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N90-13431

EXTENSION OF EULER'S THEOREM TO n-DIMENSIONAL SPACES

by

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Abstract

Euler's theorem states that any sequence of finite rotations of a rigid body can be described as a single rotation of the body about a fixed axis in three dimensional Euclidean space. The usual statement of the theorem in the literature cannot be extended to Euclidean spaces of other dimensions. Equivalent formulations of the theorem are given in this paper and proven in a way which does not limit them to the three dimensional Euclidean space. Thus, the equivalent theorems hold in other dimensions. The proof of one formulation presents an algorithm which shows how to compute an angular-difference matrix that represents a single rotation which is equivalent to the sequence of rotations that have generated the final n-D orientation. This algorithm results also in a constant angular-velocity which, when applied to the initial orientation, yields eventually the final orientation regardless of what angular velocity generated the latter. Finally, the extension of the theorem is demonstrated in a four dimensional numerical example.

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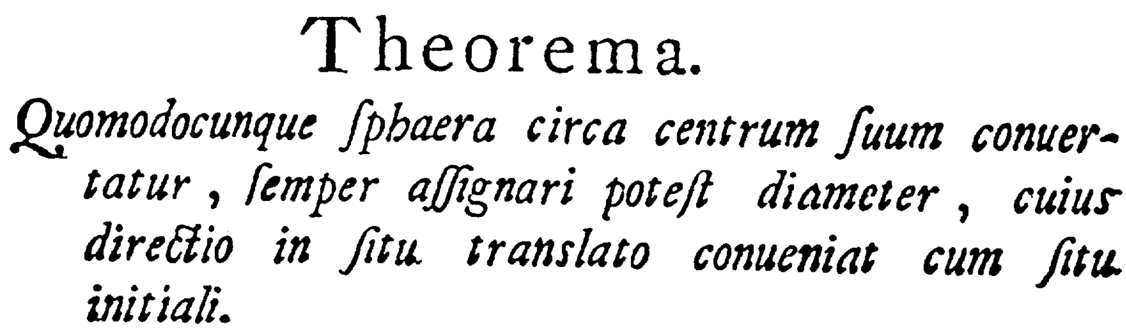
I. EULER'S THEOREM

In 1775 Euler published a paper on the rotation of rigid bodies [1]. In that paper, which was written in Latin, he



Theorema.

Quomodocunque ſphaera circa centrum ſuum conuertatur, ſemper aſſignari poteſt diameter, cuius directio in ſitu translato conueniat cum ſitu initiali.

Fig.1: Euler's theorem on the rotation of a rigid body as it appeared in the 1775 publication.

presented the theorem whose photograph is shown in fig. 1. The theorem states the following:

> In whatever way a sphere is rotated about its center, it is always possible to reckon a diameter about which a rotation brings the sphere into coincidence with its original location.

A modern formulation of this theorem states [2]:

> A body set of axes at any time t can always be obtained by a single rotation of the initial set of axes.

We prefer to formulate this theorem as follows:

> Regardless of the way a coordinate system is rotated from its original orientation, it is always possible to find a fixed axis in space about which a single rotation of the initial coordinates ends at the final orientation.

Euler's theorem serves as a cornerstone in attitude determination [3-9] and tracking [7,9,10]. In particular, if $\hat{n}$ is a unit vector along the axis of rotation and θ is the angle by which the initial coordinate system has to be rotated in order to coincide with the final one, then $D(\hat{n},\theta)$, the transformation matrix from the initial to the final coordinate system, is given by [7-10]

$$D(\hat{n},\theta) = I\cos\theta + (1-\cos\theta)\hat{n}\hat{n}^T - \sin\theta[\hat{n}x] \tag{1}$$

where I denotes the identity matrix, T denotes the matrix transpose and $[\hat{n}x]$ denotes the cross product matrix of $\hat{n}$. The relationship formulated in (1) can also be expressed as follows

$$D(\hat{n},\theta) = e^{-[\underline{\theta}x]} \tag{2}$$

where $\underline{\theta} = \hat{n}\theta$. The rate of change of the vector quantity $\underline{\theta}$ as a function of $\underline{w}$, the angular velocity at which the coordinate system rotates, is given by [9,10]

$$\dot{\underline{\theta}}(t) = \underline{w}(t) + \frac{1}{2}\underline{\theta}(t)x\underline{w}(t) + \frac{2-\theta(t)\cot[\theta(t)/2]}{2\theta(t)^2}\,\underline{\theta}(t)x[\underline{\theta}(t)x\underline{w}(t)] \tag{3}$$

II. ALTERNATE FORMULATIONS OF EULER'S THEOREM

Angular-matrix (discrete) formulation

Denote the skew-symmetric matrix $[\underline{\theta}x]$ by Θ; that is,

$$\Theta = [\underline{\theta}x] \tag{4}$$

where the explicit expression for Θ is:

$$\Theta = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \tag{5}$$

Equation (2) can be written as

$$D(\hat{n},\theta) = e^{-\Theta} \tag{6}$$

Let the initial orientation of a certain coordinate system with respect to some reference system be expressed by the attitude matrix D_o. Suppose now that this coordinate system is rotated from its initial orientation by the sequence of rotations $\underline{\theta}_1$, $\underline{\theta}_2$, ..., $\underline{\theta}_k$. Denote the cross product matrices which correspond to $\underline{\theta}_1$, $\underline{\theta}_2$, ..., $\underline{\theta}_k$ by $\underline{\Theta}_1$, $\underline{\Theta}_2$, ..., $\underline{\Theta}_k$ respectively. Then, in view of (6), the attitude matrix that transforms the reference coordinate system to the final one, and which expresses the orientation of that system, is given by

$$D_f = e^{-\Theta_k} \dots\dots e^{-\Theta_2} e^{-\Theta_1} D_o \tag{7}$$

On the other hand

$$D_f = e^{-\Theta_f} D_o \tag{8}$$

However

$$\Theta_f \neq \Theta_k + \dots\dots + \Theta_2 + \Theta_1 \tag{9}$$

We realize that the equivalence between (7) and (8) is another expression of Euler's theorem. We can, then, formulate Euler's theorem also as follows:

> **Regardless of the way a coordinate system is rotated from its original orientation, it is always possible to express the final orientation of the system by the attitude matrix D_f where**
>
> $$D_f = e^{-\Theta_f} D_o$$
>
> **and Θ_f is a skew-symmetric matrix.**

We call this formulation discrete because the rotation expressed by Θ_f is equivalent to the discrete k rotations expressed by the individual Θ_i (i=1,2, ..., k) matrices.

<u>Angular-rate (continuous) formulation</u>

Euler's theorem gave rise to (1) and (3) which indicate how to find the orientation of a coordinate system, at any given time t_f, with respect to its initial orientation at time t_o if $\underline{w}(t)$, the history of its rate of rotation, is known for $t_o<t<t_f$; namely, $\underline{w}(t)$ is used in (3) to solve for $\underline{\theta}(t)$ and then, that solution is used in (1) to obtain the required orientation specified by $D(t_f)$. The attitude matrix $D(t_f)$ can be computed in yet another way, since the rate of change of $D(t)$ as a function of $\underline{w}(t)$ is given by the well known matrix differential equation

$$\dot{D}(t) = - [\underline{w}(t)x]D(t) \qquad (10)$$

The matrix $[\underline{w}(t)x]$ is defined on the components of $\underline{w}(t)$ when the latter is resolved in the changing (final) coordinate system. We also denote this matrix by $W(t)$; that is,

$$W(t) = [\underline{w}(t)x] \qquad (11)$$

Hence (10) can be written also as

$$\dot{D}(t) = -W(t)D(t) \qquad (12)$$

The explicit expression for W(t) (or $[\underline{w}(t)x]$) is given by the **skew-symmetric** matrix:

$$W(t) = \begin{bmatrix} 0 & -w_3(t) & w_2(t) \\ w_3(t) & 0 & -w_1(t) \\ -w_2(t) & w_1(t) & 0 \end{bmatrix} \qquad (13)$$

There is, then, an equivalence between the pair (1) and (3) on the one hand, and (12) on the other hand.

Euler's theorem states, basically, that there always exists a vector $\underline{\theta}(t_f)$ which specifies the orientation regardless of which $\underline{w}(t)$ generated that $\underline{\theta}(t_f)$. Consequently, any $\underline{w}(t)$ which satisfies the following two conditions, rotates the initial coordinate system into the same orientation

$$\frac{\underline{w}(t)}{w(t)} = \hat{n} \qquad (14.a)$$

$$\int_{t_o}^{t_f} w(t)\, dt = \theta(t_f) \qquad (14.b)$$

While the truth of the last proposition is self evident, it can also be easily verified by solving (3) for any angular rate $\underline{w}(t)$ which satisfies conditions (14). Since **any** $\underline{w}(t)$ which satisfies (14) rotates the initial coordinates into the same orientation, then certainly the constant angular rate vector specified by

$$\underline{w} = \frac{\underline{\theta}(t_f)}{t_f - t_o} \tag{15}$$

rotates the initial coordinates into the same orientation and since according to Euler's theorem such $\underline{\theta}(t_f)$ always exists, then such constant $\underline{w}$ exists too. Finally, since such a constant $\underline{w}$ exists then, following (4) and (11), there also exists a corresponding constant matrix, W,

$$W = \frac{1}{t_f - t_o}\, \Theta_f \tag{16}$$

which when used in solving (12), yields the $D(t_f)$ that corresponds to $\underline{\theta}(t_f)$. Therefore, in view of the equivalence between the pair (1) and (3) on the one hand, and (12) on the other hand, we can phrase an equivalent formulation of Euler's theorem as follows:

> **Regardless of what matrix W(t) generated $D(t_f)$, it is always possible to find a constant matrix W which generates the same $D(t_f)$.**

We call this formulation continuous because it relates to the continuous change of the orientation as a result of the existence of an angular rate at which the orientation changes.

III. REPRESENTATION OF ROTATIONS IN n-D

Denote the dimension of an Euclidean space by n. The rotation matrix in n-D, being a square matrix, consists of n^2 elements.

However, the orthogonality of the matrix imposes $(n+1)n/2$ constraints on it. Consequently a rotation matrix in n-D has only $m=(n-1)n/2$ independent parameters. That is, **a rotation matrix in n-D is defined by exactly $m=(n-1)n/2$ parameters.** Consider now the 3-D rotation. As indicated by (1) and indeed as stated by Euler [1,11] the vector $\underline{\theta}(t_f)$ in its three components contains the necessary and sufficient information for specifying the 3-D rotation. Similarly the orthogonal rotation matrix $D(t_f)$ contains three independent parameters although it has nine elements. So the 3-D case is unique in that $n=m$ and the rotation can be described by either a vector or a matrix. In all other dimensions, though, $n \neq m$ and since m parameters are needed to define the rotation, **a vector with its n elements cannot specify a rotation.** Rotation matrices though, with their m independent parameters, do specify the rotation. As a consequence of this discussion, it is concluded that the original version of Euler's theorem or any of its variants presented in Section I are not extendible to n-D while the alternate formulation of Euler's theorem given in the preceding section may be extended to n-D.

IV. EULER'S THEOREM IN n-D

In view of the conclusion drawn in the last section, the general formulations of Euler's theorem in n-D are that given in Section II. Let us first address the angular-matrix (discrete) formulation and rephrase it in a more general frame by the following theorem:

Theorem 1: **Given the arbitrary unitary matrices $D_o=D(t_o)$ and $D_f=D(t_f)$, then D_f can always be expressed in the form:**

$$D_f = e^{-\Theta} D_o \tag{17}$$

where Θ is a skew Hermitian-matrix.

Since the rotation matrix is orthogonal, the angular-matrix formulation of Euler's theorem is a special case of this theorem. Although the theorem is not new [see e.g. exercise 4 on p. 346 of ref. 12], for the sake of completeness, we present here a proof of the theorem.

Proof: Define the unitary matrix

$$D = D_f D_o^+ \tag{18}$$

where + denotes the conjugate transpose of a matrix. Since D is unitary, it is also normal and as such it has n orthogonal eigenvectors (see theorem A1 in the Appendix). Define a matrix V whose columns are the eigenvectors of D. Then V is unitary. Since the eigenvectors of D form an orthonormal set, then

$$D = V\ G\ V^+ \tag{19}$$

where G is the diagonal matrix of the eigenvalues of D (see A2). Now since D is unitary, its n eigenvalues $g_1, g_2, \ldots\ldots, g_n$ lie on the unit circle of the complex plane (see A3); that is,

$$g_i = e^{j\phi_i} \qquad i=1,2, \ldots ,n$$

where $j=(-1)^{1/2}$ and ϕ_i is the phase of the i^{th} eigenvalue. Let us form a diagonal matrix Φ

$$\Phi = \mathrm{diag}\{-\phi_1, -\phi_2, \; \ldots \; , -\phi_n\} \tag{20}$$

Then, obviously,

$$G = e^{-j\Phi} \tag{21}$$

Next we define a constant matrix Θ as

$$\Theta = V \, j\Phi \, V^+ \tag{22}$$

then (see A4)

$$e^{-\Theta} = e^{-V \, j\Phi \, V^+} = V \, e^{-j\Phi} \, V^+ \tag{23}$$

Substituting (21) into (23) we obtain

$$e^{-\Theta} = V \, G \, V^+ \tag{24}$$

A comparison between (24) and (19) yields

$$D = e^{-\Theta} \tag{25}$$

Then from (18) and (25) we obtain

$$D_f = e^{-\Theta} \, D_o$$

To complete the proof we still have to show that Θ is skew-Hermitian. From (25)

$$D^+ = e^{-\Theta^+} \tag{26}$$

Also (see A5)

$$D^{-1} = e^{\Theta} \tag{27}$$

but, since D is unitary

$$D^+ = D^{-1}$$

therefore the right hand side of (26) is equal to that of (27), consequently

$$\theta^+ = -\theta$$

This completes the proof.

With Theorem 1 on-hand we are ready now to address the angular-rate (continuous) formulation of Euler's theorem. Here too we rephrase the latter in a more general frame as follows:

Theorem 2: **Given the arbitrary unitary matrices $D_o=D(t_o)$ and $D_f=D(t_f)$, then D_f can always be obtained as a solution of (12) with the initial condition D_o where W is a constant skew-Hermitian matrix.**

Note that due to the orthogonality of the rotation matrix, the angular-rate formulation of Euler's theorem constitutes a special case of the last theorem. The following proof of the latter is based on the former theorem.

Proof: From Theorem 1

$$D_f = e^{-\theta} D_o$$

On the other hand, the solution (12) when W is constant is

$$D_f = e^{-W(t_o-t_f)} D_o$$

the equality of these yields

$$W = \frac{1}{t_f-t_o} \theta \qquad (28)$$

that is; no matter what W(t) generated D_f, we can always find a constant W according to (28) for which the solution of (12) with the initial condition D_o yields D_f at t_f. Since Θ is skew-Hermitian then, from (28), W is skew-Hermitian too. This concludes the proof.

While we can consider W as an average velocity computed by (28), we note however that W is not the average of W(t); that is,

$$W \neq \frac{1}{t_f - t_o} \int_{t_o}^{t_f} W(t)\, dt \qquad (29)$$

This inequality is known in 3-D as **non-commutativity.** (Another expression for the non-commutativity of rotations is the inequality expressed in (9)).

The above theorems extend Euler's theorem in two ways. First they deal with the general n-D rather than the 3-D case and secondly they extend Euler's theorem to the unitary (complex) transformation. Euler's original formulation is, then, a special case of the above theorems.

V. NUMERICAL EXAMPLE

To demonstrate the facts pointed out in the preceding section we bring a fourth order example in which we show how D_f which is obtained as a result of the solution of (5) for a certain time varying angular velocity matrix, W(t), can be obtained by the solution of (5) with a constant angular velocity matrix, W.

For simplicity we deal with a special unitary matrix; namely, with an orthogonal one. Also for simplicity and with no loss of generality, we choose $D_o=I$. We first use the following time varying W(t) to solve (12) from $t_o=0.$ to $t_f=0.5$ sec

$$W(t) = \begin{bmatrix} 0. & 1.5t & 1.5t^2 & 0.8t^3 \\ -1.5t & 0. & -0.9\sin(6.28t) & -0.95/(1.-t) \\ -1.5t^2 & 0.9\sin(6.28t) & 0. & 0.75 \\ -0.8t^3 & 0.95/(1.-t) & -0.75 & 0. \end{bmatrix} \tag{30}$$

this yields the following solution at t_f:

$$D_f = \begin{bmatrix} 0.98130682 & -0.15805594 & -0.08266215 & -0.07226489 \\ 0.18388549 & 0.76180341 & 0.21777062 & 0.58173674 \\ 0.04691911 & -0.10221727 & 0.96379421 & -0.24176631 \\ -0.03196326 & -0.61985926 & 0.12978307 & 0.77324588 \end{bmatrix}$$

The following eigenvector matrix of D_f is obtained using the EIGZF routine of the IMSL library:

$$V = \begin{bmatrix} -0.038+j0.161 & -0.038-j0.161 & 0.687+j0. & 0.687+j0. \\ 0.683+j0. & 0.683+j0. & 0.037-j0.177 & 0.037+j0.177 \\ -0.170+j0.184 & -0.170-j0.184 & -0.052-j0.659 & -0.052+j0.659 \\ 0.056+j0.664 & 0.056-j0.664 & -0.151+j0.181 & -0.151-j0.181 \end{bmatrix}$$

The same routine also yields the following eigenvalues of D_f:

$$g_1 = 0.7452+j0.6668 = e^{j0.7300}$$

$$g_2 = 0.7452-j0.6668 = e^{-j0.7300}$$

$$g_3 = 0.9949+j0.1012 = e^{j0.1013}$$

$$g_4 = 0.9949-j0.1012 = e^{-j0.1013}$$

The diagonal matrix $j\Phi$ is computed according to (20) and used in (22) and (28) to compute the following constant angular velocity matrix:

$$W = \begin{bmatrix} 0.00000000 & 0.37147707 & 0.12387442 & 0.04981448 \\ -0.37147707 & 0.00000000 & -0.35388342 & -1.31771209 \\ -0.12387442 & 0.35388342 & 0.00000000 & 0.40459905 \\ -0.04981448 & 1.31771209 & -0.40459905 & 0.00000000 \end{bmatrix} \tag{31}$$

When now (12) is solved with the initial condition $D_o=I$ starting at $t_o=0$. then, as expected, D_f is the solution obtained at time $t=0.5$. Finally, to demonstrate the inequality presented in (29) we average $W(t)$ given in (30). The resulting average is:

$$\frac{1}{t_f-t_o}\int_{t_o}^{t_f} W(t)\, dt = \begin{bmatrix} 0. & 0.375 & 0.125 & 0.025 \\ -0.375 & 0. & -0.573 & -1.317 \\ -0.125 & 0.573 & 0. & 0.750 \\ -0.025 & 1.317 & -0.750 & 0. \end{bmatrix}$$

Obviously the latter matrix differs from the constant angular matrix, W, given in (31).

V. CONCLUSIONS

Euler's fundamental theorem on the ability to describe any orientation of a rigid body as a single rotation and the various known versions of this theorem cannot be directly extended to other dimensions because all known formulations hinge on the concept of axis of rotation which does not exist in dimensions other than three. Nevertheless, when it is recognized that the general n-D rotation is characterized not by an axis of rotation but rather by an angular-difference matrix or by an angular velocity matrix, Euler's theorem can be reformulated in 3-D in ways which are equivalent to the other known formulations and then the new formulations can be extended to n-D. In this work we presented the new formulations in 3-D and then we proved that they hold for any dimension. One of the new formulations states that no matter what was the sequence of rotations that resulted in the final orientation, it is always possible to express the rotation matrix as an exponential function of a skew-symmetric angular-difference matrix. The other new formulation of Euler's theorem states that no matter how the angular velocity matrix changes as a function of time, we can always find a constant angular-velocity matrix which will result in identical orientation change over the same time interval.

The proof of the theorems supplied the algorithm needed to compute the angular-difference matrix and the equivalent constant angular-velocity matrix once the initial and the final attitudes as well as the time interval are given. To demonstrate the new formulations of the theorem and their extendibility to dimensions other than 3 we used the algorithm to solve a 4-D example. The example clearly demonstrates the ability to reach the same final orientation using a constant angular velocity matrix.

Appendix

This appendix lists some known theorems which are used in the proof of the theorem on the extended Euler theorem.

A1: A set of n orthonormal eigenvectors can be found for an nxn normal matrix. [See p. 76 of Ref. 13].

A2: A matrix can be reduced to a diagonal matrix by a similarity transformation if and only if a set of n linearly independent eigenvectors can be found. [See p. 72 of Ref. 13].

A3: The eigenvalues of a unitary matrix have absolute value 1. [See p. 129 in Ref. 14].

A4: If $A = TJT^{-1}$, then $f(A) = Tf(J)T^{-1}$. [See p. 80 of Ref. 13].

A5: If $A = e^{B}$ then $A^{-1} = e^{-B}$.

Proof: Since B and -B commute, then $e^{B}e^{-B} = e^{B-B} = I$

hence

$$Ae^{-B}=I$$

thus

$$A^{-1}=e^{-B}$$

Acknowledgment

The author wishes to express his thanks to Dr. F. Landis Markley for reviewing this paper and for many constructive discussions and suggestions. The author also wishes thank Dr. R.J. Rowland, Jr. of the Department of Classics of the University of Maryland for his help in translating Euler's theorem from the Latin.

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