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A TORUS BIFURCATION THEOREM WITH SYMMETRY

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Abstract. A general theory for the study of degenerate Hopf bifurcation in the presence of symmetry has been carried out only in situations where the normal form equations decouple into phase/amplitude equations. In this paper we prove a theorem showing that in general we expect such degeneracies to lead to secondary torus bifurcations. We then apply this theorem to the case of degenerate Hopf bifurcation with triangular (D_3) symmetry proving that in codimension two there exist regions of parameter space where two branches of asymptotically stable 2-tori coexist but where no stable periodic solutions are present.
Although this study does not lead to a theory for degenerate Hopf bifurcations in the presence of symmetry, it does present examples that would have to be accounted for by any such general theory.

§1. Introduction

One of the more interesting features of degenerate Hopf bifurcations in the presence of symmetry is the appearance, via secondary bifurcation, of quasiperiodic motion on a torus. In this paper we concentrate on two-parameter systems of ODE and prove theorems that allow us to find and compute the direction of branching for some of these tori. The advantage of our approach is that we determine this information using only the Taylor expansion of the vector field at the point where degenerate Hopf bifurcation occurs.

The simplest form of Hopf bifurcation with symmetry group Γ occurs as follows. We assume that Γ is a compact Lie group acting absolutely irreducibly on a vector space V, that is, the only matrices on V which commute with Γ are multiples of the identity.

Let

$$dz/dt = f(z,\lambda), \quad z \in V \oplus V = V \otimes \mathbf{C} \tag{1.1}$$

be a system of ODE where f is Γ-equivariant. In complex coordinates, absolute irreducibility implies that

$$f(z,\lambda) = a(\lambda)z + \ldots \tag{1.2}$$

where $a(\lambda) \in \mathbf{C}$. We say that (1.1) has a *Hopf bifurcation* at $\lambda = 0$ if $a(0)$ is purely imaginary.

The standard Hopf theorem ($V = \mathbf{R}$, $\Gamma = 1$) states that if the eigenvalue crossing condition

$$\frac{d}{d\lambda} \mathrm{Re}(a)(0) \neq 0 \tag{1.3}$$

holds, then there exists a unique branch of periodic solutions to (1.1). Moreover, if a certain coefficient μ_2 involving the second and third order terms in f satisfies

$$\mu_2 \neq 0, \tag{1.4}$$

then the direction of branching (supercritical or subcritical) and the asymptotic stability of these periodic solutions are determined. We call a Hopf bifurcation *degenerate* if either (1.3) or (1.4) fails. Such singularities are studied by Takens [1974] and Golubitsky & Langford [1981].

In Hopf bifurcation with symmetry we have a *degeneracy* if either the direction of branching or the asymptotic stability of a branch of periodic solutions is not determined at the lowest order that it could have been. We are interested in such degeneracies because they may be unavoidable in two parameter systems. Degenerate Hopf bifurcations with $O(2)$-symmetry have been studied extensively for the past few years and the results concerning this specific case are discussed in Section 2.

We now explain why one should expect invariant tori to be produced by perturbing certain of these degeneracies. To do this we recall some of the theory of Hopf bifurcation with symmetry. We assume that f is in Birkhoff normal form, that is we assume

$$f \text{ is } \Gamma \times S^1\text{-equivariant} \tag{1.5}$$

where for $x \oplus c \in V \oplus C$ we have $(\gamma,\theta)(x\oplus c) = (\gamma x)\oplus(e^{i\theta}c)$. The S^1 symmetry comes from phase shifts. We detect branches of periodic solutions by choosing a subgroup $\Sigma \subset \Gamma \times S^1$ such that

$$\dim \mathrm{Fix}(\Sigma) = 2 \tag{1.6}$$

where

$$\mathrm{Fix}(\Sigma) = \{z \in V\oplus C\colon \sigma z = z, \ \forall\, \sigma \in \Sigma\}. \tag{1.7}$$

In normal form $f\colon \mathrm{Fix}(\Sigma)\times R \to \mathrm{Fix}(\Sigma)$. Thus, the differential equation (1.1) restricted to $\mathrm{Fix}(\Sigma)$ satisfies the hypotheses of standard Hopf bifurcation. In particular, if (1.3) and (1.4) hold for the restricted system then there exists a branch of periodic solutions for (1.1) in $\mathrm{Fix}(\Sigma)$ and the direction of branching (in λ) is determined. (Golubitsky & Stewart [1985] show that the assumption that f is in Birkhoff normal form is not needed to prove these points.)

Stability of these solutions however, is not determined by the standard Hopf theorem, since the Floquet multipliers corresponding to eigenvectors in $\mathrm{Fix}(\Sigma)^{\perp}$ also enter into this determination of stability. In this paper we focus on degeneracies produced when determining stability along known branches of periodic solutions obtained using (1.6). Specifically, suppose that one tracks along a branch of periodic solutions and that at some special value $\lambda = \lambda_0$ the periodic solution loses stability by having a simple complex conjugate pair of Floquet multipliers cross through the unit circle with nonzero speed. The torus bifurcation theorem (see Iooss [1987]) guarantees the existence of invariant tori. Suppose now that (1.1) depends on a second parameter μ and that the critical value λ_0 also depends on μ. We can imagine a situation where as μ is varied λ_0 moves into the origin, say at $\mu = 0$. When this happens we will find a degenerate Hopf bifurcation with symmetry. Moreover, it seems reasonable that the speed of the Floquet exponent that crosses through zero and the direction of branching of the branch of tori can be determined from the Taylor expansion of f at the origin and with λ and μ set to zero. Our results are summarized in Theorems 4.5 and 4.6.

We note that several authors have considered the bifurcation of tori from branches of periodic solutions. See Renardy [1982], Rand [1982] and Ruelle [1973]. An important point here is that the Floquet matrix itself commutes with the isotropy subgroup Σ and as a result the Floquet multipliers may be forced by symmetry to have high multiplicity. See Chossat & Golubitsky [1988]. Thus the assumption above that the Floquet multipliers are simple may not always be satisfied.

The scenario that we described above does happen in the case of $O(2)$ symmetry. However, as we explain in Section 2, there is a relatively simple way to analyze the resulting tori (the torus bifurcation theorem is not needed there). In addition, the resulting flow on the torus is particularly simple. Symmetry forces the flow to be linear.

A more interesting example occurs in Hopf bifurcation with D_n symmetry. Here the generic Hopf theory has been worked out (Golubitsky & Stewart [1986] and van Gils & Valkering [1986]). Because in this case $\dim V = 2$, it follows that $\dim \mathrm{Fix}(\Sigma)^{\perp} = 2$ and the Floquet multipliers discussed above must be simple. It is this example (itself motivated by considering rings of oscillators) that has motivated our theorem. In Section 3 we discuss the general results for Hopf bifurcation with D_n symmetry while in Section 5 we illustrate our theorem by explicitly calculating the direction of branching of tori in the D_3 case. Bifurcation diagrams corresponding to degenerate Hopf bifurcation with D_3 symmetry are presented in Section 6.

In Section 4 we present our hypotheses and theorems. This section can be read directly after the Introduction since explicit knowledge of the $O(2)$ and D_n examples is not needed for the general theory. In Section 5 we show how to find two-frequency motions by applying standard Hopf bifurcation results to a certain normal vector field whose existence is found in Section 4 using results of Krupa [1988].

§2. Degenerate Hopf Bifurcation with O(2) Symmetry

We begin by surveying some of the results on degenerate Hopf bifurcations with O(2) symmetry. This problem has been studied by Erneux & Matkowsky [1984], Knobloch [1986], Chossat [1986], Golubitsky & Roberts [1987], Nagata [1986], and Crawford & Knobloch [1988].

The action of O(2) on $\mathbb{R}^4 = \mathbb{C}\oplus\mathbb{C}$ is generated by

$$\begin{aligned} &(a) \quad \theta(z_1,z_2) = (e^{i\theta}z_1,\ e^{i\theta}z_2), \quad \forall\, \theta \in SO(2) \\ &(b) \quad \kappa(z_1,z_2) = (\bar{z}_1,\bar{z}_2). \end{aligned} \tag{2.1}$$

Consider the O(2)-equivariant system of ODE

$$\frac{dz}{dt} = f(z,\lambda),\ f(0,\lambda) = 0 \tag{2.2}$$

depending on a bifurcation parameter λ. We assume that (2.2) has a Hopf bifurcation at $\lambda = 0$; due to symmmetry the eigenvalues $\sigma(\lambda) \pm i\omega(\lambda)$ of $(df)_{0,\lambda}$ are each of multiplicity two. By Hopf bifurcation we mean that $\sigma(0) = 0,\ \omega(0) \equiv \omega_0 \neq 0$.

Van Gils & Mallet-Paret [1984], Chossat & Iooss [1985], Golubitsky & Stewart [1985] and others have shown that if

$$\sigma'(0) \neq 0, \tag{2.3}$$

then two branches of periodic solutions $z(t)$ bifurcate from the origin and, moreover, these solutions may be detected by their symmetry. They are:

$$\begin{aligned} &(a) \quad \text{rotating waves (RW): } \theta z(t) = z(t-\theta) \\ &(b) \quad \text{standing waves (SW): } \kappa z(t) = z(t). \end{aligned} \tag{2.4}$$

Generically, the exchange of stability for such systems goes as follows. Assume that $x = 0$ is asymptotically stable when $\lambda < 0$. Then for either branch (2.4) to consist of asymptotically stable periodic solutions, both branches must be supercritical and then precisely one branch consists of stable solutions. More precisely, there are two coefficients, derived from the third order terms of f, which determine the direction of branching of solutions (2.4) with stability being determined by which coefficient is larger.

Erneux & Matkowsky [1984] observed that when such systems depend on two parameters,

$$\frac{dz}{dt} = f(z,\lambda,\mu), \tag{2.5}$$

it is possible to arrange for a distinguished value of μ, say $\mu = 0$, where both cubic coefficients are equal. They show that invariant 2-tori exist in (2.5) for μ near 0. The types of bifurcation diagrams which may occur are shown in Figure 2.1. (The direction of branching and the stability of the 2-tori depend on fifth and seventh order terms in f. See Golubitsky & Roberts [1987].)

In retrospect the existence of these 2-tori can be understood in a relatively simple way. First, assume that (2.5) is in Birkhoff normal form which means that now f may be assumed to be $O(2) \times S^1$-equivariant (cf. Golubitsky & Stewart [1985]). In normal form, (2.5) splits into phase-amplitude equations where the amplitude equations have the form

$$\frac{d}{dt}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = p(r_1^2+r_2^2, r_1^2r_2^2, \lambda, \mu)\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + q(r_1^2+r_2^2, r_1^2r_2^2, \lambda, \mu)(r_2^2-r_1^2)\begin{bmatrix} r_1 \\ -r_2 \end{bmatrix} \tag{2.6}$$

where $r_j = |z_j|$. Nontrivial equilibria (r_1, r_2) of (2.6) correspond to standing waves $(r_1 = r_2)$, rotating waves $(r_1 r_2 = 0)$ or invariant 2-tori $(r_1 \neq r_2,\ r_1 r_2 \neq 0)$.

Thus the Erneux & Matkowsky 2-tori are on the same footing as the periodic solutions in the study of degenerate Hopf bifurcation with $O(2)$ symmetry. Swift [1984] noted that the amplitude equations (2.6) have D_4-symmetry (generated by $(r_1,r_2) \to (r_1,-r_2)$, $(-r_1,r_2)$ and (r_2,r_1)).

Therefore, degenerate $O(2)$ Hopf bifurcations can be studied using D_4-equivariant singularity theory just as degenerate Hopf bifurcation without symmetry can be studied by Z_2-equivariant singularity theory (see Golubitsky & Langford [1981]). The D_4-classification was carried out up to (topological) D_4-codimension two in Golubitsky & Roberts [1987]. See also Crawford & Knobloch [1988] or Golubitsky, Stewart & Schaeffer [1988].

It should also be noted that these 2-tori have a special structure due to the $O(2) \times S^1$ symmetry of normal form. The flow on the 2-torus is linear. Chossat [1986] has shown that this property persists, even if f is not assumed to be in Birkhoff normal form. His technique is to use a Liapunov-Schmidt reduction to look for two frequency solutions of the form

$$z(t) = e^{i\omega t} R_{\eta t} q \tag{2.7}$$

where R_θ is the rotation matrix corresponding to $\theta \in SO(2)$. The function (2.7) has two independent frequencies if ω/η is irrational.

§3. Generic Hopf Bifurcation with D_n Symmetry

When $n \geq 3$ the group D_n has two-dimensional irreducible representations. Thus, in systems with D_n symmetry, Hopf bifurcation from a D_n-invariant steady state may occur by eigenvalues of multiplicity one or two crossing the imaginary axis. In this section we review the results of Golubitsky & Stewart [1986] and van Gils & Valkering [1986] about generic D_n-Hopf bifurcation in the double eigenvalue case. See also Golubitsky, Stewart & Schaeffer [1988].

Without loss of generality we may assume that the action of D_n on $\mathbb{R}^2 \cong \mathbb{C}$ is generated by

$$\begin{aligned} &\text{(a)} \quad \gamma \cdot z = e^{i\gamma} \quad \text{where} \quad \gamma = 2\pi/n, \quad \text{and} \\ &\text{(b)} \quad \kappa(z) = \bar{z}. \end{aligned} \tag{3.1}$$

It is possible to choose coordinates on $\mathbb{C}^2$ such that the action of $D_n \times S^1$ is generated by

$$\begin{aligned} &\text{(a)} \quad \gamma \cdot (z, z_2) = (e^{i\gamma} z_1, e^{i\gamma} z_2) \\ &\text{(b)} \quad \kappa \cdot (z_1, z_2) = (z_2, z_1) \\ &\text{(c)} \quad \theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2). \end{aligned} \tag{3.2}$$

It can be shown that for each n, there are precisely three (conjugacy classes of) isotropy subgroups whose fixed point subspaces are two-dimensional. There exists a unique branch of periodic solutions for each of these subgroups in D_n-symmetric generic Hopf bifurcation. There is a discrete analogue of a 'rotating wave' and two discrete analogues of 'standing waves'.

The rotating wave has isotropy subgroup

$$\tilde{Z}_n = \{(\gamma, \gamma) : \gamma \in Z_n\}.$$

The standing waves each have isotropy subgroups isomorphic to Z_2. The symmetry of one of the standing waves is generated by κ; the symmetry of the other standing wave is generated by $(\kappa,\pi) \in D_n \times S^1$, at least when $n \neq 0$ mod 4. See Golubitsky & Stewart [1986] for details.

The exchange of stability for these branches goes as follows. Suppose the D_n-invariant steady state is stable subcritically and loses stability by having eigenvalues cross the imaginary axis with nonzero speed. When $n \neq 4$, no branch is stable unless all branches are supercritical. There is one third order term that determines whether the rotating waves are supercritical and another third order term that determines whether the standing waves are (jointly) supercritical. No branch consists of stable trajectories unless all branches are supercritical, in which case precisely one branch consists of stable solutions.

Supposing that all branches are supercritical then it can be determined at third order whether the rotating wave or one of the standing waves is stable. If a standing wave is to be stable, then which one is stable is determined by a coefficient of order

$$m = \begin{cases} n+2 & n \text{ odd} \\ \dfrac{n+2}{2} & n \text{ even} \end{cases} \tag{3.3}$$

In Section 5 we show that the conditions for our torus bifurcation theorem (Theorem 4.5) may be satisfied when, in a two-parameter system, there is an isolated value of the parameters where a Hopf bifurcation occurs and the bifurcating branches are neutrally stable at third order.

The remainder of this section is devoted to discussing these results in more detail. The notation we set here will be used in Section 5. We begin by describing the $D_n \times S^1$ invariants and equivariants.

Recall m as defined in (3.3) and define

$$
\begin{aligned}
&(a)\quad N = |z_1|^2 + |z_2|^2 \\
&(b)\quad P = |z_1|^2|z_2|^2 \\
&(c)\quad S = (z_1\bar{z}_2)^m + (\bar{z}_1 z_2)^m \\
&(d)\quad T = i\{|z_1|^2-|z_2|^2\}\{(z_1\bar{z}_2)^m-(\bar{z}_1 z_2)^m\}
\end{aligned}
\tag{3.4}
$$

Proposition 3.1: *Let* $D_n \times S^1$ *act on* $\mathbb{C}^2$ *as defined by (3.2).*

(a) Every smooth $\Gamma_n \times S^1$*-invariant germ* $f:\mathbb{C}^2 \to \mathbb{R}$ *has the form*

$$f(z_1,z_2) = h(N,P,S,T) \tag{3.5}$$

for some smooth germ $h:\mathbb{R}^4 \to \mathbb{R}$.

(b) Every smooth $D_n \times S^1$*-equivariant germ* $g:\mathbb{C}^2 \to \mathbb{C}^2$ *has the form*

$$g = A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B\begin{bmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{bmatrix} + C\begin{bmatrix} \bar{z}_1^{m-1} z_2^m \\ z_1^m \ \bar{z}_2^{m-1} \end{bmatrix} + D\begin{bmatrix} z_1^{m+1}\bar{z}_2^m \\ \bar{z}_1^m \ z_2^{m+1} \end{bmatrix} \tag{3.6}$$

where A,B,C,D are complex-valued $D_n \times S^1$*-invariant germs.*

We consider the system of ODE

$$\frac{dz}{dt} = g(z,\lambda) \tag{3.7}$$

where $g:\mathbb{C}^2\times\mathbb{R} \to \mathbb{C}^2$ is $D_n \times S^1$-equivariant. That is, we assume that we are studying Hopf bifurcation with D_n-symmetry for a system of ODE which is already in Poincaré-Birkhoff normal form to all orders.

In Table 3.1 we present the equations determining the branching of solutions for each of the three isotropy subgroups mentioned above. These results are taken from Golubitsky & Stewart [1986].

Solution Type	(z_1, z_2) Orbit Rep.	Branching Equations
Rotating Wave	$(u,0)$	$\mathrm{Re}(A + Bu^2) = 0$
Standing Wave 1	(u,u)	$\mathrm{Re}(A + Bu^2 + Cu^{2m-2} + Du^{2m}) = 0$
Standing Wave 2		
$n \neq 0 \bmod 4$	$(u,-u)$	$\mathrm{Re}(A + Bu^2 - Cu^{2m-2} - Du^{2m}) = 0$
$n = 0 \bmod 4$	$(u, e^{2\pi i/n}u)$	

Table 3.1: Branching equations for maximal isotropy.

The asymptotic stability of these solutions are determined by the signs of the real parts of the eigenvalues of dg evaluated at the solution. The actual computation of these eigenvalues is aided substantially by the existence of the $D_n \times S^1$ symmetry. In particular, for each of the three isotropy subgroups Σ, the space $\mathbb{C}^2$ may be written as

$$\mathrm{Fix}(\Sigma) \oplus \mathrm{Fix}(\Sigma)^{\perp} \tag{3.8}$$

where Σ acts nontrivially on $\mathrm{Fix}(\Sigma)^{\perp}$. Since dg evaluated at a solution commutes with Σ, it follows that both $\mathrm{Fix}(\Sigma)$ and $\mathrm{Fix}(\Sigma)^{\perp}$ are dg invariant subspaces. Thus, the eigenvalues of dg are obtained in each case by finding the eigenvalues of two 2x2 matrices. Moreover, the S^1 symmetry forces one eigenvalue of $dg|\mathrm{Fix}(\Sigma)$ to have real part zero. Also, for rotating waves when $n \neq 4$, the group $\tilde{Z}_n$ acts on $\mathrm{Fix}(\tilde{Z}_n)^{\perp}$ as nontrivial rotations and forces $dg|\mathrm{Fix}(\tilde{Z}_n)^{\perp}$ to itself be a scalar multiple of a rotation matrix.

Using this group theoretic information, it is possible to compute the signs of the real parts of the eigenvalues of dg. These results are summarized in Table 3.2.

Solution Type	EV of $dg \mid \mathrm{Fix}(\Sigma)$	EV's of $dg \mid \mathrm{Fix}(\Sigma)^{\perp}$
Rotating Wave	$-\mathrm{Re}(A_N+B) + O(u)$	$-\mathrm{Re}\, B$
Standing Wave 1	$-\mathrm{Re}(2A_N+B) + O(u)$	$\mathrm{tr} = \mathrm{Re}(B-(m+1)Cu^{2m-4}) + O(u^{2m-2})$ $\det = -\mathrm{Re}(B\bar{C}) + O(u)$
Standing Wave 2	$-\mathrm{Re}(2A_N+B) + O(u)$	$\mathrm{tr} = \mathrm{Re}(B+(m+1)Cu^{2m-4}) + O(u^{2m-2})$ $\det = \mathrm{Re}(B\bar{C}) + O(u)$

Table 3.2: Signs of Eigenvalues of dg along primary branches.

The directions of branching and the asymptotic stability of the branches discussed above follow from Tables 3.1 and 3.2 assuming that the nondegeneracy conditions

$$\begin{aligned}
&(a)\quad \mathrm{Re}(A_N+B) \neq 0\\
&(b)\quad \mathrm{Re}(B) \neq 0\\
&(c)\quad \mathrm{Re}(2A_N+B) \neq 0\\
&(d)\quad \mathrm{Re}(B\bar{C}) \neq 0\\
&(e)\quad \mathrm{Re}(A_\lambda) \neq 0
\end{aligned} \tag{3.9}$$

hold when evaluated at the origin. Observe that these branches are all neutrally stable at third order if

$$\mathrm{Re}\, B(0,0) = 0. \tag{3.10}$$

It is this coefficient that may vanish in a two-parameter system and that must be zero in order to apply our torus bifurcation theorem.

§4. The Torus Bifurcation Theorem

In this section we prove the existence, direction of branching and asymptotic stability of certain invariant 2-tori in codimension two bifurcations occurring in a class of symmetric systems of ODE. The general Theorems 4.5 and 4.6 are complemented by explicit formulas for computation that are derived in Section 5. A number of hypotheses are needed to prove our theorem and we describe them now. These hypotheses abstract properties of the D_n-equivariant systems described in Section 3. We return to D_3 symmetry in Sections 5 and 6, where the results of this section are applied.

Specifically, we consider the two-parameter system of ODE

$$\frac{dz}{dt} = f(z,\lambda,\mu) \tag{4.1}$$

where $f: \mathbb{C}^n \times \mathbb{R}^2 \to \mathbb{C}^n$ is smooth. The roles of the parameters λ and μ are distinguished as follows. We assume that (4.1) undergoes a Hopf bifurcation as the primary bifurcation parameter λ is varied and that a secondary torus bifurcates off of a primary branch of periodic solutions as λ is further varied. The role of the auxiliary parameter μ is to allow the secondary torus bifurcation to coalesce with the primary Hopf bifurcation as μ is varied. We study here the simplest instances of such a codimension two bifurcation consistent with symmetry. We now state the hypotheses needed to define this simplest setting.

Our intention is to make a preliminary and naive discussion of degenerate Hopf bifurcations in the presence of symmetry. We do not pretend to have a general theory. A general theory, however, will have to include the examples and setting we study here.

§4.1 Hypotheses on the primary Hopf bifurcation

The simplest form of Hopf bifurcation in the presence of symmetry occurs under the following hypotheses (see Golubitsky & Stewart [1985]). We let Γ be a finite subgroup of $O(n)$ and let Γ act on $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ by $\gamma(x + iy) = \gamma x + i\gamma y$. The reason for restricting Γ to be finite will be discussed in (H2) below. We assume:

(H1) Γ acts absolutely irreducibly on $\mathbb{R}^n$,

that is, the only nxn real matrices commuting with Γ are scalar multiples of the identity.

We assume that the f in (4.1) commutes with Γ, undergoes a Hopf bifurcation at $\lambda = 0$ when $\mu = 0$, and is in Birkhoff normal form. The first and third of these assumptions are summarized by:

(H2) f is $\Gamma \times S^1$ - equivariant

where the circle group S^1 is viewed as the complex numbers of modulus one acting on $\mathbb{C}^n$ by complex multiplication. Thus

$$f(\sigma z,\lambda,\mu) = \sigma f(z,\lambda,\mu) \quad \text{for all} \quad \sigma \in \Gamma \times S^1.$$

Hypotheses (H1) and (H2) imply

$$\begin{aligned} &\text{(a)} \quad f(0,\lambda,\mu) = 0 \\ &\text{(b)} \quad f(-z,\lambda,\mu) = -f(z,\lambda,\mu) \\ &\text{(c)} \quad (df)_{0,\lambda,\mu} v = A(\lambda,\mu)v \quad \text{for all} \quad v \in \mathbb{C}^n \end{aligned} \tag{4.2}$$

where $A(\lambda,\mu) \in \mathbb{C}$. Hypothesis (H1) implies that $\Gamma \times S^1$ acts irreducibly on $\mathbb{C}^n$ and hence (4.2a) is valid. Since $\pi \in S^1$ acts as -1 on $\mathbb{C}^n$, (H2) implies (4.2b). (H2) also implies that $(df)_{0,\lambda,\mu}$ commutes with $\Gamma \times S^1$, from which (4.2c) follows.

The assumption that (4.1) undergoes a Hopf bifurcation at $\lambda = \mu = 0$ implies that $A(0,0)$ is purely imaginary. We assume that periodic solutions to (4.1) are generated from this Hopf bifurcation in as simple a way as possible. We now describe this process Begin by assuming that the eigenvalue $A(\lambda,0)$ crosses the imaginary axis with nonzero speed, that is,

(H3) $\Delta_3 \equiv \frac{\partial}{\partial\lambda}(\mathrm{Re}\, A)(0,0) \neq 0.$

Assumption (H3) implies that for each μ near 0 there is a unique value $\lambda(\mu)$ at which $A(\lambda(\mu),\mu)$ is purely imaginary. For simplicity, and without loss of generality, we assume that $\lambda(\mu) \equiv 0$ so that

(H4) $A(0,\mu) = i\omega_0(\mu).$

where $\omega_0(0) \neq 0$. Thus, we assume that for each μ a Hopf bifurcation from the trivial steady-state occurs in (4.1) at $\lambda = 0$.

Let $\Sigma \subset \Gamma \times S^1$ be an isotropy subgroup. Golubitsky and Stewart [1985] show that a unique branch of periodic solutions to (4.1) with symmetry group Σ can be found when

(H5) $\dim \mathrm{Fix}(\Sigma) = 2.$

Due to the assumption of Birkhoff normal form (H2), these periodic solutions all have the form

$$z(t) = \varepsilon e^{i\omega t}p \tag{4.3}$$

where $\varepsilon > 0$ and

$$p \in \mathrm{Fix}(\Sigma) \text{ is chosen with } |p| = 1. \tag{4.4}$$

Moreover, these periodic solutions are found by solving the equation

$$f(\varepsilon p,\lambda,\mu) - i\varepsilon\omega p = 0, \tag{4.5}$$

and, assuming (H3), (4.5) can be solved uniquely for

$$\begin{aligned} &\text{(a)} \quad \omega = \omega^*(\varepsilon^2,\mu) \equiv \omega_0(\mu) + \omega_2(\mu)\varepsilon^2 + \omega_4(\mu)\varepsilon^4 + O(\varepsilon^6) \\ &\text{(b)} \quad \lambda = \lambda^*(\varepsilon^2,\mu) = \lambda_2(\mu)\varepsilon^2 + \lambda_4(\mu)\varepsilon^4 + O(\varepsilon^6). \end{aligned} \tag{4.6}$$

It follows from (4.6b) that this branch of periodic solutions is supercritical (in λ) if $\lambda_2(0) > 0$ and subcritical if $\lambda_2(0) < 0$. We assume

(H6) $\nabla_6 \equiv \frac{1}{6} \langle (\partial^3 f)_{0,0,0}(p,p,p),\ p^* \rangle \neq 0$

where (due to Birkhoff normal form) $p^* = p$ is an eigenvector of $(df)^*_{0,0,0}$ with eigenvalue $\omega_0(0)i$. A calculation shows that

$$\lambda_2(0) = -\Delta_6/\Delta_3. \tag{4.7}$$

To verify (4.7) substitute (4.6) in (4.5), set the coefficient of ε^3 in (4.6) to zero, and take the real part of the inner product with p^*.

4.2 Hypotheses on the secondary torus bifurcation

The assumption of Birkhoff normal form allows us to reduce the problem of finding periodic solutions of (4.1) to finding stationary solutions of (4.5). With this assumption the problem of finding a secondary torus bifurcation of (4.1) is reduced to finding a secondary Hopf bifurcation of (4.5). We now discuss the group theoretic restrictions on the action of Γ which will admit the possibility of purely imaginary eigenvalues occurring in the linearization of (4.5) along the nontrivial branch of stationary solutions parametrized by (4.6).

Define

$$g(z,\lambda,\mu,\omega) = f(z,\lambda,\mu) - i\omega z. \tag{4.8}$$

The linearization dg, evaluated at a solution (4.6), must commute with the isotropy subgroup $\Sigma \subset \Gamma \times S^1$. Let

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \ldots \oplus V_k \tag{4.9}$$

be the isotypic decomposition under Σ, that is, each of the V_j's are sums of isomorphic irreducibile representations under Σ and the irreducible representations of Σ appearing in distinct V_j's are themselves distinct. Since $\mathrm{Fix}(\Sigma)$ is the sum of all the trivial representations of Σ, we may take

$$V_1 = \mathrm{Fix}(\Sigma). \tag{4.10}$$

Suppose now that dg has a complex conjugate pair of purely imaginary eigenvalues. Generically, we expect the (generalized) eigenspace associated with these eigenvalues to be in some V_j, without loss of generality we can take $j = 2$. The simplest type of torus bifurcation occurs when the purely imaginary eigenvalues are simple and the simplest way to force this hypothesis to be valid is to assume

(H7) $\dim V_2 = 2$.

Let $L = dg|V_2$ restricted to the branch of solutions parametrized by (4.6). So

$$L = L(\varepsilon^2,\mu). \tag{4.11}$$

We comment on (4.11). Along the branch (4.6)

$$z = \varepsilon p, \ \lambda = \lambda^*(\varepsilon^2,\mu) \ \text{ and } \ \omega = \omega^*(\varepsilon^2,\mu).$$

Moreover, by (4.2b) and (4.8) g is odd in z. Thus dg is even in z and hence in ε and the form of L given in (4.11) is valid.

By (H7) $\mathcal{L}$ is a linear mapping on a two-dimensional space. So $\mathcal{L}$ has purely imaginary eigenvalues precisely when

$$\mathrm{tr}(\mathcal{L}) = 0 \quad \text{and} \quad \det(\mathcal{L}) > 0.$$

We want to guarantee that for some μ near 0, there are values of ε^2 for which $\mathcal{L}$ has purely imaginary eigenvalues. That is, we want to guarantee the existence of solutions to the equation $\mathrm{tr}\ \mathcal{L}(\varepsilon^2,\mu) = 0$. Now observe that

$$(dg)_{0,0,\mu,0} = (df)_{0,0,\mu} - i\omega_0(\mu)I = 0$$

by (4.2c) and (H4). Thus $\mathcal{L}(0,\mu) \equiv 0$ and Taylor's theorem allows us to write

$$\mathcal{L}(\varepsilon^2,\mu) = \varepsilon^2 L(\varepsilon^2,\mu). \tag{4.12}$$

In order to guarantee the existence of solutions to $\mathrm{tr}(\mathcal{L}) = 0$, we assume

(H8) $\mathrm{tr}\ L(0,0) = 0$,

(H9) $\Delta_9 \equiv \dfrac{\partial}{\partial \varepsilon^2} (\mathrm{tr}\ L)(0,0) \neq 0.$

Hypotheses (H8) and (H9) along with the Implicit Function Theorem guarantee a unique solution to

$$\mathrm{tr}\ L(\varepsilon^2,\mu) = 0 \tag{4.13}$$

given by

$$\varepsilon^2 = E(\mu). \tag{4.14}$$

Now solutions to (4.13) can only exist when $E(\mu) \geq 0$. Thus, to insure the existence of such solutions, we assume

(H10) $\Delta_{10} \equiv \dfrac{\partial}{\partial \mu} (\mathrm{tr}\ L)(0,0) \neq 0.$

Implicit differentiation of (4.13) shows that

$$\frac{dE}{d\mu}(0) = -\Delta_{10}/\Delta_9. \tag{4.15}$$

Thus, a unique solution to (4.13) exists when μ is near 0 and

$$\operatorname{sgn}(\mu) = -\operatorname{sgn}(\Delta_9)\ \operatorname{sgn}(\Delta_{10}). \tag{4.16}$$

Finally, in order for L to have purely imaginary eigenvalues, we must assume

(H11) $\det\ L(E(\mu),\mu) > 0$

for all μ near 0 satisfying (4.16). Hypothesis (H11) will be the subject of further discussion below.

These eleven hypotheses describe the simplest situation where a secondary torus bifurcation of the type discussed at the beginning of this section might exist. In particular, these hypotheses guarantee the existence of a Hopf bifurcation along a nontrivial branch of the steady-state equation, (4.5). However, as we shall now discuss, the existence of $\Gamma \times S^1$-symmetry in (4.1) insures that the standard Hopf bifurcation theorem does not apply since the S^1-symmetry of normal form forces one eigenvalue of $dg|\mathrm{Fix}(\Sigma)$ to be zero. Observe that the group theoretic argument which guarantees that dg has one zero eigenvalue also implies that dg has $\dim \Gamma + 1 - \dim \Sigma$ zero eigenvalues.

Next we make the observation that group-theoretically there are two types of torus bifurcation. Consider the action of Σ on V_2. The assumption that $L = dg|V_2$ can have purely imaginary eigenvalues, coupled with the fact that L commutes with Σ, places restrictions on the action of Σ.

Let $K(\Sigma)$ be the kernel of the action of Σ on V_2 and let $T(\Sigma) = \Sigma/K(\Sigma)$. As observed in Golubitsky and Stewart [1985] either

(a) $V_2 = \mathbb{R} \oplus \mathbb{R}$ where $T(\Sigma)$ acts absolutely irreducibly on $\mathbb{R}$, or

(b) $T(\Sigma)$ acts irreducibly, but not absolutely irreducibly, on V_2.

Moreover, since $V_2 \cap \mathrm{Fix}(\Sigma) = \{0\}$ we know that $T(\Sigma)$ acts nontrivially on V_2. Hence in case (b), $T(\Sigma) \cong \mathbb{Z}_q$ for some $q \geq 3$, since these are the only finite groups which act faithfully, irreducibly, and not absolutely irreducibly on $\mathbb{R}^2$, and in case (a) $T(\Sigma) \cong \mathbb{Z}_2$.

<u>Definition 4.1</u>: When $T(\Sigma) \cong \mathbb{Z}_2$ we call the tori resulting from the secondary torus bifurcation *standing tori* and when $T(\Sigma) \cong \mathbb{Z}_q$ $(q \geq 3)$ we call these *rotating tori*.

Observe that (H11) simplifies in the rotating tori case. There $L(\varepsilon^2,\mu)$ commutes with $\mathbb{Z}_q$ and is hence a multiple of a rotation. Thus group theoretic restrictions force $\det L(\varepsilon^2,\mu) \geq 0$ and (H11) simplifies to

$(H11)_r$ $\Delta^r_{11} \equiv L(0,0) \neq 0.$

The situation for standing tori is more complicated, as group restrictions may force $L(0,\mu) = 0$. In this case we find in examples that

$$L(\varepsilon^2,\mu) = \varepsilon^{2m} D(\varepsilon^2,\mu) \tag{4.17}$$

where m depends on Γ and Σ but not, in general, on the particular f in (4.1). For standing tori we replace (H11) with

$(H11)_s$ $\Delta^s_{11} \equiv \det D(0,0) > 0$

Note that (4.17) is valid for rotating waves; there $D = L$ and $m = 1$.

We end this subsection by explicitly constructing the action of Σ on V_2. Observe that each V_j is invariant under S^1, since S^1 commutes with the full group $\Gamma \times S^1$. Since $\dim V_2 = 2$ by (H7), it follows that we may identify V_2 with $\mathbb{C}$ and the action of Σ on V_2 with a subgroup of S^1 acting on $\mathbb{C}$. Thus, the action of $T(\Sigma)$ on $\mathbb{C}$ is generated by

$$z \to e^{2\pi i/q} z$$

for q as defined above. (Note that standing tori correspond to $q = 2$.)

4.3 Hopf bifurcation with zero eigenvalues

In this subsection we use results of Krupa [1988] to prove a torus bifurcation theorem for vector fields f satisfying (H1)-(H11). Recall that f is assumed to be $\Gamma \times S^1$-equivariant and to have a periodic solution $z(t) = \varepsilon e^{i\omega t} p$.

We now concentrate on determining the form of f on a neighborhood of the group orbit $X = (\Gamma \times S^1)\cdot p$. The existence of the periodic solution implies that f is tangent to X along X. We utilize two results from Krupa [1988]. Let $N(X)$ denote the $\Gamma \times S^1$-equivariant normal bundle of $X \subset \mathbb{C}^n$, let N_x denote the fiber over x and let $\pi : N(X) \to X$ be the projection.

<u>Theorem 4.2</u> There exist $\Gamma \times S^1$-equivariant vector fields f_T and f_N such that

$$f = f_T + f_N \tag{4.18}$$

where $f_T(y)$ is in the tangent space to the group orbit of $\Gamma \times S^1$ through y and $f(y) \in N_{\pi(y)}$.

<u>Theorem 4.3</u>: Let $x(t)$ be the trajectory in $\dot{x} = f(x)$ with $x(0) = y$ and let $z(t)$ be the trajectory in $\dot{z} = f_N(z)$ with $z(0) = y$. Then

$$x(t) = \delta(t)z(t) \tag{4.19}$$

for some smooth curve $\delta(t) \in \Gamma \times S^1$ with $\delta(0) =$ identity.

It follows from Theorems 4.2 and 4.3 that a Hopf bifurcation to a periodic trajectory $z(t)$ for f_N leads to a trajectory for f on the union of group orbits through $z(t)$.

<u>Remarks 4.4</u>: (a) When Γ is finite, all group orbits are circles and the flow is on a 2-torus. The S^1-action forces the flow to be conjugate to linear. To lowest order, this flow has the form $\varepsilon e^{i\omega t}z(t)$. In perturbation theory language the flow will have the form

$$\varepsilon e^{i(\omega+\theta)t}(p + h(t)) \tag{4.20}$$

where $\theta \in \mathbb{R}$ and $h(t) \in N_p$ are small. Moreover, $h(t)$ is a solution of

$$h'(t) = f_N(p + h(t),\lambda,\mu). \tag{4.21}$$

To verify (4.21) let (4.20) be a solution to (4.1) and use the decomposition (4.18).

(b) Observe that

$$g|N_p = f|N_p \tag{4.22}$$

where g is defined in (4.8). This follows since

$$g = f - i\omega z = f_N + f_T - i\omega z,$$

and f_T and $i\omega z$ vanish when restricted to N_p.

<u>Theorem 4.5</u>: Assume (H1)-(H11). Then for fixed μ satisfying (4.15), there is a branch of periodic solutions to (4.1) parametrized by λ which undergoes a torus bifurcation at $\lambda = \lambda^*$ as in (4.6b). When Γ is finite, a unique branch of two-frequency trajectories bifurcate from the branch of periodic solutions at $\lambda = \lambda^*$.

<u>Proof</u>: Under our hypotheses the complex conjugate pair of eigenvalues of L that cross through 0 as λ is varied, do so with nonzero speed. In fact, when μ is fixed,

$$\frac{\partial}{\partial\lambda}(\mathrm{tr}\ L) = \frac{\partial}{\partial\varepsilon^2}(\mathrm{tr}\ L)\ \ \frac{\partial\varepsilon^2}{\partial\lambda}\ . \tag{4.23}$$

Both of these factors are nonzero, the first by (H9) and the second by (H3), (H6) and (4.7). It remains only to show that under the hypotheses above, the normal vector field $G = f_N|N_p$ undergoes a Hopf bifurcation at $\lambda = \lambda^*$. However, it is easy to show that the eigenvalues of dG at εp are just the Floquet exponents of f at the periodic solution $\varepsilon e^{i\omega t}p$ minus $i\omega$. Thus the eigenvalues of dG at εp are the eigenvalues of dg in the direction N_p (where g is defined in (4.8)). Hence our hypotheses imply that f_N undergoes a (simple) Hopf bifurcation at $\lambda = \lambda^*$ and that the corresponding complex eigenvalues cross the imaginary axis with nonzero speed. The standard Hopf Theorem coupled with Remark (4.4a) now applies. □

4.4 Direction and Stability of the Branch of Invariant 2-tori

Next we consider the direction of branching of the branch of 2-tori by determining the direction of branching of the branch of periodic solutions in the Hopf bifurcation of $G = f_N|N_p$. Theorem 4.3 implies that asymptotic stability of the periodic solutions of G in N_p implies asymptotic stability of the invariant 2-tori in (4.1).

We review the relevant discussion from the earlier subsections. Let $y = \varepsilon(p + h)$. There is a branch of equilibria of f_N at

$$h = 0 \quad \text{and} \quad \lambda = \lambda^*(\varepsilon^2,\mu) \tag{4.24}$$

where μ^* is defined by (4.16). A Hopf bifurcation for f_N occurs along this branch of equilibria at

$$\varepsilon^2 = E(\mu) \tag{4.25}$$

as defined in (4.14), since $V_2 \subset N_p$. Note that $E(0) = 0$. In fact, $L(\varepsilon^2,\mu)$ is just $df_N|V_2$.

We assume that μ has the correct sign so that (4.16) is valid, and hence (4.25) has a solution for ε^2 when $E(\mu)$ is positive. We have assumed, moreover, that as λ varies through λ^*, the imaginary eigenvalues of L cross the imaginary axis with nonzero speed, as noted in the proof of Theorem 4.5.

The standard Hopf theorem asserts that there exists a single coefficient μ_2, depending on terms of f_N through cubic order, that determines the asymptotic stability of the periodic solutions (and their direction of branching). Moreover, μ_2 is defined at the point of Hopf bifurcation given by (4.24) and (4.25). Thus

$$\mu_2 = \mu_2(\mu) \tag{4.26}$$

<u>Theorem 4.5</u>: Under the assumptions just described

$$\mu_2(\mu) = \mu^k M(\mu)$$

k for some integer k. A formula for $M(0)$ can be determined, in principal, from the Taylor expansion of f at the origin and the sign of $M(0)$ determines the asymptotic stability and direction of branching of the branch of 2-tori for (4.1). In particular, this branch is supercritical and consists of asymptotically stable 2-tori when $M(0) > 0$ and the branch is subcritical when $M(0) < 0$.

Our final (genericity) assumption is:

(H12) $M(0) \neq 0$.

The most difficult part of any calculation of invariant 2-tori is determining $M(0)$, that is, determining the direction of branching and stability of the 2-tori. In principle, it might be possible to derive a general formula for $M(0)$ when $k = 1$ using only terms in the Taylor expansion of f. In our D_3 example in the next section we have chosen the computationally simpler route of just computing the secondary Hopf bifurcation on f_N directly. One reason is that we find that $k = 1$ is valid for the standing tori and $k = 2$ is valid for the rotating tori. At this stage we do not understand why certain isotropy subgroups force k to be greater than 1.

§5 Torus Bifurcations with Triangular Symmetry

In this section we apply our torus bifurcation theorem to a vector field

$$\frac{dz}{dt} = g(z,\lambda,\mu) \tag{5.1}$$

in $D_n \times S^1$-normal form, that is, we assume g has the form (3.6). As we discussed in Section 3, see Table 3.1, generically there exist three primary branches of periodic solutions to (5.1) corresponding to two-dimensional fixed point subspaces. We need to determine when hypotheses (H1-H12) are valid for each of these branches. Note that hypotheses (H1)-(H3), (H5) and (H7) are automatically valid in these cases.

In (3.6) the invariant functions A-D are complex-valued and we denote these functions by

$$\begin{aligned} &\text{(a)} \quad A = a + i\alpha \\ &\text{(b)} \quad B = b + i\beta \\ &\text{(c)} \quad C = c + i\gamma \\ &\text{(d)} \quad D = d + i\delta \end{aligned} \tag{5.2}$$

Hypothesis (H4) states that a Hopf bifurcation occurs at $\lambda = \mu = 0$ and that the complex eigenvalue crosses the axis with nonzero speed. We simplify our analysis here by assuming

$$a = \lambda + \{\text{a function depending on z-alone}\} \tag{5.3}$$

and that $\alpha(0) \equiv \alpha_0 > 0$. Thus, we assume that the trivial steady state $z = 0$ is asymptotically stable when $\lambda < 0$ and loses stability at $\lambda = 0$. Moreover, (H4) is valid as $\Delta_3 = 1 \neq 0$.

Σ	<u>Rotating Wave</u>	<u>Standing Wave 1</u>	<u>Standing Wave 2</u>
$\mathrm{Fix}(\Sigma)$	$(z_1,0)$	(z_1,z_1)	$(z_1,-z_1)$
$\mathrm{sgn}(\Delta_6)$	$-(a_N+b)$	$-(2a_N+b)$	$-(2a_N+b)$
V_2	$(0,z_2)$	$(z_2,-z_2)$	(z_2,z_2)
$\mathrm{tr}\ L=0$	$b=0$	$b=0$	$b=0$
$\mathrm{sgn}(\Delta_9)$	$-b_N$	b_N-2c	b_N+2c
Δ_{10}	$-b_\mu$	b_μ	b_μ
(H11)	$\beta\neq0$	$\beta\gamma<0$	$\beta\gamma>0$

<u>Table 5.1</u>: Data needed to find torus bifurcations along primary branches in degenerate $D_3\times S^1$-equivariant Hopf bifurcation. All functions are evaluated at the origin.

In our analysis, we begin by assuming that $n\neq4$. The case of square symmetry ($n=4$) is more complicated (see Golubitsky & Stewart [1986] and Swift [1988]). In particular, when $n\neq4$ the standing waves are either both supercritical or both subcritical, and there are no branches of periodic solutions corresponding to submaximal isotropy. The criticality of these branches is determined by $\mathrm{sgn}(\Delta_6)$, as noted in Table 5.1. Assuming $b\neq-a_N,\ -2a_N$ validates (H6).

The degeneracy condition needed to have a torus bifurcation, hypothesis (H8), is $\mathrm{tr}\ L=0$. For each of the branches, (H8) corresponds to $b=0$ at the origin. This could have been seen directly from the stability results in Golubitsky & Stewart [1986] since the coefficient $b(0)$ being nonzero was needed to determine which of these branches of periodic solutions could be asymptotically stable. Assuming $b(0)=0$ implies that standing waves and rotating waves must be all supercritical or subcritical, depending on the sign of $a_N(0)$, which is assumed to be nonzero.

We again simplify our analysis by assuming that

$$b(z,\mu) = \mu + \{\text{a function depending only on } z\}. \tag{5.4}$$

It then follows that $\Delta_{10} = -1$ for rotating waves and $+1$ for standing waves. Thus (H10) is valid and we will have a torus bifurcation if the complex eigenvalues in the V_2-directions are nonzero since (H10) implies that these eigenvalues will cross the imaginary axis with nonzero speed. That these eigenvalues will be nonzero and purely imaginary at the point of secondary bifurcation is governed by (H11). Thus we assume $\beta(0)\gamma(0) \neq 0$, as indicated in Table 5.1.

In our discussion in Section 4 we also assumed that we could solve uniquely for the point of the secondary bifurcation, as a function of μ, which follows from $\Delta_9 \neq 0$. So we assume $b_N(0) \neq 0, \pm 2c(0)$ and (H9) is verified. (Note that when $n \geq 5$, this condition would be $b_N(0) \neq 0$.)

We summarize our discussion by listing all conditions in Table 5.2

(a)	$a(0) = 0,\ \alpha(0) > 0$	primary Hopf bifurcation
(b)	$a_N(0) > 0$	periodic solutions subcritical
(c)	$b(0) = 0$	possibility of secondary torus bifurcation
(d)	$\beta(0) \neq 0$	torus bifurcation on rotating waves
(e)	$\beta(0)\gamma(0) < 0 \quad (\ >0\ )$	torus bifurcation on standing wave 1 (standing wave 2)
(f)	$b_N(0) \neq 0,\ \pm 2c(0)$	See (4.16): unique torus bifurcation along rotating wave when $\mathrm{sgn}(\mu) = -\mathrm{sgn}(b_N(0))$ standing wave 1 when $\mathrm{sgn}(\mu) = -\mathrm{sgn}(b_N(0)-2c(0))$ standing wave 2 when $\mathrm{sgn}(\mu) = -\mathrm{sgn}(b_N(0)+2c(0))$

Table 5.2: Conditions for Torus Bifurcation with D_3-symmetry

The final issue we must address is the direction of branching of the secondary branch of 2-tori, hypothesis (H12). At the end of Section 4, we discussed the difficulty of deriving a formula for the direction of branching of the torus branches. Because of this fact, we compute, in Subsections 5.1 and 5.2, the direction of branching of the secondary 2-tori bifurcation along the rotating and standing waves branches only in the case of D_3-symmetry. Let $\beta_0 \equiv \beta(0)$ and $\gamma_0 \equiv \gamma(0)$. We prove the following:

Theorem 5.1: In degenerate D_3-equivariant Hopf bifurcation, the direction of branching of the branch of rotating 2-tori is supercritical if

$$-2a_N \left[b_N + \frac{\gamma_0}{3\beta_0}\right] \tag{5.5}$$

is positive and subcritical if negative.

Theorem 5.2: In degenerate D_3-equivariant Hopf bifurcation, the direction of branching of the branch of standing 2-tori off of the branch of periodic standing waves 1 is supercritical if

$$-\frac{\beta_0}{3\gamma_0}\left[\operatorname{sgn}(\gamma_0)\frac{\beta_0^{2}}{2a_N\sqrt{6|\beta_0\gamma_0|}} + \frac{4a_N}{3}\right]\frac{a_N}{2c_0-b_N} \tag{5.6}$$

is positive and subcritical if negative. For secondary bifurcation off of standing waves 2 we replace (5.6) by:

$$-\frac{\beta_0}{3\gamma_0}\left[\operatorname{sgn}(\gamma_0)\frac{\beta_0^{2}}{2a_N\sqrt{6|\beta_0\gamma_0|}} - \frac{4a_N}{3}\right]\frac{a_N}{2c_0-b_N} \tag{5.7}$$

It is possible to derive the direction of branching of standing torus 2 from that of standing torus 1 using the following observation first noted in

Swift [1988], using the terminology "parameter" symmetry.

We call the mapping

$$Q(z_1,z_2) = (z_1,-z_2) \tag{5.8}$$

a *quasisymmetry* since it is in the normalizer of Γ in $O(4)$ but is not in $\Gamma = D_3 \times S^1$. It follows that the map

$$h = Qg(Qz) \tag{5.9}$$

is Γ-equivariant whenever g is Γ-equivariant. In this particular case, the quasisymmetry Q interchanges standing waves 1 and 2. Since h is vector field equivalent to g, the dynamics of h is the same as that of g. Thus, computing the direction of branching of standing torus 1 for h, gives the direction of branching of standing torus 2 for g.

It remains only to note using (3.4) and (3.6) that when $n = 3$, g transforms to h in (5.9) as follows:

$$\begin{aligned} &\text{(a)} \quad (N,P,S,T) \to (N,P,-S,-T) \\ &\text{(b)} \quad (A,B,C,D) \to (A,B,-C,-D) \end{aligned} \tag{5.10}$$

Thus (5.7) may be derived from (5.6) by transforming γ_0 to $-\gamma_0$.

5.1 Rotating Tori for D_3-Symmetry

We let Σ be the isotropy subgroup of rotating waves

$$\tilde{Z}_3 = \{(\gamma,-\gamma)\}: \gamma \in Z_3\}$$

with the two-dimensional fixed point space $\{(z_1,0)\}$. The branching equation for the periodic solutions lying in $Fix(\Sigma)$ is given by

$$A + Bu^2 = 0 \qquad u \in R. \tag{5.11}$$

From our discussion above concerning the torus bifurcation, we may assume that the system of ODE (5.1) has the form:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \{i\alpha_0 + \lambda + a_N N + a_{NN}N^2/2 + a_P P\}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
$$+ \{\mu + b_N N + i\beta_0\}\begin{bmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{bmatrix} + \{c_0 + i\gamma_0\}\begin{bmatrix} \bar{z}_1{}^2 z_2^3 \\ z_1^3\bar{z}_2{}^2 \end{bmatrix} + h.o.t \tag{5.12}$$

where $N = |z_1|^2 + |z_2|^2$ and $P = |z_1|^2|z_2|^2$. The branch of periodic solutions to (5.1) within $\tilde{Z}_3$ is given by:

$$z_\varepsilon(t) = \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\omega^* t}, \tag{5.13}$$

where $\omega = \omega^*(\varepsilon^2,u) = \alpha_0 + \beta_0\varepsilon^2 + O(\varepsilon^4)$. Along the branch of periodic solutions, the eigenvector due to translation, corresponding to the eigenvalue zero, is constant. In fact (5.12) shows that this eigenvector equals ip where $p = (1,0)^T$. We exploit this fact to explicitly reduce the vector field to the normal section. We let

$$V_p = \{z \in \mathbb{C}^2: Re\langle z, ip\rangle = 0\}. \tag{5.14}$$

By the implicit function theorem we can solve locally in (5.15a) for $\theta = \theta^*(h, \varepsilon, \mu)$ such that

$$(a) \quad \mathrm{Re}\langle f(\varepsilon(p + h), \lambda^*, \mu) - i(\omega^* + \theta)\varepsilon(p + h), ip\rangle = 0, \text{ and} \tag{5.15}$$

$$(b) \quad \theta^*(0,0,0) = 0$$

If we now let

$$g(h,\varepsilon,\mu,\theta) = f(\varepsilon(p + h), \lambda^*(\varepsilon,\mu),\mu) - i(\omega^*(\varepsilon,\mu) + \theta^*)\varepsilon(p + h) \tag{5.16}$$

then the subspace V_p is invariant under g and $g|V_p$ is the normal component of the vector field g. For the details of this reduction see Vanderbauwhede, Krupa & Golubitsky [1988] and Krupa [1988].

We note that θ^* corresponds to an element η of the Lie algebra $L(\Gamma)$ and determines the drift along the group orbits.

In the language of asymptotic expansions we are looking for solutions of the form:

$$\varepsilon e^{i(\omega^*+\varepsilon^3\theta)t}(p + h(\varepsilon^2\beta t)),$$

where h is a 2π-periodic function.

In (z_1,z_2) coordinates we write $h = (h_1,h_2)$ and $h_1 = x + iy$. The normal vectorfield then is given by

$$\frac{dx}{dt} = \varepsilon^2[2(a_N + \mu)x + a_N|h_2|^2 + O(|x|^2 + |x||h_2|^2 + |h_2|^4 + \varepsilon^2|x|)] \tag{5.17}$$

$$\frac{dh_2}{dt} = \varepsilon^2[-(\mu + i\beta_0 + i\frac{\theta^*}{\varepsilon^2})h_2 + 2a_N x h_2 + a_N h_2|h_2|^2$$

$$+ O(\varepsilon^2 + |\mu||h_2|^3 + |h_2|^5)]$$

We remark that θ^* is of order ε^2 and occurs in the second equation in such a way that its value will only influence the period of the bifurcating periodic solution, and not its stability. Therefore, we may suppress θ^*.

More importantly, we observe that the direction of branching is determined by the higher order terms. To see this, we rescale the time by letting $\bar{t} = -\varepsilon^2 \beta_0 t$, and eliminate the ε^2 which factors $L(\varepsilon,\mu)$ (see (4.11)). We obtain:

$$
\begin{aligned}
-\beta_0 \frac{dx}{dt} &= 2(a_N + \mu)x + a_N|h_2|^2 + \text{h.o.t.} \\
-\beta_0 \frac{dh_2}{dt} &= -(\mu + i\beta_0)h_2 + (2a_N x + (a_N + i\beta_0)|h_2|^2)h_2 + \text{h.o.t.}
\end{aligned}
\tag{5.18}
$$

From the first equation we see that $x = -|h_2|^2/2 + \dots$, and inserting this in the second equation, we conclude that the direction of branching is not determined. To the next order in ε^2 we get (keeping the rescaled time)

$$
\begin{aligned}
-\beta_0 \frac{dx}{dt} &= 2(a_N + \mu)x + a_N|h_2|^2 + \varepsilon^2(2(a_{NN} + 2b_N)x + (a_{NN} + b_N)|h_2|^2) \\
&\quad + O(\varepsilon^2(|x| + |h_2|^2)^2 + \varepsilon^4) \\
-\beta_0 \frac{dh_2}{dt} &= -(\mu + i\beta_0 + \varepsilon^2 b_N)h_2 + (a_N + i\beta_0)|h_2|^2 h_2 + 2a_N x h_2 \\
&\quad + \varepsilon^2(2a_{NN} x h_2 + (c_0 + i\gamma_0)\bar{h}_2^{\,2} + (a_{NN} + b_N)|h_2|^2 h_2) \\
&\quad + O(\varepsilon^2|\mu||h_2|^3 + \varepsilon^2(|x| + |h_2|^2)^2 + \varepsilon^4)
\end{aligned}
\tag{5.19}
$$

From (5.17) we derive that at a periodic solution

$$
x = \frac{-a_N + \varepsilon^2 b_N}{2a_N}|h_2|^2 + O(\varepsilon^2|h_2|^4 + \varepsilon^4).
$$

Then we find the direction of branching from (5.18)

$$
\mu = \varepsilon^2\{-b_N + 2|h_2|^2\left[\frac{\gamma_0}{3\beta_0} + b_N\right] + O(\varepsilon^2 + |h_2|^4)\}. \tag{5.19}
$$

From this equation Theorem 5.1 follows easily.

5.2 Standing Tori for D_3-Symmetry

We let Σ be the isotropy subgroup $Z_2 \oplus Z_2^c$ of Standing Waves I with two-dimensional fixed point space $\{(z_1,z_1)\}$. In order to construct V_p and θ^* as in the rotating case we first change coordinates in (5.12): $u = (z_1+z_2)/\sqrt{2}$, $v = (z_1-z_2)/\sqrt{2}$.

The construction of the reduced vector field is mutatis mutandis the same as in the case of the rotating tori. Again we will suppress θ^*, because we are only interested in the direction of branching. The effect of the change of coordinates is that the primary branch has the same form:

$$z_\varepsilon(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\omega^* t},$$

where in the standing wave case

$$\omega^*(\varepsilon^2,\mu) = \alpha_0 + \frac{1}{2}\beta_0\varepsilon^2 + O(\varepsilon^4).$$

It occurs at

$$\lambda^*(\varepsilon^2,\mu) = -(a_N + \frac{\mu}{2})\varepsilon^2 - (a_{NN} + \frac{a_P}{2} + b_N + \frac{c_0}{2})\varepsilon^4/2.$$

The reduced vector field has the form (compare with (5.12))

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon^2 f(x,\ h_2,\ \mu,\ \varepsilon^2) \\ \frac{dh_2}{dt} &= \varepsilon^2 g(x,\ h_2,\ \mu,\ \varepsilon^2) \end{aligned} \tag{5.20}$$

The linear part of g depends only on h_2. It is given by the mapping from $\mathbb{C}^2$ into itself by:

$$v \to \left[\frac{(\mu+i\beta_0)}{2} + (\frac{b_N}{2} - c_0 - i\gamma_0)\varepsilon^2\right]v + \left[\frac{(\mu-i\beta_0)}{2} + \frac{(b_N-c_0+i\gamma_0)}{2}\varepsilon^2\right]\bar{v} + O(\varepsilon^4)$$

If we consider this mapping as a mapping on $\mathbb{R}^2$, then its trace equals

$$\mu + (b_N - 2c_0)\varepsilon^2 + O(\varepsilon^4)$$

and its determinant equals

$$-\frac{3}{2}(\mu c_0 - \gamma_0\beta_0)\varepsilon^2 + O(\varepsilon)^4.$$

We apply an ε-dependent transformation to put the linear part of the vector field into normal form. Writing $h_2 = u + iv$ the transformation is:

$$u \to \varepsilon u$$

$$v \to \sqrt{-2\beta_0/3\gamma_0}\; v .$$

We rescale the time by ε, i.e. $\varepsilon t = \bar{t}$. After these transformations the linear part of the vector field has the form:

$$\frac{dx}{dt} = \varepsilon(2a_N + \mu)x$$

$$\frac{dh_2}{dt} = \varepsilon\,\frac{\mu}{2}\,(h_2 + \bar{h}_2) + i\varepsilon^2\tau_0 h_2,$$

where $\tau_0 = \sqrt{-3\beta_0\gamma_0/2}$. We rescale the time once more, $\varepsilon^2\tau_0 t = \bar{t}$. Then, the linear vector field has a circle of periodic solutions:

$$\begin{bmatrix} x \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \eta e^{i(s+\phi)} \end{bmatrix} \qquad \phi \in S^1, \eta \in \mathbb{R}^+.$$

We then put the full nonlinear vector field up to order three in (x, h_2) into Birkhoff normal form. Phrased differently, after a near identity transformation in the variables (x, h_2) and a truncation at order three, the resulting vector field has the form:

$$\varepsilon \frac{dx}{dt} = \hat{f}(x, |h_2|^2, \mu, \varepsilon)$$

$$\varepsilon^2 \frac{dh_2}{dt} = \tilde{g}(x, |h_2|^2, \mu, \varepsilon)\, h_2.$$

If we can solve the equation $\hat{f} = \tilde{g} = 0$ then we will know the direction of branching. Straightforward (but rather lengthy) computations show that

$$\tilde{f} = 2a_N\xi + \frac{1}{3\gamma_0} a_N|h_2|^2\beta_0 + O[(\varepsilon^2+|\mu|)(|x|+|h_2|) + |x||h_2| + |h_2|^3 + |x|^2]$$

where

$$\xi = x + \frac{\varepsilon i}{6\gamma}(h_2^2 - \bar{h}_2^2)\beta_0$$

Therefore, at a solution of $\tilde{f} = 0$ we will have that

$$x = -\frac{\beta_0}{6\gamma_0}\left[\,|h_2|^2 - \frac{\varepsilon i}{2a_N}(h_2^2 - \bar{h}_2^2)\right] + O(\varepsilon^2 |h_2| + |h_2|^3).$$

Substiting into the equations $\tilde{g} = 0$ then yields

$$\mu = (2c_0 - b_N)\varepsilon^2 - |h_2|^2\left(\frac{4\beta_0 a_N}{3\gamma_0} + \frac{\beta_0^3}{18\kappa a_N\gamma_0^2}\right) + O(|\varepsilon| + |h_2|)^3$$

where $\kappa = (-2\beta_0/3\gamma_0)^{1/2}$. From this equation Theorem 5.2 follows.

The (above mentioned) lengthy calculations were checked with the formula manipulation program REDUCE.

§6. The Bifurcation Diagrams

The results of Section 5 imply that when deriving the bifurcation diagrams for the torus bifurcation in degenerate Hopf bifurcation with D_3-symmetry, we may assume

$$\begin{aligned}
&\text{(a)} \quad A = \lambda + a_N N + \alpha_0 i \\
&\text{(b)} \quad B = \mu + b_N N + \beta_0 i \qquad\qquad (6.1)\\
&\text{(c)} \quad C = c_0 + \gamma_0 i \\
&\text{(d)} \quad D = 0,
\end{aligned}$$

as only these terms enter the determination of direction of branching and stability.

We assume:

$$\alpha_0 > 0, \ \beta_0 > 0, \ \gamma_0 < 0 \text{ and } a_N < 0. \qquad (6.2)$$

We make these choices for the following reasons. First, without loss of generality, the frequencies α_0 and β_0 may be assumed to be positive. Second, the quasisymmetry Q transforms γ_0 to $-\gamma_0$ (and c_0 to $-c_0$); so we may assume that γ_0 is negative (at the expense of interchanging the two branches of standing waves). Finally, we are interested mainly in those situations where asymptotically stable states may exist. Indeed, stability can occur only when the primary branches are supercritical; hence we assume $a_N < 0$.

To simplify subsequent calculations rescale time and space to obtain

$$\beta_0^2 = 1/2 \ \text{ and } \ \gamma_0 = -3\beta_0. \qquad (6.3)$$

We now find that when a torus bifurcation occurs, it occurs at

$$(a) \quad \lambda_r = \frac{a_N}{b_N}\mu + \dots \qquad \text{(rotating torus)}$$

$$(b) \quad \lambda_s = \frac{a_N}{b_N - 2c_0}\mu + \dots \qquad \text{(standing torus)} \tag{6.4}$$

and the direction of branching is determined by:

$$(a) \qquad 1 - \frac{1}{b_N} \qquad \text{(rotating torus)}$$

$$(b) \qquad \frac{1}{b_N - 2c_0}\left[4a_N - \frac{1}{4a_N}\right] \qquad \text{(standing torus)} \tag{6.5}$$

where supercritical is positive and subcritical is negative.

We note that it is possible to choose coefficients independently so that for a fixed μ:

(i) standing wave 1 is stable at the initial bifurcation (choose $\mu < 0$),

(ii) both a rotating and a standing torus bifurcate (choose $b_N > 0$ and $b_N > 2c_0$),

(iii) either torus may bifurcate first as λ increases (choose $c_0 > 0$ to have standing wave first, $c_0 < 0$ for rotating wave),

(iv) the rotating torus is either supercritical or subcritical (choose $b_N > 1$ or $b_N < 1$),

(v) the standing torus is either supercritical or subcritical (choose $a_N < 0.25$ or $0 > a_N > -0.25$).

Therefore, it is possible to choose parameters so that the bifurcation diagram pictured in Figure 6.1 occurs. Here we find the possibility of two stable 2-tori and no stable periodic solutions. Note that this phenomenon may not occur in codimension two for D_n-Hopf bifurcation when $n \geq 5$, since c_0 is then a higher order term.

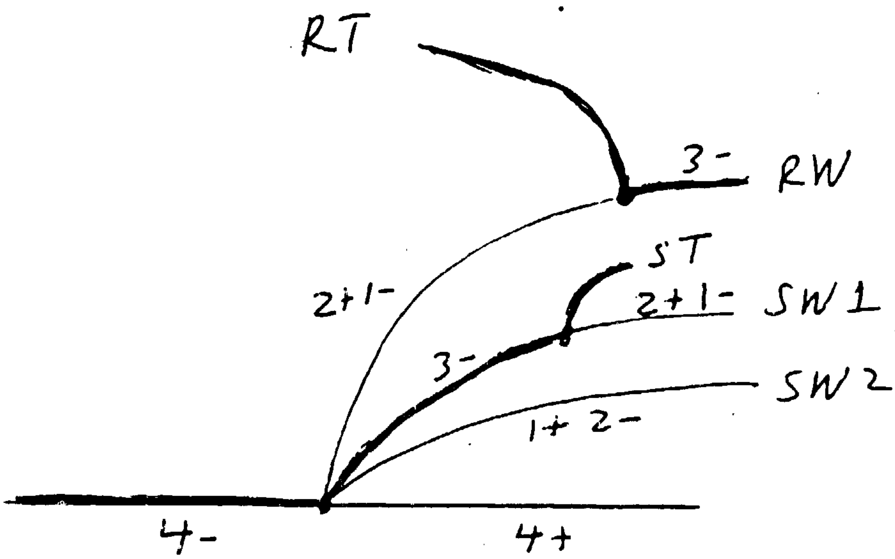

Figure 6.1: Bifurcation diagram indicating values of λ having two branches of asymptotically stable 2-tori and no stable steady-states or periodic solutions.

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