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NOTATION

b	Angular orientation at time t, not necessarily along the most direct rotational path; also used to indicate the body system
b_0	Initial angular orientation
b_f	Final angular orientation
b_s	Angular orientation, for following the most direct rotational path
$D_{u \to v}$	3×3 element direction cosine matrix (DCM), for transforming a coordinate of the u-system into the v-system.
e	Exponent of disturbance function
$\underline{e} \equiv \{e_x, e_y, e_z\}^T$	Unity vector, indicating the direction of the Euler axis for rotation from an initial system to a final one, expressed in the initial system
$M_{u \to v}$	4×4 element quaternion matrix of the quaternion for rotation from u to v
$M^*_{u \to v}$	Transmutated quaternion matrix
$\underline{q}_{u \to v} \equiv \{q_1, q_2, q_3, q_4\}^T$	Quaternion for rotation from u to v, uniquely describing the orientation of v into u
T	Transpose of a matrix; also used to indicate a column vector
t	Time
t_0	Initial time
t_f	Final time
$\underline{x}^b \equiv \{x^b, y^b, z^b\}^T$	Orthogonal right-hand object-body coordinate system; in aeronautical applications the x^b-axis is pointing forward along the aircraft body main axis, the y^b-axis is pointing outward through the right wing, and the z^b-axis is pointing downward
$\underline{x}^i \equiv \{x^i, y^i, z^i\}^T$	Orthogonal right-hand Earth-based inertial reference system, with the x^i-axis pointing to the north, the y^i-axis to the east, and the z^i-axis downward, toward the center of Earth; in the experiments the x^i-axis coincides with the direction of the viewing axis

$\underline{x}^{q0}$	Like the initial system b_0, fixed with respect to the inertial system, with the x^{q0}-axis aligned with the Euler axis and the y^{q0}-axis in the x^{b0}-o-y^{b0} plane
$\underline{x}^{q1};\underline{x}^{q2};\underline{x}^{q3}$	Intermediate systems to obtain the plane in which to rotate $\underline{\omega}^b$ in order to be disturbed
α	"Angular distance" between orientation b and b_f, i.e., the angle of rotation for the remaining most direct path rotation from b to b_f
α_0	Total angle of rotation for the most direct path rotation from initial orientation b_0 to final orientation b_f
β	"Rotational deviation" from the most direct path; it is the amount of rotation of the quaternion $\Delta\underline{q}$
$\overline{\beta}$	Averaged rotational deviation from the most direct path over the interval $t_0 \leq t \leq t_f$
γ,δ	Azimuth angle and elevation angle, respectively, which specify the orientation of $\underline{\omega}_0$ in the b_0-system
$\Delta\underline{q}$	Quaternion for the rotational deviation, i.e., for rotation from b_s to b
ν	Angle of disturbance; it is the angle over which $\underline{\omega}^b$ is rotated in order to be disturbed
ν_0	Angle of disturbance at time t_0
χ,ξ	Azimuth angle and elevation angle, respectively, which specify the orientation of $\underline{\omega}^{q0}$ in the q_0-system
ψ,θ,ϕ	Euler angles for yaw, pitch, and roll, respectively, specifying the orientation of body with respect to the inertial system
ϕ_d	Angle of plane in which $\underline{\omega}_b$ is rotated to be disturbed
$\underline{\omega}_0 \equiv \underline{\omega}_0^b$	Vector of angular rotation, for the most direct rotational path between orientation b_0 and b_f, expressed in the b_0-system
$\omega_0 \equiv \lvert\underline{\omega}_0\rvert$	Constant rotational speed
$\underline{\omega}_d^b$	Disturbed vector of angular rotation; it is the vector of rotation at which the object proceeds rotating at time t, expressed in the b-system

$\underline{\omega}_s$	Vector of angular rotation, for the most direct rotational path between orientation b_0 and b_f, rotating at reduced angular rate, expressed in the b_0-system
$\underline{\omega}^b_{i \to b} = \underline{\omega}^b$	Vector of angular rotation, for the remaining most direct rotational path between orientation b and b_f, expressed in the b-system
$\underline{\omega}^{q0}$	The vector $\underline{\omega}^b$ expressed in the $\underline{q}_0$-system
Ω	4×4 element matrix, containing the components of $\underline{\omega}^b_d$, used for computing the rotational path
Ω_s	4×4 element matrix, containing the components of $\underline{\omega}_s$, used for computing the most direct rotational path

AN ALGORITHM FOR THE SYSTEMATIC DISTURBANCE OF OPTIMAL ROTATIONAL SOLUTIONS

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SUMMARY

An algorithm for introducing a systematic rotational disturbance into an optimal (i.e., single axis) rotational trajectory is described. This disturbance introduces a motion vector orthogonal to the quaternion-defined optimal rotation axis. By altering the magnitude of this vector, the degree of non-optimality can be controlled. The metric properties of the distortion parameter are described, with analogies to two-dimensional translational motion.

This algorithm has been implemented in a motion-control program on a three-dimensional graphic workstation. It supports a series of human performance studies on the detectability of rotational trajectory optimality by naive observers.

1. INTRODUCTION

This paper describes an algorithm for generating kinematically suboptimal ("warped") rotational trajectories. First, the basic idea of creating a suboptimal trajectory is demonstrated with a two-dimensional translatory analogy. Next, the mathematical formulation for describing the rotation of an object is given for three methods: (1) Euler angles, (2) direction cosine matrix, and (3) quaternions, and the relation between these methods is discussed. Then, the rotational equivalent of the straight path for translatory motion is defined and ways for computing the rotational path and the deviation from the direct path are delineated. Finally, the method used for creating a reproducible "smoothly warped" suboptimal trajectory is outlined. This method allows both the shape of the trajectory and the magnitude of the deviation from the optimal path to be defined by two independent parameters. A flow chart is presented which summarizes the computations performed to create displays employing this algorithm.

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2. DEVIATION FROM THE OPTIMAL PATH TRAJECTORY FOR TRANSLATORY MOTION

2.1 Method for Creating a Suboptimal Trajectory

The basic idea of creating a suboptimal trajectory can best be illustrated with a simple, two-dimensional translational analogy. Consider two points on a plane as illustrated in figure 1. An object moves from the initial location b_0, at time t_0, to the final location b_f, at time t_f, at a constant velocity, V_0. The kinematically optimal trajectory between b_0 and b_f is a straight line; the traveled distance along this line is R_0. At time t, the object is at location b, which is not necessarily along this most direct path. However, the remaining most direct path from b to b_f is again given by a straight line which connects these two locations. If the velocity vector, $\underline{V}$, points from b to location b_f, the object would proceed from t until it reaches t_f along this remaining most direct path. However, in order to produce a remaining suboptimal trajectory, a disturbance is introduced by rotating $\underline{V}$ over the angle of disturbance ν, to obtain the disturbed vector, $\underline{V}_d$. The angle ν is chosen to be a simple exponential function of the range R between b and b_f, according to

$$\tan \nu(t) = \left(\frac{R(t)}{R_0} \right)^e \tan \nu_0 \tag{1}$$

where the exponent e determines the characteristics of the disturbance and therefore the trajectory shape. The parameter ν_0, which is the angle of disturbance at t_0, determines the magnitude of the disturbance.

For $e > 1$, $\nu(t)$ will reduce to zero quickly, such that the trajectory will be curved mainly at the beginning and straight toward the end, while for $0 < e \leq 1$ the opposite is the case. Typical trajectory shapes for these two ranges of e are shown in figure 2.

2.2 Measures of "Suboptimality" of Motion

Various measures can be considered for specifying the "degree of suboptimality" of motion. The first one is the difference between the traveled distance along the nondirect path, and the shortest distance, R_0. For constant velocity V_0, this is equivalent to the difference between the actual travel time $(t_f - t_0)$ and the most direct path travel time, R_0/V_0. Since this difference will be small relative to the total travel distance (or travel time), this measure will be fairly insensitive for trajectories with small deviations.

A second candidate for a measure of indirectness is the averaged angle of disturbance, computed by

$$\overline{\nu} = \frac{1}{(t_f - t_0)} \int_{t=t_0}^{t_f} \nu(t)dt \tag{2}$$

This method has the disadvantage that only the disturbance and not the actual deviation from the straight path is considered. Thus, for a straight trajectory section (like the final section of a trajectory with $e > 1$), ν is zero and does not add to the measure of indirectness, although the deviation from the most direct path definitely exists (fig. 2a).

A third measure of indirectness is the averaged deviation from the straight path. The way in which this score is computed is shown in figure 3. It is assumed that the object travels along the nondirect path with a constant velocity V_0. At each instance of time t, the distance $d(t)$ between a point on the trajectory b and the equivalent position on the optimal path b_s is computed; b_s is the position on the optimal path at which the object would arrive at time t when traveling with a reduced speed of $V_s = R_0/(t_f - t_0) \leq V_0$. The positions b and b_s are computed by solving the differential vector equations

$$\dot{\underline{x}}_b(t) = \underline{V}_d(t)\ ; \quad \dot{\underline{x}}_{b_s}(t) = \frac{R_0}{(t_f - t_0)V_0}\underline{V}_0 \tag{3}$$

with initial conditions

$$\underline{x}_b(t_0) = \underline{x}_{b_s}(t_0) = \underline{x}_{b_0}$$

where

$$\underline{x}_b \equiv \begin{bmatrix} x \\ y \end{bmatrix}_b ; \quad \underline{x}_{b_s} \equiv \begin{bmatrix} x \\ y \end{bmatrix}_{b_s}$$

and the distance $d(t)$ is

$$d(t) = |\underline{x}_b(t) - \underline{x}_{b_s}(t)| \tag{4}$$

The averaged deviation from the optimal path over the interval $t_0 \leq t \leq t_f$ is then computed by

$$\overline{d} = \frac{1}{(t_f - t_0)} \int_{t=t_0}^{t_f} d(t)dt \tag{5}$$

3. DEVIATION FROM THE OPTIMAL TRAJECTORY FOR ROTATIONS

3.1 Definition of Coordinate Systems and Object Angular Rotation

The description of the angular rotation of an object in space involves rotational transformations between coordinate systems. Two basic systems are defined: an Earth-based inertial reference system, i, and an object body coordinate system, b. The Earth-based system, defined according to the aeronautical convention, is a right-hand system with its x^i-axis toward the north, its y^i-axis to the east, and its z^i-axis pointing downward toward the center of Earth. In creating displays on the workstation, the inertial system is chosen to coincide with the observer's viewing system, with the x^i-axis being the viewing direction. The object is described in the body system. For aerospace applications, for example, this system is attached to the aircraft body, with the x^b-axis pointing forward along the aircraft body main axis, the y^b-axis pointing outward through the right wing, and the z^b-axis pointing downward. (NOTE: Coordinate systems other than the North-East-Down convention are frequently used by non-aerospace disciplines. However, by means of simple geometrical computations, coordinates expressed in one system can be transformed to another; the generality of the algorithm presented in this paper is not affected.)

The angular orientation of the body system with respect to the Earth system can be described in a variety of ways. The first one is by a set of three Euler angles, which define three successive rotations. In aerospace applications these angles are the yaw angle ψ, which sets the heading plane; the pitch angle θ, which sets the aircraft elevation angle with respect to the horizontal plane; and the roll angle ϕ, in this order (fig. 4). A coordinate defined in the inertial system, $\underline{x}^i \equiv \{x^i, y^i, z^i\}^T$, can now be expressed in the rotated body system, $x^b \equiv \{x^b, y^b, z^b\}^T$, by

$$\underline{x}^b = [\phi]\,[\theta]\,[\psi]\,\underline{x}^i \equiv D\{\psi,\theta,\phi\}\underline{x}^i \equiv D_{i\to b}\underline{x}^i \tag{6}$$

where

$$[\psi] \equiv \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix};\quad [\theta] \equiv \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix};\quad [\phi] \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \tag{7}$$

where c and s denote sine and cosine, respectively, and D is the nine-element direction cosine matrix (DCM). The superscript T denotes transpose, since a column vector is specified. Likewise, by the inverse transformation the coordinates of the b-system can be expressed in the i-system by

$$\underline{x}^i = [\psi]^T\,[\theta]^T\,[\phi]^T\,\underline{x}^b \equiv D^T\{\psi,\theta,\phi\}\underline{x}^b \equiv D_{b\to i}\underline{x}^b \tag{8}$$

where the superscript T denotes the transposed matrix and, since D is an orthonormal matrix, transpose and inverse are identical.

Thus, the second way of expressing the angular orientation of the b-system in the i-system is by means of the nine-element DCM. A third way is through the use of quaternions. Euler's theorem states that, regardless of the initial orientation (defined by the system u) and the final orientation (defined by the system v), it is always possible to find one axis about which the object can be rotated to bring it from orientation u to v. The orientation of this axis of rotation, expressed in the initial system u, is given by the unity vector $\underline{e} \equiv \{e_x, e_y, e_z\}^T$, and the amount of rotation by the angle α. The quaternion for rotation from u to v, which uniquely describes the orientation of v with respect to u, is given by

$$\underline{q}_{u\to v} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \equiv \begin{bmatrix} \cos \alpha/2 \\ e_x \sin \alpha/2 \\ e_y \sin \alpha/2 \\ e_z \sin \alpha/2 \end{bmatrix} \tag{9}$$

Since $\underline{e}$ is unity, it follows from equation (9) that

$$|\underline{q}| = \left(q_1^2 + q_2^2 + q_3^2 + q_4^2\right)^{1/2} = 1 \tag{10}$$

which is a unique property of the quaternion, and

$$|\alpha| = 2 \cos^{-1} q_1 = 2 \sin^{-1}\left(q_2^2 + q_3^2 + q_4^2\right)^{1/2} \; ; \quad 0 \le \alpha \le 180° \tag{11}$$

It also follows from equation (9) that the inverse quaternion, for rotation from v to u (i.e., the quaternion which describes the orientation of the u-system in the v-system), is given by

$$\underline{q}_{v\to u} = \underline{q}_{u\to v}^{-1} = \begin{bmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{12}$$

which simply means that the direction of rotation about the Euler axis is inverted.

It is very useful to find the quaternion for a sequence of rotations. Suppose that the angular orientation of system v with respect to u is given by $\underline{q}_{u\to v}$ and the orientation of a third system, w, with respect to v by $\underline{q}_{v\to w}$. Then, the orientation of w with respect to u is given by

$$\underline{q}_{u\to w} = \{M_{v\to w}\}\underline{q}_{u\to v} = \{M_{u\to v}\}^{*}\underline{q}_{v\to w} \tag{13}$$

with

$$M \equiv \left[\begin{array}{c:ccc} q_1 & -q_2 & -q_3 & -q_4 \\ \hdashline q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{array}\right] ; \quad M^{*} \equiv \left[\begin{array}{c:ccc} q_1 & -q_2 & -q_3 & -q_4 \\ \hdashline q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{array}\right] \tag{14}$$

where M is the quaternion matrix, composed of the elements of $\underline{q}$, and M^* is the transmutated quaternion matrix, obtained by transposing the vector kernel (or minor) of the first element, indicated by the dotted partition.

By using equation (13) in a sequence of rotations, the quaternion $\underline{q}_{i\to b}$ for rotation from inertial system i to body system b can be found as a function of the Euler angles. Following equation (9) the quaternions for the yaw, pitch, and roll rotations are given by

$$\underline{q}_{i\to u} = \begin{bmatrix} c\psi/2 \\ 0 \\ 0 \\ s\psi/2 \end{bmatrix} ; \quad \underline{q}_{u\to v} = \begin{bmatrix} c\theta/2 \\ 0 \\ s\theta/2 \\ 0 \end{bmatrix} ; \quad \underline{q}_{v\to b} = \begin{bmatrix} c\phi/2 \\ s\phi/2 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

respectively, where u and v denote the intermediate stages. Then, $\underline{q}_{i\to b}$ is given by

$$\underline{q}_{i\to b} = \{M_{v\to b}\}\{M_{u\to v}\}q_{i\to u} \tag{16}$$

and after evaluating equation (16) with equations (14) and (15)

$$\underline{q}_{i\to b} = \begin{bmatrix} c\phi/2 \;\; c\theta/2 \;\; c\psi/2 + s\phi/2 \;\; s\theta/2 \;\; s\psi/2 \\ s\phi/2 \;\; c\theta/2 \;\; c\psi/2 - c\phi/2 \;\; s\theta/2 \;\; s\psi/2 \\ c\phi/2 \;\; s\theta/2 \;\; c\psi/2 + s\phi/2 \;\; c\theta/2 \;\; s\psi/2 \\ -s\phi/2 \;\; s\theta/2 \;\; c\psi/2 + c\phi/2 \;\; c\theta/2 \;\; s\psi/2 \end{bmatrix} \tag{17}$$

The DCM $D_{i\to b}$ can be expressed in terms of $\underline{q}_{i\to b}$ as follows:

$$D_{i\to b} = \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2q_3 + q_1q_4) & 2(q_2q_4 - q_1q_3) \\ 2(q_2q_3 - q_1q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_4q_3 + q_1q_2) \\ 2(q_2q_4 + q_1q_3) & 2(q_4q_3 - q_1q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix} \tag{18}$$

3.2 Rotational Equivalent of a Translatory Straight Path

It is clear from the definition of the Euler axis that the rotational analogue of a straight path between two points in the translatory case is a rotation about the Euler axis. This rotation brings the object from an initial orientation to a final one along the most direct path. The total angle of rotation about this axis is equivalent to the distance between two points in the translatory case. Equivalently, this means that the instantaneous axis of rotation (or vector of angular velocity) is fixed in space. This also means that each coordinate of the object will describe a path, which will be the great-arc of a circle, located in a plane perpendicular to this fixed axis of rotation. For points of the object located on this axis, the radius of the circle will be zero. Mathematically, this means that the coordinates of these points are not affected by the rotational transformation.

In contrast, when the rotation is not proceeding along the optimal trajectory, the instantaneous axis of rotation will not be fixed in space and will perform a "wobbling" motion, somewhat like the nutation of a gyroscope.

3.3 Computation of the Rotational Path

Similar to translatory motion, in which the path is obtained by integrating the instantaneous velocity vector, the rotational path is obtained by integrating the components of the vector of angular velocity. For the rotational motion of the body system b, this vector is given by

$$\underline{\omega}^{b}_{i\to b}(t) = \{\omega_x, \omega_y, \omega_z\}^T \tag{19}$$

where the subscript $i\to b$ indicates the rotation of the b-system with respect to the i-system, and the superscript b indicates that this vector is expressed in the rotating b-system. The rotational path is then obtained by solving the differential equation

$$\dot{\underline{q}}_{i\to b}(t) = \{\Omega(t)\}\underline{q}_{i\to b}(t)\,; \quad \underline{q}_{i\to b}(t_0) = \underline{q}_{i\to b_0} \tag{20}$$

where b_0 indicates the orientation of b at time t_0 and

$$\Omega(t) = 1/2\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{21}$$

3.4 Computation of the "Deviation" from the Optimal Rotational Trajectory

For the optimal rotational trajectory, the vector $\underline{\omega}^b_{i\to b}(t) = \underline{\omega}^b_{i\to b}(t_0) \equiv \underline{\omega}_0 =$ constant will be fixed in space and will coincide with the Euler axis. $\omega_0 \equiv |\underline{\omega}_0|$ is the constant rotational speed and α_0 is the total angle of rotation, equivalent to the shortest distance, R_0, in the translatory case. The computation of the "deviation" from the optimal trajectory is analogous to the translatory case (fig. 3). At each instant of time the quaternion for the orientation $b, \underline{q}_{i\to b}(t)$, and the quaternion for the corresponding orientation along the optimal trajectory $b_s, \underline{q}_{i\to b_s}(t)$, are computed. b_s is the orientation on the optimal trajectory which would be reached at time t when rotating at a reduced angular rate of $\omega_s = \alpha_0/(t_f - t_0) \leq \omega_0$. The quaternion $\underline{q}_{i\to b_s}(t)$ is obtained by solving the differential equation

$$\dot{\underline{q}}_{i\to b_s}(t) = \{\Omega_s\}\underline{q}_{i\to b_s}(t)\,; \quad \underline{q}_{i\to b_s}(t_0) = \underline{q}_{i\to b_0} \tag{22}$$

where Ω_s is fixed and generated according to equation (21) with

$$\underline{\omega}_s = \frac{\alpha_0}{(t_f - t_0)\omega_0}\underline{\omega}_0 \tag{23}$$

The "rotational deviation" from the optimal trajectory (which is equivalent with the deviation d from the straight path in the translatory case) is given by the amount of rotation β of the quaternion

$$\Delta\underline{q}(t) \equiv \underline{q}_{b_s\to b}(t) = \{M_{i\to b}(t)\}\underline{q}^{-1}_{i\to b_s}(t) \tag{24}$$

where $\underline{q}^{-1}_{i\to b_s}$ is computed from $\underline{q}_{i\to b_s}$ by equation (12) and β is computed with equation (11). The averaged deviation angle over the rotation interval $t_0 \leq t \leq t_f$ is then computed by

$$\overline{\beta} = \frac{1}{(t_f - t_0)}\int_{t=t_0}^{t_f} \beta(t)dt \tag{25}$$

3.5 Introduction of the Disturbance

Consider that the b-system rotates from initial orientation b_0 at time t_0 to final orientation b_f at time t_f, not necessarily along the optimal trajectory. At time t the orientation of the b-system is b. However, analogous to the translational case (in which the direction of the remaining most direct path is $\underline{V}$), in the rotational case the remaining most direct path from time t to t_f onward is given by the Euler-axis rotation from b to b_f

$$\underline{q}_{b\to b_f}(t) = \{M_{i\to b_f}\}\underline{q}_{b\to i}(t) = \{M_{i\to b_f}\}\underline{q}^{-1}_{i\to b}(t) \tag{26}$$

where $\underline{q}^{-1}_{i\to b}$ is computed from $\underline{q}_{i\to b}$ by equation (12), $\underline{q}_{i\to b}$ is computed by solving the differential equation (eq. (20)), and $M_{i\to b_f}$ is given by the final orientation of b in i. For $0 < \alpha < 180°$, the orientation of the Euler axis for rotation from b to b_f (expressed in the b-system) is specified by

$$\underline{e}(t) = \begin{bmatrix} q_2/s \\ q_3/s \\ q_4/s \end{bmatrix}_{b\to b_f} \quad ; \quad s = \left(q_2^2 + q_3^2 + q_4^2\right)^{1/2} = \sin(\alpha/2) \tag{27}$$

and the "angular distance" from b to b_f is $\alpha = 2\sin^{-1}s$ (eq. (11)). If the instantaneous vector of rotation $\underline{\omega}^b_{i\to b}(t) \equiv \underline{\omega}^b(t)$ (the subscript $i\to b$ is henceforth omitted for clarity) would be chosen along $\underline{e}(t)$, the rotation would proceed from t onward until it reaches t_f along a remaining most direct path. However, in order to produce a remaining suboptimal trajectory, a disturbance is introduced to deviate $\underline{\omega}^b(t)$ from $\underline{e}(t)$. As in the translatory case, the deviation of $\underline{\omega}^b(t)$ from $\underline{e}(t)$ should be a function of the angular distance $\alpha(t)$. The vector $\underline{\omega}^b(t)$ is deviated from $\underline{e}(t)$ by rotating it over the angle $\nu(t)$, which is chosen to be a simple exponential function of $\alpha(t)$ according to

$$\tan\nu(t) = \left(\frac{\alpha(t)}{\alpha_0}\right)^e \tan\nu_0 \tag{28}$$

where ν_0 is the disturbance at time t_0, and e is an exponent. The parameter ν_0 determines the averaged deviation from the optimal trajectory and thus the magnitude of the disturbance, and the exponent e determines the shape of the disturbance. Since b globally approaches b_f, $\alpha(t)$ will decrease monotonically such that the time-history of $\nu(t)$ (and thus the shape of the trajectory) will be determined by the exponent e. For $e > 1$, $\nu(t)$ will reduce to zero quickly such that the trajectory will be "warped" mainly at the beginning. For $0 < e \le 1$, the greatest warping occurs toward the end of the trajectory.

It should be noted that, although the angle of disturbance $\nu(t)$ is defined, the plane in which $\underline{\omega}^b$ is rotated is still undefined. A sequence of rotational transformations is required to choose a plane which ensures a reproducible, smoothly warped rotational path. The method by which this plane is chosen is described below.

The first rotational transformation that is required to introduce the disturbance involves a transformation to the q_0-system. Like the b_0-system, the q_0-system is also fixed with respect to the inertial system, but it has the x^{q0}-axis aligned with the $\underline{\omega}_0$-axis, and the y^{q0}-axis in the x^{b0}-o-y^{b0} plane (fig. 5). Since the orientation of $\underline{\omega}_0$ in the b_0-system is given by its azimuth angle γ and elevation angle δ, the fixed DCM for rotation from b_0 to q_0 is obtained by a successive yaw rotation by the angle γ and a pitch rotation by the angle δ, following equations (6) and (7), according to

$$D_{b_0 \to q_0} = \begin{bmatrix} c\delta & 0 & -s\delta \\ 0 & 1 & 0 \\ s\delta & 0 & c\delta \end{bmatrix} \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\delta c\gamma & c\delta s\gamma & -s\delta \\ -s\gamma & c\gamma & 0 \\ s\delta c\gamma & s\delta s\gamma & c\delta \end{bmatrix} \tag{29}$$

First, $\underline{\omega}^b(t) = \omega_0 \underline{e}(t)$ for the remaining most direct path from time t until t_f, as computed by equation (27), is transformed from the b-system into the q_0-system by

$$\underline{\omega}^{q0}(t) = [D_{b \to q_0}]\underline{\omega}^b(t) \tag{30}$$

where

$$[D_{b \to q_0}] = [D_{b_0 \to q_0}][D_{i \to b_0}][D_{b \to i}] \tag{31}$$

The first two matrices on the right-hand side of equation (31) can be precomputed before starting the motion-control program, and the third matrix is computed from $\underline{q}_{b \to i} \equiv \underline{q}^T_{i \to b}$ by solving equation (20) and using equation (18).

The orientation of $\underline{\omega}^{q0}$ in the q_0-system is shown in figure 6. If the rotation were totally along the optimal trajectory, $\underline{\omega}^{q0}$ would coincide with $\underline{\omega}_0$ and thus with the x^{q0}-axis. However, its deviation from this axis can be expressed in the q_0-system by the azimuth angle χ and the elevation angle ξ. A system q_1 is now defined, with the x^{q1}-axis along $\underline{\omega}^{q0}$ and the y^{q1}-axis in the x^{q0}-o-y^{q0} horizontal plane. Similar to equation (29), the DCM for rotation from q_0 to q_1 is given by

$$D_{q_0 \to q_1} = \begin{bmatrix} c\xi c\chi & c\xi s\chi & -s\xi \\ -s\chi & c\chi & 0 \\ s\xi c\chi & s\xi s\chi & c\xi \end{bmatrix} \tag{32}$$

Since

$$\underline{\omega}^{q_0} \equiv \{\omega_x, \omega_y, \omega_z\}^T ; \quad |\underline{\omega}^{q_0}| = \omega_0 \tag{33}$$

it follows from the geometry of figure 6 that

$$\begin{aligned} \cos\chi &= \omega_x/d ; \quad \cos\xi = d/\omega_0 \\ \sin\chi &= \omega_y/d ; \quad \sin\xi = -\omega_z/\omega_0 \end{aligned} \tag{34}$$

where

$$d = \left(\omega_x^2 + \omega_y^2\right)^{1/2} \tag{35}$$

Substituting these expressions in equation (32) yields

$$D_{q_0 \to q_1} = \begin{bmatrix} \omega_x/\omega_0 & \omega_y/\omega_0 & \omega_z/\omega_0 \\ -\omega_y/d & \omega_x/d & 0 \\ -\omega_x\omega_z/(d\omega_0) & -\omega_y\omega_z/(d\omega_0) & d/\omega_0 \end{bmatrix} \tag{36}$$

Equations (35) and (36) show that $D_{q_0 \to q_1}$ can be computed simply from the components of $\underline{\omega}^{q_0}$ in the q_0-system.

Next, the q_2-system is defined now rotated about the x^{q_1}-axis by the angle ϕ_d (fig. 7). This angle is randomly chosen from a look-up table, but remains fixed throughout a trajectory calculation. The DCM for rotation from q_1 to q_2 is then given by

$$D_{q_1 \to q_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_d & s\phi_d \\ 0 & -s\phi_d & c\phi_d \end{bmatrix} \tag{37}$$

The plane x^{q_2}-o-y^{q_2} is the required plane in which the vector $\underline{\omega}$ is rotated to be disturbed (fig. 7). Last, the system q_3 is defined, rotated about the z^{q_2}-axis by the angle of disturbance $\nu(t)$, and computed by equation (28) (fig. 7). The DCM for rotation from q_2 to q_3 is then given by

$$D_{q_2 \to q_3} = \begin{bmatrix} cv & sv & 0 \\ -sv & cv & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (38)$$

With the DCMs previously computed with equations (31) and (36)-(38), the DCM for rotation from b to q_3 can be computed according to

$$[D_{b \to q_3}] = [D_{q_2 \to q_3}][D_{q_1 \to q_2}][D_{q_0 \to q_1}][D_{b \to q_0}] \quad (39)$$

The x^{q_3}-axis is now aligned with the direction of the "disturbed" vector $\underline{\omega}_d^b$. This vector, expressed in the b-system, is then computed by

$$\underline{\omega}_d^b = [D_{b \to q_3}]^T \underline{\omega}^{q_3} \quad (40)$$

where

$$\underline{\omega}^{q_3} \equiv [\omega_0, 0, 0]^T \quad (41)$$

2.5 Computer Implementation

The motion-control program involves (1) precomputation of matrices, vectors, and quaternions, which remain constant for all created trajectories, and (2) real-time computations for realizing the rotational path and computation of the deviations from the most direct path. This motion-control program is incorporated into an experimental control program for psychophysical studies of human observers' sensitivity for trajectory optimality. This experimental control program presents the motion stimuli, records observers' responses, and performs post-experiment data reduction.

A chart of the sequence of computations performed during the presentation of a rotational trajectory stimulus in a typical psychophysical experiment is shown in figure 8. The prerun computations are shown in bold blocks and the relevant input parameters for each stimulus are shown in the circles. For each stimulus, the initial orientation b_0 with respect to the inertial system i, the direction and magnitude of the most direct path vector of rotation $\underline{\omega}_0$, and the total angle of rotation v_0 are specified.

The orientation of b_0 in i is specified by the azimuth angle ψ, the elevation angle θ, and the roll angle ϕ (fig. 4). The direction of $\underline{\omega}_0$ is specified with respect to the b_0-system by the azimuth angle γ and the elevation angle δ (fig. 5). The magnitude of $\underline{\omega}_0$ is the specified rotational speed, ω_0. For each presentation the values of ψ,θ,ϕ and γ,δ are picked at random and without replacement from a look-up table.

First, the quaternion $\underline{q}_{i \to b_0}$ is precomputed with ψ, θ, and ϕ, using equation (17). The DCM $D_{i \to b_0}$ is computed simply with equation (18). Next, following the geometry of figure 5, from γ and δ, the components of $\underline{\omega}_0$ in the b_0-system can be precomputed according to

$$\underline{\omega}_0 = \omega_0 \begin{bmatrix} c\gamma c\delta \\ s\gamma c\delta \\ -s\delta \end{bmatrix} \qquad (42)$$

Since $\underline{\omega}_0^b \equiv \underline{\omega}_0$ coincides with the Euler axis for rotation from b_0 to b_f and thus has the same direction, the fixed quaternion for rotation from b_0 to b_f follows from equations (42) and (9):

$$\underline{q}_{b_0 \to b_f} = \begin{bmatrix} c\alpha_0/2 \\ c\gamma c\delta s\alpha_0/2 \\ s\gamma c\delta s\alpha_0/2 \\ -s\delta s\alpha_0/2 \end{bmatrix} \qquad (43)$$

and the corresponding quaternion matrix $M_{b_0 \to b_f}$ is computed with equation (14). The latter one, together with the quaternion of the initial orientation $q_{i \to b_0}$ can be used to precompute the quaternion of the final orientation

$$\underline{q}_{i \to b_f} = \{M_{b_0 \to b_f}\}\underline{q}_{i \to b_0} \qquad (44)$$

and the corresponding quaternion matrix $M_{i \to b_f}$ is computed with equation (14).

Last, $D_{b_0 \to q_0}$ is precomputed with γ and δ, using equation (29); $D_{q_1 \to q_2}$ with ϕ_d, using equation (37) where ϕ_d is picked at random from a table; and $\underline{\omega}^{q3}$ is defined with equation (41). The matrix Ω_s is computed using equation (21), with $\underline{\omega}_s$ computed with equation (23), and $\underline{\omega}_0$ is computed with equation (42).

The on-line computations during stimulus presentation are aimed at computing the "disturbed" vector of angular rotation $\underline{\omega}_d^b(t)$ with equation (40), which constitutes the vector of rotation at which the object proceeds rotating at time t. This requires the computation of $\underline{\omega}^b(t) \equiv \omega_0\underline{e}(t)$ with equations (26) and (27), which is the undisturbed vector of rotation for the remaining most direct path between orientation b and b_f, and the computation of various DCMs for transformations between the b-, i-, q_0-, q_1-, q_2-, and q_3-systems. It also requires the computation of the "angular distance" from b to b_f, $\alpha(t)$ with equation (11). With $\alpha(t)$, the angle of disturbance $\nu(t)$ is computed with equation (28). The disturbed vector of rotation is used to compute $\Omega(t)$ with equation (21), which in its turn, is used to compute the rotational path, described by the quaternion $\underline{q}_{i \to b}(t)$ by solving the differential equation of equation (20).

It can be shown that the averaged rotational deviation from the most direct path, $\bar{\beta}$, and the duration of the rotation, normalized with respect to the duration of the most direct path, $(t_f - t_0)\omega_0/\alpha_0$, depend on the parameters v_0, e, and α_0 only, and are not affected by the initial orientation, the orientation of the Euler axis, or the angle of the plane of disturbance, ϕ_d. Hence these values can be precomputed and tabulated. For each set of v_0, e, and α_0, the rotational path is preexecuted twice: the first time to compute the time interval $(t_f - t_0)$ and the second time, using this interval, to compute the averaged deviation, $\bar{\beta}$.

At run time the relevant parameter for specifying the magnitude of distortion of the rotational path is $\bar{\beta}$ rather than v_0. Therefore, before starting the run, the value of v_0 for realizing the trajectory with the specified $\bar{\beta}$ is computed by linear interpolation from the tabulated values. In order to verify that the specified trajectory was executed, the actual rotational deviation β is computed on line, and at the end of the presentation of the rotational path, the averaged value is compared with the specified one.

4. CONCLUSION

This report has presented a systematic method for creating suboptimal rotational trajectories. Two parameters of the distortion metric, v_0 and e, independently control the magnitude of the distortion and its "shape" (i.e., how the distortion is distributed over the time course of the trajectory). These systematic disturbances can be introduced into a motion-control program to examine human observers' sensitivity to rotational trajectory optimality. In addition, this algorithm can be used in other applications requiring systematic and reproducible disturbances of rotational trajectories.

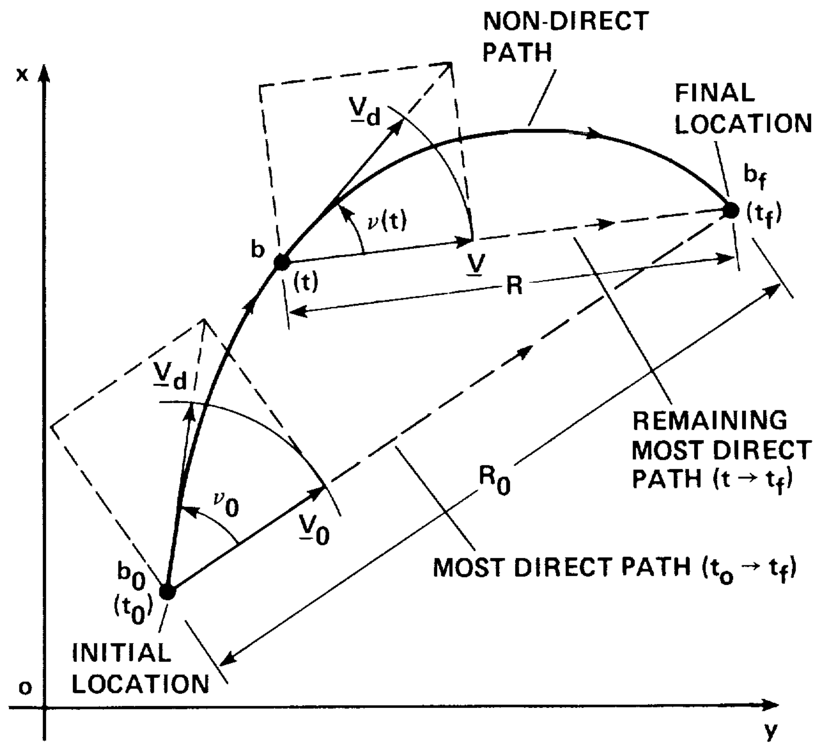

Figure 1. Method for creating a suboptimal trajectory for translational motion.

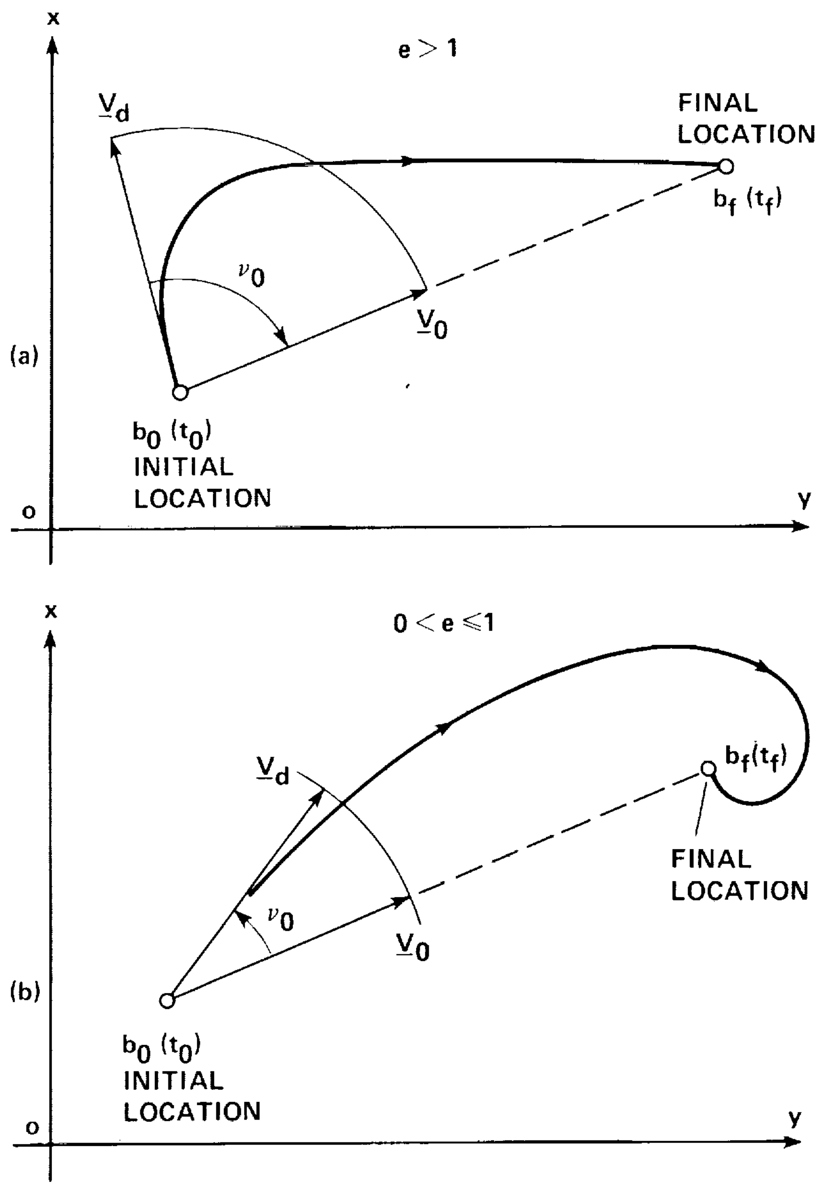

Figure 2. The effect of the exponent e on the trajectory shape.

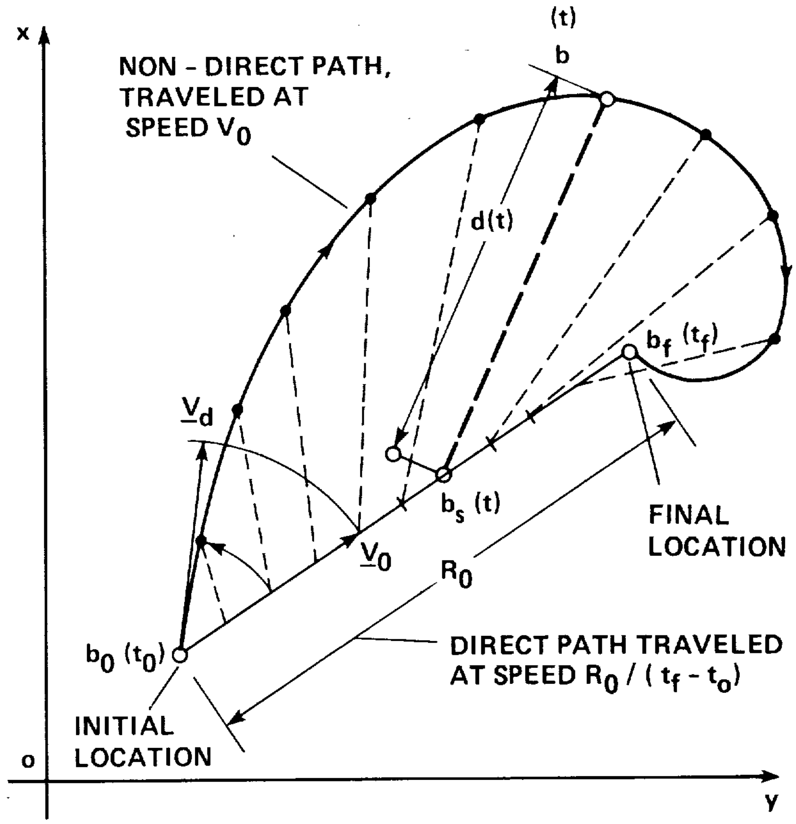

Figure 3. Method for computing the averaged deviation from the most direct path.

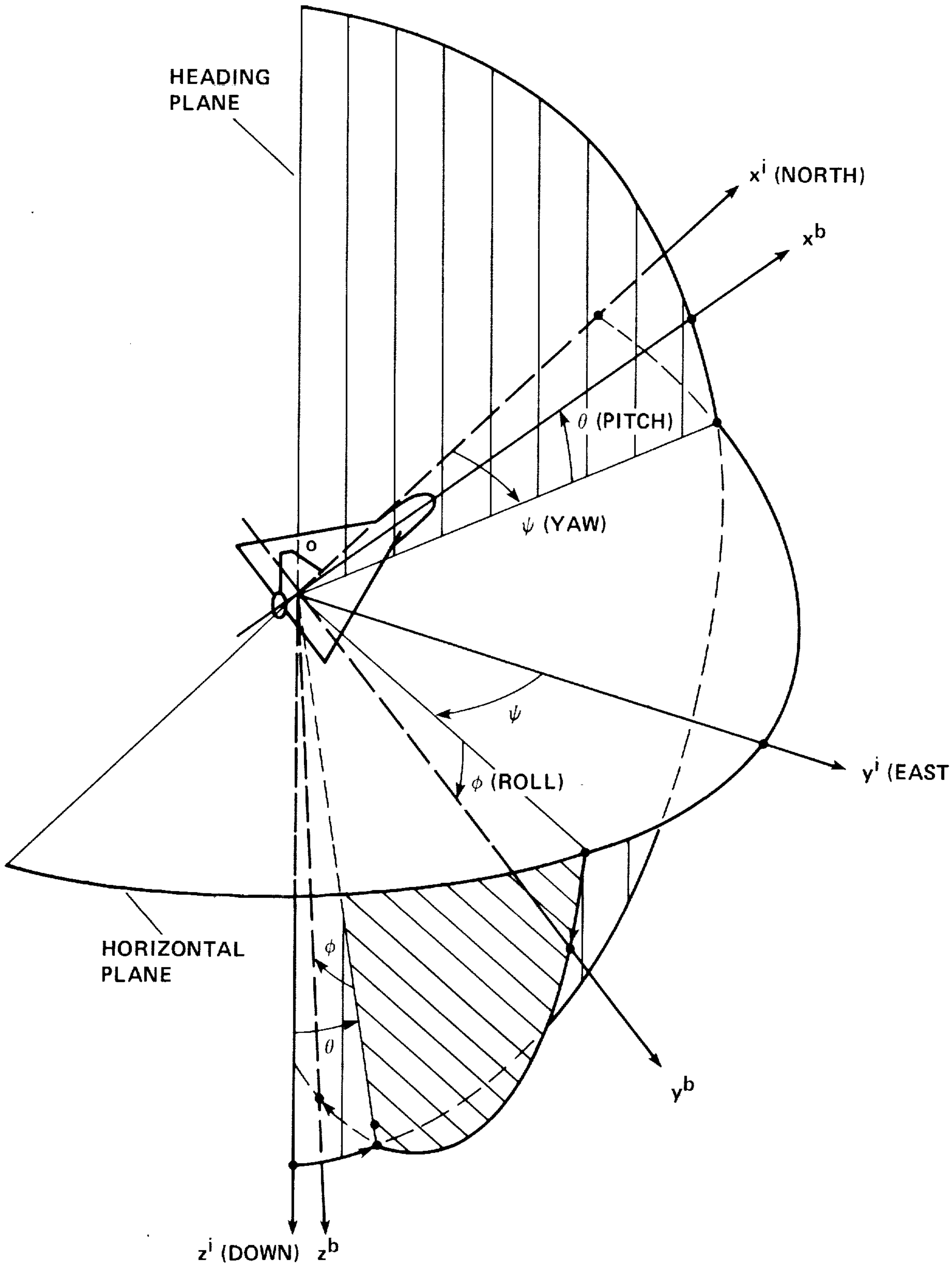

Figure 4. Definition of yaw (ψ), pitch (θ), and roll (ϕ) Euler angles.

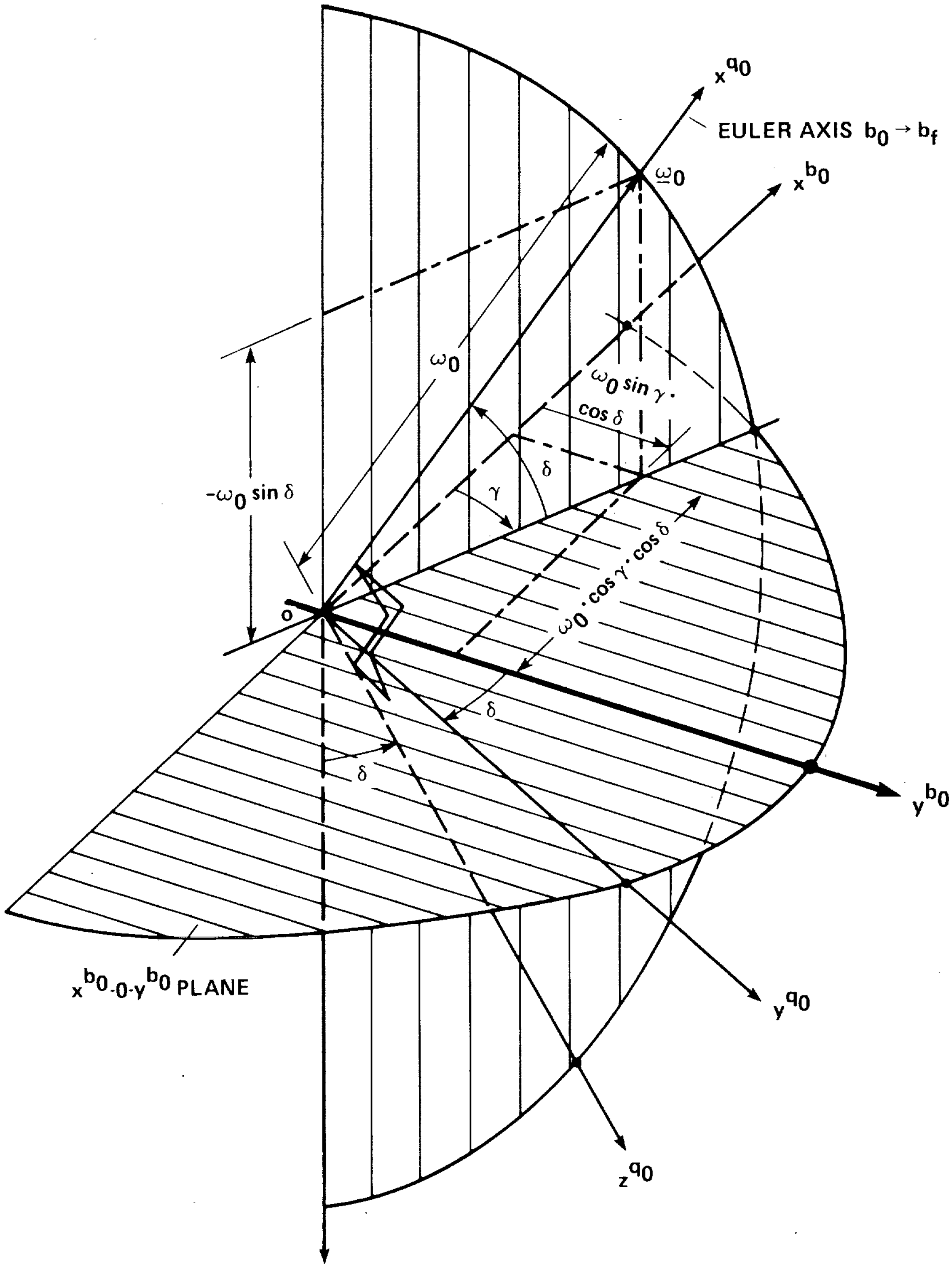

Figure 5. Definition of the q_0-system.

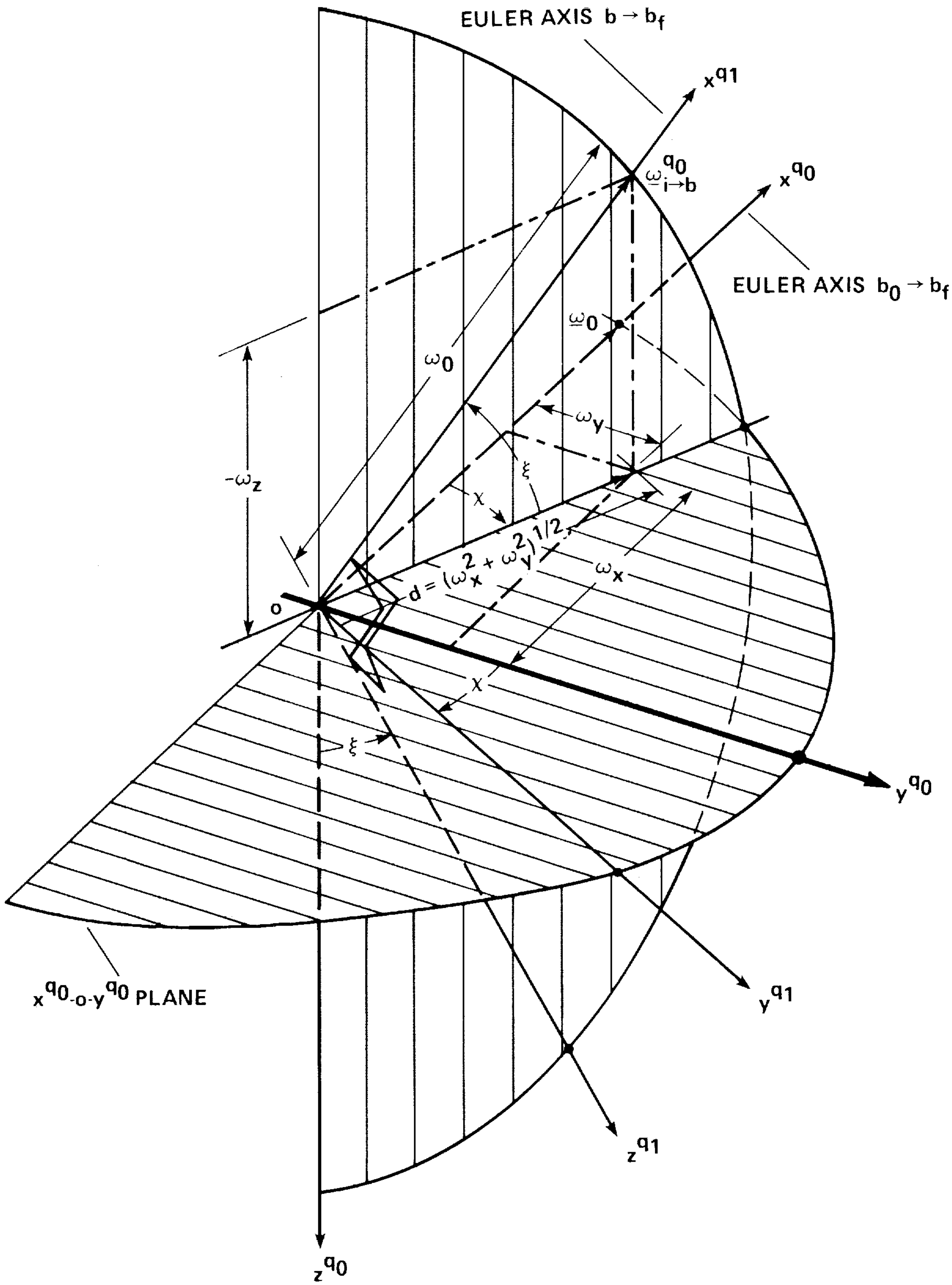

Figure 6. Definition of the q_1-system.

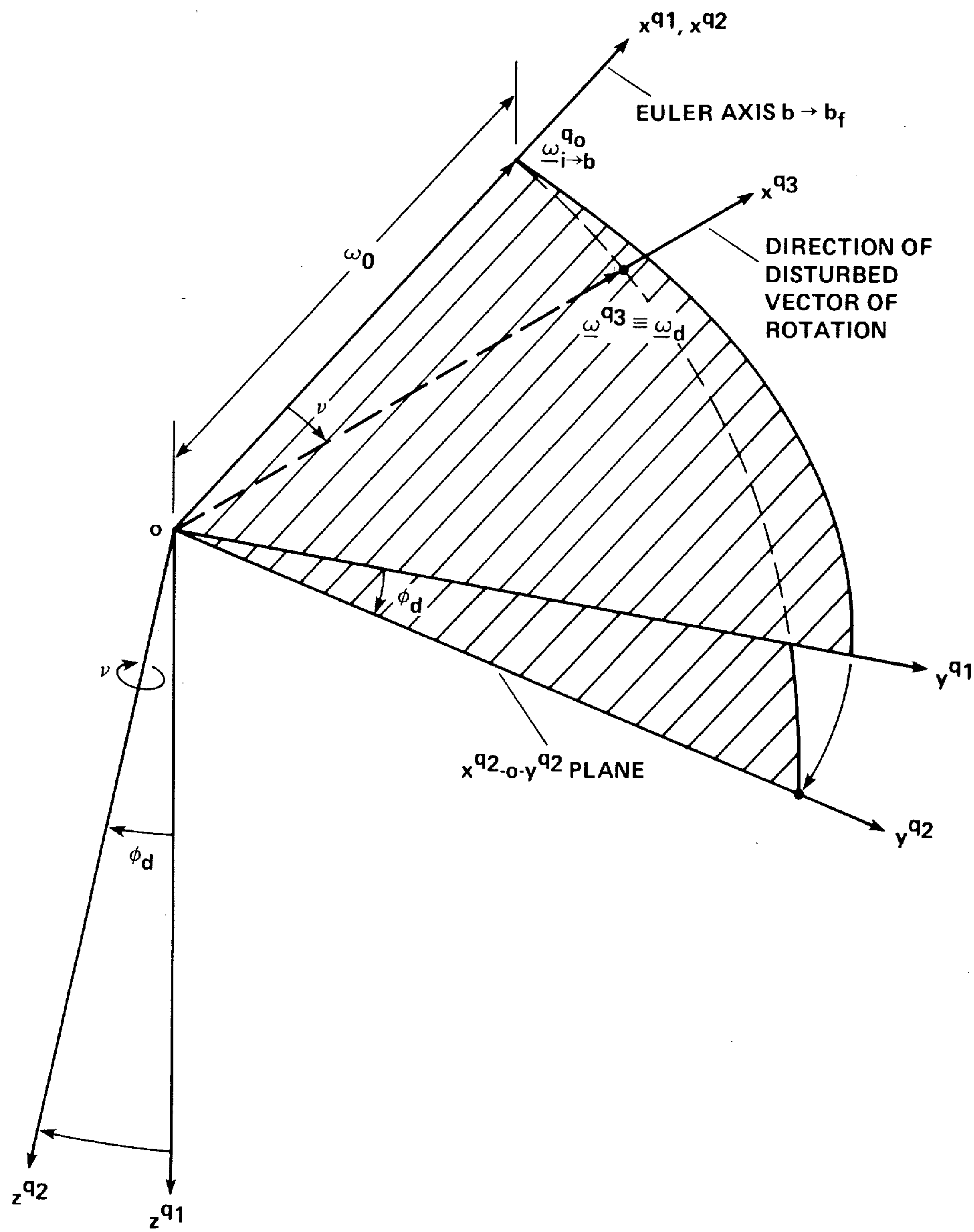

Figure 7. Definition of the q_2 and q_3-systems.

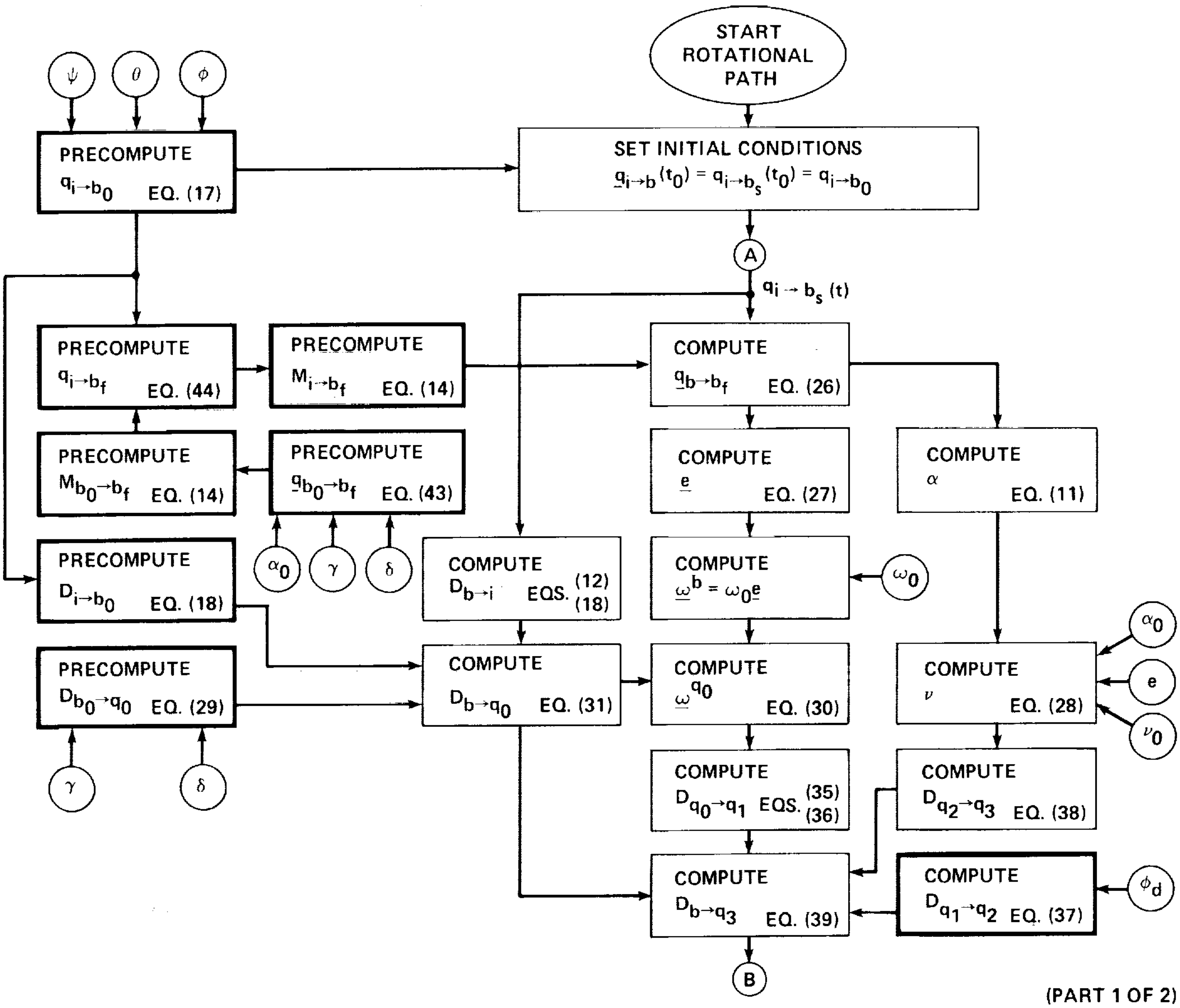

Figure 8. Flow chart of sequence of computations.

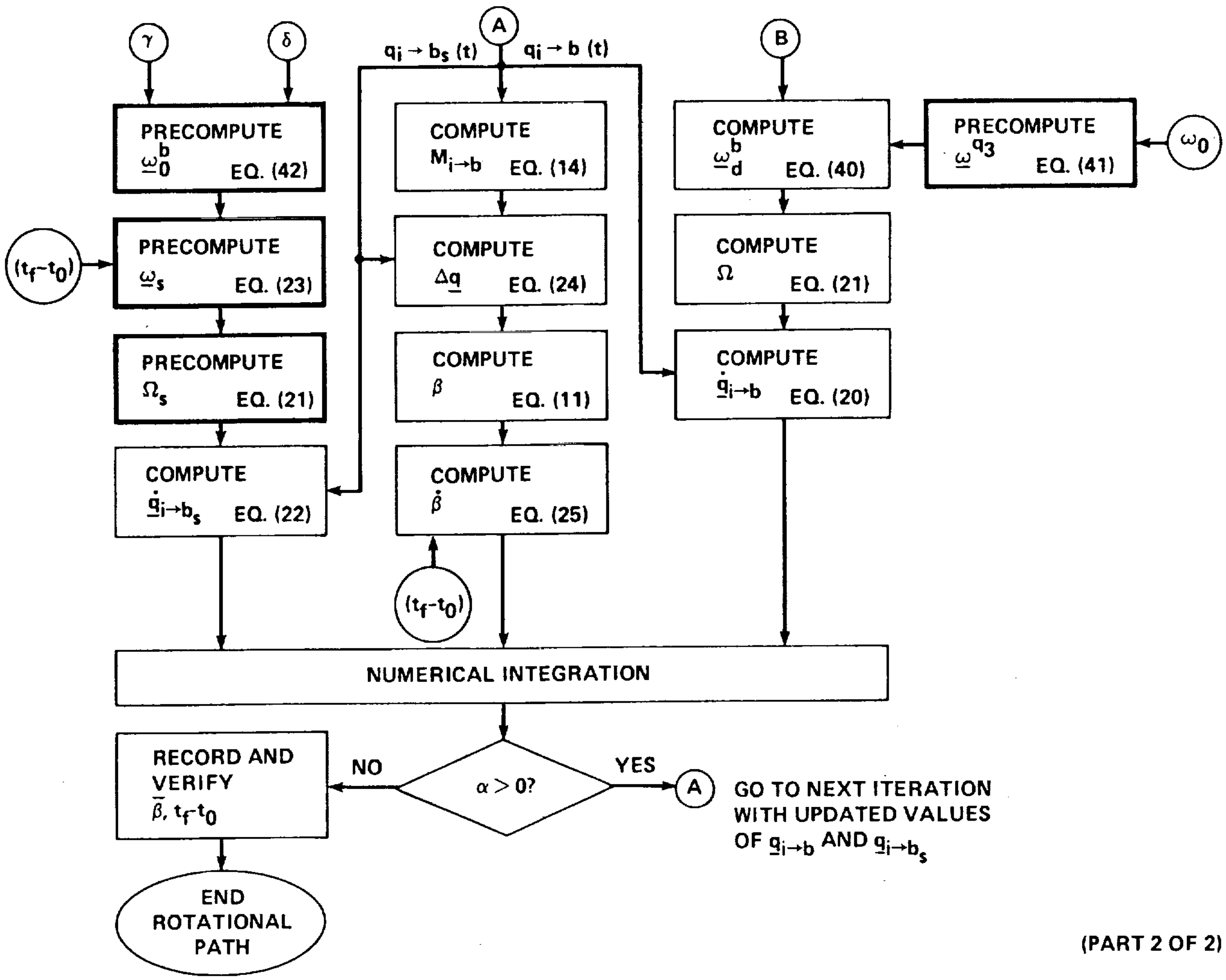

Figure 8. Concluded.

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16. Abstract

An algorithm for introducing a systematic rotational disturbance into an optimal (i.e., single axis) rotational trajectory is described. This disturbance introduces a motion vector orthogonal to the quaternion-defined optimal rotation axis. By altering the magnitude of this vector, the degree of non-optimality can be controlled. The metric properties of the distortion parameter are described, with analogies to two-dimensional translational motion.

This algorithm has been implemented in a motion-control program on a three-dimensional graphic workstation. It supports a series of human performance studies on the detectability of rotational trajectory optimality by naive observers.

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