NASA Contractor Report 181929

ICASE Report No. 89–59

ICASE

CONVERGENCE RESULTS FOR PSEUDOSPECTRAL APPROXIMATIONS OF HYPERBOLIC SYSTEMS BY A PENALTY TYPE BOUNDARY TREATMENT

Daniele Funaro

David Gottlieb

Contract No. NAS1–18605

September 1989

Institute for Computer Applications in Science and Engineering

NASA Langley Research Center

Hampton, Virginia 23665–5225

Operated by the Universities Space Research Association

NASA

National Aeronautics and Space Administration

Langley Research Center

Hampton, Virginia 23665-5225

(NASA-CR-181929) CONVERGENCE RESULTS FOR PSEUDOSPECTRAL APPROXIMATIONS OF HYPERBOLIC SYSTEMS BY A PENALTY TYPE BOUNDARY TREATMENT Final Report (ICASE) 21 p CSCL 12A

N90-12230

Unclas

G3/64 0237061

CONVERGENCE RESULTS FOR PSEUDOSPECTRAL APPROXIMATIONS OF HYPERBOLIC SYSTEMS BY A PENALTY TYPE BOUNDARY TREATMENT

Daniele Funaro

Dipartimento di Matematica
Universit'a di Pavia
Pavia, Italy

David Gottlieb[1]

Division of Applied Mathematics
Brown University
Providence, Rhode Island

ABSTRACT

A new method of imposing boundary conditions in the pseudospectral approximation of hyperbolic systems of equations is proposed. It is suggested to collocate the equations, not only at the inner grid points, but also at the boundary points and use the boundary conditions as penalty terms. In the pseudospectral Legrendre method with the new boundary treatment, a stability analysis for the case of a constant coefficient hyperbolic system is presented and error estimates are derived.

[1]This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

1. Introduction

The importance of the correct numerical implementation of the boundary conditions, in approximating hyperbolic systems of equations, is widely recognized. The pioneering works of Gustafsson, Kreiss & Sundstrom [6] and Osher [7], provide the stability theory for the boundary conditions treatment, in the framework of the finite differences method. Basically, the conditions for stability are reduced to an algebraic problem.

The rôle of boundary conditions in spectral or pseudospectral methods is even more crucial than in finite differences method. One of the reasons is that, since spectral methods are *global* methods, the behavior at one point affects the computation in the whole domain, so that the information at the boundary propagates very fast. Therefore, if on one hand the spectral algorithms do not necessitate special treatment at the boundary (thus in general there are no *numerical boundary conditions*), on the other hand a non correct specification may cause explosive instabilities.

In this paper we introduce a new method of applying boundary conditions when approximating hyperbolic systems of equations by the collocation method. The novel idea is to collocate the differential system at *all* the grid points (included the boundary points) and use the boundary conditions as a penalty term. The same idea was developed in [3] for the pseudospectral Chebyshev discretization of a scalar hyperbolic equation. Here we prove stability and show convergence estimates for the pseudospectral method based on the Legendre nodes.

The paper is structured as follows. In section 2 we describe the method in the scalar case and prove error estimates for the pseudospectral Legendre method. In section 3 we treat the system in the diagonal form with coupling through the boundary conditions. A complete convergence result is shown. In section 4 we show how to implement the method in a more general case.

2. Estimates for the scalar equation

In this section we derive error estimates for the new pseudospectral Legendre approximation to a scalar hyperbolic constant coefficients problem. Let $U \equiv U(x,t)$ be the solution to:

$$(2.1) \qquad \begin{cases} (a) \quad U_t = aU_x & x \in [-1,1], \quad 0 < t < T, \\ (b) \quad U(1,t) = h(t) & 0 < t < T, \\ (c) \quad U(x,0) = f(x) & x \in [-1,1], \end{cases}$$

where h and f are given functions and $a \in \mathbf{R},\ a > 0$.

Let N to be an integer. In the pseudospectral Legendre method we approximate U by $v \equiv v(x,t)$, which is required to be a polynomial of degree at most N in the variable x for any $0 \le t < T$. This is done by demanding that v satisfies equation (2.1)(a) at the grid points x_j $(j = 1, N)$. The points x_j $(j = 0, N)$ are the nodes of the Gauss-Lobatto quadrature formula and the extrema in [-1,1] of the N^{th} degree Legendre polynomial P_N. More precisely we take $x_0 = 1$, $x_N = -1$, while x_j $(j = 1, N-1)$ are the zeroes of P'_N in decreasing order.

The choice of this particular grid allows an accurate evaluation of integrals by summing over the grid values. Namely, let ω_j $(j = 0, N)$ be the weights of the Gauss-Lobatto formula, then for any polynomial p of degree at most $2N-1$ the following equality holds (see, for instance [2]):

$$(2.2) \qquad \int_{-1}^{1} p\,dx = \sum_{j=0}^{N} p(x_j)\omega_j.$$

In the following, we will set $\omega = \omega_0 = \omega_N = \frac{2}{N(N+1)}$.

The new method involves a different treatment of the boundary condition (2.1)(b) (see [3]), rather than imposing exactly such a condition. In fact, we are concerned in finding v such that:

$$(2.3) \qquad \begin{cases} (a) \quad v_t = av_x & \text{at } x = x_j, \quad j = 1, N, \\ (b) \quad v_t(1,t) = av_x(1,t) - \gamma(v(1,t) - h(t)), & \\ (c) \quad v(x_j, 0) = f(x_j) & j = 0, N. \end{cases}$$

The coefficient $\gamma > 0$ in (2.3)(b) will be specified later for the analysis of stability.

It is convenient to compare $v(\cdot,t)$ with some projection of the solution $U(\cdot,t)$ in the space of polynomials. To this end we introduce two projection operators denoted by I_N and Π_N respectively.

Definition 2-1 $I_N U$ is the polynomial of degree at most N that interpolates U at the points x_j $(j = 0, N)$, i.e.: $I_N U(x_j) = U(x_j)$, $j = 0, N$.

Let $H^\sigma(-1,1)$, $\sigma \in \mathbf{R}$ be the usual Sobolev spaces with $H^0(-1,1) = L^2(-1,1)$.

Definition 2-2 $\Pi_N U$ is the polynomial of degree at most N that is the best approximation of U in the $H^1(-1,1)$ norm with the condition: $\Pi_N U(\pm 1) = U(\pm 1)$.

The two following error estimates concerning the projectors I_N and Π_N can be found in [1]:

$$\|U - I_N U\|_{H^\mu(-1,1)} \le C N^{2\mu-\sigma} \|U\|_{H^\sigma(-1,1)}, \tag{2.4}$$

$$\forall U \in H^\sigma(-1,1), \quad \sigma > \frac{1}{2}, \quad 0 \le \mu \le \sigma;$$

$$\|U - \Pi_N U\|_{H^\mu(-1,1)} \le C N^{\mu-\sigma} \|U\|_{H^\sigma(-1,1)}, \tag{2.5}$$

$$\forall U \in H^\sigma(-1,1), \quad \sigma \ge 1, \quad 0 \le \mu \le \sigma.$$

In the next theorem we estimate the error $\delta = \Pi_N U - v$, where U is the solution to (2.1) and v is the solution to (2.3).

Theorem 2-1
Let $\gamma \ge \frac{a}{\omega}$ then we have:

$$\sum_{j=0}^{N} \delta^2(x_j,T)\omega_j + a\int_0^T (\delta^2(1,t) + \delta^2(-1,t))dt \le \tag{2.6}$$

$$\le e\left(\sum_{j=0}^{N}(I_N f - \Pi_N f)^2(x_j)\omega_j + T\int_0^T \sum_{j=0}^{N} Q^2(x_j,t)\omega_j dt\right),$$

where $Q = a\,[\Pi_N U_x - (\Pi_N U)_x]$.

Proof. Applying Π_N to (2.1)(a), one gets:

$$(2.7) \qquad \begin{cases} (\Pi_N U)_t = a(\Pi_N U)_x + Q & \text{at } x = x_j, \quad j = 0, N, \\ (\Pi_N U)(1,t) = h(t) & 0 < t < T, \\ (\Pi_N U)(x_j, 0) = (\Pi_N f)(x_j) & j = 0, N. \end{cases}$$

Therefore, upon subtracting (2.3) from (2.7), we have the following error equation:

$$(2.8) \qquad \begin{cases} (a) \quad \delta_t = a\delta_x + Q & \text{at } x = x_j, \ j = 1, N, \\ (b) \quad \delta_t(1,t) = a\delta_x(1,t) + Q(1,t) - \gamma\delta(1,t), & \\ (c) \quad \delta(x_j, 0) = (\Pi_N f - I_N f)(x_j) & j = 0, N. \end{cases}$$

Note that δ is a polynomial of degree at most N.
Multiplying (2.8)(a) by $\delta(x_j,t)\omega_j$ $(j = 1, N)$, (2.8)(b) by $\delta(1,t)\omega_0$, and summing, we get (using (2.2)):

$$(2.9) \qquad \sum_{j=0}^{N} (\delta_t \delta)(x_j, t)\omega_j = a \int_{-1}^{1} \delta_x \delta dx \ + \\ + \sum_{j=0}^{N} (Q\delta)(x_j,t)\omega_j - \gamma\omega\delta^2(1,t), \qquad 0 \leq t \leq T.$$

Integration by parts yields:

$$(2.10) \qquad \frac{d}{dt} \sum_{j=0}^{N} \delta^2(x_j,t)\omega_j = (a - 2\gamma\omega)\delta^2(1,t) - a\delta^2(-1,t) + 2\sum_{j=0}^{N} (Q\delta)(x_j,t)\omega_j \leq$$

$$\leq -a(\delta^2(1,t)+\delta^2(-1,t)) + \frac{1}{T}\sum_{j=0}^{N}\delta^2(x_j,t)\omega_j + T\sum_{j=0}^{N}Q^2(x_j,t)\omega_j, \quad 0\leq t\leq T,$$

where we used the relation $a - 2\gamma\omega \leq -a$. Finally the Gronwall lemma yields (2.6).○

Recall now that there exist two positive constants C_1 and C_2 such that one has (see [1], page 286):

$$C_1\int_{-1}^{1} p^2 dx \leq \sum_{j=0}^{N} p^2(x_j)\omega_j \leq C_2\int_{-1}^{1} p^2 dx, \tag{2.11}$$

for any polynomial of degree at most N. So, we can prove the following result.

Theorem 2-2
Let $U \in L^2(0,T;H^\sigma(-1,1)) \cap L^\infty(0,T;H^{\sigma-1}(-1,1)), \sigma \geq 1$ be the solution of (2.1). Let v be the solution of (2.3) with $\gamma \geq \frac{a}{\omega}$. Then we have the error bounds:

$$\|(U-v)(\cdot,T)\|_{L^2(-1,1)} \leq \tag{2.12}$$

$$\leq CN^{1-\sigma}\left(\sqrt{T}\,\|U\|_{L^2(0,T;H^\sigma(-1,1))} + \|U(\cdot,T)\|_{H^{\sigma-1}(-1,1)} + \|f\|_{H^{\sigma-1}(-1,1)}\right),$$

$$\left(\int_0^T (U-v)^2(\pm 1,t)\,dt\right)^{\frac{1}{2}} \leq \tag{2.13}$$

$$\leq CN^{1-\sigma}\left(\sqrt{T}\,\|U\|_{L^2(0,T;H^\sigma(-1,1))} + \|f\|_{H^{\sigma-1}(-1,1)}\right).$$

Proof. Note first that by (2.5) Q can be estimated in the following way:

$$\|Q\|_{L^2(-1,1)} \leq a\|\Pi_N U - U\|_{H^1(-1,1)} + a\|\Pi_N U_x - U_x\|_{L^2(-1,1)} \leq \tag{2.14}$$

$$\leq CN^{1-\sigma}\|U\|_{H^\sigma(-1,1)}, \qquad \sigma \geq 1.$$

Then, (2.12) is easily obtained by the previous theorem, by (2.11), (2.4) and by the triangle inequality:

$$\|U - v\|_{L^2(-1,1)} \leq \|U - \Pi_N U\|_{L^2(-1,1)} + \|\Pi_N U - v\|_{L^2(-1,1)}. \tag{2.15}$$

To show (2.13) it is sufficient to observe that, by definition, $\Pi_N U$ coincides with U at $x = \pm 1$ for any t, so that:

$$\int_0^T (\Pi_N U - U)^2(\pm 1, t)dt = 0, \tag{2.16}$$

and then use the estimates shown above. ○

Note that the same results hold when $a < 0$ and the rôle of the boundary points $x = 1$ and $x = -1$ is interchanged. We note also that, as in the Chebyshev case (see [3]), γ has to be proportional to N^2 in order to prove a stability result.

3. Error analysis for a system of equations

Let k_1, k_2 be two integers and denote by $A^{(i)}$ the following $k_i \times k_i$ $(i = 1,2)$ diagonal matrices:

$$A^{(i)} = diag\{a_1^{(i)}, \cdots, a_j^{(i)}, \cdots, a_{k_i}^{(i)}\};$$

where $a_j^{(1)} > 0,\ \ j = 1, k_1$ and $a_j^{(2)} > 0,\ \ j = 1, k_2$.
Let R be a $k_2 \times k_1$ matrix and L a $k_1 \times k_2$ matrix. Let r and l be the norms of the operators R and L respectively, i.e.:

$$r = \|R\|_{\mathcal{L}(\mathbf{R}^{k_1}, \mathbf{R}^{k_2})}, \qquad l = \|L\|_{\mathcal{L}(\mathbf{R}^{k_2}, \mathbf{R}^{k_1})}.$$

In the following we assume that r and l satisfy the relation:

$$0 < rl < 1. \tag{3.1}$$

Denote by $< \cdot, \cdot >_i$ and by $\| \cdot \|_i$ the scalar product and the norm in $\mathbf{R}^{k_i}$ $(i = 1,2)$ respectively. For a given positive definite $k_i \times k_i$ diagonal matrix $M^{(i)}$, we shall denote by $\sqrt{M^{(i)}}$ the corresponding positive definite $k_i \times k_i$ diagonal matrix whose entries are the square roots of the entries of $M^{(i)}$. Note that we have the equalities:

$$< M^{(i)}\phi, \psi >_i = < \phi, M^{(i)}\psi >_i = < \sqrt{M^{(i)}}\phi, \sqrt{M^{(i)}}\psi >_i , \tag{3.2}$$

$$\forall \phi, \psi \in \mathbf{R}^{k_i}, \quad i = 1,2.$$

We are now ready to state the differential problem. We are concerned with finding the functions vector $U \equiv U(x,t)$, where $U \equiv (U^{(1)}, U^{(2)})$ and $U^{(i)}(x,t) \in \mathbf{R}^{k_i}$, $\forall x \in [-1,1]$, $\forall t \in [0,T]$, $i = 1,2$; such that:

$$\begin{cases} U_t^{(1)} = -A^{(1)} U_x^{(1)} \\ U_t^{(2)} = A^{(2)} U_x^{(2)}, \quad x \in [-1,1], \ \ 0 < t < T, \end{cases} \tag{3.3}$$

$$
(3.4) \qquad \begin{cases} U^{(1)}(-1,t) = LU^{(2)}(-1,t) + g_1(t) \\ U^{(2)}(1,t) = RU^{(1)}(1,t) + g_2(t), \qquad 0 < t < T, \end{cases}
$$

$$
(3.5) \qquad \begin{cases} U^{(1)}(x,0) = f_1(x) \\ U^{(2)}(x,0) = f_2(x), \qquad x \in [-1,1], \end{cases}
$$

where $f_i, g_i \;\; (i = 1,2)$ are given functions.
In the Legendre collocation method we seek $u \equiv u(x,t)$ with $u \equiv (u^{(1)}, u^{(2)}), \;\; u^{(i)}(x,t) \in \mathbf{R}^{k_i}, \;\; \forall x \in [-1,1], \;\; \forall t \in [0,T], \; i = 1,2$. The vectors $u^{(i)}, \; i = 1,2$, whose components are polynomials in the variable x of degree at most N, are determined according to the following collocation scheme:

$$
(3.6) \qquad \begin{cases} u_t^{(1)} = -A^{(1)} u_x^{(1)} & \text{at } x = x_j, \;\; j = 0, N-1, \\ u_t^{(2)} = A^{(2)} u_x^{(2)} & \text{at } x = x_j, \;\; j = 1, N, \end{cases}
$$

$$
(3.7) \qquad \begin{cases} u_t^{(1)}(-1,t) = -A^{(1)} u_x^{(1)}(-1,t) - \Gamma^{(1)}\{u^{(1)}(-1,t) - Lu^{(2)}(-1,t) - g_1(t)\}, \\ u_t^{(2)}(1,t) = A^{(2)} u_x^{(2)}(1,t) - \Gamma^{(2)}\{u^{(2)}(1,t) - Ru^{(1)}(1,t) - g_2(t)\}, \end{cases}
$$

$$
(3.8) \qquad \begin{cases} u^{(1)}(x_j,0) = f_1(x_j) & j = 0, N, \\ u^{(2)}(x_j,0) = f_2(x_j) & j = 0, N. \end{cases}
$$

Here $\Gamma^{(i)}, \;\; i = 1,2$ are positive definite $k_i \times k_i$ diagonal matrices to be specified later. Note that we took into account at the points $x_0 = 1$ and $x_N = 1$ both the equations and the boundary conditions.

In order to study the convergence of u to U when N tends to $+\infty$, we introduce another collocation scheme. Namely, we look for $v \equiv v(x,t),\ v \equiv (v^{(1)}, v^{(2)})$, such that:

$$\text{(3.9)} \qquad \begin{cases} v_t^{(1)} = -A^{(1)} v_x^{(1)} & \text{at } x = x_j, \quad j = 0, N-1, \\ v_t^{(2)} = A^{(2)} v_x^{(2)} & \text{at } x = x_j, \quad j = 1, N, \end{cases}$$

$$\text{(3.10)} \qquad \begin{cases} v_t^{(1)}(-1,t) = -A^{(1)} v_x^{(1)}(-1,t) - \Gamma^{(1)}\{v^{(1)}(-1,t) - U^{(1)}(-1,t)\}, \\ v_t^{(2)}(1,t) = A^{(2)} v_x^{(2)}(1,t) - \Gamma^{(2)}\{v^{(2)}(1,t) - U^{(2)}(1,t)\}, \end{cases}$$

$$\text{(3.11)} \qquad \begin{cases} v^{(1)}(x_j,0) = f_1(x_j) & j = 0, N, \\ v^{(2)}(x_j,0) = f_2(x_j) & j = 0, N. \end{cases}$$

This time the system is totally uncoupled. Instead of comparing U and u we estimate the error between u and v. The scalar analysis will enable us to estimate the difference $U - v$.

Theorem 3-1
Let $\Gamma^{(i)} = A^{(i)} \frac{1}{\omega\sqrt{rl}}$, *then we have the estimate:*

$$\text{(3.12)} \qquad \sum_{j=0}^{N} \left(r \, \|B^{(1)} w^{(1)}(x_j,T)\|_1^2 \; + \; l \, \|B^{(2)} w^{(2)}(x_j,T)\|_2^2 \right) \omega_j \le$$

$$\le \frac{rl}{2(1-\sqrt{rl})} \int_0^T \left(\sqrt{\frac{r}{l}} \, \|(U^{(1)} - v^{(1)})(1,t)\|_1^2 \; + \; \sqrt{\frac{l}{r}} \, \|(U^{(2)} - v^{(2)})(-1,t)\|_2^2 \right) dt,$$

where $w = u - v$ *and* $B^{(i)} = \sqrt{[A^{(i)}]^{-1}}$, $i = 1, 2$.

Proof. From (3.6)-(3.11) it is clear that w satisfies the equations:

$$(3.13) \qquad \begin{cases} w_t^{(1)} = -A^{(1)} w_x^{(1)} & \text{at } x = x_j, \quad j = 0, N-1, \\ w_t^{(2)} = A^{(2)} w_x^{(2)} & \text{at } x = x_j, \quad j = 1, N, \end{cases}$$

(3.14)

$$\begin{cases} w_t^{(1)}(-1,t) = -A^{(1)} w_x^{(1)}(-1,t) - \Gamma^{(1)}\{w^{(1)}(-1,t) - Lw^{(2)}(-1,t) + L\epsilon^{(2)}(-1,t)\}, \\ w_t^{(2)}(1,t) = A^{(2)} w_x^{(2)}(1,t) - \Gamma^{(2)}\{w^{(2)}(1,t) - Rw^{(1)}(1,t) + R\epsilon^{(1)}(1,t)\}, \end{cases}$$

$$(3.15) \qquad \begin{cases} w^{(1)}(x_j, 0) = 0 & j = 0, N, \\ w^{(2)}(x_j, 0) = 0 & j = 0, N, \end{cases}$$

where $\epsilon^{(i)} = U^{(i)} - v^{(i)}$, $i = 1, 2$. Note that (3.4) has been used to eliminate $U^{(1)}(-1,t)$ and $U^{(2)}(1,t)$.

Now,let $D^{(i)}$, $i = 1, 2$ two diagonal positive definite $k_i \times k_i$ matrices to be specified later. Multiplying the first set of equations in (3.13) and (3.14) by $(D^{(1)} w^{(1)})(x_j, t)\omega_j$ and summing up on $j = 0, N$, we get by (2.2):

$$(3.16) \quad \sum_{j=0}^{N} < w_t^{(1)}, D^{(1)} w^{(1)} >_1 (x_j, t)\omega_j = -\int_{-1}^{1} < A^{(1)} w_x^{(1)}, D^{(1)} w^{(1)} >_1 (x,t)dx \ +$$

$$-\omega < \Gamma^{(1)} w^{(1)} - \Gamma^{(1)} L w^{(2)}, D^{(1)} w^{(1)} >_1 (-1,t) - \omega < \Gamma^{(1)} L\epsilon^{(2)}, D^{(1)} w^{(1)} >_1 (-1,t) \leq$$

$$\leq -\frac{1}{2} < A^{(1)} w^{(1)}, D^{(1)} w^{(1)} >_1 (1,t) \ + \ \frac{1}{2} < A^{(1)} w^{(1)}, D^{(1)} w^{(1)} >_1 (-1,t) \ +$$

$$-\omega < \Gamma^{(1)} w^{(1)} - \Gamma^{(1)} L w^{(2)}, D^{(1)} w^{(1)} >_1 (-1,t) +$$

$$+ \frac{\eta}{2}\omega < \Gamma^{(1)} w^{(1)}, D^{(1)} w^{(1)} >_1 (-1,t) \; + \; \frac{\omega}{2\eta} \|\sqrt{D^{(1)}\Gamma^{(1)}} \; L\epsilon^{(2)}(-1,t)\|_1^2,$$

where $0 < \eta < 2$. In the same way we have:

$$\sum_{j=0}^{N} < w_t^{(2)}, D^{(2)} w^{(2)} >_2 (x_j,t)\omega_j \leq \tag{3.17}$$

$$\leq \frac{1}{2} < A^{(2)} w^{(2)}, D^{(2)} w^{(2)} >_2 (1,t) \; - \; \frac{1}{2} < A^{(2)} w^{(2)}, D^{(2)} w^{(2)} >_2 (-1,t) +$$

$$-\omega < \Gamma^{(2)} w^{(2)} - \Gamma^{(2)} R w^{(1)}, D^{(2)} w^{(2)} >_2 (1,t) +$$

$$+ \frac{\eta}{2}\omega < \Gamma^{(2)} w^{(2)}, D^{(2)} w^{(2)} >_2 (1,t) \; + \; \frac{\omega}{2\eta} \|\sqrt{D^{(2)}\Gamma^{(2)}} \; R\epsilon^{(1)}(1,t)\|_2^2.$$

Note that η, $D^{(1)}$ and $D^{(2)}$ are still to be specified. We start by setting $\eta = 2(1-\sqrt{rl})$. Recalling the hypotheses on $\Gamma^{(i)}$, by summing (3.16) and (3.17) one obtains:

$$\frac{d}{dt}\left(\sum_{j=0}^{N}\left[\|\sqrt{D^{(1)}}w^{(1)}\|_1^2 \; + \; \|\sqrt{D^{(2)}}w^{(2)}\|_2^2\right](x_j,t)\omega_j\right) = \tag{3.18}$$

$$= -\left[\|\sqrt{A^{(1)}D^{(1)}}w^{(1)}\|_1^2 - \frac{2}{\sqrt{rl}} < A^{(2)} R w^{(1)}, D^{(2)} w^{(2)} >_2 + \|\sqrt{A^{(2)}D^{(2)}}w^{(2)}\|_2^2\right](1,t) +$$

$$-\Big[\|\sqrt{A^{(1)}D^{(1)}}w^{(1)}\|_1^2 - \frac{2}{\sqrt{rl}} < A^{(1)}Lw^{(2)}, D^{(1)}w^{(1)} >_1 + \|\sqrt{A^{(2)}D^{(2)}}w^{(2)}\|_2^2\Big](-1,t) +$$

$$+\frac{1}{2(1-\sqrt{rl})\sqrt{rl}}\left(\|\sqrt{D^{(2)}A^{(2)}}\,R\epsilon^{(1)}(1,t)\|_2^2 + \|\sqrt{D^{(1)}A^{(1)}}\,L\epsilon^{(2)}(-1,t)\|_1^2\right).$$

At this point we want the two first terms in brackets on the RHS of (3.18) to be positive. This is true if we choose:

$$D^{(1)} = \left(A^{(1)}\right)^{-1} r^2, \qquad D^{(2)} = \left(A^{(2)}\right)^{-1} rl.$$

In fact, the first of those terms becomes:

$$r^2\|w^{(1)}\|_1^2 \; - \; 2\sqrt{rl} < Rw^{(1)}, w^{(2)} >_2 \; + rl\|w^{(2)}\|_2^2,$$

and this is positive since:

$$|2\sqrt{rl} < Rw^{(1)}, w^{(2)} >_2| \le \|Rw^{(1)}\|_2^2 \; + \; rl\|w^{(2)}\|_2^2 \le r^2\|w^{(1)}\|_1^2 \; + \; rl\|w^{(2)}\|_2^2.$$

Similar arguments hold for the second term. This allows us to write the following inequality:

$$\text{(3.19)} \qquad \frac{d}{dt}\left(\sum_{j=0}^{N}\Big[r\|B^{(1)}w^{(1)}(x_j,t)\|_1^2 \; + \; l\|B^{(2)}w^{(2)}(x_j,t)\|_2^2\Big]\omega_j\right) \le$$

$$\le \frac{1}{2(1-\sqrt{rl})\sqrt{rl}}\left(l\|R\epsilon^{(1)}(1,t)\|_2^2 \; + \; r\|L\epsilon^{(2)}(-1,t)\|_1^2\right) \le$$

$$\leq \frac{rl}{2(1-\sqrt{rl})}\left(\sqrt{\frac{r}{l}}\|\epsilon^{(1)}(1,t)\|_1^2 + \sqrt{\frac{l}{r}}\|\epsilon^{(2)}(-1,t)\|_2^2\right).$$

Finally, by integrating in time, we obtain (3.12).○

Using the results of the previous section we can finally prove our main convergence theorem.

Theorem 3-2

Let U be the solution of (3.3)-(3.4)-(3.5) and let u be the solution to (3.6)-(3.7)-(3.8) with $\Gamma^{(i)} = A^{(i)}\frac{N(N+1)}{2\sqrt{rl}}$. Then one has:

$$\left\| \left(\sum_{i=1}^{2}\|U^{(i)}-u^{(i)}\|_i^2\right)^{\frac{1}{2}}(\cdot,T)\right\|_{L^2(-1,1)} \leq \tag{3.20}$$

$$\leq CN^{1-\sigma}T\left\| \left(\sum_{i=1}^{2}\|U^{(i)}\|_i^2\right)^{\frac{1}{2}}\right\|_{C^0([0,T];H^\sigma(-1,1))},$$

where C only depends on l, r, σ and $A^{(i)}$, $i = 1, 2$.

Proof. We first write: $U - u = U - v - w$. Then a bound for each component of $U^{(i)} - v^{(i)}$, $i = 1, 2$ is given by (2.12), while for w we have:

$$\left\| \left(\sum_{i=1}^{2}\|w^{(i)}\|_i^2\right)^{\frac{1}{2}}(\cdot,T)\right\|_{L^2(-1,1)} = \tag{3.21}$$

$$= \left\| \left(\sum_{i=1}^{2}\|\left[B^{(i)}\right]^{-1}B^{(i)}w^{(i)}\|_i^2\right)^{\frac{1}{2}}(\cdot,T)\right\|_{L^2(-1,1)}^2 \leq$$

$$\leq \max_{i=1,2}\left\{\left\| \left[B^{(i)}\right]^{-1} \right\|^2_{\mathcal{L}(\mathbf{R}^{k_i},\mathbf{R}^{k_i})}\right\} \left\| \left(\sum_{i=1}^{2} \|B^{(i)} w^{(i)}\|_i^2\right)^{\frac{1}{2}} (\cdot, T) \right\|^2_{L^2(-1,1)} \leq$$

$$\leq C(r,l) \max_{i=1,2} \max_{1\leq j\leq k_i} \{(a_j^{(i)})^2\} \left(\int_0^T \|U^{(1)} - v^{(1)}\|_1^2(1,t)dt + \int_0^T \|U^{(2)} - v^{(2)}\|_2^2(-1,t)dt \right)$$

where we used theorem 3-1. Finally the last term in (3.21) is estimated as in (2.13).○

4. Suggestions for the implementation of non diagonal systems

In this section we discuss the implementation of our new approach to the general hyperbolic system:

$$Z_t = HZ_x, \tag{4.1}$$

where $Z \equiv Z(x,t)$ is a k component vector and H is a constant coefficients $k \times k$ matrix with k_1 negative eigenvalues and k_2 positive eigenvalues $(k = k_1 + k_2)$.
The following boundary conditions are imposed at $x = -1$:

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix} Z(-1,t) = \begin{pmatrix} h_1(t) \\ 0 \end{pmatrix}, \tag{4.2}$$

where B_{11} is a $k_1 \times k_1$ matrix, B_{12} is a $k_1 \times k_2$ matrix and h_1 is a given k_1 components vector. Besides, the following boundary conditions are imposed at $x = 1$:

$$\begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix} Z(1,t) = \begin{pmatrix} 0 \\ h_2(t) \end{pmatrix}, \tag{4.3}$$

where B_{21} is a $k_2 \times k_1$ matrix, B_{22} is a $k_2 \times k_2$ matrix and h_2 is a given k_2 components vector.
Suppose that there exists a nonsingular matrix:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where T_{ij} is a $k_i \times k_j$ matrix, such that the change of variables $Z = TU$ diagonalizes the system (4.1). Thus, we get:

$$\frac{\partial U}{\partial t} = T^{-1}HT\,\frac{\partial U}{\partial x} = \begin{pmatrix} -A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \frac{\partial U}{\partial x}, \tag{4.4}$$

where U and $A^{(i)}$, $i = 1,2$ have been defined in section 3.

The boundary conditions are respectively transformed as follows:

$$(4.5)\qquad (B_{11}T_{11}+B_{12}T_{21})U^{(1)}(-1,t)+(B_{11}T_{12}+B_{12}T_{22})U^{(2)}(-1,t)=h_1(t),$$

$$(4.6)\qquad (B_{21}T_{11}+B_{22}T_{21})U^{(1)}(1,t)+(B_{21}T_{12}+B_{22}T_{22})U^{(2)}(1,t)=h_2(t).$$

Therefore (4.2) and (4.3) are equivalent to (3.4) and (3.5) if and only if the matrices $B_{11}T_{11}+B_{12}T_{21}$ and $B_{21}T_{12}+B_{22}T_{22}$ are invertible. In this case we have:

$$L=-(B_{11}T_{11}+B_{12}T_{21})^{-1}(B_{11}T_{12}+B_{12}T_{22}),$$

$$R=-(B_{21}T_{12}+B_{22}T_{22})^{-1}(B_{21}T_{11}+B_{22}T_{21}),$$

$$g_1=(B_{11}T_{11}+B_{12}T_{21})^{-1}h_1,$$

$$g_2=(B_{21}T_{12}+B_{22}T_{22})^{-1}h_2.$$

We would like to show how to apply the scheme (3.6)-(3.8) directly to the system (4.1)-(4.3). Assuming the hypotheses of theorem 3-2 we have $\Gamma^{(i)}=\beta A^{(i)}$, $i=1,2$, where $\beta=N(N+1)/2\sqrt{rl}$. We also set (denoting by u the approximation to U):

$$\Xi(x,t)=\begin{pmatrix} u^{(1)}(x,t)-Lu^{(2)}(x,t)-g_1(t)\\ u^{(2)}(x,t)-Ru^{(1)}(x,t)-g_2(t)\end{pmatrix}=$$

$$=\begin{pmatrix} I & -L\\ -R & I\end{pmatrix}u(x,t)-\begin{pmatrix} g_1\\ g_2\end{pmatrix}(t).$$

Thus (3.6)-(3.7) can be written in the form:

$$(4.7)\qquad \frac{\partial u}{\partial t}(x_j,t)=A\frac{\partial u}{\partial x}(x_j,t)+\beta\delta_{Nj}A^{(-)}\Xi(-1,t)-\beta\delta_{0j}A^{(+)}\Xi(1,t),\quad j=0,N,$$

where $A^{(-)} = \begin{pmatrix} -A^{(1)} & 0 \\ 0 & 0 \end{pmatrix}$, $\quad A^{(+)} = \begin{pmatrix} 0 & 0 \\ 0 & A^{(2)} \end{pmatrix}$, and $\quad A = A^{(-)} + A^{(+)}$.

Let us now define $z = Tu$. The function z satisfies:

$$\frac{\partial z}{\partial t}(x_j, t) = H\frac{\partial z}{\partial x}(x_j, t) + \beta\delta_{Nj}(TA^{(-)}T^{-1})T\Xi(-1, t) + \tag{4.8}$$

$$-\beta\delta_{0j}(TA^{(+)}T^{-1})T\Xi(1, t), \qquad j = 0, N.$$

Defining $\quad B = T\begin{pmatrix} I & -L \\ -R & I \end{pmatrix} T^{-1}$, we get:

$$T\Xi = Bz - T\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{4.9}$$

Finally, taking $H^{(\pm)} = TA^{(\pm)}T^{-1}$ and substituting in (4.8), we obtain the pseudospectral scheme to approximate (4.1)-(4.3), namely:

$$\frac{\partial z}{\partial t}(x_j, t) = H\frac{\partial z}{\partial x}(x_j, t) + \beta\delta_{Nj}H^{(-)}\left[Bz(-1, t) - T\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t)\right] + \tag{4.10}$$

$$-\beta\delta_{0j}H^{(+)}\left[Bz(1, t) - T\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t)\right], \qquad j = 0, N.$$

This is equivalent to collocate the equation (4.1) at all the points with some suitable penalty terms, deriving from the boundary conditions, added at the points $x = \pm 1$. It is clear that the same convergence estimates of theorem 3-2 also apply for the error $Z - z =$ $= T(U - u)$.

References

[1] C.Canuto, M.Y.Hussaini, A.Quarteroni & T.A.Zang, *Spectral Methods in Fluid Dynamics* , Springer, New York (1987).

[2] P.J.Davis & P.Rabinowitz, *Methods in Numerical Integration*, Academic Press, New York (1984).

[3] D.Funaro & D.Gottlieb, A new method of imposing boundary conditions in pseudospectral approximations of hyperbolic equations, Math. of Comput., Vol.51, n.184 (1988), pp.599-613.

[4] D.Gottlieb, L.Lustman & E.Tadmor, Stability analysis of spectral methods for hyperbolic initial-boundary value systems, SIAM J. Numer. Anal. 24(1987), pp.241-256.

[5] D.Gottlieb, L.Lustman & E.Tadmor, Convergence of spectral methods for hyperbolic initial-boundary value systems, SIAM J. Numer. Anal. 24(1987), pp.532-537.

[6] B.Gustafsson, H.O.Kreiss & A.Sundström, Stability theory of difference approximatios for mixed initial boundary value problems II, Math. of Comput., Vol.26, n.119 (1972), pp.649-686.

[7] S.Osher, Stability of difference approximations of dissipative type for mixed initial-boundary value problems I, Math. of Comput., Vol. 23 (1969), pp.335-340.

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No. NASA CR-181929 ICASE Report No. 89-59	2. Government Accession No.	3. Recipient's Catalog No.

4. Title and Subtitle

CONVERGENCE RESULTS FOR PSEUDOSPECTRAL APPROXIMATIONS OF HYPERBOLIC SYSTEMS BY A PENALTY TYPE BOUNDARY TREATMENT

5. Report Date

September 1989

6. Performing Organization Code

7. Author(s)

Daniele Funaro
David Gottlieb

8. Performing Organization Report No.

89-59

10. Work Unit No.

505-90-21-01

9. Performing Organization Name and Address

Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, V A 23665-5225

11. Contract or Grant No.

NAS1-18605

12. Sponsoring Agency Name and Address

National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665-5225

13. Type of Report and Period Covered

Contractor Report

14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor:
Richard W. Barnwell

Mathematics of Computation

Final Report

16. Abstract

A new method of imposing boundary conditions in the pseudospectral approximation of hyperbolic systems of equations is proposed. It is suggested to collocate the equations, not only at the inner grid points, but also at the boundary points and use the boundary conditions as penalty terms. In the pseudo-spectral Legrendre method with the new boundary treatment, a stability analysis for the case of a constant coefficient hyperbolic system is presented and error estimates are derived.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
spectral approximations, boundary conditions, error estimates	64 Numerical Analysis Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	20	A03

NASA FORM 1626 OCT 86

NASA-Langley, 1989