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A Procedure for Computing Surface Wave Trajectories on an Inhomogeneous Surface

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η	surface variable along orthogonal trajectory of rays
θ	angle, relative to $\widehat{N}$, at which ray is launched from surface
κ_{g1}	geodesic curvature of ray
κ_{g2}	geodesic curvature of orthogonal trajectory of rays
κ_l	component of curvature vector of surface wave front in direction $\hat{p}$
κ_{n1}	normal curvature of ray
κ_{n2}	normal curvature of orthogonal trajectory of rays
$\boldsymbol{\kappa}_w$	curvature vector of surface wave front
λ	direction $\frac{dv}{du}$ on surface
λ_n	direction $\frac{dv}{du}$ orthogonal to rays
μ	general ray parameter
ξ	specific surface parameter
ρ	parameter determined by densities of materials on two sides of surface
σ	arc length along ray
ϕ	$= \frac{\pi}{2} - \theta$
ψ	angle that surface paraxial ray makes with $\widehat{N}, \widehat{T}$ plane

Subscripts:

f	base point of surface
i	initial point of surface
m	conditions in medium surrounding surface
o	initial point on surface ray

Basic Considerations

The analysis utilizes vector terminology consistent with such fundamental differential geometry texts as references 7 and 8. The section "Analysis" includes a derivation of the required equations and a description of the procedure for computing the surface rays and the ray strip width, which is analogous to ray tube area for volume waves. These quantities are required to obtain the phase and amplitude of the surface wave.

For purposes of analysis, it is assumed that the surface can be described by a vector equation in two surface variables u, v. That is, a point on the surface is given by

$$\boldsymbol{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k} \qquad (1)$$

A class of practical shapes that can be modeled in this form is discussed in the section "Analytically Lofted Surfaces." The section "Sample Calculations" illustrates how the wave equations are used to compute surface waves on such surfaces. The final section treats some details of computing the wave field and considers the problem of waves launched from the surface into a surrounding medium that will support waves.

The analysis does not include derivation of the wave speeds associated with the various types of surface waves. These derivations are included in the basic works on those waves, as for example, reference 9. If the surface is such that the wave speed is uniform, it follows from Fermat's principle that the surface wave propagates along geodesic lines; that is, the ray paths are geodesics.

If the wave speed is not uniform but is given as a function of position, the rays no longer follow geodesic paths; however, the surface wave fronts are still orthogonal trajectories of the rays. Problems for which the wave speed is determined as a function of direction are not treated. Important problems in this category include surface waves on anisotropic crystals, Rayleigh waves on surfaces having local radii of curvature not much larger than the wave length, and certain Lamb waves in relatively thick shells with relatively small radii of curvature.

Analysis

With the surface described in terms of the surface variables u, v by equation (1), the element of arc length is given by

$$ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = E\,du^2 + 2F\,du\,dv + G\,dv^2 \qquad (2)$$

where E, F, G are the metric coefficients

$$E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u \qquad (3a)$$

$$F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v \qquad (3b)$$

$$G = \boldsymbol{r}_u \cdot \boldsymbol{r}_v \qquad (3c)$$

A direction on the surface is determined by assigning a value λ to the derivative $\frac{dv}{du}$. Thus, if an arbitrary increment du is taken, the corresponding arc increment ds in the λ direction is

$$ds = \sqrt{E + 2F\,\frac{dv}{du} + G\,\left(\frac{dv}{du}\right)^2}\,du$$

$$= \sqrt{E + 2F\lambda + G\lambda^2}\,du \tag{4}$$

The u, v variables are the basic geometric parameters used to define the surface, and calculations are normally performed in this system. However, since the wave fronts are orthogonal trajectories of the rays, it is convenient analytically to utilize an alternate set of coordinates taken, respectively, along the rays and along the orthogonal trajectories. Along the rays, we take time t as the coordinate and denote arc length along the ray by σ so that

$$d\sigma = c\,dt = \sqrt{E_o}\,dt \tag{5}$$

The coordinate normal to the rays is denoted by η. Thus η determines which ray of the family is being considered. (See fig. 1.) Denote by dn the incremental arc length normal to the rays:

$$dn = \sqrt{G_o}\,d\eta \tag{6}$$

Thus, the general arc length element is

$$ds^2 = E_o\,dt^2 + G_o\,d\eta^2 \tag{7}$$

where $E_o = c^2$.

We could obtain a differential equation for the rays by minimizing the time integral

$$t - t_1 = \int_{\sigma_1}^{\sigma} \frac{d\sigma}{c} \tag{8}$$

utilizing a procedure analogous to that used to derive the equations of a geodesic by minimizing the distance integral (ref. 7, p. 140). However, a briefer and simpler approach is available because of the orthogonality of the t, η coordinate system. It is known (ref. 7, p. 130) that the geodesic curvature of the first set of coordinate lines (η = Constant) is given by the formula

$$\kappa_{g1} = -\frac{1}{2E_o\sqrt{G_o}}\,\frac{dE_o}{d\eta}$$

$$= -\frac{1}{c}\,\frac{dc}{dn} \tag{9}$$

This equation states that the geodesic curvature of a ray is proportional to the derivative of log c in the direction normal to ray. This is an intrinsic relation and therefore is true in any coordinate system. For purposes of calculation it must be expressed in terms of the u, v coordinates that define the surface. Toward this end, assume first that an initial point u_o, v_o is known on the ray, along with an initial direction λ_o and the distribution of wave speed $c(u, v)$.

The direction normal to the ray is obtained from the relation

$$\frac{\delta v}{\delta u} \equiv \lambda_n = -\left(\frac{E + F\lambda}{F + G\lambda}\right) \tag{10}$$

(See ref. 7, p. 59, eq. (2–10).) Then $\frac{du}{dn}$ is obtained from equation (4):

$$\frac{du}{dn} = \frac{1}{\sqrt{E + 2F\lambda_n + G\lambda_n^2}} \tag{11}$$

and

$$\frac{dv}{dn} = \frac{\delta v}{\delta u}\frac{du}{dn} = \lambda_n \frac{du}{dn} \tag{12}$$

With equations (11) and (12), the derivative

$$\frac{dc}{dn} = \frac{\partial c}{\partial u}\frac{du}{dn} + \frac{\partial c}{\partial v}\frac{dv}{dn} \tag{13}$$

in equation (9) is expressed in terms of u and v. The geodesic curvature κ_{g1} can also be expressed in terms of the original coordinates (u, v) and metric coefficients. This relation is (see ref. 7, p. 128, eq. (1–6))

$$\kappa_{g1} = \left[\Gamma_{11}^2 (u')^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1)(u')^2 v' + (\Gamma_{22}^2 - 2\Gamma_{12}^1) u' (v')^2 \right.$$
$$\left. -\Gamma_{22}^1 (v')^3 + (u'v'' - u''v')\right] H \tag{14}$$

where primes indicate differentiation with respect to σ, and

$$H = \sqrt{EG - F^2} \tag{15}$$

The Γ's are Christoffel symbols defined as follows (ref. 7, p. 107)

$$2H^2\Gamma_{11}^1 = GE_u - 2FF_u + FE_v \tag{16a}$$

$$2H^2\Gamma_{12}^1 = GE_v - FG_u \tag{16b}$$

$$2H^2\Gamma_{22}^1 = 2GF_v - GG_u - FG_v \tag{16c}$$

$$2H^2\Gamma_{11}^2 = 2EF_u - EE_v - FE_u \tag{16d}$$

$$2H^2\Gamma_{12}^2 = EG_u - FE_v \tag{16e}$$

$$2H^2\Gamma_{22}^2 = EG_v - 2FF_v + FG_u \tag{16f}$$

The differential equation for the ray path in the u, v system is now obtained by substituting relation (14) into equation (9). To simplify the resulting equation, we first change the ray parameter from arc length σ to a general ray parameter μ. This is accomplished by multiplying through by $\left(\frac{d\sigma}{d\mu}\right)^3$:

$$\left(\frac{d\sigma}{d\mu}\right)^3 \Big[\Gamma_{11}^2(u')^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1)(u')^2 v' + (\Gamma_{22}^2 - 2\Gamma_{12}^1)u'(v')^2$$

$$- \Gamma_{22}^1(v')^3 + (u'v'' - u''v')\Big] H = -\left(\frac{d\sigma}{d\mu}\right)^3 \frac{1}{c}\frac{dc}{dn} \quad (17)$$

Now, since

$$\left(\frac{d\sigma}{d\mu}\right)^3 \frac{du}{d\sigma}\frac{d^2v}{d\sigma^2} = \frac{du}{d\mu}\frac{d^2v}{d\mu^2} \quad (18a)$$

and similarly

$$\left(\frac{d\sigma}{d\mu}\right)^3 \frac{d^2u}{d\sigma^2}\frac{dv}{d\sigma} = \frac{d^2u}{d\mu^2}\frac{dv}{d\mu} \quad (18b)$$

it follows that equation (17) can be written as

$$\Big[\Gamma_{11}^2(u')^3 + (2\Gamma_{12}^1 - \Gamma_{11}^1)(u')^2 v' + (\Gamma_{22}^2 - 2\Gamma_{12}^1)u'(v')^2 - \Gamma_{22}^1(v')^3$$

$$+ (u'v'' - u''v')\Big] H = -\left(\frac{d\sigma}{d\mu}\right)^3 \frac{1}{c}\frac{dc}{dn} \quad (19)$$

where now the primes denote differentiation with respect to μ. Finally, we choose for the arc parameter the coordinate variable u, so that $u' = 1, u'' = 0, v' = \frac{dv}{du} = \lambda$, the local ray direction. With these substitutions, equation (19) can be written as an equation for the local change in ray direction.

$$\frac{d\lambda}{du} = \Gamma_{22}^1\lambda^3 + (2\Gamma_{12}^1 - \Gamma_{22}^2)\lambda^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2)\lambda - \Gamma_{11}^2$$

$$- \frac{1}{cH}\frac{dc}{dn}\left(\frac{d\sigma}{du}\right)^3 \quad (20)$$

where, by equation (4),

$$\frac{d\sigma}{du} = \sqrt{E + 2F\lambda + G\lambda^2} \quad (21)$$

Equation (20) is used to compute a ray as follows. Starting at the initial point u_o, v_o one takes a step in the initially specified direction λ_o by incrementing u_o: $u_1 = u_o + du$; computing the corresponding increment in v: $dv = \lambda\ du, v_1 = v_o + dv$; and then computing $c, \frac{dc}{dn}, H, \frac{d\sigma}{du}$, and the Γ's at u_1, v_1. Substituting these quantities into the right-hand side of equation (20) and multiplying by du yields the change in direction $d\lambda$ for the next step. Then λ is incremented by this amount and a step is taken in the new direction; this process is continued to generate the ray. The arc length on each step is calculated from the elapsed time (eq. (5)). Accuracy of the procedure can be checked by duplicating a calculation with different step sizes. Computation time for 1000 steps is at most a few seconds on a modern computer.

To determine the amplitude variation, the spreading of the wave must be calculated. This spreading is measured by the distance between the rays: dn, in equation (6). Since for a pair of rays, $d\eta$ is fixed and since $dn = \sqrt{G_o}\ d\eta$, the problem is to determine G_o as a function of σ.

When the surface does not have a uniform wave speed, the simplest way to approach this problem, both conceptually and analytically, is to perform the calculation directly. We simply compute two adjacent rays simultaneously. Equal increments in time are specified, so that in equation (20) the increments du are determined by the relation

$$du(t) = \frac{du}{d\sigma}\frac{d\sigma}{dt}\,dt$$

$$= \frac{c\ dt}{\sqrt{E + 2F\lambda + G\lambda^2}} \quad (22)$$

Thus, at each step we have a point u_1, v_1 on the first ray and a corresponding point u_2, v_2 on the second so that

$$dn(t) = \sqrt{E(\Delta u)^2 + 2F\ \Delta u\ \Delta v\ + G(\Delta v)^2} \quad (23)$$

and

$$\sqrt{G_o(t)} = \frac{dn(t)}{d\eta} \quad (24)$$

where $d\eta$ is the constant increment distinguishing the two rays.

It is possible to compute the ray strip width without calculating an adjacent ray independently. Since this kind of development may be required for analytical purposes, the theory will be outlined here as a matter of academic interest.

The variation of $\sqrt{G_o}$ with t, $\frac{d\sqrt{G_o}}{dt}$, is related to $\frac{dE_o}{dn}$ and to the local total curvature K by Gauss's equation, which for this case, is (ref. 7, p. 113)

$$\frac{d^2\sqrt{G_o}}{dt^2} + \frac{1}{\sqrt{E_o}}\frac{d^2E_o}{dn^2} + K\sqrt{G_o} = 0 \tag{25}$$

where the total, or Gaussian, curvature K is

$$K = \frac{eg - f^2}{H^2} \tag{26}$$

The quantities e, f, g are the curvature coefficients or coefficients of the second fundamental form. They are computed by the formulas

$$\left.\begin{aligned} e &= \boldsymbol{r}_{uu} \cdot \widehat{N} \\ f &= \frac{\boldsymbol{r}}{x_{uv}} \cdot \widehat{N} \\ g &= \frac{\boldsymbol{r}}{x_{vv}} \cdot \widehat{N} \end{aligned}\right\} \tag{27}$$

where $\widehat{N}$ is the local surface normal

$$\widehat{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{H} \tag{28}$$

Thus, K can be computed at each point of the ray, and since $\sqrt{E_o} = c$, equation (25) can be written

$$\frac{d^2\sqrt{G_o}}{dt^2} + K\sqrt{G_o} = -\frac{1}{c}\frac{d^2c}{dn^2} \tag{29}$$

For a surface with uniform wave speed, the right-hand side of equation (29) vanishes, and $\sqrt{G_o}$ can be calculated in a straightforward step-by-step manner as the points on the ray (a geodesic in this case) are computed. (Application of the WKB method (ref. 10) would involve complications, since K changes sign when the ray traverses an inflected region.) In this case, this procedure may well be preferable to the alternate method of computing an adjacent ray.

However, if the right-hand side of equation (29) does not vanish, we have

$$\frac{d^2c}{dn^2} = \frac{d^2c}{du^2}\left(\frac{du}{dn}\right)^2 + \frac{d^2c}{dv^2}\left(\frac{dv}{dn}\right)^2 + \frac{dc}{du}\frac{d^2c}{dn^2} + \frac{dc}{dv}\frac{d^2v}{dn^2} \tag{30}$$

The derivatives of c with respect to u and v can be calculated from the given distribution $c(u, v)$. The first derivatives $\frac{du}{dn}, \frac{dv}{dn}$ are calculated by equations (10) through (12). The problem lies with the second derivatives, $\frac{d^2u}{dn^2}, \frac{d^2v}{dn^2}$. They are related to the geodesic curvature κ_{g2} of the wave fronts by an equation similar to equation (14). The ray spreading $\sqrt{G_o}$ is related to κ_{g2} by the formula

$$\kappa_{g2} = \frac{1}{\sqrt{G_o}}\frac{d\sqrt{G_o}}{d\sigma} = \frac{1}{c\sqrt{G_o}}\frac{d\sqrt{G_o}}{dt} \tag{31}$$

(ref. 7, p. 130, eqs. (1)-(10)).

Thus, given the local geodesic curvature of the wave front at the initial point, one can compute $\frac{d^2u}{dn^2}$ and $\frac{d^2v}{dn^2}$ in a manner similar to that applied to equation (14). Then equation (29) can be integrated one step to obtain $\frac{d\sqrt{G_o}}{d\sigma}$ and $\sqrt{G_o}$ at the next point. Next, κ_{g2} can be calculated by equation (31), and thus the process is continued. Thus, it is seen that, with a nonuniform wave speed, this procedure is more difficult than simply computing an adjacent ray in theory, in programming, and in calculation time.

Analytically Lofted Surfaces

The procedure described in the previous section can be applied to any surface that is given in the form of equation (1). The example given here will utilize a class of analytically lofted surfaces described in reference 6. These surfaces are defined by specifying the shape of an initial cross section in a plane $x = x_i$ and a final, or terminating, cross section shape at $x = x_f$ and then requiring the intermediate cross section to represent a smooth transition from the initial to the final shape. The shapes of the two extreme cross sections $y(\xi), z(\xi)$ are described parametrically in terms of an arbitrary parameter ξ. This parameter might be for example the normalized y coordinate, the normalized arc length, the angle variable in polar coordinates, or the ellipse angle parameter.

Let $\alpha(x)$ be a function, varying smoothly and monotonically from 0 to 1 as x varies from x_i to x_f (homotopy function). Examples are $(\frac{x-x_i}{x_f-x_i})^n$ and $\left[\sin(\frac{x-x_i}{x_f-x_i}\frac{\pi}{2})\right]^n$. Then, for $x_i \le x \le x_f$ a cross-section shape is defined by

$$\tilde{y}(x,\xi) = [1-\alpha(x)]\, y_i(\xi) + \alpha(x)\, y_f(\xi) \tag{32a}$$

$$\tilde{z}(x,\xi) = [1-\alpha(x)]\, z_i(\xi) + \alpha(x)\, z_f(\xi) \tag{32b}$$

These shapes vary smoothly from the initial shape at x_i to the terminating shape at x_f. By taking ξ as a normalized coordinate, the size variation of the cross sections can be specified by prescribing a scaling function $\beta(x)$, which varies smoothly but not necessarily monotonically from 0 to 1 as x varies from

x_i to x_f. The scaled variables are defined by

$$y(x,\xi) = \beta(x)\ \tilde{y}(x,\xi) \tag{33a}$$

$$z(x,\xi) = \beta(x)\ \tilde{z}(x,\xi) \tag{33b}$$

The vector equation of this transition surface is

$$\boldsymbol{r}(x,\xi) = x\hat{\imath} + y(x,\xi)\hat{\jmath} + z(x,\xi)\hat{k} \tag{34}$$

which has the required form (eq. (1)) where the surface variables u, v are x, ξ, respectively. For modeling complicated shapes, such as blended wing-body combinations, β must be allowed to vary with ξ as well as x, but for illustrative purposes equations (33) describe a sufficiently general class of surfaces. The x = Constant lines describe the cross-section shapes, and the ξ = Constant lines are the lofting lines.

The first derivatives are

$$\begin{aligned} \boldsymbol{r}_x = \hat{\imath} &+ \left[\frac{d\beta}{dx}\ \tilde{y} + \beta\ \frac{d\alpha}{dx}\ (y_f - y_i)\right]\hat{\jmath} \\ &+ \left[\frac{\partial\beta}{\partial x}\ \tilde{z} + \beta\ \frac{d\alpha}{dx}\ (z_f - z_i)\right]\hat{k} \end{aligned} \tag{35a}$$

$$\begin{aligned} \boldsymbol{r}_\xi = \beta&\left[(1-\alpha)\ \frac{dy_i}{d\xi} + \alpha\ \frac{dy_f}{d\xi}\right]\hat{\jmath} \\ &+ \beta\left[(1-\alpha)\ \frac{dz_i}{d\xi} + \alpha\ \frac{dz_f}{d\xi}\right]\hat{k} \end{aligned} \tag{35b}$$

From these formulas, the metric coefficients can be calculated as well as H and, if needed, the surface normal $\widehat{N}$. Equations (35) can be differentiated again if the curvature coefficients are required.

Sample Calculations

For purposes of illustration, a surface is defined as follows. The initial and final cross sections are defined to be ellipses:

$$y_i(\xi) = \cos\xi$$

$$z_i(\xi) = \frac{1}{3}\sin\xi$$

$$y_f(\xi) = \cos\xi$$

$$z_f(\xi) = 0.58\sin\xi$$

With

$$R(x) \equiv \frac{x - x_i}{x_f - x_i}$$

the lofting functions were taken as

$$\alpha(x) = R$$

$$\beta(x) = R^{3/4}(1-R)^{1/2}\left(\frac{x_f - x_i}{2}\right)$$

A top view of this surface is shown in figure 2.

To illustrate the wave calculation, we assume a source located on the surface at

$$x_o = 0.55(x_f - x_i), \eta_o = \pi/2$$

First, we assume a uniform wave speed

$$c_o = 50(x_f - x_i)\ \text{sec}^{-1}$$

and compute a segment of the wave emanating from x_o, η_o. Figure 3(a), in which the surface is displayed from the side, shows the rays and wave fronts for this segment.

Next, it is assumed that the surface is no longer uniform, but is such that the wave speed increased in the x-direction according to the formula

$$c(x) = c_o[1 + 0.008(x - x_o)]$$

Figure 3(b) shows the influence of the gradient on the rays that propagate from x_o, η_o initially with same directions as those in figure 3(a).

Figure 4 demonstrates the effect of the local surface total curvature on ray spreading. In figure 4(a), the surface is the same as that used in the previous example. It is elliptic ($K > 0$) everywhere. Now, however, the rays do not spread from a surface source, but are initially parallel as if launched by grazing rays of a plane wave. Examples of surface waves initiated in this way are Franz and Stoneley waves. As the wave propagates over the surface, the rays converge and actually pass through a focus.

Figure 4(b) shows a surface that is hyperbolic everywhere ($K < 0$). Its nose cross section is an elliptic whose major-to-minor axis ratio is 2:1, and the base cross section is a circle. The lofting functions are

$$\alpha(x) = R(x)$$

$$\beta(x) = 0.375(0.1R + 0.9R^2)(x_f - x_i)$$

Again, the rays are initially parallel, but the ray strip rapidly diverges.

Application of Results to Wave Field Calculation

As noted earlier, the phase is determined by the time along the ray, for harmonic motion of fixed frequency. It is related to the ray path arc length σ by

$$\sigma = \int_{t_o}^{t} c\ dt \tag{36}$$

The amplitude A of the surface wave is determined according to an energy principle in reference 2 by the formula

$$A = \frac{\text{Constant}}{\sqrt{\rho c} G_o^{1/4}} \tag{37}$$

where ρ is a parameter that depends on the densities on the two sides of the surface. If ρ and c are given, then A is fully determined once the deviation $\sqrt{G_o}$ of the rays has been computed by one of the two methods previously described. Equation (37) may be compared with the formula given in reference 3 for a surface disturbance U that satisfies the wave equations and the impedance boundary condition

$$\frac{dU}{dn} + kYU = 0 \tag{38}$$

where k is the reference wave number, and Y is proportional to the surface impedance, which may vary over the surface. In reference 3, Y is taken to be real and positive. Instead of treating a surface ray strip, reference 3 treats an actual tube of rays adjacent to the surface. The resulting amplitude formula is

$$A = \text{Constant}\frac{\sqrt{cY}}{G_o^{1/4}} \tag{39}$$

If the surface wave is of a type that ripples the surface and if the surrounding medium will support a wave, then the surface wave may launch a head wave into this medium. If the sound speed c_m in the medium is less than the surface wave speed, then the rays leave the surface at an angle θ to the surface normal such that

$$\sin\theta = \frac{c_m}{c} \tag{40}$$

The surface wave is "supersonic" relative to the surrounding medium, and the head wave is the Mach wave associated with that motion. If the wave speed c_m is constant, the rays in the surrounding medium are straight. In this case, the variation of ray tube area in the medium depends only on the incremental differences in the ray directions as they leave the surface. To determine these incremental angles, we study a ray strip associated with a particular axial, or central, ray. First, we consider the variation in ray directions along the ray (in the $\widehat{N}, \widehat{T}$ plane), and then in the direction orthogonal to the ray (in the $\widehat{N}, \hat{n}$) plane.

If $\phi = \pi/2 - \theta$ is the angle at which a launched ray leaves the surface at a distance σ along the ray, then the angle at which the ray at $\sigma + d\sigma$ is launched is $\phi + \frac{d\phi}{d\sigma}\ d\sigma$ so that

$$d\phi = \frac{d\phi}{d\sigma}\ d\sigma \tag{41}$$

If c is constant along the ray path, θ is constant, and therefore $\frac{d\phi}{d\sigma}$ results from the normal curvature of the ray;

$$d\phi = \kappa_{n1}\ d\sigma \tag{42}$$

However, if c varies along the ray, there is also a corresponding variation in θ, which is obtained by differentiating equation (40) along σ:

$$\frac{d\theta}{d\sigma} = -\frac{c_m}{c^2\cos\theta}\frac{dc}{d\sigma} \tag{43}$$

Consequently, the total incremental angle is

$$d\phi = \left(\kappa_{n1} - \frac{c_m}{c^2\cos\theta}\frac{dc}{d\sigma}\right)\ d\sigma \tag{44}$$

Next, we consider the variation in launch angle normal to the $\widehat{N}, \widehat{T}$ plane. Let $\hat{p}$ be a unit vector in the launch direction:

$$\hat{p} = \sin\theta\widehat{T} + \cos\theta\widehat{N} \tag{45}$$

Denote the curvature vector of the surface wave front by $\boldsymbol{\kappa}_\omega$. Its component in the launch direction is

$$\begin{aligned} \kappa_l &= \boldsymbol{\kappa}_\omega \cdot \hat{p} \\ &= (\kappa_{n2}\widehat{N} + \kappa_{g2}\widehat{T}) \cdot \hat{p} \\ &= \kappa_{n2}\cos\theta + \kappa_{g2}\sin\theta \end{aligned} \tag{46}$$

The angular ray spreading orthogonal to the $\widehat{N}, \widehat{T}$ plane is

$$d\psi = \kappa_l\ dn \tag{47}$$

Finally, the ray tube spreading for the launched rays is proportional to $\frac{d\psi}{d\eta}\frac{d\phi}{d\sigma}$, which, by using equations (44) and (47), yields

$$\text{Spreading} = \text{Constant}\left(\kappa_{n1} - \frac{cm}{c^2\cos\theta}\frac{dc}{d\sigma}\right)(\kappa_{n2}\cos\theta + \kappa_{g2}\sin\theta) \tag{48}$$

In this equation, κ_{g2} is obtained from equation (31). The normal curvature κ_{n1} along the ray direction $\lambda = \frac{dv}{du}$ is

$$\kappa_{n1} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} \tag{49}$$

while κ_{n2} is computed from the same formula with λ replaced by λ_n. For Franz and Stoneley waves, $\theta = 90°$; therefore $c = c_m$ and $\frac{d\theta}{d\sigma} = 0$ in equation (43), and equation (48) reduces to

$$\text{Spreading} = \text{Constant}\ (\kappa_{n1}\kappa_{g2}) \tag{50}$$

Concluding Remarks

Equations have been derived for computing surface waves, including surfaces with a nonuniform wave speed. The prior literature has dealt primarily with the theoretical development with little consideration given to computational methods, and examples have been limited to waves on surfaces of simple analytic description such as cones, spheres, cylinders. The computational procedure presented herein is a relatively general method. Sample calculations illustrated the procedure for a class of practical shapes of the type that include aerodynamic and hydrodynamic surfaces. Equations for spreading of rays launched from the surface into a surrounding medium that will support waves have also been included.

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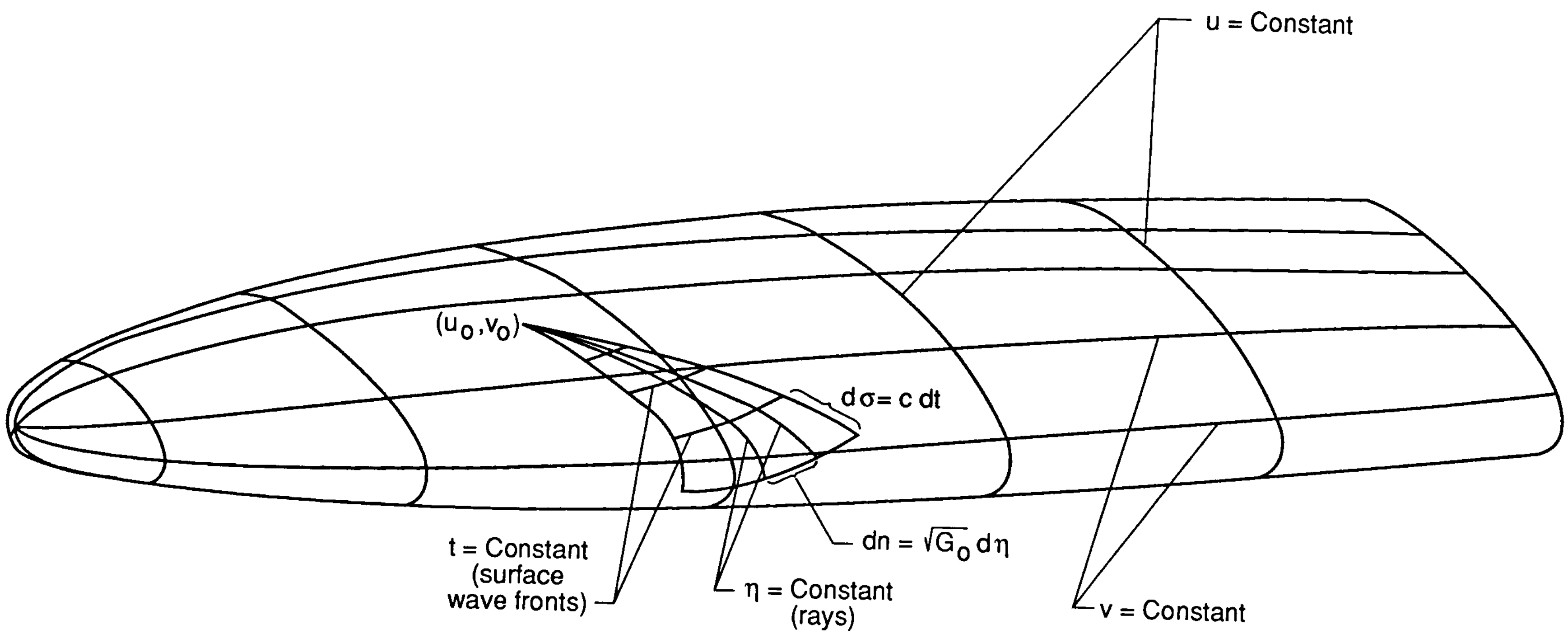


Figure 1. Basic geometry illustrating surface parameters u, v, t, and η.

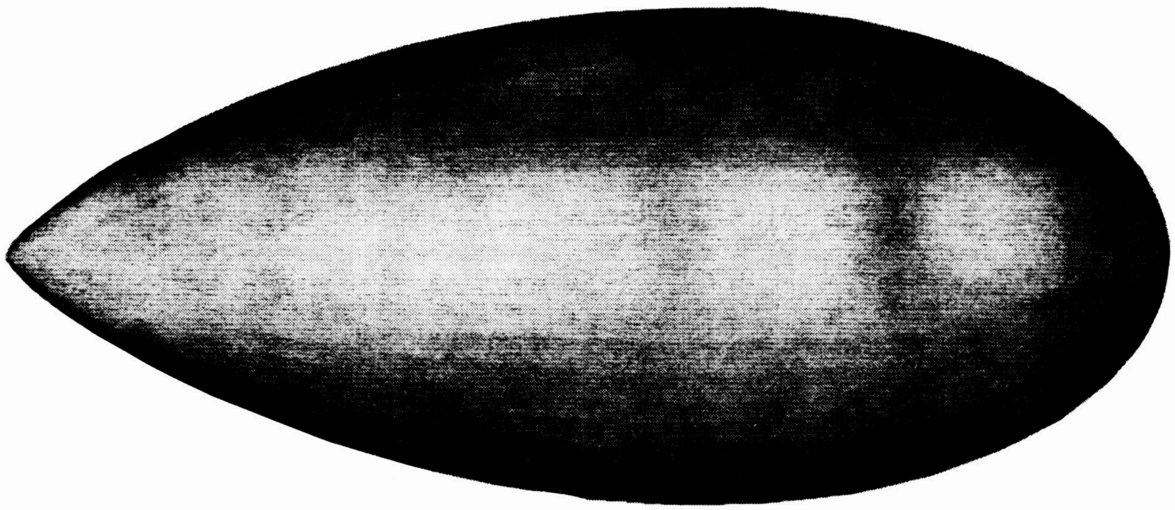
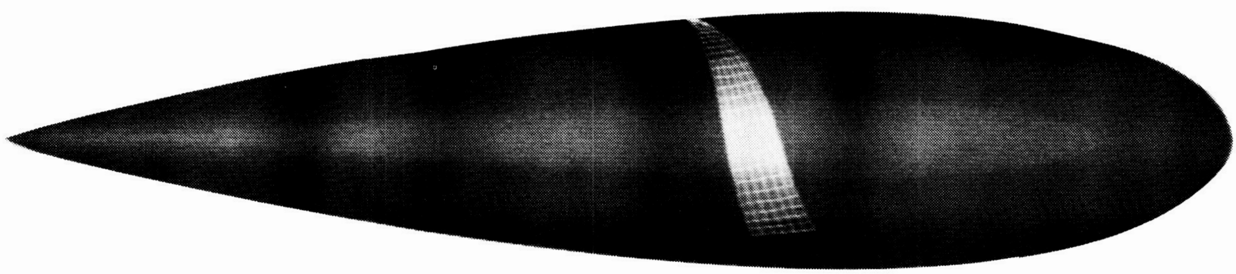
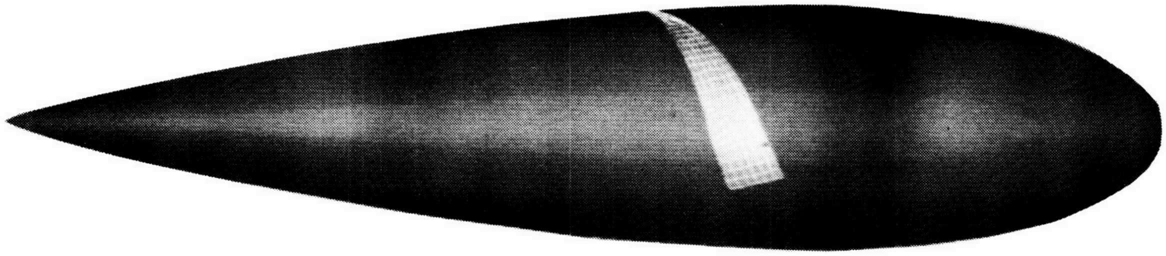


Figure 2. Top view of surface used for sample calculation.

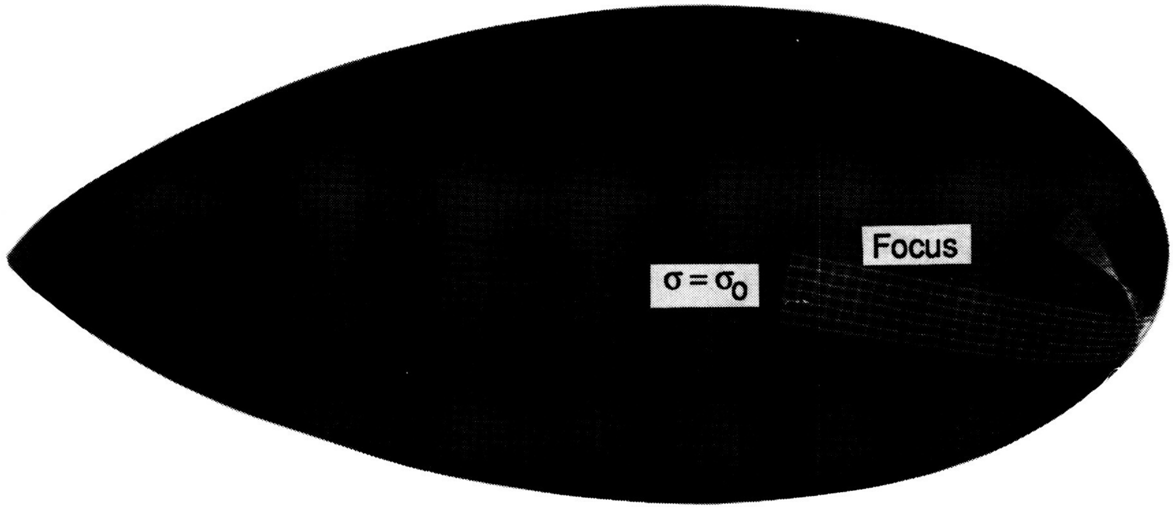


(a) Constant wave speed.

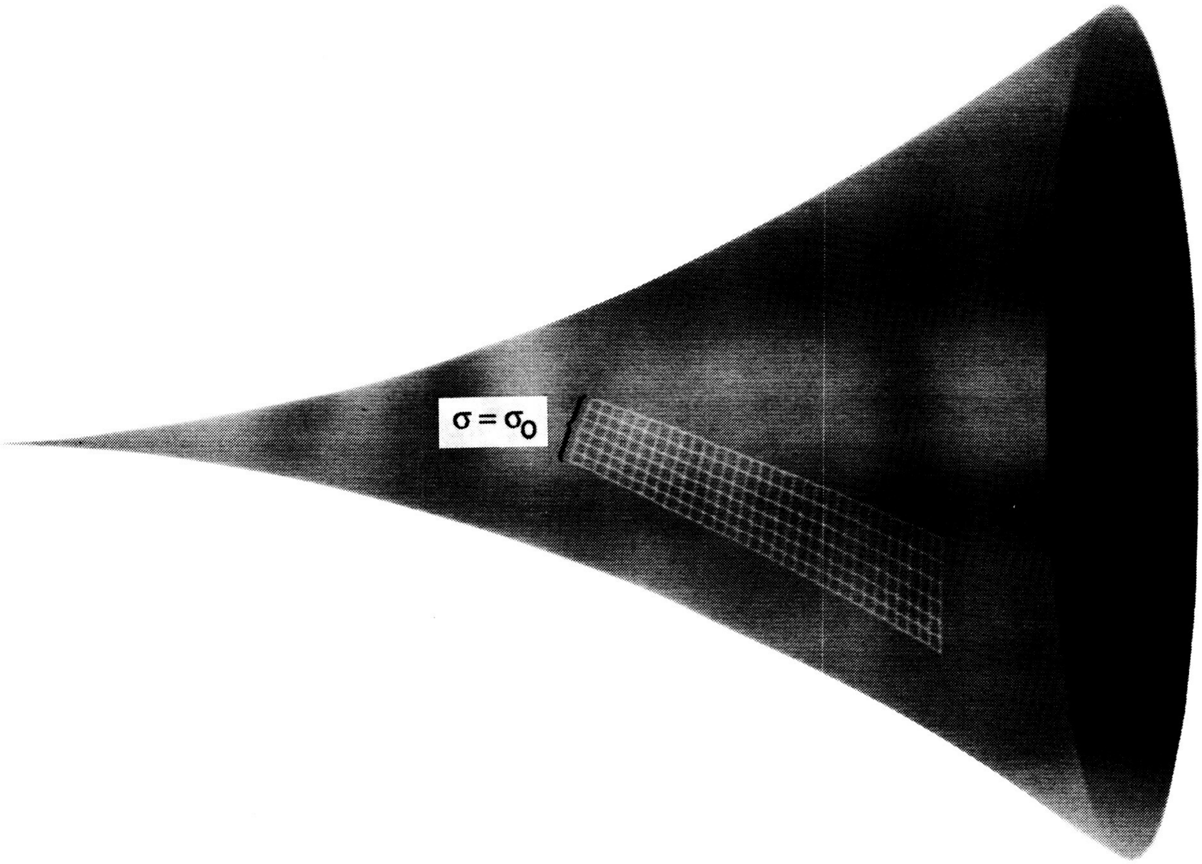


(b) Small positive wave speed gradient.

Figure 3. Surface for sample calculation.



(a) Elliptic surface.



(b) Hyperbolic surface.

Figure 4. Effect of surface shape on ray spreading.

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16. Abstract

Equations are derived for computing surface waves on smooth surfaces, including surfaces with a nonuniform wave speed. The prior literature has dealt primarily with the theoretical development with little consideration given to computational methods, and examples have been limited to waves on surfaces of simple analytic description such as cones, spheres, cylinders. The computational procedure presented herein is a relatively general method. Sample calculations illustrate the procedure for a class of practical shapes of the type that include aerodynamic and hydrodynamic surfaces. Equations are also included for computing the spreading of rays into a surrounding medium that will support waves.

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