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VERY RESTRICTED FOUR-BODY PROBLEM

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VERY RESTRICTED FOUR-BODY PROBLEM

by

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SUMMARY

First, a state of motion of three finite bodies m_1, m_2, m_3 is idealized by an approximation to the law of mechanics such that m_2 and m_3 revolve around each other in circular orbits and that their center of mass revolves around m_1 also in a circular orbit. The motion of a fourth body of an infinitesimal mass is then studied in a similar manner, as in the restricted three-body problem.

CONTENTS

VERY RESTRICTED FOUR-BODY PROBLEM

INTRODUCTION

In another paper* the general behavior of an artificial satellite in the earth-moon-sun system was studied in terms of two three-body problems. In the present paper some justification will be provided for that approach by treating dynamically an idealized case of motion of an infinitesimal body of mass m in a system of three bodies m_1, m_2, and m_3 so arranged that the center of mass, $0'$, of m_2 and m_3 is revolving around the center of mass, 0, of the entire system in a circular orbit and m_2 and m_3 themselves are revolving around $0'$ also in circular orbits. Such a state of motion of the three bodies is obviously possible only in the form of approximation. However, if

$$m_1 >> m_2 + m_3 >> m$$

and if the separation A between m_1 and $0'$ is very much greater than that separation a between m_2 and m_3, both of these two conditions being true in the case of an artificial satellite in the earth-moon-sun (m_2-m_3-m_1) system, the approximation will deviate from the actual solution of mechanics very little.

AN INTEGRAL OF THE EQUATION OF MOTION FOR THE FOURTH BODY

Further assume that the three bodies m_1, m_2, m_3 always remain in the same plane and let the distances of m_1 and $0'$ from 0 be A_1 and A_2, and those of m_2 and m_3 from $0'$ be a_1 and a_2. Now choose a rectangular coordinate system with its origin at 0 and its three axes ξ, η, ζ fixed in space, the ζ-axis being perpendicular to the plane of the three finite bodies. Hence the coordinates of the four bodies may be written as $m_1(\xi_1, \eta_1, 0)$, $m_2(\xi_1, \eta_2, 0)$, $m_3(\xi_3, \eta_3, 0)$, and $m(\xi, \eta, \zeta)$. The equations of motion of the infinitesimal body m are given by

$$\frac{d^2\xi}{dt^2} = -Gm_1 \frac{\xi - \xi_1}{r_1^3} - Gm_2 \frac{\xi - \xi_2}{r_2^3} - Gm_3 \frac{\xi - \xi_3}{r_3^3} , \tag{1}$$

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*Huang, S.-S., "Some Dynamical Properties of the Natural and Artificial Satellites," NASA Technical Note D-502.

$$\frac{d^2\eta}{dt^2} = -Gm_1 \frac{\eta - \eta_1}{r_1^3} - Gm_2 \frac{\eta - \eta_2}{r_2^3} - Gm_3 \frac{\eta - \eta_3}{r_3^3} , \tag{2}$$

$$\frac{d^2\zeta}{dt^2} = -Gm_1 \frac{\zeta}{r_1^3} - Gm_2 \frac{\zeta}{r_2^3} - Gm_3 \frac{\zeta}{r_3^3} , \tag{3}$$

where r_1, r_2, r_3 represent the distances of the infinitesimal body m from m_1, m_2, and m_3, respectively.

Let Ω_1 be the angular velocity with which the m_2-m_3 system revolves around 0, and Ω_2 that with which m_3 revolves around $0'$. It can be easily seen that

$$\xi_1 = -A_1 \cos \Omega_1 t , \qquad \eta_1 = -A_1 \sin \Omega_1 t ; \tag{4}$$

$$\xi_2 = A_2 \cos \Omega_1 t - a_1 \cos \Omega_2 t ,$$

$$\eta_2 = A_2 \sin \Omega_1 t - a_1 \sin \Omega_2 t ; \tag{5}$$

and

$$\xi_3 = A_2 \cos \Omega_1 t + a_2 \cos \Omega_2 t ,$$

$$\eta_3 = A_2 \sin \Omega_1 t + a_2 \sin \Omega_2 t . \tag{6}$$

To the approximation involved in the assumption of circular motions of the three finite bodies, we have, from the result of the two-body problem,

$$\Omega_1 = \left[\frac{G(m_1 + m_2 + m_3)}{A^3}\right]^{\frac{1}{2}} , \tag{7}$$

and

$$\Omega_2 = \left[\frac{G(m_2 + m_3)}{a^3}\right]^{\frac{1}{2}} . \tag{8}$$

Obviously,

$$A = A_1 + A_2 , \quad \text{and} \quad a = a_1 + a_2 . \tag{9}$$

Next, choose a new rectangular coordinate system xyz centered at $0'$ with the x-axis revolving with m_2 and m_3 and with the z-axis parallel to the ζ-axis. The equations of transformation from the old one to the new one can easily be found to be

$$\xi = A_2 \cos \Omega_1 t + x \cos \Omega_2 t - y \sin \Omega_2 t , \tag{10}$$

$$\eta = A_2 \sin \Omega_1 t + x \sin \Omega_2 t + y \cos \Omega_2 t , \tag{11}$$

$$\zeta = z . \tag{12}$$

The coordinates of m_1, m_2, and m_3 in the new system are given by

$$x_1 = -A\cos(\Omega_2 - \Omega_1)t, \quad y_1 = A\sin(\Omega_2 - \Omega_1)t, \quad z_1 = 0; \tag{13}$$

$$x_2 = -a_1, \quad y_2 = 0, \quad z_2 = 0; \tag{14}$$

and

$$x_3 = a_2, \quad y_3 = 0, \quad z_3 = 0. \tag{15}$$

Substituting Equations 4 through 6 and 10 through 12 in Equations 1 to 3, and utilizing Equations 9 and 13 through 15 give the following result:

$$\frac{d^2x}{dt^2} - 2\Omega_2\frac{dy}{dt} = \Omega_2^2\left(x - \frac{A_2}{A}\frac{\Omega_1^2}{\Omega_2^2}x_1\right) - Gm_1\frac{x - x_1}{r_1^3} - Gm_2\frac{x + a_1}{r_2^3} - Gm_3\frac{x - a_2}{r_3^3}, \tag{16}$$

$$\frac{d^2y}{dt^2} + 2\Omega_2\frac{dx}{dt} = \Omega_2^2\left(y - \frac{A_2}{A}\frac{\Omega_1^2}{\Omega_2^2}y_1\right) - Gm_1\frac{y - y_1}{r_1^3} - Gm_2\frac{y}{r_2^3} - Gm_3\frac{y}{r_3^3}, \tag{17}$$

$$\frac{d^2z}{dt^2} = -Gm_1\frac{z}{r_1^3} - Gm_2\frac{z}{r_2^3} - Gm_3\frac{z}{r_3^3}, \tag{18}$$

where

$$\left.\begin{aligned} r_1^2 &= (x - x_1)^2 + (y - y_1)^2 + z^2, \\ r_2^2 &= (x + a_1)^2 + y^2 + z^2, \\ r_3^2 &= (x - a_2)^2 + y^2 + z^2. \end{aligned}\right\} \tag{19}$$

If a function U is defined as

$$U(x, y, z) \equiv \Omega_2^2\left[\frac{1}{2}(x^2 + y^2) - \frac{A_2}{A}\frac{\Omega_1^2}{\Omega_2^2}(x_1x + y_1y)\right] + \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \frac{Gm_3}{r_3}, \tag{20}$$

then Equations 16 through 18 assume the simplified form

$$\frac{d^2x}{dt^2} - 2\Omega_2\frac{dy}{dt} = \frac{\partial U}{\partial x}, \tag{21}$$

$$\frac{d^2y}{dt^2} + 2\Omega_2\frac{dx}{dt} = \frac{\partial U}{\partial y}, \tag{22}$$

$$\frac{d^2z}{dt^2} = \frac{\partial U}{\partial z}, \tag{23}$$

which can be integrated to give, for each epoch of an infinitesimal time-interval,

$$V^2 = 2U + \text{constant} , \tag{24}$$

where V is the magnitude of velocity in the xyz system of reference. Equation 24 plays a role in the present problem, just as Jacobi's integral in the restricted three-body problem.

ZERO-VELOCITY SURFACES

It follows from Equations 20 and 24 that the zero-velocity surface can be defined by

$$\frac{\Omega_2^2}{2}(x^2+y^2) - \frac{A_2}{A}\Omega_1^2(x_1x+y_1y) + \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \frac{Gm_3}{r_3} = \text{constant}. \tag{25}$$

Since $z_1 = 0$,

$$x_1x + y_1y = rA\cos\Theta , \tag{26}$$

where Θ is the angle between $0'm$ and $0'm_1$. In the case of the earth-moon-sun system, it is the angle subtended by the artificial satellite and the sun at $0'$. Thus, the zero-velocity surfaces are not fixed even in the rotating coordinate system; rather, they change with the position of m_1. However, an instantaneous (or osculating) zero-velocity surface can be defined for each position of m_1. It is in this sense that zero-velocity surfaces will be discussed. Indeed, the general behavior of the motion in the very restricted four-body problem can be understood by these osculating zero-velocity surfaces just as that of the motion in the restricted three-body problem by the zero-velocity surfaces themselves.

With the aid of Equations 7, 8, and 26, Equation 25 becomes

$$\frac{1}{2}\frac{(m_2+m_3)(x^2+y^2)}{a^3} - \frac{m_1r}{A^2}\cos\Theta + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} = \text{constant} . \tag{27}$$

Since the motion of m is of interest only when $r \ll A$, the term $1/r_1$ in Equation 27 can be expanded in terms of spherical harmonics, $P_n(\cos\Theta)$; that is,

$$\frac{1}{r_1} = \frac{1}{A}\sum_{n=0}^{\infty}\left(\frac{r}{A}\right)^n P_n(\cos\Theta) . \tag{28}$$

Taking only the first three terms in the right side of Equation 28 and substituting them in the place of $1/r_1$ in Equation 27, we obtain

$$\frac{(m_2 + m_3)(x^2 + y^2)}{a^3} + \frac{m_1 r^2}{A^3}(3 \cos \Theta - 1) + \frac{2m_2}{r_2} + \frac{2m_3}{r_3} = \text{constant} , \qquad (29)$$

where the term $2m_1/A$ has been absorbed in the constant term.

If a is now taken as the unit of length and $m_2 + m_3$ as the unit of mass, Equation 29 reduces to

$$x^2 + y^2 + \frac{m_1 r^2}{A^3} (3 \cos^2 \Theta - 1) + \frac{2(1-\mu)}{r_2} + \frac{2\mu}{r_3} = C , \qquad (30)$$

where

$$\mu = \frac{m_3}{m_2 + m_3} \qquad (31)$$

and C is a constant of integration. This differs from the zero-velocity surfaces of the restricted three-body problem only by the addition of a small perturbing term that contains the factor m_1/A^3.

DOUBLE POINTS OF THE SURFACES

Consider the change in position of the three double points L_1, L_2, L_3 which are located on the x-axis when the perturbing term vanishes. Since this is now limited to the xy plane,

$$\Theta = \theta - \theta_0 , \qquad (32)$$

where θ and θ_0 are the respective angles that the positive x-axis makes with the vectors $0'm$ and $0'm_1$. In the case of the earth-moon-sun system, θ_0 changes from 0 to 2π in a period of the lunar month. Therefore, in a time-scale of a few hours θ_0 may be regarded as constant.

Substituting Equation 32 into Equation 30, we obtain after reduction

$$F(x,y) \equiv (1 + \beta)(x^2 + y^2) + 3\beta \left[(x^2 - y^2) \cos 2\theta_0 + 2xy \sin 2\theta_0\right] + \frac{2(1-\mu)}{r_2} + \frac{2\mu}{r_3} - C = 0 , \qquad (33)$$

where

$$\beta = \frac{1}{2} \frac{m_1}{A^3} . \qquad (34)$$

The conditions for double points are

$$\frac{1}{2}\frac{\partial F}{\partial x} = \left[1 + (1 + 3\cos 2\theta_0)\beta\right]x + 3(\beta\sin 2\theta_0)y$$

$$- \frac{(1-\mu)(x-x_2)}{r_2^{\,3}} - \frac{\mu(x-x_3)}{r_3^{\,3}} = 0 \qquad (35)$$

and

$$\frac{1}{2}\frac{\partial F}{\partial y} = \left[1 + (1 - 3\cos 2\theta_0)\beta\right]y + 3(\beta\sin 2\theta_0)x$$

$$- \frac{(1-\mu)y}{r_2^{\,3}} - \frac{\mu y}{r_3^{\,3}} = 0 \ , \qquad (36)$$

from which all five double points can be determined on the xy plane. Double points are no longer the particular solution of the problem because they change with θ_0 and also because the problem is being treated only approximately (by taking only the first few terms in the series expansion of $1/r_1$, etc.).

Since β is small, its second and higher orders can be neglected. It appears from Equation 36 that the three double points which approach the x-axis when $\beta = 0$ have their y-coordinates of the order of $\beta\sin 2\theta_0$ when $\beta \neq 0$. Thus, the term $3(\beta\sin 2\theta_0)y$ in Equation 35 is of the order of $(\beta\sin 2\theta_0)^2$ and can be neglected. Hence, Equation 35 is reduced to

$$\left[1 + (1 + 3\cos 2\theta_0)\beta\right]x - \frac{(1-\mu)(x-x_2)}{r_2^{\,3}} - \frac{\mu(x-x_3)}{r_3^{\,3}} = 0 \ , \qquad (37)$$

which differs from its counterpart in the restricted three-body problem only by the factor $(1 + 3\cos 2\theta_0)\beta$.

Once the x-coordinates of the three double points are derived from the solution of Equation 37, their y-coordinates can be obtained by

$$\left(\frac{dy}{d\beta}\right)_{\beta=0} = F_i \sin 2\theta_0 \ , \qquad (38)$$

where

$$F_i = \left[\frac{3x}{\frac{1-\mu}{r_2^{\,3}} + \frac{\mu}{r_3^{\,3}} - 1}\right]_{L_i} \ ,$$

which follows directly from Equation 36 except that now the values of x, r_2, and r_3 are taken at one of the three double points $L_i (i = 1, 2, 3)$ for the case $\beta = 0$.

The change in position of the three double points with β and θ_0 can be most conveniently seen by first taking the derivatives of their coordinates with respect to β and setting $\beta = 0$. Consider the three points separately: (1) L_2 between $+\infty$ and x_3, (2) L_1 between x_3 and x_2, and (3) L_3 between x_2 and $-\infty$.

(1) Let the distance from m_3 in the x-direction to the double point L_2 be represented by ρ. Then Equation 35 becomes, by neglecting the second and higher orders of $\beta \sin 2\theta_0$,

$$\left[1 + (1 + 3 \cos 2\theta_0)\beta\right](1 - \mu + \rho) - \frac{1 - \mu}{(1 + \rho)^2} - \frac{\mu}{\rho^2} = 0 . \tag{39}$$

Differentiating Equation 39 with respect to β and setting $\beta = 0$ give

$$\left(\frac{d\rho}{d\beta}\right)_{\beta=0} = E_2(1 + 3 \cos 2\theta_0) , \tag{40}$$

where

$$E_2 = - \frac{(1 - \mu + \rho_0)(1 + \rho_0)\rho_0^3}{3\rho_0^3(1 + \rho_0) + 2\mu(1 - \rho_0^3)} . \tag{41}$$

The symbol ρ_0 at the right side of Equation 41 is the solution of ρ for Equation 39 with $\beta = 0$. Similarly the change in value of C which corresponds to the variation in position of L_2 can be computed. Differentiating Equation 33 with respect to β and setting $\beta = 0$ afterwards give

$$\left(\frac{dC}{d\beta}\right)_{\beta=0} = (1 - \mu + \rho_0)^2 (1 + 3 \cos 2\theta_0) , \tag{42}$$

ρ_0 in Equation 42 having the same meaning as that in Equation 41.

(2) Let the distance in the x-direction from m_3 to the double point L_1 be represented by ρ. Then by a similar approximation, as before, Equation 35 reduces to

$$\left[1 + (1 + 3 \cos 2\theta_0)\beta\right](1 - \mu - \rho) - \frac{1 - \mu}{(1 - \rho)^2} + \frac{\mu}{\rho^2} = 0 . \tag{43}$$

By exactly the same procedure as before, we obtain

$$\left(\frac{d\rho}{d\beta}\right)_{\beta=0} = E_1(1 + 3 \cos 2\theta_0) , \tag{44}$$

where

$$E_1 = \frac{(1 - \mu - \rho_0)\rho_0^3}{3\rho_0^3 + 2\mu(1 + \rho_0 + \rho_0^2)} , \tag{45}$$

$$\left(\frac{dC}{d\beta}\right)_{\beta=0} = (1 - \rho_0 - \mu)^2 (1 + 3 \cos 2\theta_0) , \tag{46}$$

and ρ_0 is the solution of Equation 43 with $\beta = 0$.

(3) Let the distance in the x-direction from m_2 to the double point L_3 be represented by $1 - \rho$. Then Equation 35 reduces by approximation to

$$\left[1 + (1 + 3\cos 2\theta_0)\beta\right](1 + \mu - \rho) - \frac{1-\mu}{(1-\rho)^2} - \frac{\mu}{(2-\rho)^2} = 0 . \quad (47)$$

From this is derived

$$\left(\frac{d\rho}{d\beta}\right)_{\beta=0} = E_3(1 + 3\cos 2\theta_0) , \quad (48)$$

where

$$E_3 = \frac{(1 + \mu - \rho_0)(2 - \rho_0)^3}{3(2 - \rho_0)^3 + 2\mu\left[(2-\rho_0)^2 + (2 - \rho_0) + 1\right]} ; \quad (49)$$

and

$$\left(\frac{dC}{d\beta}\right)_{\beta=0} = (1 - \rho_0 + \mu)^2 (1 + 3\cos 2\theta_0) , \quad (50)$$

where ρ_0 is the solution of Equation 47 with $\beta = 0$.

Now if the positions and their corresponding values of C of the three double points on the x-axis are known for the case $\beta = 0$, their positions and the corresponding values of C for the case of small β may be derived by

$$\rho = \rho_0 + \beta\left(\frac{d\rho}{d\beta}\right)_{\beta=0} , \quad (51)$$

and

$$C = C_0 + \beta\left(\frac{dC}{d\beta}\right)_{\beta=0} . \quad (52)$$

The change in the ordinates of these points is given by Equation 38.

For the earth-moon system, $\mu = 0.01216$. The values for the relevant quantities in this case are listed in Table 1. The derivative of the coordinates of these three points

Table 1
Values of ρ, C, E_i, and F_i for the changes in position of L_1, L_2, L_3 with $\beta = 0$
[$\mu = 0.01216$]

	$L_1(i = 1)$	$L_2(i = 2)$	$L_3(i = 3)$
ρ	0.1510	0.1679	0.00709
C	3.18843	3.17223	3.01216
E_i	0.0741	-0.1566	0.3326
F_i	0.6053	1.5831	-0.3820

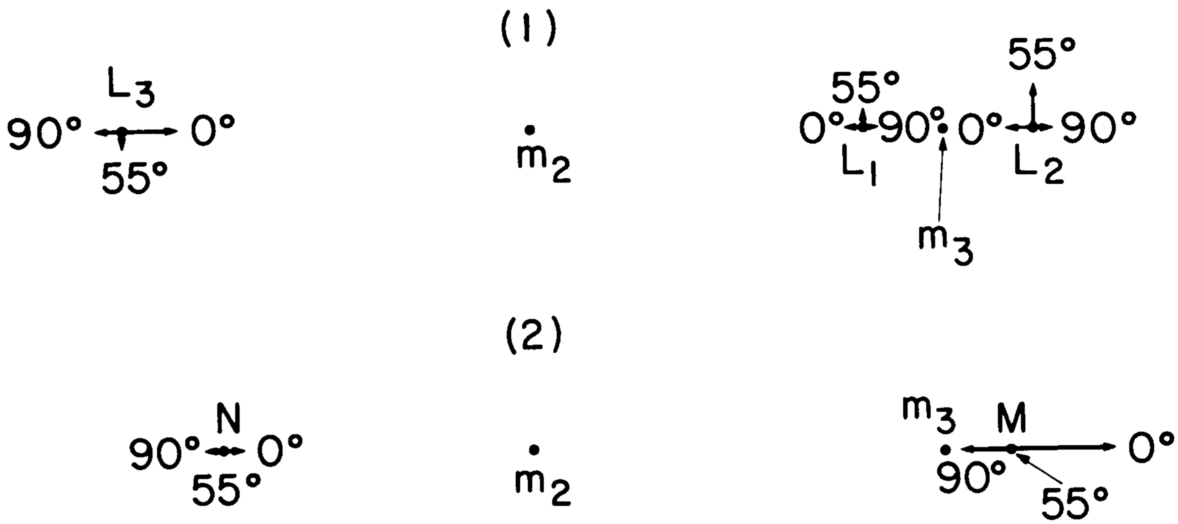

Figure 1 - Changes in position with β at $\beta = 0$ of (1) the three double points, L_1, L_2, L_3, and of (2) the intersecting points M and N with the x-axis of the zero-velocity surface passing through L_1. Notice the opposite directions of the changes in position of L_2 and M .

with respect to β at their normal positions ($\beta = 0$) are furthermore illustrated in Figure 1 for three positions of the sun ($\theta_o = 0$, $\pi/4$, $\pi/2$). The directions and magnitudes of the arrows in the figure indicate the derivatives of the coordinates of these points with respect to β for three values of $\theta_0 (0, \pi/4, \pi/2)$.

In order to examine the change in position of the double points L_4 and L_5, which make two equilateral triangles with m_2 and m_3 when $\beta = 0$, we must resort to the original Equations 35 and 36. Differentiate them with respect to β, and set in the resulting equations:

$$\beta = 0, \quad r_2 = r_3 = 1, \quad x = \frac{1}{2} - \mu, \quad y = \pm \frac{\sqrt{3}}{2}.$$

The required quantities $(dx/d\beta)_{\beta=0}$ and $(dy/d\beta)_{\beta=0}$ are derived by solving the equations simultaneously.

DEGENERATION OF THE CRITICAL ZERO-VELOCITY SURFACES

When $\beta = 0$, which corresponds to the restricted three-body problem, the zero-velocity surface that passes through L_1 is frequently known as the inner contact surface and that which passes through L_2, the outermost contact surface. The former intersects the x-axis at two more points besides L_1. Call the intersecting point on the positive x-axis M and that on the negative x-axis N. From the change in position of M and N with β, the general behavior of the system of zero-velocity surfaces can be inferred.

(1) Point M: Let its distance from m_3 be σ. Thus, from Equation 33,

$$C = (1 - \mu + \sigma)^2 \left[1 + (1 + 3 \cos 2\theta_0)\beta\right] + \frac{2(1 - \mu)}{1 + \sigma} + \frac{2\mu}{\sigma} . \tag{53}$$

Differentiating Equation 53 with respect to β and utilizing the relation given by Equation 46 give

$$\left(\frac{d\sigma}{d\beta}\right)_{\beta=0} = E_m(1 + 3 \cos 2\theta_0) , \tag{54}$$

where

$$E_m = - \frac{(\rho_0 + \sigma_0)(2 - 2\mu + \sigma_0 - \rho_0)}{2\left[1 - \mu + \sigma_0 - \frac{1 - \mu}{(1 + \sigma_0)^2} - \frac{\mu}{\sigma_0^2}\right]} , \tag{55}$$

in which σ_0 is the solution of Equation 53 with $\beta = 0$ and $C = C_0$, corresponding to the inner contact surface of the restricted three-body problem.

(2) Point N: Let its distance from m_2 be σ. Following the same procedure as before, we derive

$$\left(\frac{d\sigma}{d\beta}\right)_{\beta=0} = E_n(1 + 3 \cos 2\theta_0) , \tag{56}$$

where

$$E_n = \frac{(1 - 2\mu - \rho_0 - \sigma_0)(1 + \rho_0 + \sigma_0)}{2\left[\mu + \sigma_0 - \frac{1 - \mu}{\sigma_0^2} - \frac{\mu}{(1 + \sigma_0)^2}\right]} , \tag{57}$$

where σ_0 is σ of N when $\beta = 0$. In both Equations 55 and 57, ρ_0 is the solution of Equation 43 with $\beta = 0$.

For $\mu = 0.01216$,

$$E_m = 0.6217, \quad \text{and} \quad E_n = 0.0589 . \tag{58}$$

The changes in position of M and N are illustrated in Figure 1, from which it is seen that M moves out while L_2 moves in as $\beta(1 + 3 \cos 2\theta_0)$ increases. In other words, the inner contact surface will eventually meet the outermost contact surface at a certain value of $\beta(1 + 3 \cos 2\theta_0)$. When this happens, the two surfaces degenerate into one surface. It is evident from Equations 40, 54, 58 and Table 1 that the smallest value of β for which the critical surfaces become degenerated occurs at $\theta_0 = 0$. This threshold value of β (denoted by β_c hereafter) can be determined in the following way: First calculate the two points L_1 and L_2 by Equation 35 with $\theta_0 = 0$. Denote the distances of L_1 and L_2 from m_3 by ρ_1 and ρ_2, from which the corresponding values of C (denoted by C_1 and C_2, respectively) can be obtained from Equation 33. The degenerated case is given by the condition

$$C_1 = C_2 , \tag{59}$$

which gives the required value β_c. In Table 2 there is computed for the case $\mu = 0.01216$ three sets of values from which we obtain

$$\beta_c = 0.0064 \tag{60}$$

by graphical interpolation. The corresponding values for ρ_1, $\rho_2 = \sigma_M$, and $C_1 = C_2$ are given in the last row of the table. Figure 2 illustrates the degenerated zero-velocity surface that passes through both L_1 and L_2.

Table 2
Determination of β_c
[μ = 0.01216]

β	ρ_1	C_1	ρ_2	C_2	σ_M
0.0	0.15097	3.18843	0.16788	3.17223	0.12580
0.0025	0.15171	3.19542	0.16672	3.18557	0.13306
0.005	0.15246	3.20241	0.16482	3.19888	0.14442
0.0075	0.15321	3.20938	0.16334	3.21214	(No solution)
β_c = 0.0064	β_c = 0.15288	β_c = 3.20632	β_c = 0.16398	β_c = 3.20632	β_c = 0.16398

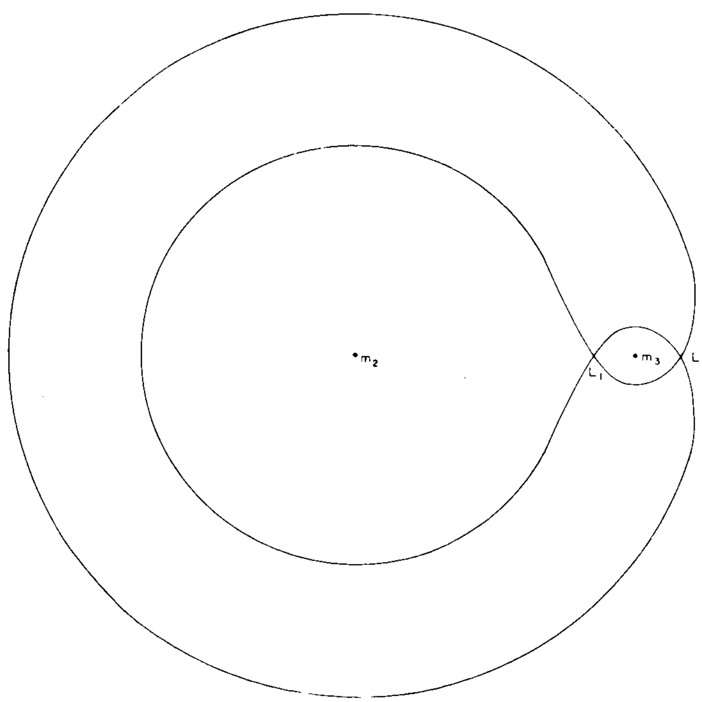

Figure 2 - Degenerate zero-velocity surface passing through both L_1 and L_2; plotted for the case $\mu = 0.01216$ corresponding to the earth-moon system

Further increase in β causes a fundamental change in shape of the critical zero-velocity surfaces. The surface that passes through L_1 opens up at M as can be seen from Equation 53, which does not yield any significant solution when $\beta > \beta_c$. The general behavior of the critical zero-velocity surfaces for the cases $\beta > \beta_c$ is illustrated in Figure 3. In general the zero-velocity surfaces in the very restricted four-body problem are not symmetric even with respect to the x-axis. However, when $\theta_0 = 0$, they become symmetrical as is shown in both Figures 2 and 3.

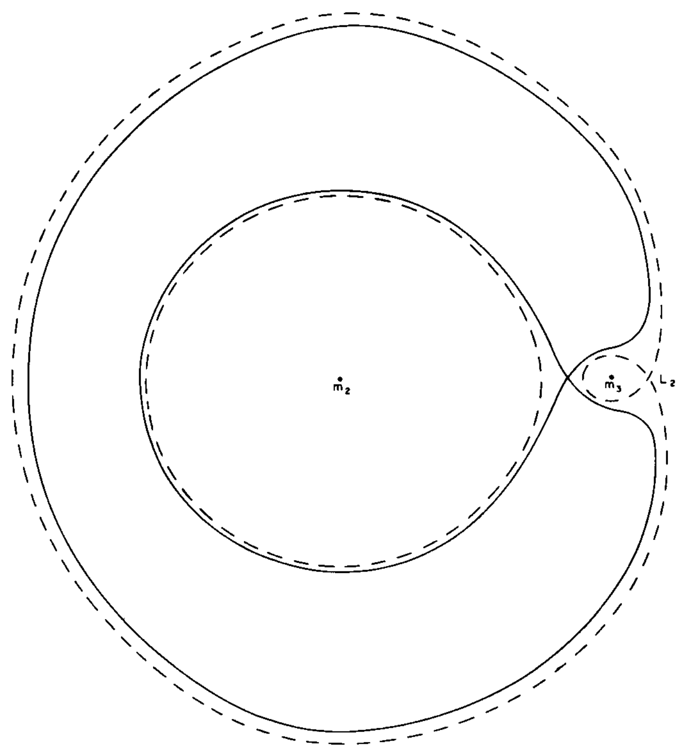

Figure 3 - Zero-velocity surfaces passing through L_1 and L_2, respectively, when $\beta > \beta_c$; plotted for the case $\mu = 0.01216$, $\beta = 0.025$, and $\theta_0 = 0$ (figure is symmetric with respect to the x-axis only because $\theta_0 = 0$)

DISCUSSION

For the earth-moon-sun system, from Equation 34,

$$\beta = 0.0028 , \tag{61}$$

which is smaller than the threshold value for degeneracy as given by Equation 60. Thus, the critical surfaces will never become degenerated for any value of θ_0. In other words, the inner and outermost contact surfaces for the earth-moon system can still be defined in spite of the presence of the sun, justifying the treatment in the previous paper of satellites in the earth-moon-sun system as restricted three-body problems.

For $\beta < \beta_c$, the inner and outermost contact surfaces may be regarded as oscillating when θ_0 varies periodically. It follows from Equation 46 that a positive value of $\beta(1 + 3 \cos 2\theta_0)$ makes a satellite escape easier than does a negative value. For example, a satellite with $C \geq 3.18843$ will not escape from the neighborhood of the earth (or of the moon) in the framework of the restricted three-body problem (i.e., $\beta = 0$). By the introduction of the fourth body (m_1), a satellite will be retained inside the inner contact surface permanently only if $C \geq 3.19625$. Similarly, the limiting value of C for retaining a satellite inside the outermost contact surface is now 3.18714, against 3.17223 in the restricted three-body problem.

Although the present method of approach does not give the perturbation of orbital elements of artificial satellites, it gives a general idea of where they could or could not go under given initial conditions when they are no longer very near to the earth.

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