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**A Transient Response  
Method for Linear  
Coupled Substructures**

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## NOMENCLATURE

|               |   |
|---------------|---|
| $[\hat{C}]$   | interface compatibility coefficient matrix      |
| $[D]$         | force coefficient matrix                        |
| $\{F\}$       | force vector                                    |
| $\{G_j\}$     | term $j$ in expansion of interface force vector |
| $[K]$         | stiffness matrix                                |
| $[M]$         | mass matrix                                     |
| $\{q\}$       | modal coordinate vector                         |
| $\{\bar{q}\}$ | modal coordinate vector due to applied forces   |
| $\{\hat{q}\}$ | modal coordinate vector due to interface forces |
| $t$           | time  |
| $t_i$         | time at time increment $i$                      |
| $\{X\}$       | physical degrees of freedom                     |
| $\Delta t$    | length of integration time step                 |
| $\zeta$       | percent critical modal damping                  |
| $[\phi]$      | mode shape matrix                               |
| $[\omega]$    | diagonal matrix of circular frequencies         |
| $[\omega^2]$  | diagonal matrix of eigenvalues                  |

### Subscripts

$a, b$  substructure  $a, b$ , respectively

### Superscripts

$I$  interface



## TECHNICAL PAPER

# A TRANSIENT RESPONSE METHOD FOR LINEAR COUPLED SUBSTRUCTURES

## INTRODUCTION

The successful design of a modern aerospace structure requires an accurate determination of the internal loads it will experience during its useful lifetime. Often the internal loads that impact the design the most are those caused by the transient response of the structure due to suddenly applied external forces. For example, the ignition and thrust buildup of launch vehicle engines and the subsequent lift-off of the vehicle results in a transient response that can cause the maximum internal loads the vehicle structure will experience during its lifetime. Transient response analyses of structures has been the subject of study for many years because of its effect on the design of structures.

The equations of motion representing a linear structural system are a set of coupled second-order differential equations. Even though a number of numerical methods are available for the direct solution of such equations, it is often not practical because of the large number of equations involved. Models of structures generated by finite-element computer codes often can have several thousand degrees of freedom (DOF). The standard approach to overcome the problem of too many DOF is to first compute the vibration modes and frequencies of the structure by solving an eigenvalue problem. The structural vibration modes are then used to transform from the original discrete physical coordinates to a set of new modal coordinates. This permits the number of equations to be reduced by truncating the higher frequency modal coordinates without significantly reducing the accuracy of the results. Also, the transformation results in a set of uncoupled rather than a coupled set of equations. This approach yields a set of equations of form and size that are easily solved on present day computers. The difficulty of performing a vibration analysis to determine the modes and frequencies of the structure prior to solving the set of equations expressed in modal coordinates is still a major problem due to the large order of the eigenvalue problem that must be solved.

The vibration analysis of structures has been the subject of many papers. Two approaches have been used to make the vibration analysis of large structural models tractable. One approach is to develop efficient algorithms for the solution of large eigenvalue problems directly, and the other approach is to make judicious approximations to reduce the size of the eigenvalue problem without significantly affecting the results. One example of the first is based on the repeated application of the Rayleigh-Ritz method of vibration analysis in which each solution is an improvement over the previous. This method was originally called an iterative Rayleigh-Ritz technique [1] and later renamed subspace iteration method [2,3]. The other approach, called substructure synthesis, results in a reduction in size of the eigenvalue problem. Many vibration analysis methods [4-11] are based on this approach. Substructure synthesis is a method where the structure is considered to be made up of a number of substructures. The motion of each substructure is represented by a set of displacement vectors. The vectors may or may not be substructure modes. The number of vectors required to represent the motion of a substructure is much less than the number of physical DOF's required to represent its motion. As a result of this approximation, the size of the eigenvalue problem is greatly reduced.

Other transient response methods have been studied that circumvent the need to solve a large eigenvalue problem and still reduce the number of DOF in the equations of motion to a manageable number. These methods [12-14] are based on the direct integration of the equations of motion expressed in coordinates other than system discrete coordinates or system modal coordinates. They use a set of coordinates that contains both interface discrete coordinates and substructure modal coordinates.

In this paper, a method is developed for the transient response analysis of structures that is different from the approaches described above. The structure is considered to be composed of substructures. The method is based on approximating the interface forces between the substructures as power series in time. The equations of motion of each substructure are integrated for a time step, with all external forces applied including the interface forces with unknown coefficients. The unknown coefficients in the power series are evaluated by satisfying the interface compatibility relationships at the end of the time step. Once the coefficients in the power series are known, the interface forces and the response characteristics at the end of the time step can be computed. This procedure is repeated as many times as required to span the time interval of interest. The implementation of the method on a computer is quite simple and permits enforcing the interface compatibility relationships by a simple matrix multiplication. Satisfying boundary conditions between substructures in this manner allows the boundary conditions to be changed during the transient response by simply changing the matrix used to enforce the compatibility relationships. Therefore, this method is ideally suited for the transient response analysis of structures that have interface boundary conditions between substructures that change during the response. The transient response analysis of launch vehicles during the lift-off event is an example of this type of problem, since the boundary conditions between the vehicle/pad change as the vehicle lifts off.

## GOVERNING EQUATIONS

For clarity, the governing equations are formulated for a structural system composed of two substructures. However, the extension of the equations for systems consisting of an arbitrary number of substructures or bodies is evident. The derivation of the equations begins by writing the equations of motion of the two bodies and the associated interface compatibility equations for the system shown in Figure 1. The interface compatibility conditions for the two bodies are expressed as

$$\{X_a^I\} \equiv \{X_b^I\} \quad \text{and} \quad \{F_a^I\} + \{F_b^I\} = \{0\} \quad (1)$$

For simplicity the interface force compatibility will be rewritten as

$$\{F_a^I\} = \{F^I\} \quad \text{and} \quad \{F_b^I\} = -\{F^I\} \quad (2)$$

The equations of motion for the substructures were derived by using Lagrange's equation with undetermined multipliers [15]. The undamped equations of motion of the substructures are written as



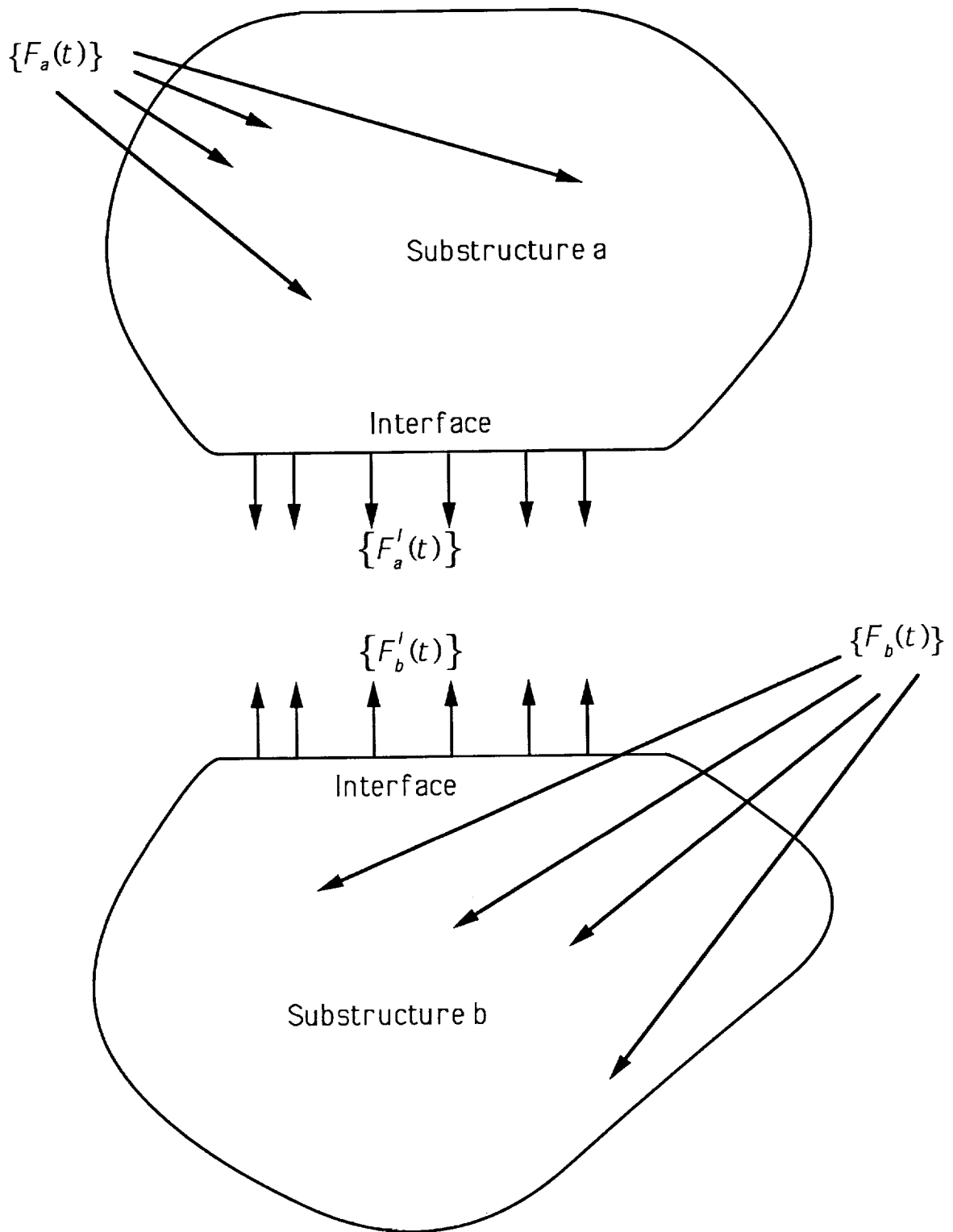


Figure 1. Free body diagram.

$$[M_a] \{\ddot{X}_a\} + [K_a] \{X_a\} = \{F_a(t)\} + \begin{Bmatrix} \{0\} \\ \{F^I(t)\} \end{Bmatrix} , \quad (3a)$$

$$[M_b] \{\ddot{X}_b\} + [K_b] \{X_b\} = \{F_b(t)\} - \begin{Bmatrix} \{0\} \\ \{F^I(t)\} \end{Bmatrix} . \quad (3b)$$

The interface forces are the undetermined multipliers. The equations of motion expressed by equations (3a) and (3b) are in discrete coordinates with the interface coordinates last. The equations can be solved in this form, but for large systems the computer storage requirement can be excessive. This problem is overcome by using the substructures vibration modes to transform the equations of motion from discrete coordinates to normal mode coordinates. This transformation uncouples the equations and results in a system of equations much simpler to solve from a computational point of view. Modal damping of the substructures can be introduced at this point. Equations (3a) and (3b) transformed to modal coordinates are

$$\{\ddot{q}_a\} + 2\zeta_a[\omega_a] \{\dot{q}_a\} + [\omega_a^2] \{q_a\} = [D_a] \{F_a(t)\} + [D_a^I] \{F^I(t)\} , \quad (4a)$$

$$\{\ddot{q}_b\} + 2\zeta_b[\omega_b] \{\dot{q}_b\} + [\omega_b^2] \{q_b\} = [D_b] \{F_b(t)\} + [D_b^I] \{F^I(t)\} , \quad (4b)$$

where

$$\{X_a\} = [\phi_a] \{q_a\} \quad \text{and} \quad \{X_b\} = [\phi_b] \{q_b\} . \quad (5)$$

Since the interface forces  $\{F^I(t)\}$  result from the motion of a structure that is governed by second order differential equations, they will be of trigonometric form. If equations (4a) and (4b) are solved in a step-wise fashion, it is appropriate to approximate the interface forces by a power series valid for a time step of length  $\Delta t$ . This is expressed as

$$\{F^I(t)\} = \sum_{j=0}^{\infty} \{G_j\} (t - t_i)^j \quad t_i \leq t \leq t_i + \Delta t . \quad (6)$$

This series is expected to converge very rapidly for time steps of the size normally used in integration of equations of motion. Therefore, very few terms need to be retained in the series to achieve a good approximation of the interface forces. Terms up through the third order will be retained in this development. Truncating equation (6) and substituting in equations (4a) and (4b) yields

$$\begin{aligned} \{\ddot{q}_a\} + 2\zeta_a[\omega_a] \{\dot{q}_a\} + [\omega_a^2] \{q_a\} &= [D_a] \{F_a(t)\} + [D_a^I] \{\{G_0\} + \{G_1\}(t-t_i) \\ &+ \{G_2\} (t-t_i)^2 + \{G_3\} (t-t_i)^3\} \quad ; \quad t_i \leq t \leq t_i + \Delta t \quad , \end{aligned} \quad (7a)$$

$$\begin{aligned} \{\ddot{q}_b\} + 2\zeta_b[\omega_b] \{\dot{q}_b\} + [\omega_b^2] \{q_b\} &= [D_b] \{F_b(t)\} - [D_b^I] \{\{G_0\} + \{G_1\}(t-t_i) \\ &+ \{G_2\} (t-t_i)^2 + \{G_3\} (t-t_i)^3\} \quad ; \quad t_i \leq t \leq t_i + \Delta t \quad . \end{aligned} \quad (7b)$$

Since equations (7a) and (7b) are linear differential equations, superposition of their solutions are permitted. Therefore, let us define the following

$$\{q_a\} = \{\bar{q}_a\} + \{\tilde{q}_a\} \quad ; \quad \{\dot{q}_a\} = \{\dot{\bar{q}}_a\} + \{\dot{\tilde{q}}_a\} \quad ; \quad \{\ddot{q}_a\} = \{\ddot{\bar{q}}_a\} + \{\ddot{\tilde{q}}_a\} \quad , \quad (8a)$$

$$\{q_b\} = \{\bar{q}_b\} + \{\tilde{q}_b\} \quad ; \quad \{\dot{q}_b\} = \{\dot{\bar{q}}_b\} + \{\dot{\tilde{q}}_b\} \quad ; \quad \{\ddot{q}_b\} = \{\ddot{\bar{q}}_b\} + \{\ddot{\tilde{q}}_b\} \quad , \quad (8b)$$

where  $\{\bar{q}_a\}$ ,  $\{\tilde{q}_a\}$ ,  $\{\bar{q}_b\}$ ,  $\{\tilde{q}_b\}$ , and their derivatives are obtained by solving the following equations for  $t_i \leq t \leq t_i + \Delta t$

$$\{\ddot{\bar{q}}_a\} + 2\zeta_a[\omega_a] \{\dot{\bar{q}}_a\} + [\omega_a^2] \{\bar{q}_a\} = [D_a] \{F_a(t)\} + [D_a^I] \{G_0\} \quad , \quad (9a)$$

$$\{\ddot{\tilde{q}}_a\} + 2\zeta_a[\omega_a] \{\dot{\tilde{q}}_a\} + [\omega_a^2] \{\tilde{q}_a\} = [D_a^I] \{\{G_1\} (t-t_i) + \{G_2\} (t-t_i)^2 + \{G_3\} (t-t_i)^3\} \quad , \quad (9b)$$

$$\{\ddot{\bar{q}}_b\} + 2\zeta_b[\omega_b] \{\dot{\bar{q}}_b\} + [\omega_b^2] \{\bar{q}_b\} = [D_b] \{F_b(t)\} - [D_b^I] \{G_0\} \quad , \quad (9c)$$

$$\{\ddot{\tilde{q}}_b\} + 2\zeta_b[\omega_b] \{\dot{\tilde{q}}_b\} + [\omega_b^2] \{\tilde{q}_b\} = [D_b^I] \{\{G_1\} (t-t_i) + \{G_2\} (t-t_i)^2 + \{G_3\} (t-t_i)^3\} \quad , \quad (9d)$$

where the initial conditions for equations (9a) through (9d) are

$$\{\bar{q}_a(t_i)\} = \{q_a(t_i)\} \quad , \quad \{\dot{\bar{q}}_a(t_i)\} = \{\dot{q}_a(t_i)\} \quad , \quad (10a)$$

$$\{\tilde{q}_a(t_i)\} = \{0\} \quad , \quad \{\dot{\tilde{q}}_a(t_i)\} = \{0\} \quad , \quad (10b)$$

$$\{\bar{q}_b(t_i)\} = \{q_b(t_i)\} \quad , \quad \{\dot{\bar{q}}_b(t_i)\} = \{\dot{q}_b(t_i)\} \quad , \quad (10c)$$

$$\{\tilde{q}_b(t_i)\} = \{0\} \quad , \quad \{\dot{\tilde{q}}_b(t_i)\} = \{0\} \quad , \quad (10d)$$

and  $\{G_0\}$ , the first time in the power series, is determined by substituting  $t_i$  in equation (6).

$$\{G_0\} = \{F^I(t_i)\} \quad . \quad (11)$$

Equations (9a) and (9c) can also be solved numerically or closed form to obtain the solutions at  $t_{i+1} = t_i + \Delta t$ . Closed-form solutions are usually possible since the applied forces over a time increment are often simple expressions. These solutions are

$$\{\bar{q}_a(t_{i+1})\} \quad ; \quad \{\dot{\bar{q}}_a(t_{i+1})\} \quad ; \quad \{\ddot{\bar{q}}_a(t_{i+1})\} \quad , \quad (12a)$$

$$\{\bar{q}_b(t_{i+1})\} \quad ; \quad \{\dot{\bar{q}}_b(t_{i+1})\} \quad ; \quad \{\ddot{\bar{q}}_b(t_{i+1})\} \quad . \quad (12b)$$

Equations (9b) and (9d) can be solved in closed form, however, the solutions contain  $\{G_1\}$ ,  $\{G_2\}$ , and  $\{G_3\}$  as unknowns. The use of superposition of solutions of linear differential equations is the most practical approach for getting the solutions in a form suitable for evaluating the unknowns  $\{G_1\}$ ,  $\{G_2\}$ , and  $\{G_3\}$ . This is done by solving equations (9b) and (9d), with the elements in  $\{G_1\}$ ,  $\{G_2\}$ , and  $\{G_3\}$  being assigned a unit value, one at a time, and summing the solutions. This can be expressed in matrix form as

$$\{\tilde{q}_a(t_{i+1})\} = [C_{a1}] \{G_1\} + [C_{a2}] \{G_2\} + [C_{a3}] \{G_3\} \quad , \quad (13a)$$

$$\{\dot{\tilde{q}}_a(t_{i+1})\} = [\dot{C}_{a1}] \{G_1\} + [\dot{C}_{a2}] \{G_2\} + [\dot{C}_{a3}] \{G_3\} \quad , \quad (13b)$$

$$\{\ddot{\tilde{q}}_a(t_{i+1})\} = [\ddot{C}_{a1}] \{G_1\} + [\ddot{C}_{a2}] \{G_2\} + [\ddot{C}_{a3}] \{G_3\} \quad , \quad (13c)$$

$$\{\tilde{q}_b(t_{i+1})\} = [C_{b1}] \{G_1\} + [C_{b2}] \{G_2\} + [C_{b3}] \{G_3\} \quad , \quad (14a)$$

$$\{\dot{\tilde{q}}_b(t_{i+1})\} = [\dot{C}_{b1}] \{G_1\} + [\dot{C}_{b2}] \{G_2\} + [\dot{C}_{b3}] \{G_3\} \quad , \quad (14b)$$

$$\{\ddot{\tilde{q}}_b(t_{i+1})\} = [\ddot{C}_{b1}] \{G_1\} + [\ddot{C}_{b2}] \{G_2\} + [\ddot{C}_{b3}] \{G_3\} \quad . \quad (14c)$$

The interface compatibility conditions will be used to evaluate  $\{G_1\}$ ,  $\{G_2\}$ , and  $\{G_3\}$ . Since the solutions of the differential equations are going to be evaluated each  $\Delta t$ , the compatibility conditions will be approximately satisfied by requiring the interface displacement, velocity, and acceleration of both bodies be equal at the end of each  $\Delta t$ . That is

$$\{X_a^I(t_{i+1})\} = \{X_b^I(t_{i+1})\} \quad ; \quad \{\dot{X}_a^I(t_{i+1})\} = \{\dot{X}_b^I(t_{i+1})\} \quad ; \quad \{\ddot{X}_a^I(t_{i+1})\} = \{\ddot{X}_b^I(t_{i+1})\} \quad . \quad (15)$$

Substituting equations (5), applied to the interface coordinates, and equations (8) in equations (15) yields

$$[\phi_a^I] \{\bar{q}_a(t_{i+1})\} - [\phi_b^I] \{\bar{q}_b(t_{i+1})\} = [\phi_b^I] \{\tilde{q}_b(t_{i+1})\} - [\phi_a^I] \{\tilde{q}_a(t_{i+1})\} \quad , \quad (16a)$$

$$[\phi_a^I] \{\dot{\bar{q}}_a(t_{i+1})\} - [\phi_b^I] \{\dot{\bar{q}}_b(t_{i+1})\} = [\phi_b^I] \{\dot{\tilde{q}}_b(t_{i+1})\} - [\phi_a^I] \{\dot{\tilde{q}}_a(t_{i+1})\} \quad , \quad (16b)$$

$$[\phi_a^I] \{\ddot{\bar{q}}_a(t_{i+1})\} - [\phi_b^I] \{\ddot{\bar{q}}_b(t_{i+1})\} = [\phi_b^I] \{\ddot{\tilde{q}}_b(t_{i+1})\} - [\phi_a^I] \{\ddot{\tilde{q}}_a(t_{i+1})\} \quad . \quad (16c)$$

The terms on the left hand side of equations (16) represent the difference in displacements, velocities, and accelerations of the interface DOF's of substructures a and b, due to the external applied forces. These terms are written as

$$\{\bar{\delta}(t_{i+1})\} = [\phi_a^I] \{\bar{q}_a(t_{i+1})\} - [\phi_b^I] \{\bar{q}_b(t_{i+1})\} \quad , \quad (17a)$$

$$\{\dot{\bar{\delta}}(t_{i+1})\} = [\phi_a^I] \{\dot{\bar{q}}_a(t_{i+1})\} - [\phi_b^I] \{\dot{\bar{q}}_b(t_{i+1})\} \quad , \quad (17b)$$

$$\{\ddot{\bar{\delta}}(t_{i+1})\} = [\phi_a^I] \{\ddot{\bar{q}}_a(t_{i+1})\} - [\phi_b^I] \{\ddot{\bar{q}}_b(t_{i+1})\} \quad . \quad (17c)$$

Substituting equations (13), (14), and (17) into equations (16) gives

$$\begin{aligned} \{\bar{\delta}(t_{i+1})\} &= \{[\phi_b^1][C_{b1}] - [\phi_a^1][C_{a1}]\} \{G_1\} + \{[\phi_b^1][C_{b2}] - [\phi_a^1][C_{a2}]\} \{G_2\} \\ &+ \{[\phi_b^1][C_{b3}] - [\phi_a^1][C_{a3}]\} \{G_3\} \quad , \end{aligned} \quad (18a)$$

$$\begin{aligned} \{\dot{\delta}(t_{i+1})\} &= \{[\phi_b^1][\dot{C}_{b1}] - [\phi_a^1][\dot{C}_{a1}]\} \{G_1\} + \{[\phi_b^1][\dot{C}_{b2}] - [\phi_a^1][\dot{C}_{a2}]\} \{G_2\} \\ &+ \{[\phi_b^1][\dot{C}_{b3}] - [\phi_a^1][\dot{C}_{a3}]\} \{G_3\} \quad , \end{aligned} \quad (18b)$$

$$\begin{aligned} \{\ddot{\delta}(t_{i+1})\} &= \{[\phi_b^1][\ddot{C}_{b1}] - [\phi_a^1][\ddot{C}_{a1}]\} \{G_1\} + \{[\phi_b^1][\ddot{C}_{b2}] - [\phi_a^1][\ddot{C}_{a2}]\} \{G_2\} \\ &+ \{[\phi_b^1][\ddot{C}_{b3}] - [\phi_a^1][\ddot{C}_{a3}]\} \{G_3\} \quad . \end{aligned} \quad (18c)$$

Equations (18) can be put in a single matrix equation in partitioned form. This results in the following

$$\begin{bmatrix} \{\bar{\delta}(t_{i+1})\} \\ \{\dot{\delta}(t_{i+1})\} \\ \{\ddot{\delta}(t_{i+1})\} \end{bmatrix} = \begin{bmatrix} [\bar{C}_1] & [\bar{C}_2] & [\bar{C}_3] \\ [\dot{C}_1] & [\dot{C}_2] & [\dot{C}_3] \\ [\ddot{C}_1] & [\ddot{C}_2] & [\ddot{C}_3] \end{bmatrix} \begin{bmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{bmatrix} \quad , \quad (19)$$

or

$$\begin{bmatrix} \{\bar{\delta}(t_{i+1})\} \\ \{\dot{\delta}(t_{i+1})\} \\ \{\ddot{\delta}(t_{i+1})\} \end{bmatrix} = [\hat{C}] \begin{bmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{bmatrix} \quad (20)$$

$\{G_1\}$ ,  $\{G_2\}$ , and  $\{G_3\}$  can be obtained from equation (20) by inverting the coefficient matrix  $[\hat{C}]$ .

$$\begin{bmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{bmatrix} = [\hat{C}]^{-1} \begin{bmatrix} \{\bar{\delta}(t_{i+1})\} \\ \{\dot{\delta}(t_{i+1})\} \\ \{\ddot{\delta}(t_{i+1})\} \end{bmatrix} \quad (21)$$

It should be observed that the coefficient matrix in equation (21) does not depend on anything related to time except the time step  $\Delta t$ . Therefore, if the  $\Delta t$  is held constant during the integration, the coefficient matrix used to compute  $\{G_1\}$ ,  $\{G_2\}$ , and  $\{G_3\}$  only needs to be computed once at the start of the integration.

The essential equations for implementing the proposed transient response method have been developed. The task at hand now is to use these equations to form an efficient and practical algorithm for the solution of transient response equations.

## COMPUTATION PROCEDURE

The computation procedure for this transient response method is best explained by providing a list of steps in the order they are to be implemented. The procedure consists of five initial computation steps that are executed once followed by eight steps that are executed once each integration time step. The steps are as follows:

- Step 1 – Compute free modes and frequencies of each substructure.
- Step 2 – Select an integration time step  $\Delta t$  that is consistent with the highest substructure free frequency.
- Step 3 – Compute the interface compatibility matrix  $[\hat{C}]$ , as defined in equation (20), and its inverse.
- Step 4 – Set  $t_i =$  integration start time, with  $i = 1$ .
- Step 5 – Compute initial conditions at integration start time. These include substructure modal displacements and velocities, along with initial interface forces.
- Step 6 – Set  $t_{i+1} = t_i + \Delta t$ .
- Step 7 – Compute response of substructures due to applied loads at time  $= t_{i+1}$  by solving equations (9a) and (9c) with  $\{G_0\}$  defined by equation (11).
- Step 8 – Compute interface displacements, velocities, and accelerations due to applied loads at time  $= t_{i+1}$  using equations (17).
- Step 9 – Compute the coefficients of the power series as expressed in equations (6) at time  $= t_{i+1}$  using equations (21).
- Step 10 – Compute the response of the substructures due to the interface forces at time  $= t_{i+1}$  using equations (13) and (14).
- Step 11 – Compute the total response of the substructures at time  $= t_{i+1}$  by adding the response due to applied forces and the response due to interface forces using equations (8).

Step 12 – Compute the interface forces using equations (6) and reset initial conditions for next time step using equations (10) and (11).

Step 13 – Set  $t_i = t_{i+1}$  and return to step 6.

Care must be exercised in carrying out step 1 to prevent an unacceptable loss of accuracy. If one simply computes the free modes and frequencies from the discrete coordinate substructure models and truncates the number of modes based on a cut-off frequency, the motion and forces at the interface will not be represented very well. This is the same problem that occurs when free modes are used in modal synthesis methods. This loss of accuracy can be eliminated by introducing an additional step. The step consists of transforming the discrete coordinate models to the Craig-Bampton [6] form using the interface DOF's as boundary DOF's. The modal truncation is introduced in the formation of the Craig-Bampton model by truncating the fixed boundary modes based on a cut-off frequency that is consistent with the applied forcing function. The free modes and frequencies are computed from the Craig-Bampton models and are not truncated. This procedure assures the motion and forces at the interface are accurately represented.

## DEMONSTRATION PROBLEM

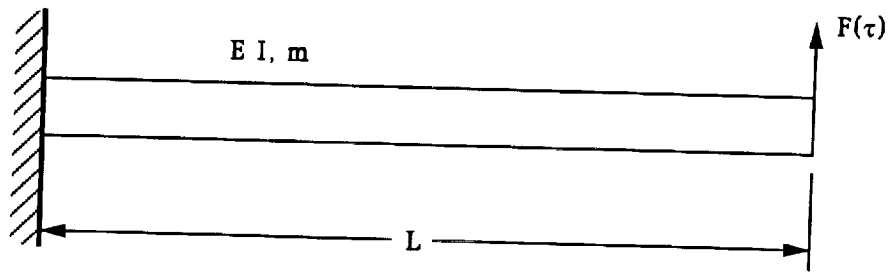
To demonstrate the transient response method for linear coupled substructures, a cantilevered beam with an applied tip force was selected. The cantilevered beam's material and geometric properties are defined in Figure 2a. The applied force at the tip of the cantilevered beam is defined in Figure 2b. The cantilevered beam was discretized into a two-dimensional finite element model composed of two substructures (body A and body B) of equal length. A total of 20 DOF's (10 translations and 10 rotations) and 10 elements were used to model the two substructures (Fig. 2c). A modal reduction of the finite element models was accomplished using the Craig-Bampton [6] technique. The Craig-Bampton constrained normal modes were retained up to 100 Hz. Interface DOF's were retained in discrete form. The free modes and frequencies of the substructures were computed by an eigenvalue analysis of the Craig-Bampton models. This eigenvalue analysis produced frequencies in the range of 4 to 286 Hz for body A and 0 to 286 Hz for body B. All modes from the eigenvalue analysis were retained for this study.

A benchmark model composed of the two reduced substructures was developed to investigate the coupled substructure transient response method. The benchmark model was formulated by coupling the two reduced beam substructures together using the direct stiffness method. The coupled equations of motion for the benchmark model were then uncoupled by an eigenvalue analysis. The uncoupled equations of motion resulted in a frequency range of 1 to 185 Hz, with all modes retained.

Damping is normally introduced in the uncoupled equations of motion in terms of a percent of critical damping. For this study, a value of zero percent of critical damping was used in order to compare the benchmark closed-form solution to the coupled substructure solution.

To solve the transient response analysis of the two-body differential equations, a Fortran computer routine was written utilizing the computational procedure described in the previous section. For the benchmark differential equations, a standard closed-form transient response routine was used. The applied force defined in Figure 2b was linear interpolated for both the closed-form analysis and the





$$\begin{aligned}
 EI &= 52080 \text{ lbs-in.}^2 \\
 m &= .00307169 \text{ lbs-sec.}^2/\text{in.} \\
 L &= 48.0 \text{ in.}
 \end{aligned}$$

Figure 2a. Cantilever beam with applied load  $F(\tau)$ .

$$F(\tau) = \begin{cases} 1.0 \sin(\pi \tau / 0.2) & 0 \leq \tau \leq 0.2 \\ 0.0 & \tau > 0.2 \end{cases}$$

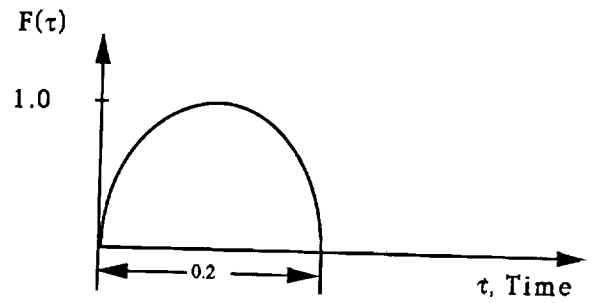


Figure 2b. Definition of applied force.

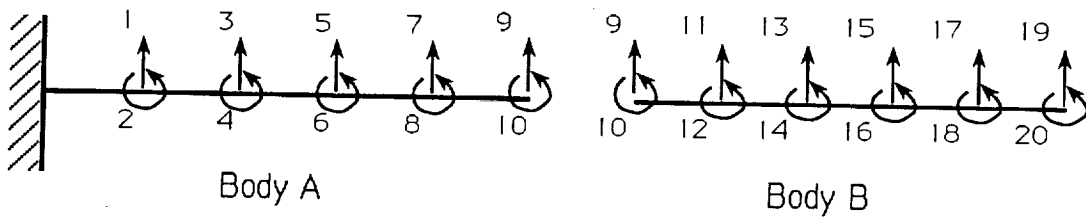


Figure 2c. Two-dimensional finite element model of cantilevered beam.

method presented. The highest frequency of the two substructures is 286 Hz. This gives a minimum period of 0.0035 seconds. In order to achieve accurate results in transient analysis, a time step must be chosen below the minimum period of the problem being solved. For this study, a range of time steps was chosen to research the accuracy of the proposed method. The time steps chosen range from 0.0003 to 0.0042 seconds. The benchmark transient response was computed using the same time steps used in the substructures transient response method. All transient response analyses were run from 0 to 1.0 seconds.

Interface shear force and moment were recovered from the analysis of the two substructures and were compared to the benchmark results. Also compared were the tip acceleration, velocity, and displacement. Using a time step of 0.001 seconds, the tip acceleration, velocity, and displacement, obtained from the response analysis of the coupled substructures and the benchmark response analysis, were plotted versus time in Figures 3a, 3b, and 3c. Within the fidelity of the plots, the results fell on top of each other.

The solutions from the benchmark closed-form transient response are compared to the substructure method by the relative error defined in the following equation:

$$\text{RMS Error} = \frac{1}{\text{Big}} \sqrt{\frac{\sum (X_b - X_s)^2}{NT}} \quad \text{NT}$$

where  $X_b$  is the benchmark time vector result (i.e., force, acceleration, etc.);  $X_s$  is the substructures time vector result (i.e., force, acceleration, etc.); NT is the total number of time points computed; and Big is the absolute largest benchmark result.

The RMS error results are plotted versus the time step used in the transient response analysis. The RMS errors for the tip acceleration, velocity, and displacement results are shown in Figure 4a. RMS errors for the interface shear force and moment are shown in Figure 4b. A correlation exists between the order of the time derivative results to that of the RMS error in Figures 4a and 4b. For example, the tip acceleration is an order of time derivative higher than the tip velocity, and the RMS error of the tip acceleration is an order of magnitude higher than that of the tip velocity. Likewise, the tip velocity RMS error is higher compared to the tip displacement. This can also be seen between the interface shear force and moment where an order of one exists between the time derivatives.

For a reasonable time step, the coupled transient response method presented gives accurate results of the demonstration program. As the time step gets closer to the minimum period, a leveling off of the RMS error occurs. This effect corresponds to accuracy of the integration for a given time step used in the analysis. Using a time step larger than the minimum period results in erroneous results.

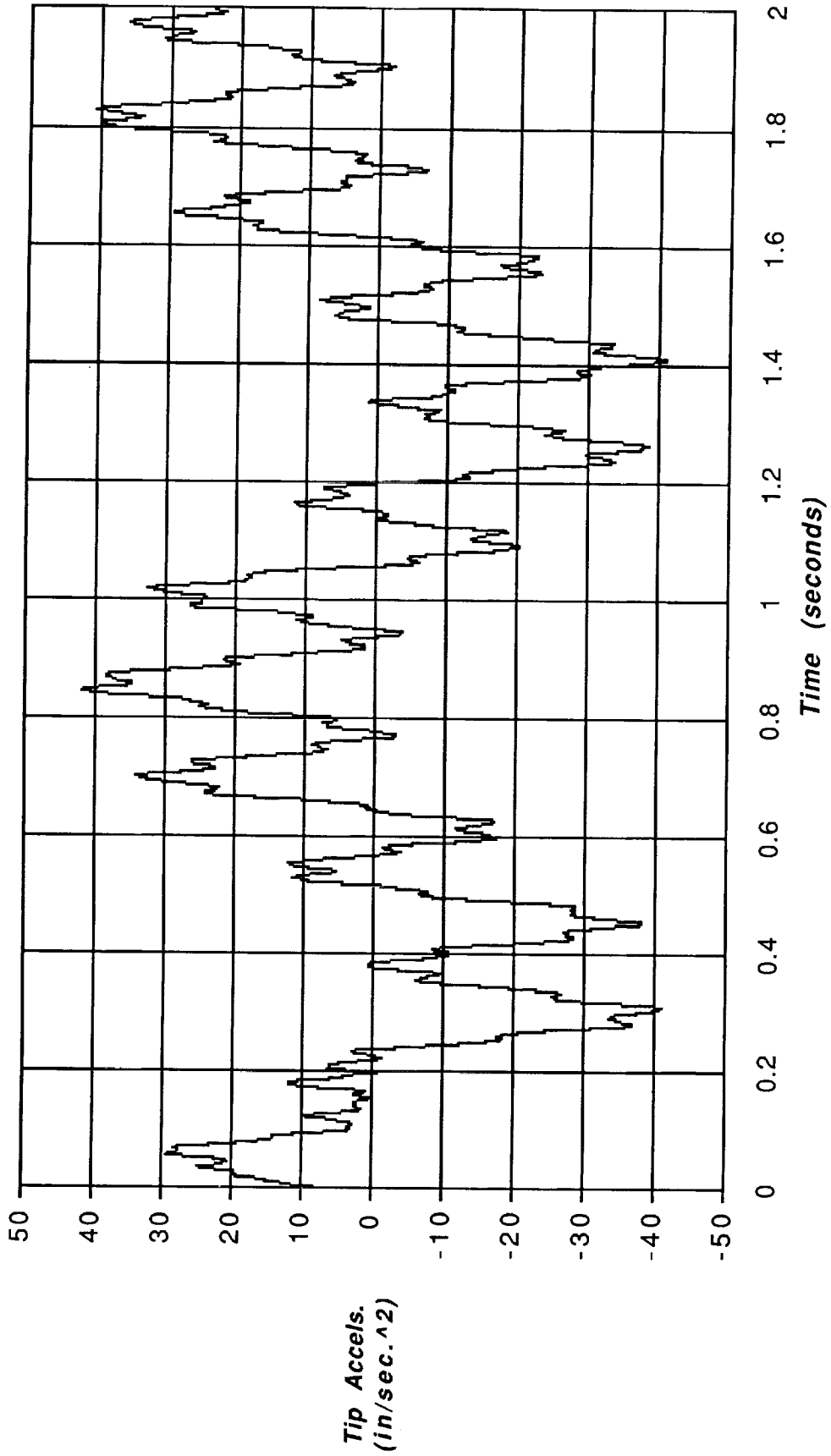


Figure 3a. Tip acceleration versus time (delta time = 0.001).

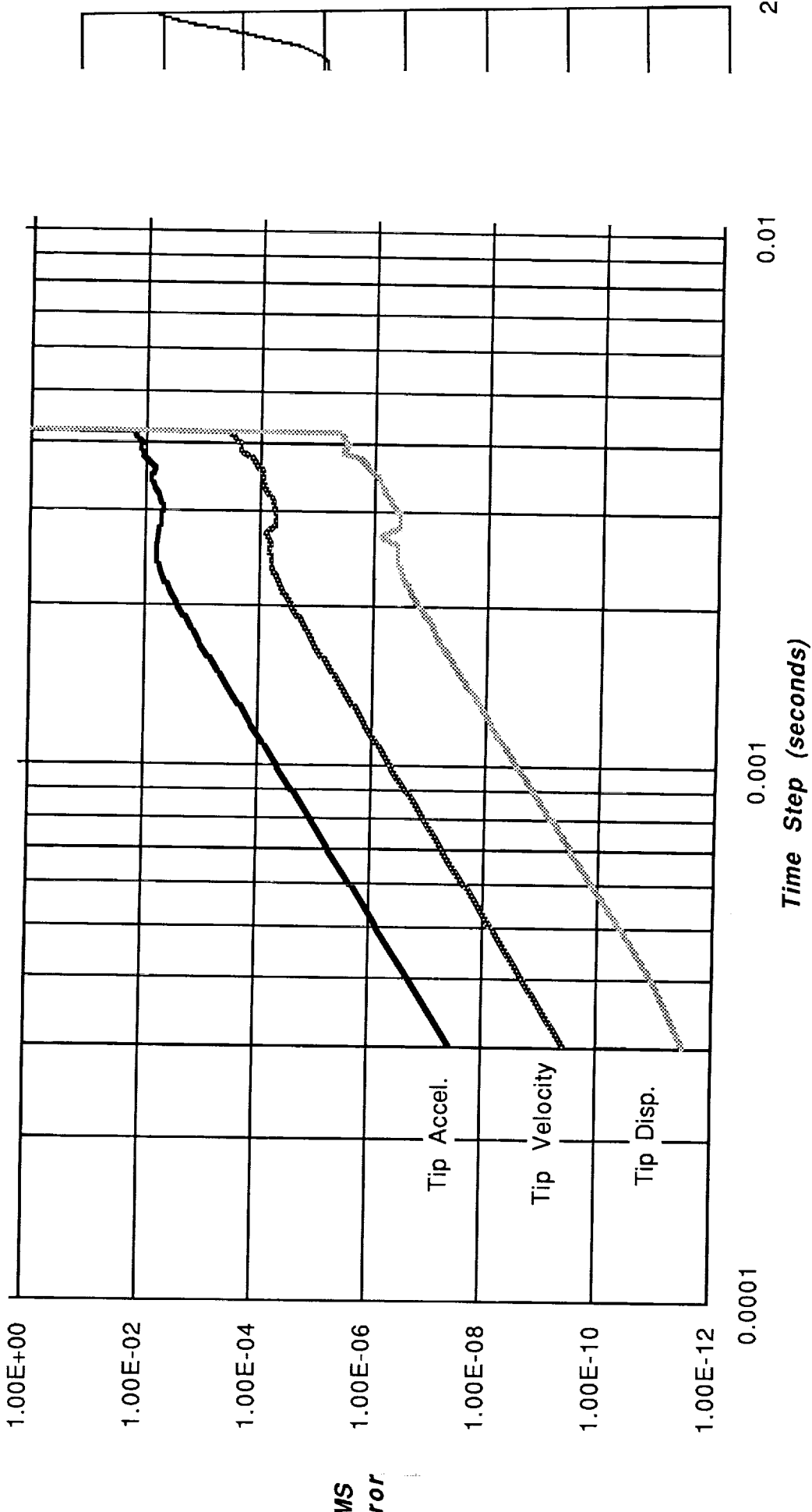


Figure 4a. RMS error versus time step.

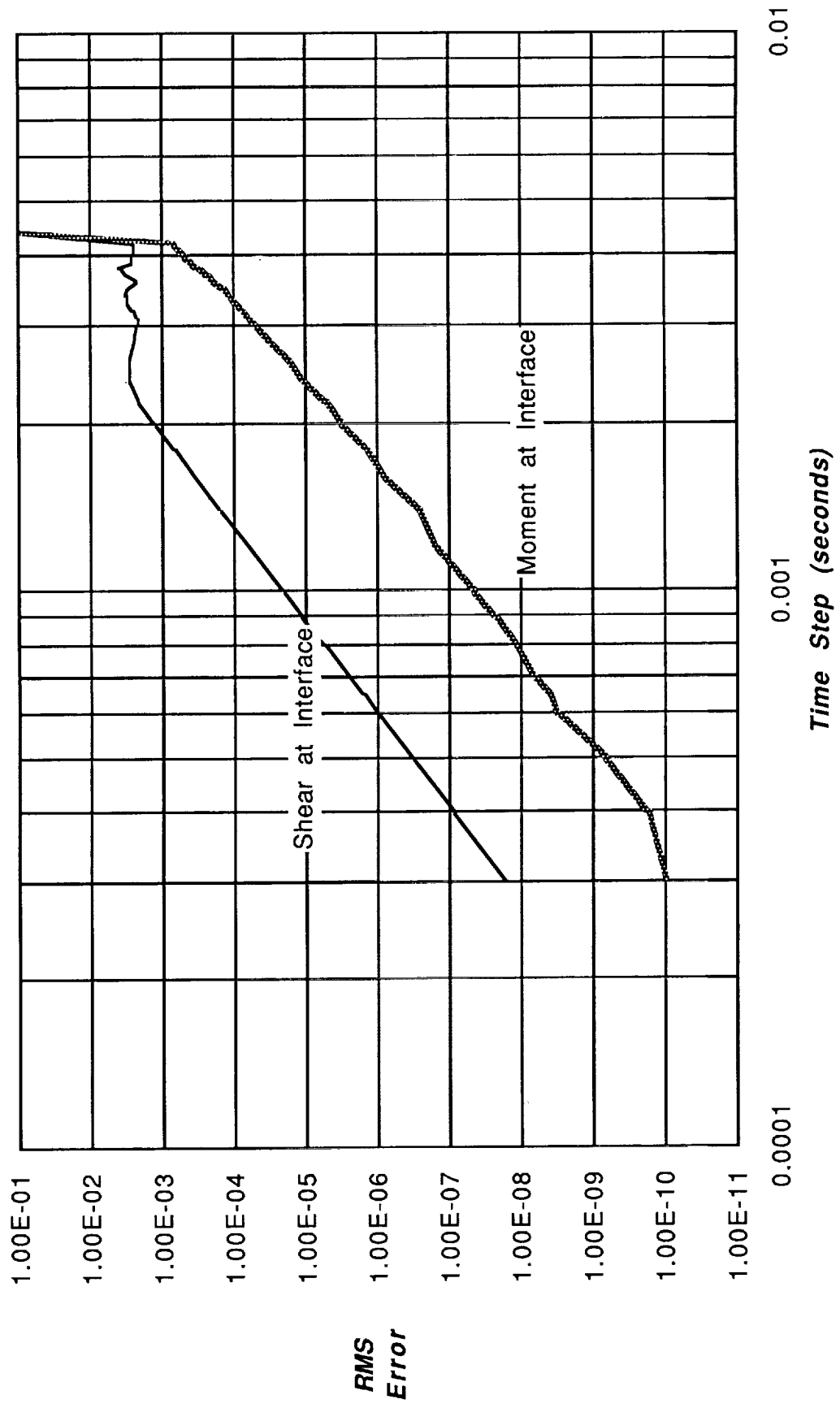


Figure 4b. RMS error versus time step.

## SUMMARY

A method for transient response analysis of linear coupled substructures has been presented. The equations of motion are solved in a stepwise fashion with the interface forces between substructures being approximated by a third-order power series in time. The coefficients in the power series are evaluated for each time step by enforcing compatibility of displacements, velocities, and accelerations at the substructure interfaces at the end of each time step. The method is approximate because the interface compatibility is enforced only at the end points of a time step rather than continuously. Therefore, as in most numerical integration methods, the smaller the time step the more accurate the results. The evaluation of the coefficients in the interface force power series is accomplished by a matrix multiplication. The only dependence on time in the interface compatibility coefficient matrix is the integration time step. Therefore, the matrix is generated only once unless the time step size is changed. The accuracy of the method has been validated by comparing the results with a closed-form response analysis of cantilever beam.

The interface compatibility between substructures is maintained by the use of the interface compatibility coefficient matrix. Therefore, this method is especially appealing for the response analysis of structural systems that have substructure interface boundary conditions that change during the response run. The coefficient matrix is simply changed or regenerated to reflect the change in boundary conditions between the substructures. The application of this method to the response analysis of a launch vehicle while lifting off from the launch pad was mentioned earlier. Another application is for structures that exhibit slip-stick motion at interfaces caused by friction. The appropriate interface compatibility matrix is selected by monitoring the interface force and determining if the interface is slipping or sticking.

Since the equations of motion of each substructure are solved independently for each time step, it appears the method would be ideally suited for parallel processing.

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| 16. Abstract<br><p>This paper presents a new method for determining the transient response of a discrete coordinate model of a linear structural system composed of substructures. The method is applicable to systems consisting of any number of substructures, both determinate and indeterminate interface boundaries, and any topological arrangement of the substructures. The method is simple to implement from a computational point of view because the equations of motion of each of the substructures are solved independently, and the interface boundary compatibility conditions are enforced at each integration time step by a matrix multiplication. The method is demonstrated for a structural system consisting of two beam segments and acted upon by a time dependent force. The numerical results from the demonstration problem validates the accuracy of the method. The application of this method to structural systems with changing interface boundary conditions between substructures is discussed.</p> |  |   |  |  |                         |
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