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Multigrid Method for Stability Problems

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Abstract

The problem of calculating the stability of steady state solutions of differential equations is treated. Leading eigenvalues (i.e., having maximal real part) of large matrices that arise from discretization are to be calculated. An efficient multigrid method for solving these problems is presented. The method begins by obtaining an initial approximation for the dominant subspace on a coarse level using a damped Jacobi relaxation. This proceeds until enough accuracy for the dominant subspace has been obtained. The resulting grid functions are then used as an initial approximation for appropriate eigenvalue problems. These problems are being solved first on coarse levels, followed by refinement until a desired accuracy for the eigenvalues has been achieved. The method employs local relaxation on all levels together with a global change on the coarsest level only, which is designed to separate the different eigenfunctions as well as to update their corresponding eigenvalues. Coarsening is done using the FAS formulation in a non-standard way in which the right hand side of the coarse grid equations involves unknown parameters to be solved for on the coarse grid. This in particular leads to a new multigrid method for calculating the eigenvalues of symmetric problems. Numerical experiments with a model problem demonstrate the effectiveness of the method proposed. Using an FMG algorithm a solution to the level of discretization errors is obtained in just a few work units (less than 10), where a work unit is the work involved in one Jacobi relaxation on the finest level.

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1 Introduction

Stability analysis plays an important part in hydrodynamics [4] and in other physical sciences [6]. Basically, one is interested in the growth (decay) rate of infinitesimal perturbations to steady state solutions. In rectangular geometries with simple enough steady state solutions one can use Fourier transform techniques to reduce the resulting stability problem to an eigenvalue problem for an ODE with parameter(s) [4]. General geometries or complicated solution structure do not permit this and eigenvalue problems in two and three space dimensions need to be solved.

A problem related to the stability of a given steady state solution is the problem of transition, when a nonlinear problem with a parameter is given and the value of that parameter at which steady state solutions loose their stability is required. In fact, in cases of marginal stability this is the relevant problem instead of finding the stability of a given steady state solution, since in such cases the discretization (in space) might have shifted the eigenvalue from, say, a stable region to an unstable one.

Steady state solutions are often obtained by time marching algorithms. In such cases the stability to small perturbations can be studied during the solution process. However, efficient multigrid solvers for steady state problems do not involve usually any time marching. The solver is designed for solving the steady state problem directly. This allows obtaining solutions which are physically unstable. A way to recover the information about the stability of a given steady state solution is therefore needed.

One way of obtaining that information is to use time marching for the time dependent problem obtained by linearizing the original problem, around the steady state solution, using a random initial condition. The discretization has to be such that it preserves stability. That is, if a certain component is decaying oscillating or growing, the same should happen for the discretization. Crank-Nickolson time stepping has this property.

The effectiveness of such a method depends on the closeness of the dominant eigenvalues to the imaginary axis. The closer they are the more time steps are needed to determine the growth rate of small perturbations to the steady state solution under study. For symmetric problems the situation is much better than the non-symmetric ones, since the time steps can be taken as larger and larger for larger times. In the non-symmetric case the time stepping depends also on the imaginary part of the dominant eigenvalues, and this can be much larger than the real part. In

other words, the time stepping can depend on the oscillation rate rather than the decay rate which may be much smaller. An improvement of the above method is to use time marching until the subspace of dominant eigenvalues has been determined to a desired accuracy, then obtaining from it the set of eigenfunctions and eigenvalues as in [7].

The above approach is inappropriate when the steady state solution has been obtained by an efficient multigrid solver since it is more expensive than getting the steady state solution using a multigrid method. Moreover for transition problems it can be very time consuming and a different approach should be used.

In this paper an alternative way for calculating the stability of steady state solutions by solving directly, using a multigrid algorithm, an appropriate eigenvalue problem. This approach could be modified for transition problems. The difference is only in a global step (explained later) that is performed on the coarsest level. The exact treatment of transition problems by multigrid, together with some physical examples will be described elsewhere.

Since the eigenvalue with maximal real part is required, the multigrid processes must not contradict this requirement. The relaxation should be such that it will relatively damp all the non-dominant eigenfunctions. Jacobi or a damped Jacobi relaxation achieves this purpose. Kaczmarz on the other hand, should not be used since it damps more the eigenfunctions whose eigenvalues are away from zero. As in [2] a single grid method which involves a relaxation method and a step for updating the eigenvalue is constructed first. In non-symmetric problems the relaxation (Jacobi) couples the eigenfunctions that correspond to complex conjugate eigenvalues. Therefore, the step for updating the eigenvalue, (the global step) performs also a projection onto one of the subspaces of the two conjugate eigenvalues, say, with non-negative imaginary part.

When the dominant eigenvalue is not separated enough from the others, in the sense that their real parts are close, one has to include more eigenfunctions and to solve for the dominant eigenspace which is of higher dimension.

The coarsening strategy for the multigrid method is done using an FAS formulation since the eigenvalue problem is a non-linear one. However, since the global change done on the fine level affects the coarse grid right hand side significantly in this problem, the standard FAS should be modified to include this change of the right hand side. The resulting FAS is such that the right hand side of the coarse grid equations involves some parameters to be solved on that level. The

resulting algorithm has fast algebraic convergence rates. Moreover, it leads to a new multigrid method for symmetric eigenvalue problems.

The algorithm is performed in an FMG version, where the problem is solved first on coarse levels using the approximation obtained there as an initial guess for the solution of finer grid problems. On each level a fixed number of basic multigrid cycles are performed, yielding a solution to the level of discretization errors in just a few work units (about 10), where a work unit is the work involved in one Jacobi relaxation on the finest level.

Numerical experiments with a two dimensional model problem are given. The results demonstrate the effectiveness of the proposed method. Solution to the level of discretization errors is obtained with the 1-FMG algorithm (which uses one basic multigrid cycle per refinement and cost about 10 WU).

2 The Problem

Let $\mathcal{A}$ be a linear real valued operator. We wish to compute its eigenvalues with maximal real part. That is

$$\mathcal{A}\Psi \quad = \quad \Lambda\Psi \qquad with \quad Real(\Lambda) \quad maximized. \tag{2.1}$$

In general, for asymmetric operators, the eigenvalues and eigenfunctions are complex. The above eigenvalue problem can then be written in terms of real quantities as

$$(\mathcal{A} - \lambda I)\phi = 0, \qquad \|\phi\|^2 = 1 \tag{2.2}$$

or as

$$\begin{pmatrix} \mathcal{A} - \lambda I & \mu I \\ -\mu I & \mathcal{A} - \lambda I \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \|\phi\|^2 = \|\psi\|^2 = 1 \tag{2.3}$$

depending on whether the dominant eigenvalue is real or complex. In this notation $\lambda = Real(\Lambda)$, $\mu = Imag(\Lambda)$, $\phi = Real(\Psi)$, $\psi = Imag(\Psi)$.

Since the operator $\mathcal{A}$ is real its eigenvalues come in pairs of complex conjugates. Hence, if (λ, μ) is a solution to (2.3), also $(\lambda, -\mu)$ is a solution. Moreover, each of these eigenvalues is double. If

$(\phi,\psi)^T$ corresponds to (λ,μ) then so does $(\psi,-\phi)^T$. Similarly, $(\psi,\phi)^T$ and $(\phi,-\psi)^T$ correspond to $(\lambda,-\mu)$.

3 Single Grid Algorithm

One way to get functions in the subspace spanned by the eigenfunctions with eigenvalue of maximal real part is to simulate a time dependent process, that is, solving approximately the initial value problem

$$U_t = \mathcal{A}U \tag{3.1}$$

$$U(0) = random \tag{3.2}$$

See (I. Goldhirsch et al 1987). The dominant part of the solution at large time is a linear combination of the desired eigenfunctions. This leads us to the following.

3.1 Initial Approximation

The initial approximation for the eigenvalue problem is obtained from the following process applied repeatedly,

(a) Relax using a damped Jacobi relaxation $\mathcal{A}W = 0$.

(b) $W \leftarrow W/\|W\|$.

The stopping criterion for the process is as follows. Let $\phi_1 = W, \quad \phi_j = \mathcal{A}\phi_{j-1} \quad (j > 1)$, and let $D^{(n)}$ be an nxn matrix whose (i,j) element is $< \phi_i, \phi_j >$. Define l to be the minimal positive integer for which $\det D^{(l+1)} \rightarrow 0$. The dominant subspace $\mathcal{H}$ is of dimension l, and the process can be stopped when $\det D^{(l+1)}$ is small enough. The dimension of $\mathcal{H}$ determines how many eigenfunctions are to be simultaneously solved for in order to obtain fast convergence.

After an initial approximation for the dominant subspace has been obtained a solution of the eigenvalue problem can be started. The algorithm consists of two steps which are performed successively. The first is a local relaxation process on equation (2.2) or (2.3) depending on the eigenfunction. Its role is to damp the error in the components corresponding to non-dominant eigenvalues. The analogy of damped Jacobi relaxation with explicit time stepping suggests that using it with an appropriate under-relaxation parameter may serve our purpose. The local relaxation is then followed by a step referred to as the global step which is explained in detail next.

The Global Step

(a) dim $\mathcal{H} = 1$. In this simple situation λ can be updated according to

$$\lambda \leftarrow \frac{< A\phi, \phi >}{< \phi, \phi >}, \qquad \phi \leftarrow \phi / \|\phi\| \tag{3.3}$$

(b) dim $\mathcal{H} = 2$. Assume for simplicity of exposition that the dominant subspace is spanned by a pair of complex conjugate functions. In that case the approximation we get at an intermediate step will be a linear combination of the four eigenfunctions mentioned in section 2, and the eigenfunctions corresponding to other λ's. The latter converge to zero in the relaxation process (relative to these four), while these four corresponding to the maximal λ are coupled by the relaxation process. If our approximate solution is in the subspace spanned by the two eigenfunctions corresponding to one of the μ's, say, the non-negative one, which belong to maximal λ, then λ and μ can be calculated by certain inner products. Since in general this is not the case, we have to perform a projection onto the two dimensional subspace that corresponds to one of the μ's, say, the nonnegative one.

Let $(\bar{\phi}, \bar{\psi})^T$ be an approximate solution of (2.3) which lies mainly in the relevant four dimensional subspace. A projection into a two dimensional subspace which correspond to (λ, μ) is to be done: Each of the two components of an eigenfunction is approximately a linear combination of the current $\bar{\phi}$ and $\bar{\psi}$. That is,

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \approx \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \bar{\phi} \\ \bar{\psi} \end{pmatrix} \tag{3.4}$$

Substituting this in (2.3) and taking inner products of each of the two equations $\bar{\phi}$ and $\bar{\psi}$, we get a 4x4 system given below

$$\begin{pmatrix} C - \lambda D & \mu D \\ -\mu D & C - \lambda D \end{pmatrix} \begin{pmatrix} \underline{\alpha_1} \\ \underline{\alpha_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.5}$$

$$\|\alpha_{11}\bar{\phi} + \alpha_{12}\bar{\psi}\|^2 = 1 \tag{3.6}$$

$$\|\alpha_{21}\bar{\phi} + \alpha_{22}\bar{\psi}\|^2 = 1 \tag{3.7}$$

where C and D are given by

$$C = \begin{pmatrix} < A\phi, \phi > & < A\psi, \phi > \\ < A\phi, \psi > & < A\psi, \psi > \end{pmatrix} \tag{3.8}$$

$$D = \begin{pmatrix} < \phi, \phi > & < \psi, \phi > \\ < \phi, \psi > & < \psi, \psi > \end{pmatrix} \tag{3.9}$$

and $\underline{\alpha}_1 = (\alpha_{11}, \alpha_{12})^T$, $\underline{\alpha}_2 = (\alpha_{21}, \alpha_{22})^T$. This is equivalent to the following (complex valued) generalized 2x2 eigenvalue problem

$$(C - \rho D)\,\underline{\alpha} = 0 \tag{3.10}$$

$$\|Real[\underline{\alpha}(\phi + i\psi)]\|^2 = 1 \tag{3.11}$$

$$\|Imag[\underline{\alpha}(\phi + i\psi)]\|^2 = 1 \tag{3.12}$$

where $\rho = \lambda \pm i\mu$. The solution of the above eigenvalue problem gives $\lambda = Real(\rho), \mu = Imag(\rho)$ and $\underline{\alpha}_1 = Real(\underline{\alpha}), \underline{\alpha}_2 = Imag(\underline{\alpha})$. Under our assumption the matrix D is nonsingular since ϕ and ψ are linearly independent. Hence, the above 2x2 eigenvalue problem is a standard one, and has two eigenvalues.

(c) dim $\mathcal{H} = n$.

Let $\mathcal{H}$ be spanned by the set of real valued functions $(\phi_1, \ldots, \phi_n)$. A projection into the subspace corresponding to non-negative imaginary part $(\phi_1, \ldots, \phi_n)^T \leftarrow E(\phi_1, \ldots, \phi_n)^T$ is described. Let C be an nxn matrix whose (i,j) element is $< A\phi_i, \phi_j >$, and D is an nxn matrix whose elements are $< \phi_i, \phi_j >$. D is non-singular since the set of $\phi's$ is linearly independent. The projection step is done by solving the nxn eigenvalue problem (3.10) with the new definition for C and D together with the constraints $\|(E\underline{\phi})_i\|^2 = 1, (i = 1, \ldots, n)$. It gives n eigenvalues where each complex valued one appears together with its conjugate. With each real eigenvalue we can associate a real eigenfunction that solves a real eigenvalue problem like (2.2), while each complex one, together with its complex eigenvector, defines a pair of functions, corresponding to the real and the imaginary part of the complex eigenfunction associated with it, and a problem like (2.3). In an analogous

way we define a set of n real vectors, constructed from the eigenvectors of (3.10), which define an nxn real valued matrix whose rows are the above real n valued vectors and whose action on the set $(\phi_1, \ldots, \phi_n)$ defines a new set of functions which is a better approximation to the corresponding set of eigenvalue problems.

4 Multigrid Algorithm

From the above single grid method we construct a multigrid algorithm. The relaxation defined for the single grid algorithm involved a local step and a global step. The local step which uses the damped Jacobi relaxation has the property of damping errors that belong to non-dominant eigenfunctions on that grid. These functions are mainly high frequency functions which cannot be represented on a coarser grid. The local step serves therefore as a smoother for our problem. The global step updates the global quantities (the eigenvalues), as well as performing a projection of the solution onto an appropriate subspace that corresponds to the eigenvalues under consideration. This projection step is designed to control the important smooth eigenfunction to be calculated.

In trying to construct a multigrid algorithm we have a few options in treating the global step. The first is to employ it on all levels of discretization. Such an approach is in contrast to a basic principle in multigrid; that is, global changes should be performed on coarse grids only [1]. Note that these global changes involve in most problems the convergence of the smooth components and therefore it makes sense to perform them on coarse grids. The effect of the global step on the non-smooth components is not important since these components should converge to zero as the algorithm proceeds. However, one has to make sure that the size of the non-smooth components is small relative to the relevant smooth ones before performing the global step, since small errors in the highly oscillatory components can cause large errors in the global quantities. This is another reason why it is preferred to perform the global step on coarse grids, where small errors in the high frequency functions have smaller effect on the outcome.

4.1 Coarse Grid Correction

For clarity assume first that $\mathcal{H}$ is spanned by a pair of complex conjugate eigenfunctions. In that case the relevant eigenvalue problem is a discretization of (2.3). The standard coarse grid equations

in a multigrid algorithm for that case using the FAS formulation are

$$\begin{pmatrix} \mathcal{A}^H - \lambda I & \mu I \\ -\mu I & \mathcal{A}^H - \lambda I \end{pmatrix} \begin{pmatrix} \phi^H \\ \psi^H \end{pmatrix} = \begin{pmatrix} \tau_h^H(\phi^h) \\ \tau_h^H(\psi^h) \end{pmatrix} \tag{4.1}$$

$$\|\phi^h\|_H^2 = r_\phi^H, \quad \|\psi^h\|_H^2 = r_\psi^H \tag{4.2}$$

where

$$\tau_h^H(\omega^h) = \mathcal{A}^H I_h^H \omega^h - I_h^H \mathcal{A}^h \omega^h \tag{4.3}$$

$$r_\omega^H = r_\omega^h + \|I_h^H \omega^h\|_H^2 - \|\omega^h\|_h^2 \tag{4.4}$$

where ω^h is any fine grid function.

The right hand side of the coarse grid equation depends heavily on the current approximation on the fine level, hence it would be different if a global step was carried out on the fine grid. In other words, the fine grid change

$$\begin{pmatrix} \phi^h \\ \psi^h \end{pmatrix} \leftarrow \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \phi^h \\ \psi^h \end{pmatrix} \tag{4.5}$$

would change the right hand side of the coarse grid problem to

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \tau_h^H(\phi^h) \\ \tau_h^H(\psi^h) \end{pmatrix}. \tag{4.6}$$

Since the global step may change the solution on the fine grid significantly, and as a result also the right hand side of the coarse grid equations, it is better if the coarse grid right hand side takes this global change into account. The role of the equations $\|\phi^h\|_H^2 = r_\phi^H$, and $\|\psi^h\|_H^2 = r_\psi^H$ is to prevent us from getting a zero solution to our problem. Any other equation that will have the same effect is equally good. Since the projection step leads to complicated equations for these constraints, we deviate here for these constraints, from the standard FAS. We basically ask that the coarse grid solution is of norm one for both the components under study. This leads to a significant simplification. This together with taking into account the global step as if done on the fine level leads us to the following coarse grid equations.

$$\begin{pmatrix} \mathcal{A}^H - \lambda I & \mu I \\ -\mu I & \mathcal{A}^H - \lambda I \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \phi^H \\ \psi^H \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \tau_h^H(\phi^h) \\ \tau_h^H(\psi^h) \end{pmatrix} \tag{4.7}$$

$$\|\alpha_{11}\phi^H + \alpha_{12}\psi^H\|_H^2 = 1, \quad \|\alpha_{21}\phi^H + \alpha_{22}\psi^H\|_H^2 = 1 \tag{4.8}$$

In this equation the unknowns are $\phi^H, \psi^H, \lambda, \mu$ and $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$. The iterative method for solving this problem will involve Jacobi relaxation for ϕ^H, ψ^H keeping the rest of the unknowns fixed, followed by a global step. The global step requires the solution of an eigenvalue problem like (3.10) with the following C and D

$$C = \begin{pmatrix} < \mathcal{A}^H\phi^H - \tau_h^H(\phi^h), \phi^H > & < \mathcal{A}^H\psi^H - \tau_h^H(\psi^H), \phi^H > \\ < \mathcal{A}^H\phi^H - \tau_h^H(\phi^h), \psi^H > & < \mathcal{A}^H\psi^H - \tau_h^H(\psi^h), \psi^H > \end{pmatrix} \tag{4.9}$$

$$D = \begin{pmatrix} < \phi^H, \phi^H > & < \psi^H, \phi^H > \\ < \phi^H, \psi^H > & < \psi^H, \psi^H > \end{pmatrix}. \tag{4.10}$$

A corresponding eigenvector defines the four $\alpha's$ as in the single grid method. Having solved this problem approximately for $\phi^H, \psi^H, \lambda, \mu$ and $\alpha_{11}, \alpha_{12}, \alpha_{21}\alpha_{22}$ the interpolation to be used is

$$\begin{pmatrix} \phi^h \\ \psi^h \end{pmatrix} \leftarrow \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \phi^h + (\phi^H - I_h^H\phi^h) \\ \psi^h + (\psi^H - I_h^H\psi^h) \end{pmatrix}. \tag{4.11}$$

We now come to the general case.

dim $\mathcal{H}$ = n. Assume that there are r complex eigenvalues and n-r real eigenvalues corresponding to this subspace. In this case the fine grid problem is of the form

$$(\underline{\mathcal{A}}^h - \underline{\Gamma}^h)\underline{\phi}^h = 0 \tag{4.12}$$

where $\underline{\phi} = (\phi_1, \ldots, \phi_n)^T$,

$$\underline{\mathcal{A}}^h = \begin{pmatrix} \mathcal{A}^h & & \\ & \ddots & \\ & & \mathcal{A}^h \end{pmatrix}_{nxn} \tag{4.13}$$

(4.14)

and

$$\underline{\Gamma}^h = \begin{pmatrix} \lambda_1 & \mu_1 & & & & & \\ -\mu_1 & \lambda_1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_r & \mu_r & & \\ & & & -\mu_r & \lambda_r & & \\ & & & & & \lambda_{r+1} & \\ & & & & & & \ddots \\ & & & & & & & \lambda_n \end{pmatrix}_{nxn} . \tag{4.15}$$

Following the previous arguments, the coarse grid problem is then

$$(\underline{A}^H - \underline{\Gamma}) E \underline{\phi}^H = E \underline{\tau}_h^H(\underline{\phi}^h) \tag{4.16}$$

$$\|(E\underline{\phi}^H)_i\|^2 = 1, \quad (i = 1, \ldots, n) \tag{4.17}$$

where

$$\underline{\tau}_h^H(\underline{\phi}^h) = \begin{pmatrix} \tau_h^H(\phi_1^h) & & \\ & \ddots & \\ & & \tau_h^H(\phi_n^h) \end{pmatrix}_{nxn} \tag{4.18}$$

and E is an nxn real valued matrix, whose rows are obtained from a solution of the following eigenvalue problem

$$\det(C - \rho D) = 0, \tag{4.19}$$

where D is an nxn matrix whose elements are $< \phi_i^H, \phi_j^H >$ and C is an nxn matrix whose elements are $< A^H \phi_i^H - \tau_h^H(\phi_j^h), \phi_j^H >$. Since D is invertible there are n eigenvalues where complex ones appear in pairs of conjugates. A complex eigenvalue ρ correspond to a pair (λ_j, μ_j) for some $j \leq l$ while a real one corresponds to a real λ_j for a $j > l$. The real and the imaginary part of a complex eigenvector will define two rows in E that correspond to the corresponding pair (λ_j, μ_j), while a real valued eigenvector will define one row in E in a place corresponding to that of the eigenvalue associated with it.

Having solved the coarse grid problem approximately the following interpolation is to be used:

$$\underline{\phi}^h \leftarrow E\{\underline{\phi}^h + I_H^h(\underline{\phi}^H - \underline{I}_h^H \underline{\phi}^h)\}. \tag{4.20}$$

4.2 Multigrid Algorithm

Consider a sequence of grids $\Omega_k (k \leq m)$ with mesh sizes h_k satisfying $2h_{k+1} = h_k$. Suppose on each grid operators $\mathcal{A}^k$ are given in such a way that $\mathcal{A}^k$, $(k < m)$ is an approximation to $\mathcal{A}^{k+1}$. Assume also that interpolation operators I_{k-1}^k, from coarse to fine grids , and a restriction operator I_k^{k-1}, from fine to coarse grids, are given. The problem is to solve the following equation given on the finest level

$$(\underline{\mathcal{A}}^m - \underline{\Lambda}^m)\underline{\phi}^m = 0, \qquad \|I_m^1(\underline{\phi}^m)_i\|^2 = 1 \quad (i = 1, \ldots, n). \tag{4.21}$$

On levels $(k < m)$ the equations to be solved are of the form

$$(\underline{\mathcal{A}}^k - \underline{\Lambda}^k)E\underline{\phi}^k = E\underline{r}_m^k(\underline{\phi}^{k+1}), \qquad \|I_k^1(\underline{\phi}^k)_i\|^2 = 1 \quad (i = 1, \ldots, n). \tag{4.22}$$

where

$$\underline{r}_m^k(\underline{\omega}^{k+1}) = \underline{\mathcal{A}}^k I_{k+1}^k \underline{\omega}^{k+1} + I_{k+1}^k\{\underline{r}_m^{k+1}(\underline{\omega}^{k+2}) - \underline{\mathcal{A}}^{k+1}\underline{\omega}^{k+1}\} \quad (k < M) \tag{4.23}$$

$$\underline{r}_m^m(\underline{\omega}^m) = 0 \tag{4.24}$$

where $\underline{\omega}^{k+1}$ is any fine grid n-vector function.

Note that the finest grid problem, i.e., for k=m, can be written as the general k-grid one with the choice E=I, the nxn identity matrix.

Given an approximate solution $\{(\phi^m, \psi^m)^T, \lambda, \mu, E\}$ to the problem (4.22) the multigrid cycle for improving it is denoted by

$$(\phi^k, \psi^k, \lambda, \mu, E) \quad \leftarrow \quad MG(k, \phi^k, \psi^k, \lambda, \mu, E) \tag{4.25}$$

and is defined recursively as follows:

if k=1 **then** solve (4.22) by enough local relaxations together with global steps

else

- Perform ν_1 local relaxation sweeps on (4.22), starting with $(\phi^k, \psi^k, \lambda, \mu, E)$ and resulting in a new approximation $(\bar{\phi}^k, \bar{\psi}^k, \lambda, \mu, E)$
- Starting with $\phi^{k-1} = I_k^{k-1}\bar{\phi}^k, \psi^{k-1} = \bar{I}_k^{k-1}\bar{\psi}^k$ make γ successive cycles of the type

$$(\phi^{k-1}, \psi^{k-1}, \lambda, \mu, E) \quad \leftarrow \quad MG(k-1, \phi^{k-1}, \psi^{k-1}, \lambda, \mu, E)$$

- Interpolate correction according to the formula $\bar{\phi}^k = \bar{\phi}^k + I_{k-1}^k(\phi^{k-1} - I_k^{k-1}\bar{\phi}^k)$.
- If $k = m$ make the change $\bar{\phi}^m \leftarrow E\bar{\phi}^m$, then set $E = I$.
- Perform ν_2 relaxation sweeps on (4.22) starting with $(\bar{\phi}^k, \bar{\psi}, \lambda, \mu, E)$ and yielding $(\phi^k, \psi^k\lambda, \mu, E)$, the final result of (4.25).

FMG Algorithm

In order to obtain full efficiency, the first approximation on a given level is obtained from a solution of the same problem on the next coarser level, which itself has been calculated in a similar way. The resulting algorithm is called (FMG) and is described next. Let Π_{k-1}^k be an interpolation operator (usually of higher order than I_{k-1}^k). Given the problem (4.21) , the N-FMG solution of that problem is:

N-FMG Algorithm

Set $m = 1$.

Obtain initial approximation as in section (3.1).

Calculate $(\phi^1, \psi^1, \lambda, \mu)$ the solution of (4.21) for m=1 by several relaxations.

for $m = 2, \ldots, M$ **do:**

- Calculate $\phi^m \leftarrow \Pi_{m-1}^m\phi^{m-1}, \quad \psi^m \leftarrow \Pi_{m-1}^m\psi^{m-1}$
- Perform the cycle $(\phi^m, \psi^m, \lambda, \mu, E) \quad \leftarrow \quad MG(m, \phi^m, \psi^m, \lambda, \mu, E)$ N times.

5 Numerical Examples

Numerical experiments were carried out with the following operator

$$\mathcal{A} = \begin{pmatrix} \Delta + c^2 & a_0 \\ b_0 & \Delta + c^2 \end{pmatrix} \tag{5.1}$$

in (0,1)x(0,1), with $a_0 \times b_0 < 0$.

The eigenvalues of this operator are

$$\lambda \pm \imath\mu$$

where λ is an eigenvalue of Δ and $\mu = \sqrt{(-a_0 \times b_0)}$.

Uniform discretization was used, where Δ was discretized using the standard 5-point Laplacian. Π_m^{k+1} was bi-cubic interpolation, I_k^{k+1} was bi–linear interpolation and I_{k+1}^k was the 9-point full weighting operator. The following parameters were used in the multigrid cycling: $\nu_1 = 2, \nu_2 = 1, \gamma = 1$. Initial approximation on the coarsest level was obtained as in section (3.1). Results for the 2-FMG algorithm are given in Tables 1-4. The L_2 norm of the residuals as well as the values of λ^h and μ^h are given at the end of each cycle. Although a 2-FMG algorithm was used the results clearly show that 1-FMG algorithm solves the problem to the level of discretization errors. Observe that the imaginary part μ of the eigenvalue is exact on coarse grids since the truncation errors in approximating the off diagonal elements of A are zero, and only those elements contribute to the non symmetry of the problem. Also note the insensitivity of the results to the closeness of the eigenvalue to the imaginary axis.

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level	cycle	$\|residuals\|_2$	λ^h	μ^h
2	1	.279e+00	-19.4809	1.00000
	2	.150e-01	-19.4863	1.00000
3	1	.298e-01	-19.6737	1.00000
	2	.151e-02	-19.6757	1.00000
4	1	.312e-02	-19.7228	1.00000
	2	.203e-03	-19.7233	1.00000

Table 1: $a_0 = 1.\quad b_0 = -1.\quad c^2 = 0.$

level	cycle	$\|residuals\|_2$	λ^h	μ^h
2	1	.279e+00	-19.4809	3.16227
	2	.150e-01	-19.4863	3.16227
3	1	.298e-01	-19.6737	3.16227
	2	.151e-02	-19.6757	3.16227
4	1	.312e-02	-19.7228	3.16227
	2	.203e-03	-19.7233	3.16227

Table 2: $a_0 = 10.\quad b_0 = -1.\quad c^2 = 0.$

level	cycle	$\|residuals\|_2$	λ^h	μ^h
2	1	.279e+00	0.51912	3.16227
	2	.150e-01	0.51370	3.16227
3	1	.298e-01	0.32634	3.16227
	2	.151e-02	0.32421	3.16227
4	1	.312e-02	0.27714	3.16227
	2	.203e-03	0.27766	3.16227

Table 3: $a_0 = 10. \quad b_0 = -1. \quad c^2 = 20.$

level	cycle	$\|residuals\|_2$	λ^h	μ^h
2	1	.279e+00	0.51912	1.00000
	2	.150e-01	0.51370	1.00000
3	1	.298e-01	0.32634	1.00000
	2	.151e-02	0.32421	1.00000
4	1	.312e-02	0.27714	1.00000
	2	.203e-03	0.27766	1.00000

Table 4: $a_0 = 1. \quad b_0 = -1. \quad c^2 = 20.$

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16. Abstract

The problem of calculating the stability of steady state solutions of differential equations is treated. Leading eigenvalues (i.e., having maximal real part) of large matrices that arise from discretization are to be calculated. An efficient multigrid method for solving these problems is presented. The method begins by obtaining an initial approximation for the dominant subspace on a coarse level using a damped Jacobi relaxation. This proceeds until enough accuracy for the dominant subspace has been obtained. The resulting grid functions are then used as an initial approximation for appropriate eigenvalue problems. These problems are being solved first on coarse levels, followed by refinement until a desired accuracy for the eigenvalues has been achieved. The method employs local relaxation on all levels together with a global change on the coarsest level only, which is designed to separate the different eigenfunctions as well as to update their corresponding eigenvalues. Coarsening is done using the FAS formulation in a non-standard way in which the right hand side of the coarse grid equations involves unknown parameters to be solved for on the coarse grid. This in particular leads to a new multigrid method for calculating the eigenvalues of symmetric problems. Numerical experiments with a model problem demonstrate the effectiveness of the method proposed. Using an FMG algorithm a solution to the level of discretization errors is obtained in just a few work units (less than 10), where a work unit is the work involved in one Jacobi relaxation on the finest level.

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