^2} \left(\frac{\varepsilon}{M}\right)^2 \left| \frac{G_{l,l}}{c_0} \right|^2 \frac{\sinh^2(2\pi\delta) e^{-\pi(l-\delta)}}{\pi \cosh \pi(l-\delta) \cosh^2 \pi(l+\delta)} \\ &\quad \times \left[ (l-\delta) + (l+\delta) \frac{\cosh \pi(l+\delta)}{\cosh \pi(l-\delta)} e^{-2\pi\delta} \right]. \end{aligned} \quad (3.23)$$

Thus the sign of  $(dq_r/du)_l$  depends on the magnitude of  $\delta$ .

For  $l \leq 6$ , one can evaluate  $\delta$  approximately by using Table 1 of Press and Teukolsky.<sup>[17]</sup> One finds  $\delta_{l=2} \approx 2.05 > 2$ , while  $\delta < l$  for  $l \geq 3$ . The latter fact implies  $(dq_r/du)_l$  is positive for  $l \geq 3$ . As for  $l = 2$ , a direct evaluation of the square bracket term in the last line of Eq.(3.23) shows it is also positive. Hence we conclude that  $dq_r/du$  is positive, i.e., the radiation reaction works in favor of the *increase* in  $q$ . We mention that since  $dq_r/du \propto u^{-2}$ , its integral  $\Delta q_r = \int^{\infty} (dq_r/du) du$  is finite, hence can be computed in principle if the test particle trajectory is given. Then the result should take the form,

$$\Delta q_r = \eta \frac{\epsilon^2}{M^2}, \quad (3.24)$$

where  $\eta$  is the efficiency factor which is a function of parameters of the trajectory. For  $a < 1$  it has been shown that  $\Delta q_r$  is negative for all infalling orbits.<sup>[4]</sup> Thus the above result shows a completely distinct nature of an extreme Kerr black hole.

#### 4. Conclusions

In the lowest order of  $\epsilon/M$ , it is known that a test particle falling into an extreme Kerr black hole cannot increase the value of  $a/M (\equiv q)$  above unity,<sup>[16]</sup> no matter what the energy  $\epsilon$  and the angular momentum  $j$  of the test particle are, provided  $\epsilon/M \ll 1$  and  $j/M^2 \ll 1$ . This fact is in accordance with the cosmic censorship hypothesis<sup>[18]</sup> and with Hawking's area theorem,<sup>[19]</sup> since the area of the horizon  $A = 8\pi M r_+$  can increase only if  $q$  decreases, if it is unity in the beginning. Hence as long as the contribution of the test particle to the change of  $q$  is  $O(\epsilon/M)$ , we have  $\Delta q < 0$ . In the case of purely gravitational perturbations without a test particle, it is known that the area theorem holds for any value of  $q < 1$ .<sup>[13]</sup> Taking the limit  $q \rightarrow 1$  from below, one can extend this result and show that  $dq/du < 0$  holds for  $q = 1$ . However, for certain test particle trajectories ( $j=2M\epsilon$ ), the contribution to  $\Delta q$  can be  $O[(\epsilon/M)^2]$ . In such a case, the result

in §3 shows that the energy and angular momentum radiated away, which are of  $O[(\epsilon/M)^2]$ , is important to determine the value of  $q$ . If  $q$  decreases eventually, the black hole will settle down to a stable non-extreme Kerr black hole and the logarithmic divergence of the total flux would not have any physical significance. On the other hand, if the result is opposite, there arises a possibility of violating the cosmic censorship, or at least a possibility of maintaining the value of  $q$  very close to unity for sufficiently long time. Given the evidences that the cosmic censorship does not fail in the both cases of the test particle argument and purely gravitational perturbations, perhaps we should be skeptical about the former possibility.

As we noted in the introduction the total angular momentum of the coalescing binary neutron stars before the collapse is much larger than that of the extreme Kerr black hole. In a sense one can say that the total angular momentum of the extreme Kerr black hole is the lowest angular momentum for the binary. Unless the system loses its angular momentum down to the angular momentum of the extreme Kerr black hole, the system cannot settle down to a single Kerr black hole. The system before the final Kerr black hole probably consists of a Kerr black hole in the center and a radiating envelope around it. So there may exist a case in which the value of  $q$  can remain sufficiently close to unity during the radiating process and consequently the efficiency of gravitational radiation can be very high. Pushing this thought further boldly, one may speculate that during the gravitational collapse of a rapidly rotating star or a binary system, an extreme Kerr black hole is effectively formed and is kept extreme throughout the dynamical stage, emitting a substantial fraction of the rest mass energy in the form of gravitational waves.

If the above conjecture is correct, we can make some astrophysical implications of our results. Clark and Eardley<sup>[20]</sup> studied the evolution of a binary neutron star system consists of neutron stars of mass  $1.3M_{\odot}$  and  $0.8M_{\odot}$  and estimated the gravitational waves emitted in the stable mass stripping phase. They obtained the total energy of the gravitational waves of  $6 \times 10^{52}$  ergs which

is 1.5% of the rest mass. Recently Oohara and Nakamura<sup>[21]</sup> performed Newtonian 3D hydrodynamics calculations of coalescence of binary neutron stars of each mass  $1.4M_{\odot}$ . They are using  $140^3$  grids in the  $(x, y, z)$  coordinates. They began the simulation when the two neutron stars just touch with each other and continued the simulation up to  $t = 2$  msec. The total radiated energy estimated by using the quadrupole formula was 2.6% of the rest mass in 2 msec. This efficiency can be compared with that extrapolated from perturbation calculations for a particle falling into a Kerr black hole with  $a \leq 0.99M$ , which can be as high as 9% in an extreme case.<sup>[4,7,8]</sup> As noted in Introduction, a plausible final destiny for coalescence of binary neutron stars is an extreme Kerr black hole. So the above conjecture suggests that the efficiency of the gravitational waves is much larger than the Newtonian simulations due to the contribution from the QNMs for the extreme Kerr black hole. So there is a possibility that  $\geq 10\%$  efficiency is achieved in reality.

Finally to confirm all the theoretical conjectures in this paper, the construction of a 3D fully general relativistic code is strongly needed.

Acknowledgements: This work was supported in part by the DOE and by the NASA (the grant number NAGW-1340) at Fermilab. One of us (MS) would like to thank Astrophysics/Particle Theory Group at Fermilab for warm hospitality while part of this work was done. MS would like to thank also R. Wald for valuable discussions and W. Press for a useful correspondence.

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## APPENDIX A

Here, we derive the relation between the amplitudes  $A_{in}$  and  $Z_{in}$ . For convenience, we first list some of the coordinate variables which are frequently used here and in the text:

$$\begin{aligned}
 dr^* &= \frac{r^2 + a^2}{\Delta} dr, \\
 dv &= dt + dr^*, \\
 d\bar{\varphi} &= d\varphi + \frac{a}{\Delta} dr, \\
 \Delta &= r^2 - 2Mr + a^2, \\
 K &= (r^2 + a^2)\omega - am, \\
 J_{\pm} &= \frac{d}{dr} \pm i \frac{K}{\Delta},
 \end{aligned} \tag{A.1}$$

where  $t$ ,  $r$  and  $\varphi$  are the standard Boyer-Lindquist coordinates.<sup>[23]</sup> Although we concentrate on the case  $a = M$  in this paper, the following discussion is valid even for  $a \neq M$  as long as  $(M - a)/M \ll 1$ .

The radial function used in Appendix A of Teukolsky and Press<sup>[13]</sup> is of  $\Omega$ , in the coordinates  $(v, \bar{\varphi})$ , which we denote by  $\bar{R}_s$ . The radial function of the standard Teukolsky equation<sup>[22]</sup> is of  $\Upsilon_s$  in the coordinates  $(t, \varphi)$ , which we denote

by  $R_s$ . The variables  $\Omega_s$  and  $\Upsilon_s$  are related to each other by

$$\Upsilon_{-s} = \left(\frac{\Delta}{2}\right)^s \Omega_s. \quad (\text{A.2})$$

Hence the corresponding radial functions are related to each other by

$$R_{-s} \propto \left(\frac{\Delta}{2}\right)^s \tilde{R}_s e^{-i \int \frac{K}{\Delta} dr}. \quad (\text{A.3})$$

Further, for  $|s| = 2$ , there is a simple relation between  $R_2$  and  $R_{-2}$ ;

$$R_{-2} \Delta^2 \propto (J_+)^4 (\Delta^2 R_2). \quad (\text{A.4})$$

From Eqs.(A.3) and (A.4), one has

$$R_{-2} = f_0 \Delta^2 \frac{d^4}{dr^4} \left( \tilde{R}_{-2} \right) e^{-i \int \frac{K}{\Delta} dr}, \quad (\text{A.5})$$

where  $f_0$  is a constant which depends on the normalization of  $R_{-2}$  and  $\tilde{R}_{-2}$ . In what follows, we denote  $R_{-2}$  by  $R$  and  $\tilde{R}_{-2}$  by  $\tilde{R}$  for notational simplicity.

From Eqs.(2.9) and (2.13) of Sasaki and Nakamura<sup>[15]</sup>, our radial function  $X$  for the regularized equation is related to  $R$  by

$$\begin{aligned} X &= \frac{f(r^2 + a^2)^{3/2}}{gh} J_- \left( h J_- \left( \frac{gR}{r^2 + a^2} \right) \right) \\ &= \frac{f(r^2 + a^2)^{3/2}}{gh} \frac{d}{dr} \left( h \frac{d}{dr} \left( \frac{g}{r^2 + a^2} R e^{-i \int \frac{K}{\Delta} dr} \right) \right) e^{i \int \frac{K}{\Delta} dr}, \end{aligned} \quad (\text{A.6})$$

where  $f$ ,  $g$  and  $h$  are, to an extent, arbitrary functions of  $r$  except for certain boundary conditions they must satisfy, but for definiteness we choose  $f = h = \text{const.}$  and  $g = (r^2 + a^2)/r^2$  as given in Appendix B of Ref.15. Then  $X$  is expressed in terms of  $\tilde{R}$  as

$$X = f_1 r^2 (r^2 + a^2)^{1/2} \frac{d^2}{dr^2} \left( \frac{\Delta^2}{r^2} \left( \frac{d^4}{dr^4} \tilde{R} \right) e^{-2i \int \frac{K}{\Delta} dr} \right) e^{i \int \frac{K}{\Delta} dr}, \quad (\text{A.7})$$

where  $f_1$  is a constant.

The asymptotic behavior of the "in" mode solution for  $X$  (denoted by  $X_{in}$  in the text, but here we omit the suffix) is

$$X \rightarrow \begin{cases} e^{-i(\omega - m\omega_+)r^*} & \text{for } r \rightarrow r_+; \\ A_{in} e^{-i\omega r^*} + A_{out} e^{i\omega r^*} & \text{for } r \rightarrow \infty, \end{cases} \quad (\text{A.8})$$

and that for  $\tilde{R}$  is

$$\tilde{R} \rightarrow \begin{cases} 1 & \text{for } r \rightarrow r_+; \\ Z_{in} (r/r_+)^3 + Z_{out} (r_+/r) e^{2i\omega r^*} & \text{for } r \rightarrow \infty, \end{cases} \quad (\text{A.9})$$

where  $r_+ = M + \sqrt{M^2 - a^2}$  is the radius of the horizon and  $\omega_+ = a/(2Mr_+)$  is the angular frequency of the horizon. Equations (A.8) and (A.9) fix the normalization of the radial functions.

Now, from Eqn.(A.7)  $\sim$  (A.9) and using the solution given in Appendix A of Ref.13, it is straightforward to show that  $A_{in}$  is related to  $Z_{in}$  by

$$A_{in} = \sqrt{8} \frac{r_+^2}{\hat{\omega}^2} Z_{in}, \quad (\text{A.10})$$

which we quoted in Eq.(2.3).

## APPENDIX B

Here we evaluate the function  $F$  defined in Eq.(3.9);

$$F(\kappa, \nu; u) = \int_0^{\infty} dx \exp[-\kappa x - \nu u e^{-x}]. \quad (\text{B.1})$$

Expanding the integrant, we obtain

$$\begin{aligned} F(u) &= \int_0^{\infty} dx e^{-\kappa x} \sum_{n=0}^{\infty} \frac{(-1)^n (\nu u)^n e^{-nx}}{n!} \\ &= (\nu u)^{-\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n (\nu u)^{\kappa+n}}{(\kappa+n) n!} \\ &= (\nu u)^{-\kappa} \gamma(\kappa, \nu u), \end{aligned} \quad (\text{B.2})$$

where  $\gamma(\kappa, z)$  is the incomplete gamma function. In particular, we have

$$F(u) \rightarrow (\nu u)^{-\kappa} \Gamma(\kappa); u \rightarrow \infty. \quad (\text{B.3})$$