# THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES - XII 

## EINAL REPORT

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AESTRACT

The rapid two-dimensional slewina and vibrational control of the unsymmetrical flexible scole (spacecraft Control Laboratory Experiment) with multi-bounded controls has been considered. Fontryagin's Maximum Frinciple has been applied to the nonlinear equations of the system to derive the necessary conditions for the optimal control. The resulting two-point boundary-value problem is then solved by using the quasilinearization technique, and the near-minimum time is obtained by sequentially shortening the slewing time until the controls are near the bang-bang type. The trade-off between the minimum time and the minimum flexible amplitude requirements has been discussed. The numerical results show that the responses of the nonlinear system are significantly different from those of the 1 inearized syatem for rapid slewing. The gCole stationkeeping closed-loop dynamics arere-zxamined by zmploying a elightly different method for developing the equations of motion in which higher order terms in the expressions for the mest modal shape functions are now included. If no force actuators are mounted on the beam, the modal amplitude rewponses are hore easily excited than when these actuators are included. Eystem reshonses are dependent on both the tome actustor acationz as well as the atasa and oontrol weighting matrix elements. a prelininary ardy on the

space systems is conducted. A numerical example based on a coupled two-mass two-spring system illustrates the effect of changes caused in the mass and stitfness matrices on the closed-loop system eigenvalues. In certain cases the need for redesigning control laws previously synthesized, but not accounting for actuator masses, is indicated.

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## I. INTRODUCTION

The present grant, NSG-1414, supplement 11, continues the research effort initiated in May 1977 and accomplished in the previous grant years (May 1977 - May 1988) as reported in Refs. 1-15*. This research has concentrated on the control of the orientation and the shape of very large, inherently flexible proposed future spacecraft systems. Fossible future applications of such large spacecraft systems (LSS) include: large scale multi-beam antenna communication systems; Earth observation and resource senzing systems; orbitally based electronic mail transmission; as platforms for orbital based telescope systems; and as in-orbit test models designed to compare the performance of flexible Les systems with that predicted based on computer simulations and/or scale model Earth-based laboratory experiments. In recent years the grant research has focused on the orbital model of the spacecraft control Laboratory Experiment (sCOLE) first proposed by Taylor and Ealakrishnan ${ }^{16}$ in 1983.

The present report is divided into five chapters. Chapter II is based on a oaper presented at the 1989 AAS/AIAA Astrodynamics conference and describes rapid two-

* References cited in this report are listed separately at the end of each chapter.
dimensional slewing and vibration control of the asymmetrical ecole configuration where the beam flexibility is included in the model. Fontryagin's maximum principle has been applied to the nonlinear equations of the system to derive the necessary conditions for the optimal control where the shuttle mast, and reflector (multiple-bounded) controls are considered. The resulting two-point boundary value problem is then solved by uping the quasilinearization technique, and the near minimum time is obtained by sequentially shortening the slewing time until the controlis are nearly of the bang-bang type. The trade-off between the minimum flexible amplitude and minimum slewing time are discussed.

In the next chapter (Chapter III) a slightly different method for developing the equations of motion for the sCOLE system during stationkeeping is presented involving a more direct approach in matrix manipulation, and including higher order terms in the expressions foi the mast modal shape functions. Closed-loop responses for the system modeled by this approach are compared with similar responses as presented in Ref. 14 (based on the fh. L. thesis of c.M. Diarra) for the same ranges of the state and control penalty matrices. Further emphasis is piaced on evaluating how the flexible modes of the scole mast are Excited during representative stationkepping operations.

A preliminary study of the effect of actuator mass on the design of control laws for large space systems is the subject of chapter IV. A numerical example based on a coupled two-mass two-spring system is selected to illustrate the effects of varying the masses and stiffnesses (one at a time) on the closed-loop eigenvalues, and to determine what changes should be incorporated into the control laws previously desianed, but nct accounting for actuator masses. Finally, chapter $V$ describes the main general conclusions together with general recommendations. At the end of the grant year reported here and after submission of our proposal for the 1989-90 grant year ${ }^{17 \text {, the thrust of }}$ this research has been redirected to provide more direct support to the new Nasa controls/etructures Interaction (CSI) program.

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## II. RAPID IN-PLANE MANEUVERING OF THE FLEXI BLE orbi ti Ng SGOLE

> The rapid two-dimensional slewing and vibrational control of the unsymmetrical flexible scoLE (Spacecraft Control Laboratory Experiment) with multi-bounded controls has been considered. Pontryagin s Maximum Principle has been applied to the nonlinear equations of the system to derive the necessary conditions for the optimal control. The resulting two-point boundary-value problem is then solved by using the quasilinearization technique, and the near-minimum time is obtained by sequentially shortening the slewing time until the controls are near the bang-bang type. The trade-off between the minimum time and the minimum flexible amplitude requirements has been discussed. The numerical results show that the responses of the nonlinear system are significantly different from those of the linearized system for rapid slewing.

## I NTRODUCTI ON

The large-angle manedvering and vibrational control problem of a flexible spacecraft has been the subject of considerable research by many authors through different

- approaches to various structural models. - ${ }^{4}$ Among them, many authors placed their efforts on different control strategies
while using rather simplified spacecraft dynamic models. A few investigators have considered different and yet
complicated structural models $3^{-4}$ Among all the control strategies used, Pontryagin's Maximum Principle is an important and a basic method to such a coupled nonlinear dynamics and control problem. Although this method usually produces open-loop control strategies, it has the advantage of being able to handle control problems of more complicated structures (nonlinear dynamics and control), and it may prove to be useful in control-structure interaction problems. Unfortunately, most of the applications of this method to the slewing problem have been restricted to some simplified model, for example, a central hub with two or four symmetrically connected beams. Numerical problems appear to have limited the extension of the techniques based on the Maximum Principle to more complex system models.

However, by considering such extensions, we may encounter many interesting phenomena and produce many useful results. In this paper, we aim at using the Maximum Principle for a slightly more complicated structural model, namely, the 2 -dimensional orbiting SCOLE ${ }^{5}$. The complexity of the present problem stems from three considerations: (1) more nonlinear terms than before included in the dynamical equations; (2) more control variables used in this system; and (3) the rapid slewing or near-minimum time slewing which may produce large flexible modal amplitudes. We hope, through the present analysis, to reveal, to some extent, how the nonlinear system is different from the linearized system, and how some parameters, such as the slewing time, and the weighting elements on the controls, affect the responses of the system.

This paper consists of three parts: formulation of the system equations by using Lagrange's formula; derivation of the optimal control problem which results in the two-point boundary-value problem (TPBVP); and simulation of slews for different boundary conditions and control variables.


Fig. 1 Configuration of the Planar Orbiting SCOLE

## FORMULATI ON OF THE STATE EOUATI ONS

## System Configuration.

The Shuttle-beam-reflector system discussed in this paper is shown in Eig. 1. The Shuttle and the reflector are considered to be rigid bodies. The beam is assumed connected to the Shuttle at its mass center, 0 , In addition, the reflector is attached to the beam at an offset point, $a_{r}$, which is $x_{r}$ away from the mass center of the reflector, $o_{r}$. Both beam ends are considered to be fixed.

Fig. 1 shows the structure in the pitch plane, since our present purpose is to analyze the planar motion of the system. The equations of motion in this plane are also valid for the motion in the roll plane, except for that case the inertia parameters are different.

Three coordinate systems are used in Eig. 1: ( $k_{0}, i_{0}$ ), the orbit's axes; ( $\mathbf{k}, \mathbf{i}$, ), the Shuttle fixed coordinates;
and ( $k_{r}, i_{r}$ ), the reflector fixed coordinates. $\theta$ is the rotation angle of the Shuttle with respect to the orbit coordinates. The transverse displacement of the beam from its undeformed position is $w(z, t)$, where $z$ is the coordinate along the $k_{s}$ axis, and $t$ is time. If the displacement is assumed to be small, then, an approxmate expression for the rotation angle of the cross section of the beam is, $\phi(z, t)=\partial w(z, t) / \partial z$.

The free vibration of this structure can be considered as a free-free beam (Bernoulli-Euler type) vibration problem with boundary conditions including the masses and moments of inertia of the Shuttle and the reflector. The partial differential equation for this problem can be solved by using the separation of variable method, in which w(z,t) is assumed as

$$
\begin{equation*}
w(z, t)=\sum_{i=1}^{\infty} \psi_{i}(z) \eta_{i}(t) \tag{1}
\end{equation*}
$$

where $\psi_{i}(z)$ is the ith mode function (shape) and $\eta_{i}(t)$ is the associated amplitude of the ith mode. The natural frequencies and mode shapes for the pitch and roll motions are listed in Ref. 5, and will be used in this paper.

If the first $n$ modes of the flexible system are used in the formulatiom of the dynamical equations of the system, the expression in Eq. (1) can be rewritten as

$$
\begin{equation*}
w(z, t)=\sum_{i=1}^{n} \psi_{i}(z) \eta_{i}(t)=\psi^{\top}(z) \eta(t) \tag{2}
\end{equation*}
$$

where $\psi^{\top}=\left[\begin{array}{lll}\psi_{1} & \cdots & \psi_{n}\end{array}\right], \eta=\left[\begin{array}{lll}\eta_{1} & \cdots & \eta_{n}\end{array}\right]^{\top}$. Then, we have,

$$
\begin{align*}
& \dot{w}=\partial w / \partial t=\psi^{\top} \dot{\eta}  \tag{3}\\
& w^{\prime}=\phi(z, t)=\left(d_{\psi}^{\top} / d z\right) \eta=\psi^{\top} \eta  \tag{4}\\
& \partial^{2} w / \partial z^{2}=\left(d^{2} \psi^{\top} / d z^{2}\right) \eta=\psi^{\top} \eta  \tag{5}\\
& \delta w_{j}=\psi^{\top}\left(z_{j}\right) \delta \eta \tag{6}
\end{align*}
$$

$\theta$ and $\eta$ are the generalized coordinates of the system.

## Kinetic Energy

The kinetic energy of the system, $T$, consists of three parts, $T_{s}, T_{b}, T_{r}$, representing the kinetic energy of the Shuttle, the beam, and the reflector, respectively,

$$
\begin{equation*}
T=T_{s}+T_{b}+T_{r} \tag{7}
\end{equation*}
$$

where

$$
\begin{align*}
& T_{s}= \frac{1}{2} I_{s} \dot{\theta}^{2}  \tag{8}\\
& T_{b}=\frac{1}{2} \int_{0}^{L} \rho\left[\left(w^{2}+z^{2}\right) \dot{\theta}^{2}+\dot{w}^{2}+2 \dot{z} \dot{\theta}\right] d z  \tag{9}\\
& T_{r}= \frac{1}{2} I_{r}\left(\dot{\theta}+\dot{\phi}_{r}\right)^{2}+\frac{1}{2} m_{r}\left\{\left(\dot{w}_{r}^{2}+L^{2}\right) \dot{\theta}^{2}+2 L \dot{\theta} \dot{w}_{r}+\dot{w}_{r}^{2}\right. \\
&\left.\quad-2 x_{r}\left(\dot{\theta}+\dot{\phi}_{r}\right)\left[\left(\dot{w}_{r}+L \dot{\theta}\right) \sin \phi_{r}-\dot{\theta} w_{r} \cos \phi_{r}\right]\right\} \tag{10}
\end{align*}
$$

where $I_{s}$ and $I_{r}$ are the moments of inertia of the Shuttle and the reflector with respect to the attatchment points, respectively, $m_{r}$ is the mass of the reflector, $L$ is the length of the beam, $w_{r}=w(L, t)$, and $\phi_{r}=w^{\prime}(L, t)$.

## Potential Energy

The elastic potential energy of the beam is

$$
\begin{equation*}
V=\frac{E I}{2} \int_{0}^{L}\left(\frac{\partial^{2} u}{\partial z^{2}}\right)^{2} d z \tag{11}
\end{equation*}
$$

where EI is the constant flexural rigidity of the cross section of the beam.

## Generalized Forces

The virtual work done by the controls is

$$
\begin{equation*}
\delta W=u_{1} \delta \theta+\sum_{j=2}^{4} u_{j} \cdot \delta r_{j} \tag{12}
\end{equation*}
$$

where $u_{1}$ is the control torque on the Shuttle, and $u_{2}$ and $u_{3}$ are the actuator force vectors on the beam, and $u_{4}$ is the control force vector on the center of the reflector. The $\delta \theta$, and $\delta \mathbf{r}_{j}$ are the associated virtual displacements.

From Fig. 1, we have,

$$
\begin{aligned}
& \mathbf{u}_{j}=u_{j}\left[\cos \left(\theta+\phi_{j}\right) \mathbf{i}_{0}-\sin \left(\theta+\phi_{j}\right) \mathbf{k}_{0}\right] \\
& \mathbf{r}_{j}=\left(z_{j} \cos \theta-w_{j} \sin \theta\right) \mathbf{k}_{0}+\left(z_{j} \sin \theta+\ldots \cos \theta\right) \mathbf{i}_{0}, j=2,3,4 .
\end{aligned}
$$

where $u_{j}$ is the magnitude of $u_{j} ; z_{j}$ is the location of $u_{j}$ along the $k_{s}$ axis; $w_{j}=w\left(z_{j}, t\right)$; and $\phi_{j}=\phi\left(z_{j}, t\right)$. In this paper, $z_{2}=\left[/ 3, z_{3}=2 L / 3\right.$, and $z_{4}=[$. After substituting these expressions into Eq. (12), and noting the expression for $\delta w$ in Eq. (6), we can get,

$$
\begin{align*}
\delta W & =\left[u_{i}+\sum_{j=2}^{4} u_{j}\left(z_{j} \cos \phi_{j}+w_{j} \sin \phi_{j}\right)\right] \delta \theta+\sum_{j=2}^{4} \psi^{\top}\left(z_{j}\right) u_{j} \cos \phi_{j} \delta \eta \\
& =Q_{\theta} \delta \theta+Q_{\eta} \delta \eta \tag{13}
\end{align*}
$$

where $Q_{\theta}$ and $Q_{0}$ are the generalized forces associated with $\theta$ and $\eta$, respectively.

## Dynamical Equations

After substituting the expressions (2-5) into the kinetic energy in Eqs. $(7-10)$ and the potential energy in Eq. (11), and using the following maxtrix/vector notations,

$$
\begin{aligned}
& \int_{0}^{L} \rho w^{2} d z+m_{r} w_{r}^{2}=\eta^{\top}\left[\int_{0}^{L} \rho \psi \psi \psi^{\top} d z+m_{r} \psi(L) \psi(L)^{\top}\right] \eta=\eta^{\top} M_{2} \eta \\
& \int_{0}^{L} \rho \dot{w}^{2} d z+m_{r} \dot{w}_{r}^{2}+I_{r} \dot{\phi}^{2}=\dot{D}^{\top}\left[M_{2}+I_{r} \psi^{\prime}(L) \psi^{\prime}(L)^{\top}\right] \dot{D}_{D}=\dot{\eta}^{\top} M_{3} \dot{D} \\
& \int_{0}^{L} E I\left(\frac{\partial^{2} u}{\partial z^{2}}\right)^{2} d z=\eta^{\top}\left(\int_{0}^{L} E I \psi \psi^{\prime} \psi^{\top} d z\right) \eta=\eta^{\top} K \eta \\
& \int_{0}^{L} \rho \mathbf{z} w d z+I_{r} \dot{\phi}+m_{r} \dot{L}_{r} \dot{w}_{r} \dot{D}^{\top}\left[\int_{0}^{L} \rho z \psi d z+I_{r} \psi^{\prime}(L)+m_{r} L \psi(L)\right]=\dot{\eta}^{\top} m_{2}
\end{aligned}
$$

we can obtain the Lagrangian of the system,

$$
\begin{align*}
L & =\frac{1}{2} \dot{\theta}^{2}\left[I+\eta^{\top} M_{2} \eta+2 m_{r} x_{r}\left(w_{r} \cos \phi_{r}-L \sin \phi_{r}\right)\right] \\
& +\dot{\theta}\left[\dot{\eta}^{\top} m_{2}+m_{r} x_{r}\left(\dot{\phi}_{r} w_{r} \cos \phi_{r}-\dot{w}_{r} \sin \phi_{r}-\left[\dot{\phi}_{r} \sin \phi_{r}\right)\right]\right. \\
& +\frac{1}{2} \dot{\eta}^{\top} M_{3} \dot{\eta}-m_{r} x_{r} \dot{\phi}_{r} \dot{w}_{r} \sin \phi_{r}-\frac{1}{2} \eta \eta^{\top} K \eta \tag{14}
\end{align*}
$$

where $I=I_{s}+I_{r}+\int_{0}^{L} \rho z^{2} d z+m_{r} L^{2}$ is the total moment of inertia of the undeformed system. The Lagrange equations,

$$
\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\mathrm{Q}_{9}, \quad \frac{\mathrm{~d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\eta}}\right)-\frac{\partial L}{\partial \eta}=Q_{\eta}
$$

of the system can be obtained in the following matrix form,

$$
M(\eta)\left[\begin{array}{l}
\ddot{\theta}  \tag{15}\\
\ddot{\eta}
\end{array}\right]=F(\dot{\theta}, \eta, \dot{\eta})+B(\eta) u
$$

with

$$
\begin{align*}
& M(\eta)=\left[\begin{array}{lr}
\left.I+\eta^{\top} M_{2} \eta+2 m_{r} x_{r}(1)_{r} \cos \phi_{r}-L \sin \phi_{r}\right) & \text { symmetry } \\
m_{2}+M_{1} \eta \cos \phi_{r}-m_{s} \sin \phi_{r} & M_{3}-M_{s} \sin \phi_{r}
\end{array}\right]  \tag{16}\\
& E=\left[\begin{array}{l}
-2 \dot{\theta} \dot{\eta}^{\top}\left(M_{2} \eta+m_{2} \cos \phi_{r}-M_{1} \eta \sin \phi_{r}\right)+m_{r} \dot{\phi}^{2}\left(L \cos \phi_{r}+\psi_{r} \sin \phi_{r}\right) \\
\dot{\theta}^{2}\left(M_{2} \eta+m_{a} \cos \phi_{r}-M_{1} \eta \sin \phi_{r}\right)+\left(\dot{\phi}_{r} M_{1}^{\top}-2 \dot{\theta} M_{a}\right) \dot{\eta} \cos \phi_{r}-K_{\eta}
\end{array}\right]  \tag{17}\\
& B=\left[\begin{array}{cccc}
1 & z_{2} \cos \phi_{2}-w_{2} \sin \phi_{2} & z_{3} \cos \phi_{3}-w_{3} \sin \phi_{3} & z_{4} \cos \phi_{4}-w_{4} \sin \phi_{4} \\
\underline{0} & \psi\left(z_{2}\right) \cos \phi_{2} & \psi\left(z_{3}\right) \cos \phi_{3} & \psi\left(z_{4}\right) \cos \phi_{4}
\end{array}\right] \tag{18}
\end{align*}
$$

where $u=\left[\begin{array}{llll}u_{1} & u_{2} & u_{3} & u_{4}\end{array}\right]^{\top}$ is the control vector. Other notations used in these equations are

$$
\begin{align*}
& M_{1}=m_{r} x_{r} \psi^{\prime}(L) \psi^{\top}(L), \quad M_{s}=M_{1}+M_{1}^{\top}, \quad M_{a}=M_{1}-M_{1}^{\top} ; \\
& M_{4}=m_{r} x_{r} \psi^{\prime}(L) \psi^{\top}(L) ; \\
& m_{s}=m_{r} x_{r}\left[\psi(L)+L \psi^{\prime}(L)\right], \quad m_{a}=m_{r} x_{r}\left[\psi(L)-L \psi^{\prime}(L\right. \tag{L}
\end{align*}
$$

We need the following linearized version of Eqs. (15) to compare the responses of the two systems.

$$
\left[\begin{array}{c:c}
I & m^{\top}  \tag{19}\\
\hdashline m_{2} & M_{3}
\end{array}\right]\left[\begin{array}{l}
\ddot{\theta} \\
\ddot{\eta}
\end{array}\right]=-\left[\begin{array}{c:c}
0 & 0^{\top} \\
\hdashline \underline{O} & K
\end{array}\right]\left[\begin{array}{l}
\theta \\
\eta
\end{array}\right]+\left[\begin{array}{cccc}
1 & z_{2} & z_{3} & L \\
\hdashline \underline{0} & \psi\left(z_{2}\right) & \psi\left(z_{3}\right) & \psi(\bar{L})
\end{array}\right] u
$$

Eor convience, by introducing the notations

$$
Y_{1}^{\top}=\left[\theta, \quad \eta^{\top}\right]=\left[y_{11}, \ldots, Y_{1 k}\right], \quad k=n+1, \quad y_{2}=\dot{y}_{1}, \quad y^{\top}=\left[y_{1}^{\top}, y_{2}^{\top}\right]
$$

Eqs. (15) can be rewritten in the state form

$$
\begin{align*}
& \dot{y}_{1}=y_{2} \\
& \dot{y}_{2}=M^{-1}(\eta)\left[F\left(\eta, y_{2}\right)+B(\eta) u\right] \tag{20}
\end{align*}
$$

DERI VATI ON OF THE OPTI MAL GONTROL PROBLEM

## objective

The purpose of this paper is to find the optimal controls which rapidly drive the system from an initial state, $y(t=0)$, to a final required state, $y\left(t=t_{f}\right)$. Since the magnitudes of these controls are, in pratice, bounded, the optimal controls for the minimum time slewing problem are usually of the bang-bang type. However, this kind of control will generally introduce large flexible amplitudes. Therefore, a near-minimum-time slew is of primary interest to us.

## Necessary Conditions

Instead of starting from the minimum time control problem, we set out to deal with the optimal control problem with a quadratic cost function,

$$
\begin{equation*}
\dot{j}=\frac{1}{2} \int_{0}^{t_{f}}\left(Y_{1}^{\top} Q_{1} Y_{1}+y_{2}^{\top} Q_{2} Y_{2}+u^{\top} R u\right) d t \tag{21}
\end{equation*}
$$

where $Q_{1}, Q_{2}$, and $R$ are weighting matrices, $t_{f}$ is the given slewing time. This kind of problem has been considered by a list of authors. However, in their analysis, $t_{f}$ is fixed and there is no limitation on the magnitude of the controls. On the contrary, in the present problem, the slewing time $t_{\hat{f}}$ is no longer fixed, because we want to find a rapid slew or a near-minimum-time slew. The magnitudes of the concrols, $u$, are also bounded,

$$
\begin{equation*}
\left|u_{i}\right| \leq u_{i, b}, \quad i=1,2,3,4 \tag{22}
\end{equation*}
$$

Our strategies to solve this problem are described in the following. Eirst, the necessary conditions based on Eqs. (20-21) are derived. Then, the costraints, Eq. (22), are imposed on these necessary conditions to modify the controls. Finally, in the solution process of the resulting TPBVP, the slewing time is shortened sequentially, in order to find the near-minimum-time slewing. As we have discussed in Ref. 6, when the slewing time is shortened, the optimal control, will approach the optimal control of the minimum time slewing problem, that is, becoming the bang-bang type. It is clear that, when the controls approach the bang-bang type, the value of the index $I$ in Eq. (21) will increase and approach its maximum value.

The Hamiltonian of the system is,

$$
\begin{equation*}
H=\frac{1}{2}\left(Y_{1}^{\top} Q_{1} Y_{1}+Y_{2}^{\top} Q_{2} Y_{2}+u^{\top} R u\right)+\lambda_{1}^{\top} Y_{1}+\lambda_{2}^{\top} M^{-1}(F+B u) \tag{23}
\end{equation*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are the costate vectors associated with $Y_{1}$ and $Y_{2}$, respectively. By using the Maximum Principle, the necessary conditions for the unresticted optimal control problem are the dynamical equations (20) plus the following differential equations for the costates,

$$
\begin{align*}
& \dot{\lambda}_{1}=-\frac{\partial H}{\partial Y_{1}}=-Q_{1} Y_{1}-\frac{\partial E^{\top}}{\partial Y_{1}} M^{-1} \lambda_{2}-\left\langle\lambda_{2}^{\top} \frac{\partial M^{-1}}{\partial Y_{1}}>(F+B u)-\left\langle\lambda_{2}^{\top} M^{-1} \frac{\partial B}{\partial Y_{1}}>u\right.\right.  \tag{24}\\
& \dot{\lambda}_{2}=-\frac{\partial H}{\partial Y_{2}}=-Q_{2} Y_{2}-\lambda_{1}-\frac{\partial E^{\top}}{\partial Y_{2}} M^{-1} \lambda_{2} \tag{25}
\end{align*}
$$

where $\left\langle\lambda_{2}^{\top} \frac{\partial M^{-1}}{\partial Y_{1}}\right\rangle=\left[\frac{\partial M^{-}}{\partial Y_{1}} \lambda_{2}, \frac{\partial M^{-}}{\partial Y_{1}} \lambda_{2}^{1}, \ldots, \frac{\partial M^{-1}}{\partial Y_{1}} \lambda_{2}\right]^{\top}$ represents a special matrix (similar for $\left\langle\lambda_{2}^{\top} M^{-1} \frac{\partial B}{\partial Y_{1}}\right\rangle$ ); as well as the expressions for the optimal control,

$$
\begin{equation*}
\frac{\partial H}{\partial u}=0, \quad u=-R^{-1} B^{\top} M^{-1} \lambda_{2} \tag{26}
\end{equation*}
$$

The control rules in Eq. (26) are then modified by the following expressions?,

$$
u_{i,}=\left\{\begin{align*}
-u_{i b}, & \text { if } u_{i,}<-u_{i b}  \tag{27}\\
u_{i \varepsilon}, & \text { if }\left|u_{i, c}\right|<u_{i b}, \\
u_{i b}, & \text { if } u_{i, c}>u_{i, b}
\end{aligned}\right\} \quad \begin{aligned}
& \text { where } \\
& u_{i, c}=-\left(R^{-1} B^{\top} M^{-1} \lambda_{2}\right)_{i}, \\
& i=1,2,3,4
\end{align*}
$$

By substituting the control expressions into the dynamical equations (20) and the costate equations (24), we can obtain a set of $4(n+1)$ differential equations for the states and the costates. To obtain the control, $u$, we need to solve this set of differential equations with the $4(n+1)$ given boundary conditions: $y(t=0)$ and $y\left(t=t_{f}\right)$. This problem is called TPBVP because the B.C.'s are specified at the two ends of the slewing period.

## Solution of the TPBVP

The quasilinearization algorithm and the method of particular solutions are used to solve this nonlinear TPBVPs.

## NUMERI GAL RESULTS

Some common parameters of the SCOLE used in this paper are,

$$
\begin{aligned}
& E I=4 \times 10^{7} \mathrm{lb}-\mathrm{ft}^{2}, \rho=0.09554 \text { slug } / \mathrm{ft}, L=130 \mathrm{ft}, \\
& \mathrm{~m}_{s}=6366.46 \text { slug, } \mathrm{m}_{\mathrm{r}}=12.42 \mathrm{slug}, \\
& u_{1 b}=10,000 \mathrm{ft}-\mathrm{lb}, u_{2 b}=u_{3 b}=10 \mathrm{lb}, u_{a b}=800 \mathrm{lb} .
\end{aligned}
$$

Other different structural parameters are listed in Table 1.
Table 1
STRUGTURAL PARAMETERS OF THE 2-D SCOLE

|  | Roll-Axis | Pitch-Axis |  |
| :--- | ---: | ---: | :--- |
| $I_{s}$ | 905,443 | $6,789,100$ | slug-ft |
| $I_{r}$ | 18,000 | 9,336 | slug-ft |
| $\mathbf{x}_{r}$ | 32.5 | 18.75 | ft |
| $\omega_{1}$ | 0.319954 | 0.295016 | hz |
| $\omega_{2}$ | 1.287843 | 1.645292 | hz |
| $\omega_{3}$ | 4.800117 | 4.974182 | hz |

All the numerical tests done in this paper are rest-to-rest slews, that is, they use the same boundary coditions for the states: $\eta(t=0)=0, \eta\left(t=t_{\hat{f}}\right)=0 ; \theta(t=0)=0$, and $\theta\left(t=t_{f}\right)=\theta^{*}$, where $\theta^{*}$ is the required slewing angle, ranging from 20 deg to 180 deg . All these slewings can be divided into the following 3 groups.

## Group 1

In this group, only the Shuttle control torque has been used, i.e., $u=u_{1}$. The weighting matrices $Q_{1}=Q_{2}=\underline{0}$ and the weighting on $u_{1}, r_{1}=10^{-5}$. Figs. 2 show the near-minimum-time slewing about the roll axis, through 90 deg (Fig. 2A). The near-minimum-time, $t_{f}$, has been calculated to be 27.8 sec . The control torque is near the bang-bang type (Fig. 2F). The maximum amplitude of the first mode of the linearized system is 9.2 ft (Fig. 2B), which is less than $10 \%$ of the total length of the beam. The first modal amplitude response of the nonlinear system has a shape similar to that for the linearized system, but with a shifting of the amplitude. The second mode and the third mode of the nonlinear system have quite different time histories from their linearized
counterparts (Figs. 2C-D). The rotation angle, $\Phi_{r}$, and the displacement, $w_{r}$, at the reflector end of the beam are also plotted in Figs. 2A and 2E. They have shapes similar to the amplitude of mode 1 , because the first mode dominates the deformation of the beam for this slew.

The slewing about the pitch axis has responses similar to those about the roll axis. To make a comparison, the results of many other slewings in this group are listed in Table 2. $\eta_{1 \text { max }}$ is the maximum value of the first modal amplitude of the linearized system. Note that the number of vibrational cycles of the first mode increases as the slew angle, $\theta^{*}$, increases.

Table 2
RESULTS OF GROUP 1
Roll Axis
$\frac{\theta^{*}(\mathrm{deg})}{20}$

180
40.14
$\frac{\eta_{1 \max }(\mathrm{ft})}{7.5^{a}}$
$9.5^{a}$
$9.2^{a}$
$9.5^{b}$

| Pitch Axis |  |
| :---: | :---: |
| $\frac{t_{f}(s)}{31.85}$ | $\frac{\eta_{1 \max }(f t)}{2.8^{a}}$ |
| 48.29 | $2.8^{b}$ |
| 67.05 | $2.8^{\mathrm{c}}$ |
| 95.23 | $2.8^{d}$ |

${ }^{a}$ One cycle. ${ }^{\text {b }}$ Two cycles with one big peak and one small
 two big equal peaks and one small peak.

## Group 2

In this group, the force on the reflector, $u_{4}$, is added to the system. The weighting on the states, $Q_{1}$ and $Q_{2}$, are still chosen to be zero, and $r_{1}=10^{-5}$. The effect on the slewing responses of adding the control force $u_{4}$ may be analyzed by changing the values of $t_{f}$ and/or $r_{4}$, the weighting on $u_{4}$. Since the first modal amplitude dominates the deflection of the beam, our main concern will concentrate on the variation of the first modal amplitude.

To illustrate the effect of the parameters, $t_{f}$ and $r_{4}$ on the time response of the first modal amplitude, let's
consider a special case without lose of generality, i.e., the 90 deg slewing about the roll axis, the same case plotted in Eigs. 2 but with the control $u_{4}$ added. In Eig. $2 B$, the time response of the first modal amplitude can be approximately expressed as $-\eta_{1 \max } \sin \left(2 \pi t / t_{f}\right)$. This response is $180^{\circ}$ out-of phase with the control $u_{1}$ (Fig. 2F), because of the inertia effect of the flexible beam. However, when $u_{4}$ is added to the system, the torque produced by $u_{4}$ will accelerate the slew and balance the deflection of the beam produced by $u_{1}$.

It is not hard to imagine, from the physical point of view, that when $u_{4}$ increases to a large value, the response of the first modal amplitude may be in-phase with $u_{1}$ (or $u_{4}$ ), i.e., $\eta_{1}(t) \approx \eta_{1 \max } \sin \left(2 \pi t / t_{f}\right)$. Therefore, between the small values and the large values of $u_{4}$, there must exist a critical value at which the phase of the first modal response changes from out-of-phase to in-phase. It is also expected that during the "phase-change" period, the maximum value of the first amplitude becomes a minimum. This conjecture, fortunatly, has been proved to be true in our calculations.

One way to change the value of $u_{4}$ is to change the value of $r_{4}$, for fixed slewing time $t_{f}$. Another is to change $t_{f}$ while mantaining $r_{4}$ fixed. These results are plotted in Figs. 3A-3B. We should point out that for large values of $r_{4}$ (Fig. 3A) or large values of $t_{f}$ (Eig. 3B), $u_{4}$ is small and the response of the first modal amplitude is out-of-phase. On the contrary, small $r_{4}$ or $t_{f}$ results in large $u_{4}$ and, therefore, in-phase response. In each of these cases, a minimum value of $\eta_{1}$ max exists. It is also interesting to know that, at these critical values of $r_{4}$ or $t_{f}, \eta(t)$ experiences two oscillation cycles with two equal peak values (or valley values) of the linearized system, i.e., $\eta(t) \approx \eta_{1 \text { max }} \sin \left(4 \pi t / t_{\hat{f}}\right)$. The dotted lines in Eigs. 3A-B represent the nonlinear system responses. The nonlinear response has a shift from the linear response, especially when $t_{f}$ or $r_{4}$ is reduced. Also, we have observed that, at the critical points, although the amplitudes are small in value, the linear and the nonlinear systems have quite different time response histories.

A complete relationship between the three parameters, $\eta_{1 \text { max }}, t_{f}$, and $r_{4}$ can be investigated through the
3-dimensional surface in Fig. 4. The lower ditch on this surface represents the minimum value area of $\eta_{1 \text { max }}$. Although the global minimum value of $\eta_{1, m a x}$ occurs when $t_{f}$ is quite large, there exists a local minimum value, $\eta_{1 \max }=0.41 \mathrm{ft}$, around the middle of the ditch, where $t_{f}=23.881 \mathrm{sec}$ and $r_{4}=0.86$. This important point can be chosen as the trade-off point between rapid slew and small amplitude requirements, at which neither $t_{\hat{f}}$ nor $\eta_{1 \text { max }}$ is too large. The response shapes of the first modal amplitude for the different values of $t_{f}$ and $r_{4}$ are different. In the hilltop areas, only one vibrational cycle of $\eta_{1}(t)$ exists, but along the deep valleys of the ditch, $D_{1}(t)$ has two vibrational cycles with two equal peaks. More surprisingly, at the local minimum point mentioned above, $\eta_{1}(t)$ experiences three vibrational cycles with three equal peaks. The responses for this case are shown in Figs. 5, where the linear and nonlinear systems are quite different in spite of the small modal amplitudes.

## Group 3

Based on the example shown in Figs. 5, the controls, $u_{2}$ and $u_{3}$ are added to the control system in this group. The associated weightings on these controls are $r_{2}=10.0$ and $r_{3}=20.0$. Also, the weightings on $\eta_{1}$ and $\eta_{2}$ are selected as 200.0 and 1000.0 , respectively, to show the further reduction of the modal amplitudes. These results are plotted in Figs. 6. Compared with the results in Figs. 5, the modal amplitudes have been slightly reduced and the maximum value of $u_{4}$ has been reduced due to the addition of $u_{2}$ and $u_{3}$. Note that $u_{1}$ is not shown in Figs. 6 because of its similarity to that in Figs. 5.

## CONC:LUSI ON

The Maximum Principle has been applied to the rapid slewing problem of the planar flexible orbiting SCOLE. The dynamical equations used contain more nonlinear terms than those used by other authors, and the responses indicate the large differences between the nonlinear and the linearized systems, not only in the rapid slews where large modal amplitudes are involved, but also in the small-amplitude slews. The analysis between the relationship of the
parameters, $\eta_{1 \max }, t_{f}$, and $r_{4}$, indicates that the conflict between the rapid slew and the small flexible amplitude requirements may be compromised for multi-input control systems. The effects of these parameters on the 3-dimensional SCOLE model slewing responses need to be investigated.

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Figs. 280 Deg Roll Ads Slew, Using Shuttle Torque Only. (continued)



Figs. 3 Variation of Mode 1 vs. r4, and Tf.


Fig. 4 Relation between First Modal Amplitude, Reflector Control Weight (r4), and Slewing Time.

NONLINEAR


LINEAR
Figs. 580 Deg Roll Axds


## A. Formulation

In order to complete the calculation of the elements of the state and control influence matrices for the orbiting SCOLE system linearized about the nominal station keeping motion, we list all equations of the system which are based on the formulation of Refl1] as follows:

1. Generic Modal Equations of the Beam

$$
\begin{align*}
\ddot{A}_{\mathrm{n}} & +\omega_{\mathrm{n}}^{2} A_{\mathrm{n}}-\frac{1}{L} G_{1}\left(\beta_{\mathrm{n}}\right) \ddot{\eta}_{2}+\frac{1}{L} G_{2}\left(\beta_{\mathrm{n}}\right) \ddot{\eta}_{1}+\frac{2}{L}{ }^{(1)}{ }_{0} G_{3}\left(\beta_{\mathrm{n}}^{\prime}\right) \dot{\eta}_{2} \\
& -\frac{2 \omega_{0}}{L} G_{2}\left(\beta_{\mathrm{n}}\right) \dot{\eta}_{3}+\frac{4}{L} \omega_{0}^{2} G_{2}\left(\beta_{\mathrm{n}}\right) \eta_{1}-\frac{3}{L} \omega_{0}^{2} G_{1}\left(\beta_{1}\right) \eta_{2}=F \tag{1}
\end{align*}
$$

where
$A_{11}(n=1,2,3,4)$ is a time dependent amplitude of the $n$th mode.
$\eta_{i}(i=1,2,3)$ are angular displacements about roll, pitch and yaw axes.

$$
\begin{aligned}
& G_{1}\left(\beta_{n}\right)=f_{3}\left(\beta_{n}\right) A_{1 n}+f_{4}\left(\beta_{n}\right) B_{1 n}+f_{5}\left(\beta_{n}\right) C_{1 n}+f_{6}\left(\beta_{n}\right) D_{1 n} \\
& G_{2}\left(\beta_{n}\right)=f_{3}\left(\beta_{n}\right) A_{2 n}+f_{4}\left(\beta_{n}\right) B_{2 n}+f_{5}\left(\beta_{n}\right) C_{2 n}+f_{6}\left(\beta_{n}\right) D_{2 n} \\
& G_{3}\left(\beta_{n}^{\prime}\right)=f_{3}\left(\beta_{n}^{\prime}\right) A_{3 n}+f_{4}\left(\beta_{n}^{\prime}\right) B_{3 n} \\
& f_{3}\left(\beta_{n}\right)=-\frac{\sin \left(\beta_{n} L\right)}{\beta_{n}^{2}}+\frac{L \cos \left(\beta_{n} L\right)}{\beta_{n}} \\
& f_{4}\left(\beta_{n}\right)=\frac{\cos \left(\beta_{n} L\right)}{\beta_{n}^{2}}+\frac{L \sin \left(\beta_{n} L\right)}{\beta_{n}}-\frac{1}{\beta_{n}^{2}} \\
& f_{5}\left(\beta_{n}\right)=-\frac{L \cosh \left(\beta_{n} L\right)}{\beta_{n}}+\frac{\sinh \left(\beta_{n} L\right)}{\beta_{n}^{2}} \\
& f_{6}\left(\beta_{n}\right)=\frac{L \sinh \left(\beta_{n} L\right)}{\beta_{n}}-\frac{\cosh \left(\beta_{n} L\right)}{\beta_{n}^{2}}+\frac{1}{\beta_{n}^{2}} \\
& F=F_{x} \mid V_{3 x} S_{x n}(-L)+V_{2 x} S_{x n}(-2 L / 3)+V_{1 x} S_{x n}^{(-L / 3) \mid} \\
& +F_{y} \mid V_{3 y} S_{y n}(-L)+V_{2 y} S_{y n}(-2 L / 3)+V_{1 y} S_{y n}^{(-1 / / 3) \mid}
\end{aligned}
$$

$$
\begin{aligned}
& S_{x n}(Z)=A_{1 n} \sin \left(\beta_{n} Z\right)+B_{1 n} \cos \left(\beta_{n} Z\right)+C_{1 n} \sinh \left(\beta_{11} Z\right)+D_{1 n} \cosh \left(\beta_{n} Z\right) \\
& S_{y n}(Z)=A_{2 n} \sin \left(\beta_{n} Z\right)+B_{2 n} \cos \left(\beta_{n} Z\right)+C_{2 n} \sinh \left(\beta_{11} Z\right)+D_{2 n} \cosh \left(\beta_{n} Z\right) \\
& \theta_{n}(Z)=A_{3 n} \sin \left(\beta_{n}^{\prime} Z\right)+B_{3 n} \cos \left(\beta_{n}^{\prime} Z\right) \\
& \beta_{n}^{\prime}=\beta_{n}^{2} \sqrt{\frac{E I}{G A}}
\end{aligned}
$$

2. System Equations without Flexibility and External Forces

$$
\begin{align*}
\ddot{\eta}_{1} I_{x x}-\ddot{\eta}_{2} I_{x y} & -\ddot{\eta}_{3} I_{x z}-\omega_{0} \dot{\eta}_{3}\left(I_{x x}-I_{y y}+I_{7 z}\right)-\omega_{11} \dot{\eta}_{2} I_{y} \\
& -4 \omega_{0}^{2} \eta_{1}\left(I_{z z}-I_{y y}\right)-\omega_{0}^{2} \eta_{3} I_{x z}-3\left(\omega_{0}^{2} \eta_{2} I_{x y}=0\right.  \tag{2}\\
\ddot{\eta}_{2} I_{y y}+\ddot{\eta}_{1} I_{x y} & +\ddot{\eta}_{3} I_{y z}-\omega_{0} \dot{\eta}_{1} I_{y z}+\omega_{0} \dot{\eta}_{3} I_{x y}-3 \omega_{0}^{2} \eta_{1} I_{x y} \\
& +\omega_{0}^{2} \eta_{3} I_{y z}+3 \omega_{0}^{2} \eta_{2}\left(I_{x x}-I_{z z}\right)=0  \tag{3}\\
\ddot{\eta}_{3} I_{z z}-\ddot{\eta}_{1} I_{x z} & -\ddot{\eta}_{2} I_{y z}+\omega_{0} \dot{\eta}_{1}\left(I_{x x}-I_{y y}+I_{z z}\right)-\omega_{0} \dot{\eta}_{2} I_{x y} \\
& -4 \omega_{0}^{2} \eta_{1} I_{x z}+3 \omega_{0}^{2} \eta_{2} I_{y z}-\omega_{0}^{2} \eta_{3}\left(I_{\ldots y y}-I_{y y}\right)=0 \tag{4}
\end{align*}
$$

where

$$
\begin{aligned}
& I_{x x}=I_{s 1}+I_{R 1}+\frac{M L^{2}}{3}+M_{R}\left(L^{2}+Y^{2}\right) \\
& I_{y y}=I_{s 2}+I_{R 2}+\frac{M L^{2}}{3}+M_{R}\left(L^{2}+X^{2}\right) \\
& I_{z z}=I_{s 3}+I_{R 3}+M_{R}\left(X^{2}+Y^{2}\right) \\
& I_{x y}=M_{R} X Y \\
& I_{x Z}=I_{s 4}+M_{R} X L \\
& I_{y z}=M_{R} Y L
\end{aligned}
$$

3. System Equations with the First Four Flexible Modes

$$
\begin{align*}
& \ddot{\eta}_{1} I_{x x}-\ddot{\eta}_{2} I_{x y}-\ddot{\eta}_{3} I_{x z}-\omega_{0} \dot{\eta}_{3}\left(I_{x x}-I_{y y}+I_{z \sigma}\right)-\omega_{0} \dot{\eta}_{2} I_{y}, \\
& -4 \omega_{0}^{2} \eta_{1}\left(I_{z Z}-I_{y y}\right)-\omega_{0}^{2} \eta_{3} I_{x Z}-3 \omega_{0}^{2} \eta_{2} I_{Y Y} \\
& +\sum_{n=1}^{4} \ddot{A}_{n} d_{1 n}-\sum_{n=1}^{4} \dot{A}_{n} d_{2 n}+\sum_{n=1}^{4} A_{n} d_{3 n}=T_{x}  \tag{5}\\
& \ddot{\eta}_{2} I_{y y}+\ddot{\eta}_{1} I_{x y}+\ddot{\eta}_{3} I_{y z}-\dot{\eta}_{1} \omega_{0} I_{y z}+\dot{\eta}_{3} \omega_{0} I_{x y}-3 \omega_{0}^{2} \eta_{1} I_{v v} \\
& +\omega_{0}^{2} \eta_{3} I_{y z}+3 \omega_{0}^{2} \eta_{2}\left(I_{x x}-I_{z z}\right)+\sum_{n=1}^{4} \ddot{A}_{n} d_{4 n} \quad \sum_{n=1}^{4} A_{n} d_{5 n}=T_{y}  \tag{6}\\
& \ddot{\eta}_{3} \mathrm{I}_{z z}-\ddot{\eta}_{1} \mathrm{I}_{\mathrm{xz}}-\ddot{\eta}_{2} \mathrm{I}_{\mathrm{yz}}+\dot{\eta}_{1} \omega_{0}\left(\mathrm{I}_{\mathrm{xx}}-\mathrm{I}_{\mathrm{yy}}+\mathrm{I}_{7 \mathrm{z}}\right)-\omega_{0} \dot{\eta}_{2} \mathrm{I}_{\mathrm{vx}} \\
& -4 \omega_{0}^{2} \eta_{1} I_{x z}+3 \omega_{0}^{2} \eta_{2} I_{y z}-\omega_{0}^{2} \eta_{3}\left(I_{x x}-I_{y y}\right) \text {. } \\
& +\sum_{n=1}^{4} \ddot{A}_{n} d_{6 n}+\sum_{n=1}^{4} \dot{A}_{n} d_{7 n}+\sum_{n=1}^{4} A_{n} d_{8_{n}}=T_{7} \tag{7}
\end{align*}
$$

where

$$
\begin{aligned}
& \mathrm{I}_{\mathrm{xx}}, \mathrm{I}_{\mathrm{yy}}, \mathrm{I}_{z \mathrm{z}}, \mathrm{I}_{\mathrm{xy}}, \mathrm{I}_{\mathrm{yz}}, \mathrm{I}_{\mathrm{x} z} \text { are same as in } 2 . \\
& \mathrm{T}_{\mathrm{x}}=\mathrm{M}_{\mathrm{x}} \mathrm{U}_{\mathrm{x}}+\mathrm{F}_{\mathrm{y}} L\left[\mathrm{~V}_{1 \mathrm{y}} / 3+2 \mathrm{~V}_{2 \mathrm{y}} / 3+\mathrm{V}_{3 \mathrm{y}} \mid\right. \\
& \mathrm{T}_{\mathrm{y}}=\mathrm{M}_{\mathrm{y}} \mathrm{U}_{\mathrm{y}}+\mathrm{F}_{\mathrm{x}} L\left[\mathrm{~V}_{1 \mathrm{x}} / 3+2 \mathrm{~V}_{2 \mathrm{x}} / 3+\mathrm{V}_{3 \mathrm{x}} \mid\right. \\
& \mathrm{T}_{\mathrm{z}}=\mathrm{M}_{\mathrm{z}} \mathrm{U}_{\mathrm{z}}+X \mathrm{~F}_{\mathrm{y}} \mathrm{~V}_{3 \mathrm{y}}-\mathrm{YF}_{\mathrm{x}} V_{3 \mathrm{x}}
\end{aligned}
$$

## B. System State Equations

In this section we recast all system equations ( $1-7$ ) into matrix form.
Let

$$
x=\left[\eta_{1} \eta_{2} \eta_{3} A_{1} A_{2} A_{3} A_{4} \dot{\eta}_{1} \dot{\eta}_{2} \dot{\eta}_{3} \dot{A}_{1} \dot{\lambda}_{2} \hat{A}_{1} i_{1} \mid\right.
$$

as a state vector and

$$
\mathbf{U}=\left[\begin{array}{lllllllll}
V_{1 x} & V_{1 y} & V_{2 x} & V_{2 y} & V_{3 x} & V_{3 y} & U_{x} & U_{y} & U_{7}
\end{array} l^{T}\right.
$$

as a control input. We then set up the system state equations bv lwo different methods and get the state matrix and influence matrix, respectively.

1. Method of $\operatorname{Ref}[1]$

Generic modal equations of the beam:

$$
\left[\begin{array}{l}
\ddot{A}_{1}  \tag{8}\\
\ddot{A}_{2} \\
\ddot{A}_{3} \\
\ddot{A}_{4}
\end{array}\right]+\left[D_{1}\left[\begin{array}{l}
A_{1} \\
A_{2} \\
A_{3} \\
A_{4}
\end{array}\right]+\left[D_{2}\left[\begin{array}{l}
\ddot{\eta}_{1} \\
\ddot{\eta}_{2} \\
\ddot{\eta}_{3}
\end{array}\right]+\left[D_{3}\left[\begin{array}{l}
\dot{\eta}_{1} \\
\dot{\eta}_{2} \\
\dot{\eta}_{3}
\end{array}\right]+\left[D_{4}\left[\begin{array}{l}
\eta_{1} \\
\eta_{2} \\
\eta_{3}
\end{array}\right]=[F]\left[\begin{array}{l}
V_{1 x} \\
\mathrm{~V} 1 \mathrm{y} \\
\mathrm{~V} 2 x \\
\mathrm{~V} 2 \mathrm{y} \\
\mathrm{~V} 3 \mathrm{x} \\
\mathrm{~V} 3 \mathrm{y}
\end{array}\right]\right.\right.\right.\right.
$$

System equations without flexibility and external forces:

$$
\left[E_{1} 1\left[\begin{array}{l}
\ddot{\eta}_{1}  \tag{9}\\
\ddot{\eta}_{2} \\
\ddot{\eta}_{3}
\end{array}\right]+\left[E_{3}\right]\left[\begin{array}{l}
\dot{\eta}_{1} \\
\dot{\eta}_{2} \\
\dot{\eta}_{3}
\end{array}\right]+\left[E_{5}\right]\left[\begin{array}{l}
\eta_{1} \\
\eta_{2} \\
\eta_{3}
\end{array}\right]=0\right.
$$

System equations with the first four ilexible modes:

$$
\begin{align*}
& {\left[\mathrm{E}_{4}\left[\begin{array}{l}
\ddot{\eta}_{1} \\
\ddot{\eta}_{2} \\
\ddot{\eta}_{3}
\end{array}\right]+\left[\mathrm{E}_{2}\right]\left[\begin{array}{l}
\ddot{A}_{1} \\
\ddot{A}_{1} \\
\ddot{A}_{3} \\
\ddot{A}_{4}
\end{array}\right]+\left[\mathrm{E}_{3}\left[\begin{array}{l}
\dot{r}_{1} \\
\dot{\eta}_{2} \\
\dot{\eta}_{3}
\end{array}\right]+\left[\mathrm{E}_{4}\left[\begin{array}{l}
\dot{A}_{1} \\
\dot{A}_{2} \\
\dot{A}_{3} \\
\dot{A}_{4}
\end{array}\right]+\left[\mathrm{E}_{5}\left[\begin{array}{l}
\eta_{1} \\
\eta_{2} \\
\eta_{3}
\end{array}\right]\right.\right.\right.\right.} \\
& +\left[E_{6}\left[\begin{array}{l}
A_{1} \\
A_{2} \\
A_{3} \\
A_{4}
\end{array}\right]=\left[M_{P}\right]\left[\begin{array}{c}
V_{1 x} \\
V_{1 y} \\
V_{2 x} \\
V_{2 y} \\
V_{3 x} \\
V_{3 y}
\end{array}\right]+\left[M_{2}\right]\left[\begin{array}{c}
U_{x} \\
U_{y} \\
U_{z}
\end{array}\right]\right. \tag{10}
\end{align*}
$$

We then recasi eq(9) by inverting the matrix $\left[E_{1}\right]^{-1}$.

$$
\left[\begin{array}{l}
\ddot{\eta}_{1}  \tag{11}\\
\ddot{\eta}_{2} \\
\ddot{\eta}_{3}
\end{array}\right]=-\left[\begin{array}{lll}
E_{1}^{-1} & E_{3}
\end{array}\right]\left[\begin{array}{l}
\dot{\eta}_{1} \\
\dot{\eta}_{2} \\
\dot{\eta}_{3}
\end{array}\right]-\left[\begin{array}{ll}
E_{1}^{-1} & E_{5}
\end{array}\right]\left[\begin{array}{l}
\eta_{1} \\
\eta_{2} \\
\eta_{3}
\end{array}\right]
$$

After substituting eq (11) into eq (8), the result is,

$$
\left[\begin{array}{l}
\ddot{A}_{1}  \tag{12}\\
\ddot{A}_{2} \\
\ddot{A}_{3} \\
\ddot{A}_{4}
\end{array}\right]=-\left[D_{1}\left[\begin{array}{l}
A_{1} \\
A_{2} \\
A_{3} \\
A_{4}
\end{array}\right]-\left[\begin{array}{lll}
D_{3}-D_{2} & E_{1} & E_{3}
\end{array}\right]\left[\begin{array}{l}
\dot{\eta}_{1} \\
\dot{\eta}_{2} \\
\dot{\eta}_{3}
\end{array}\right]-\left[\begin{array}{lll}
D_{4}-D_{2} & F_{1} & E_{5}
\end{array}\right]\left[\begin{array}{l}
\eta_{1} \\
\eta_{2} \\
\eta_{3}
\end{array}\right]+\left[\begin{array}{l}
F
\end{array}\right]\left[\begin{array}{c}
\mathrm{V}_{1 x} \\
\mathrm{~V} 1 \mathrm{y} \\
\mathrm{~V} 2 x \\
\mathrm{~V} 2 \mathrm{y} \\
\mathrm{~V} 3 \mathrm{x} \\
\mathrm{~V} 3 y
\end{array}\right]\right.
$$

or briefly

$$
\begin{equation*}
\ddot{A}=\left[C_{1}\right] A+\left[C_{2}\right] \dot{\eta}+\left[C_{3}\right] \eta+[F] v \tag{13}
\end{equation*}
$$

where

$$
\begin{aligned}
& A=\left[\begin{array}{llll}
A_{1} & A_{2} & A_{3} & A_{4}
\end{array}\right]^{T} \\
& \ddot{A}=\left[\begin{array}{llll}
\ddot{A}_{1} & \ddot{A}_{2} & \ddot{A}_{3} & \ddot{A}_{4}
\end{array}\right]^{T} \\
& \eta=\left[\begin{array}{lll}
\eta & \eta_{2} & \eta_{3}
\end{array}\right]^{T} \\
& { }_{1} \\
& \eta=\left[\begin{array}{lll}
\eta_{1} & \eta_{2} & \eta_{3}
\end{array}\right]^{T} \\
& V=\left[\begin{array}{llll}
V_{1 x} & V_{1 y} & V_{2 x} & V_{2 y} \\
V_{3 x} & V_{3 y}
\end{array}\right]^{\mathrm{T}} \\
& {\left[C_{1}\right]=-\left[\begin{array}{ll}
D_{1}
\end{array}\right]} \\
& {\left[C_{2}\right]=-\left[\begin{array}{lll}
D_{3}-D_{2} & E_{1}^{-1} & E_{3}
\end{array}\right]} \\
& {\left[C_{3}\right]=-\left[\begin{array}{lll}
D_{4}-D_{2} & E_{1}^{-1} & E_{5}
\end{array}\right]}
\end{aligned}
$$

Then eq (13) without the external forces is substituted into cq(10) with the result

$$
\begin{equation*}
\left.\ddot{\eta}=\left[C_{5}\right] \dot{A}+\left[C_{4}\right] \dot{\eta}+\left[C_{6}\right] \eta+\left[C_{7}\right] A+\left[M_{7}\right] \prod_{1}\right] M^{\prime} \tag{14}
\end{equation*}
$$

where

$$
\begin{aligned}
& \ddot{\eta}=\left[\begin{array}{lll}
\ddot{\eta}_{n} & \ddot{\eta}_{2} & \ddot{\eta}_{3} \\
& 1 & \\
2 & &
\end{array}\right]^{T} \\
& \dot{A}=\left[\begin{array}{llll}
\dot{A}_{1} & \dot{A}_{2} & \dot{A}_{3} & \dot{A}_{4}
\end{array}\right]^{\mathrm{T}} \\
& \mathrm{U}^{\prime}=\left[\begin{array}{lll}
\mathrm{U}_{\mathrm{x}} & \mathrm{U}_{\mathrm{y}} & \mathrm{U}_{\mathrm{z}}
\end{array}\right]^{\mathrm{T}} \\
& {\left[C_{4}\right]=-\left[E_{1}^{-1}\left(E_{3}+E_{2} C_{2}\right)\right]} \\
& {\left[\mathrm{C}_{5}\right]=-\left[\mathrm{E}_{1}^{-1} \mathrm{E}_{4}\right]} \\
& {\left[C_{6}\right]=-\left[E_{1}^{-1}\left(E_{5}+E_{2} C_{3}\right)\right]} \\
& {\left[C_{7}\right]=-\left[E_{1}^{-1}\left(E_{6}+E_{2} C_{1}\right)\right]} \\
& {\left[M_{3}\right]=\left[E_{1}^{-1} M_{1}\right]} \\
& {\left[\mathrm{M}_{4}\right]=\left[\mathrm{E}_{1}^{-1} \mathrm{M}_{2}\right]}
\end{aligned}
$$

Eqs(14) and (13) may be combined as follows:

$$
\left[\begin{array}{l}
.  \tag{15}\\
\eta \\
\ddot{A}
\end{array}\right]=\left[\begin{array}{ll}
\mathrm{C}_{4} & \mathrm{C}_{5} \\
\mathrm{C}_{4} & 0
\end{array}\right]\left[\begin{array}{l}
\dot{\eta} \\
\dot{A}
\end{array}\right]+\left[\begin{array}{ll}
\mathrm{C}_{6} & \mathrm{C}_{7} \\
\mathrm{C}_{3} & \mathrm{C}_{1}
\end{array}\right]\left[\begin{array}{l}
\eta \\
\mathrm{A}
\end{array}\right]+\left[\begin{array}{ll}
\mathrm{M}_{3} & \mathrm{M}_{4} \\
\mathrm{~F} & 0
\end{array}\right]\left[\begin{array}{l}
\mathrm{V} \\
\mathrm{U}^{\prime}
\end{array}\right]
$$

The system state equation becomes

$$
\dot{\mathbf{X}}=[\mathbf{A}] \mathbf{X}+[\mathbf{B}] \mathbf{U}
$$

where

## 2. Direct Method

The generic modal equation (eq(8)) and system equation (cq(1))) may be directly combined to yield:

$$
\left[\begin{array}{ll}
E_{1} & E_{2}  \tag{16}\\
D_{2} & I
\end{array}\right]\left[\begin{array}{c}
\ddot{\eta} \\
\ddot{A}
\end{array}\right]=\left[\begin{array}{cc}
-E_{3} & -E_{4} \\
-D_{3} & 0
\end{array}\right]\left[\begin{array}{l}
\dot{\eta} \\
\dot{A}
\end{array}\right]+\left[\begin{array}{ll}
-E_{5} & -E_{6} \\
-D_{4} & D_{1}
\end{array}\right]\left[\begin{array}{l}
\eta \\
A
\end{array}\right]+\left[\begin{array}{ll}
M_{1} & M_{2} \\
F & 0
\end{array}\right]\left[\begin{array}{l}
V \\
U^{\prime}
\end{array}\right]
$$

$\mathrm{Eq}(16)$ may be rewritten, following the inversion of the acceleration coefficient matrix, as

$$
\begin{align*}
{\left[\begin{array}{c}
. . \\
\eta \\
\ddot{A}
\end{array}\right]=} & {\left[\begin{array}{ll}
E_{1} & E_{2} \\
D_{2} & I
\end{array}\right]^{-1}\left[\begin{array}{ll}
-E_{3} & -E_{4} \\
-D_{3} & 0
\end{array}\right]\left[\begin{array}{l}
\dot{\eta} \\
\dot{A}
\end{array}\right]+\left[\begin{array}{ll}
E_{1} & E_{2} \\
D_{2} & I
\end{array}\right]^{-1}\left[\begin{array}{ll}
-E_{5} & -E_{6} \\
-D_{4} & D_{1}
\end{array}\right]\left[\begin{array}{l}
\eta \\
A
\end{array}\right] } \\
& +\left[\begin{array}{ll}
E_{1} & E_{2} \\
D_{2} & I
\end{array}\right]^{-1}\left[\begin{array}{ll}
M_{1} & M_{2} \\
F & 0
\end{array}\right]\left[\begin{array}{l}
V \\
U
\end{array}\right] \tag{17}
\end{align*}
$$

or briefly

$$
\left[\begin{array}{c}
. \ddot{\eta}  \tag{18}\\
. \\
A
\end{array}\right]=\left[A^{\prime}\right]\left[\begin{array}{c}
\dot{\eta} \\
\dot{A}
\end{array}\right]+\left[A^{\prime \prime}\right]\left[\begin{array}{l}
\eta \\
A
\end{array}\right]+\left[B^{\prime}\right]\left[\begin{array}{l}
\mathrm{V} \\
\mathrm{U}^{\prime}
\end{array}\right]
$$

where

$$
\begin{aligned}
& {\left[A^{\prime}\right]=\left[\begin{array}{ll}
E_{1} & E_{2} \\
D_{2} & I
\end{array}\right]^{-1}\left[\begin{array}{ll}
-E_{3} & -E_{4} \\
-D_{3} & 0
\end{array}\right]} \\
& {\left[A^{\prime \prime}\right]=\left[\begin{array}{ll}
E_{1} & E_{2} \\
D_{2} & I
\end{array}\right]^{-1}\left[\begin{array}{cc}
-E_{5} & -E_{6} \\
-D_{4} & D_{1}
\end{array}\right]} \\
& {\left[B^{\prime}\right]=\left[\begin{array}{ll}
E_{1} & E_{2} \\
D_{2} & I
\end{array}\right]^{-1}\left[\begin{array}{cc}
M_{1} & M_{2} \\
F & 0
\end{array}\right]}
\end{aligned}
$$

We can get the system state equation from $\mathrm{eq}(18)$, that is

$$
\begin{equation*}
\dot{\mathbf{X}}=[\mathbf{A}] \mathbf{X}+[\mathbf{B}] \mathbf{U} \tag{19}
\end{equation*}
$$

where

$$
[\mathbf{A}]=\left[\begin{array}{ccc}
\vdots & \vdots & \mathrm{I} \\
\hdashline \ldots \ldots & \ldots \ldots \\
\hdashline \mathrm{~A} & : & \mathrm{A}
\end{array}\right] \quad \text { and } \quad[\mathbf{B}]=\left[\begin{array}{c}
0 \\
\ldots \ldots . \\
\mathbf{B}
\end{array}\right]
$$

$\therefore$ Control synthesis－LQR
The sustem state equation can be rearesented as

$$
\begin{equation*}
\dot{x}=[A] x+[B] 1 \tag{20}
\end{equation*}
$$

An LQR cost function is selected as follows：
$J=\int_{0}^{\infty}\left(X^{\top} Q X U U^{\top} R J\right) d t$
The optimal control，u，based on the LQR theory is given by

$$
\begin{equation*}
U=-\left[R^{-\mid E} \mid F\right] x \tag{22}
\end{equation*}
$$

where $f$ is the positive definite solution of the steady state Ricatti matrix equation：

$$
\begin{equation*}
F A+A^{T} F-P B F^{-1} B^{\top} P+Q=0 \tag{23}
\end{equation*}
$$

The closed loop system equation becomes

$$
\begin{equation*}
\dot{x}=[A-E K] x \tag{24}
\end{equation*}
$$

Let $x(0)$ be an initial state vector．Eased on some assumed $Q$ and $R$ penalty matrices，the olosed loop dynamic responaes can be otmulated az

$$
\begin{equation*}
x(t)=E[A-E K] t \times(0) \tag{25}
\end{equation*}
$$

which iz based on the teedback control given by

$$
\begin{equation*}
u(t)=-k x(t) \tag{26}
\end{equation*}
$$

The total torrue impulse about the three axes are

$$
\begin{array}{ll}
T_{y}=\int_{0}^{\infty}\left|T_{x}(t)\right| d t & (27) \\
T_{y}=\int_{0}^{\infty}\left|T_{y}(t)\right| d t & (23) \\
T_{z}=\int_{0}^{\infty}\left|T_{z}(t)\right| d t & (29)
\end{array}
$$

Wんまにも

$$
\left[\begin{array}{c}
T_{x}(t) \\
T_{y}(t) \\
T_{y}(t)
\end{array}\right]=\left[\begin{array}{c}
y(t) \\
-- \\
1(t)
\end{array}\right]
$$

D. Numerical Results

The ordcle control software in the IEM computer system Was used to calculate the state matrix $[A]$ and influence matrix [B] and to simulate the closed loop system responses as well as the total torque of the system for given sets of initial conditions.

We select the force factors. $F_{x}=F_{y}=1$ and torque factors $M_{x}=M_{y}=M_{z}=1$, which means that the components of $v$ and 1 reflect the actual actuator force and shutti= torque values. According to the gcole configuration and parameter values (iisted in the Appendix), the [A] and [E] matrix values of the method of Ref. [1] and the direct method ale listed in Tables $1,2,3$, and 4.
we select the initial states
$x_{1}(0)=0$ Jegreas
$x_{2}(0)=x_{2}(0)=x_{4}(0)=x_{5}(0)=x_{6}(0)=x_{7}(0)=0$
and the diagonal weighting matrices as:
trace $\mathrm{Q}^{2}=\left[10^{7} \cdot 10^{7} \cdot 10^{7} \cdot 5 \times 10^{5} \cdot 10^{5} \cdot 5 \times 10^{5} \cdot 5 \times 10^{7}\right.$.
$10.10,10,10,10,10.101$
$t a c e r=1100,100.100,100,100,100,0.001,0.001,0.0011$
The simulation of the optima clused loon syaten
responses, using both the method of fet. I if and the direct method. ane p!ottra in fige. 1. 2. 3 and is.
rhe total contol torque-muulaze of the zersem


method. The torques needed about the other two axes are much less than the components about the roll axis. Also, the maximum torques of the system are $6,525 \mathrm{ft}-1 \mathrm{~b}$ for the method of ref. [1] and 5,802 tt-11 in the direct method.
E. Conclusions

1. By comparing the results of the method of Ref. [1] and the direct method, it is seen that the results are similar to each other.
2. In the responses resulting from the direct method for the same initial displacement about the roll axis, it is seen that the first four flexible modes are generally excited more than for the results of the method of Ref. [1]. 3. If no force actagtors are adced to the beam and reflector complete damping of the modal responsee requires a much longer time (Fig. 6) than when the force actuators are utilized together with the shuttle torquers (Fige. 2 and 4). However, the use of force actuators resultis in initially larger overohoots as compared with the gase depicted in fig. 5.
3. The system responses are dependent on the force actuator locatione and the weighting metrices (Q.e) values. Suitable values of the penaliy matrioes and actuator locations should $b=$ eeleoted so thet the zustom control becomes votima:.
```
B. From the shatem analysis, we rind the tisibility of the
```


keeping operations. Svstem responses and the total torque impulses needed are similar to the rigidized sCOLE system (see Ref. [1]).
F. Reference

1. Cheick M. Diarra, "On the Dynamics and Control of the Spacecratt control Laboratorv Experiment (scole) Class of Offset Flexible Eystems," Fh.C. dissertation, Howard University, 1988. Also contract Report. Fart II, NASA Grant NSG-1414, Suppl. 9, January 1988.
 ? - 807
 ${ }^{20} 0_{\text {TIME (SEC) }}{ }^{30} \quad 40 \quad 50$
Transient Responses about Roll, Pitch, and Yaw axes (Method of Ref.(1))
 Fig. 1 :




$\square$





## Table 1:

Table 2: Submatrices of B matrix of Method of Ref|1|
ORIGINAL PASE : 3
OF POOR QUALITY

> $?$
0
+
2
$\vdots$
0
0
0
0
$?$
$0.82716 L-08$ $-3.82132 \mathrm{D}-08$ $0.662290-08$
$-0.144660-06$ $\begin{array}{ccc}0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1\end{array}$ 0
$: 1$
$:-1$
-0
$\pm$
$\vdots$
$\vdots$
0 J． 2 ब $34 \angle \omega-08$ $-0.270540-08$ $-3.42569 N-36$ $0.13229 u-06$ $\begin{array}{ll}0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 4 \\ 0 & 1 \\ 5 & 0 \\ = & 0 \\ 0 & 3 \\ 1 & 1\end{array}$ 0
0
1
0
7
0
0
0
0
0
1
poynow no．！！jo x！new $\forall$ jo son！ıeuqns ：$\varepsilon$ opqe」

|  |  | $A^{\prime \prime}$ | Submatrix（7x |  |  | $\checkmark$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| U．7＋ら．ju－jy |  | $0.15420 \mathrm{~L}-09$ | $0.457<50-03$ | $0.15382 v-02$ | $0.34 \operatorname{cosi}-02$ | －1）．9621＜D－03 |
| －j．3！＋7－－ | 3．13＜u4D－u7 | $0.6 \mathrm{bj} 57 \mathrm{u}-11$ | 0.202111504 | －0．308570－04 | －0．10．21＋5－02 | －0．152450－02 |
| リ．judctu－う） | U．1y＋l｜i－1u | －3．431 $54 \mathrm{~L}-30$ | －0．171751－04 | $0.17780 \mathrm{D}-04$ | $0.3 y ¢ 0<U-03$ | $0.30513 \mathrm{~L}-03$ |
|  | J． 0000410 －－ $0^{\text {a }}$ | $-0.21335 \mathrm{C}-\mathrm{Cu}$ | $-9.297170+01$ | －C．180611）－01 | －0．0y148u－01 | $0.10586 \mathrm{~L}-01$ |
| U．1Lけらせ | J．staryu－01 | U．0Y3530－0y | $0.23710 \cup-02$ | －0．40s790＋01 | U．152010－01 | －0．946030－02 |
|  | J． $1(10+i)$－vo | $-5.01346 \mathrm{E}-05$ | $-j .2 y+39 u-02$ | － $0.725010-02$ | $-0.221240+02$ | －0．25951L－02 |
| $-0.0230 .0-6$ | 3． 124000000 |  | $-3.12 \div 5 \times 0-01$ | －J． 2 yóju－01 | $-0.114450400$ | －3．600 $111 \mathrm{~L}+02$ |
|  |  | $A^{\prime}$ | Submatrix（ $7 \times$ |  |  |  |
| $-j .13+j 211-0)$ | J．j1くしいいーCり | U． $14454 \mathrm{D}-\mathrm{C} 4$ | － $0.227550-08$ | $0.417520-00$ | $0.2449 i j-0 j$ | $0.62716 \mathrm{~L}-08$ |
| －j．0330くいーい寺 | $-0 .+4<301-$ Co | U．21050c－60 | －3． $71816 \mathrm{~L}-10$ | ．）． $307300-08$ | J． $23342 \mathrm{u}-08$ | －3． $821320-08$ |
| －0．12u1／u－v4 | －0．121730－30 | －J．13ち buL゙こう | 0．305 1／0－C\％ | $-3.102320-08$ | －0．27054u－08 | U． $662<90-08$ |
| U．31\％）10－34 | －3．483．30u－94 | J． $12347 \mathrm{~L}-02$ | 3． 3 y 509007 | －0．123181－07 | －）． $425691-36$ | －3．144660－06 |
| －u． 11210 ט－的 | 0． $1555+0-04$ | －0．35794C－03 | －0．11730u－07 | $0.304360-07$ | $0.132290-06$ | $0.16973 \mathrm{~L}-07$ |
| j． 374 usumb | －0．03155L－04 | 1）． $48 \div 5410-05$ | $0.15036 \mathrm{~N}-07$ | $-0.9 \mathrm{dd19v-30}$ | －0．14985U－06 | －0． $10565 \mathrm{~L}-06$ |
| 0．45d＋（0－）4 | －0．100c2u－03 | $0.20223 \mathrm{~L}-02$ | $0.644830-07$ | － $0.106360-06$ | －0．665940－06 | －0．20dd40－06 |

G. APPENDIX

1. Format of Submatrices

$$
\begin{aligned}
& E_{1}=\left[\begin{array}{lll}
I_{x x} & I_{x y} & I_{x z} \\
I_{x y} & I_{y y} & I_{y z} \\
I_{z x} & I_{z y} & I_{z z}
\end{array}\right] \\
& E_{2}=\left[\begin{array}{llll}
d_{11} & d_{12} & d_{13} & d_{14} \\
d_{41} & d_{42} & d_{43} & d_{44} \\
d_{51} & d_{52} & d_{53} & d_{54}
\end{array}\right] \\
& E_{3}=\left[\begin{array}{lrr}
0 & -\omega_{0} I_{y z} & -\omega_{0}\left(I_{x x}-I_{y y}+I_{z z}\right) \\
-\omega_{0} I_{y z} & 0 & \omega_{0} I_{x y} \\
\omega_{0}\left(I_{x x}-I_{y y}+I_{z z}\right) & -\omega_{0} I_{x y} & 0
\end{array}\right]
\end{aligned}
$$

$$
E_{4}=\left[\begin{array}{llll}
d_{21} & d_{22} & d_{23} & d_{24} \\
d_{31} & d_{52} & d_{53} & d_{54} \\
d_{71} & d_{72} & d_{73} & d_{74}
\end{array}\right]
$$

$$
E_{5}=\left[\begin{array}{lcr}
-4 \omega_{0}^{2}\left(I_{z z}-I_{y y}\right) & -3 \omega_{0}^{2} I_{x y} & -\omega_{0}^{2} I_{x z} \\
-3 \omega_{0}^{2} I_{x y} & 3 \omega_{0}^{2}\left(I_{x x}-I_{z z}\right) & \omega_{0}^{2} I_{y z} \\
-4 \omega_{0}^{2} I_{z x} & 3 \omega_{0}^{2} I_{y z} & -\omega_{0}^{2}\left({ }_{x x}-I_{y y}\right)
\end{array}\right]
$$

$$
E_{5}=\left[\begin{array}{cccc}
d_{31} & d_{32} & d_{33} & d_{34} \\
0 & 0 & 0 & 0 \\
d_{81} & d_{82} & d_{83} & d_{84}
\end{array}\right]
$$

$$
D_{1}=\left[\begin{array}{cccc}
\omega_{1}^{2} & 0 & 0 & 0 \\
0 & \omega_{2}^{2} & 0 & 0 \\
0 & 0 & \omega_{3}^{2} & 0 \\
0 & 0 & 0 & \omega_{4}^{2}
\end{array}\right]
$$

$$
D_{2}=\left[\begin{array}{lll}
G_{2}\left(\beta_{1}\right) / L & -G_{1}\left(\beta_{1}\right) / L & 0 \\
G_{2}\left(\beta_{2}\right) / L & -G_{1}\left(\beta_{2}\right) / L & 0 \\
G_{2}\left(\beta_{3}\right) / L & -G_{1}\left(\beta_{3}\right) / L & 0 \\
G_{2}\left(B_{4}\right) / L & -G_{1}\left(\beta_{4}\right) / L & 0
\end{array}\right]
$$

$$
D_{3}=\left[\begin{array}{lll}
0 & 2 \omega_{0} G_{3}\left(B_{1}^{\prime}\right) / L & -2 \omega_{0} G_{2}\left(B_{1}\right) / L \\
0 & 2 \omega_{0} G_{3}\left(B_{2}^{\prime}\right) / L & -2 \omega_{0} G_{2}\left(B_{2}\right) / L \\
0 & 2 \omega_{0} G_{3}\left(B_{3}^{\prime}\right) / L & -2 \omega_{0} G_{2}\left(B_{3}\right) / L \\
0 & 2 \omega_{0} G_{3}\left(B_{4}^{\prime}\right) / L & -2 \omega_{0} G_{2}\left(B_{4}\right) / L
\end{array}\right]
$$

$$
D_{4}=\left[\begin{array}{lll}
4 \omega_{0}^{2} G_{2}\left(B_{1}\right) / L & -3 \omega_{0}^{2} G_{1}\left(B_{1}\right) / L & 0 \\
4 \omega_{0}^{2} G_{2}\left(\beta_{2}\right) / L & -3 \omega_{0}^{2} G_{1}\left(B_{2}\right) / L & 0 \\
4 \omega_{0}^{2} G_{2}\left(B_{3}\right) / L & -3 \omega_{0}^{2} G_{1}\left(B_{3}\right) / L & 0 \\
4 \omega_{0}^{2} G_{2}\left(B_{4}\right) / L & -3 \omega_{0}^{2} G_{1}\left(B_{4}\right) / L & 0
\end{array}\right]
$$

$$
F=\left[\begin{array}{lll}
F_{x} S_{x_{1}}(-L / 3) & F_{y} S_{y_{1}}(-L / 3) & F_{x} S_{x_{1}}(-2 L / 3) \\
F_{x} S_{x_{2}}(-L / 3) & F_{y} S_{y_{1}}(-L / 3) & F_{x} S_{x_{2}}(-2 L / 3) \\
F_{x} S_{x_{3}}(-L / 3) & F_{y} S_{y_{3}}(-L / 3) & F_{x} S_{x_{3}}(-2 L / 3) \\
F_{x} S_{x_{4}}(-L / 3) & F_{y} S_{y_{4}}(-L / 3) & F_{x} S_{x_{4}}(-2 L / 3)
\end{array}\right.
$$

$$
\left.\begin{array}{lll}
F_{y} S_{y_{1}}(-2 L / 3) & E_{x} S_{x_{1}}(-L) & F_{y} S_{y_{1}}(-L) \\
F_{y} S_{y_{2}}(-2 L / 3) & F_{x} S_{x_{2}}(-L) & F_{y} S_{y}(-L) \\
F_{y} S_{y_{3}}(-2 L / 3) & F_{x} S_{x_{3}}(-L) & E_{y} S_{y_{y}}(-L) \\
E_{y} S_{y_{4}}(-2 L / 3) & F_{x} S_{x_{4}}(-L) & F_{y} S_{y_{4}}(-L)
\end{array}\right]
$$

$$
\begin{aligned}
& M_{1}=\left[\begin{array}{cccccc}
0 & E_{y} L / 3 & 0 & F_{y} 2 L / 3 & 0 & F_{y} L \\
-F_{x} L / 3 & 0 & -E_{x} 2 L / 3 & 0 & -E_{x} L & 0 \\
0 & 0 & 0 & 0 & -Y F_{x} & X F_{y}
\end{array}\right] \\
& M_{2}=\left[\begin{array}{ccc}
M_{x} & 0 & 0 \\
0 & M_{y} & 0 \\
0 & 0 & M_{z}
\end{array}\right]
\end{aligned}
$$

2. System Flexible Mode Shapes
(1) Method of Ref.[1] Equation (For nth mode)

$$
\begin{aligned}
& d_{1 n}=M_{R}\left[-L S_{n y}(-L)-X Y \Theta_{n}(-L)\right]+M f_{2}\left(\beta_{n}\right) / L \\
& d_{2 n}=M_{R} \omega_{0}\left[Y S_{n x}(-L)\right] \\
& d_{3 n}=\omega_{0}^{2}\left[M f_{2}\left(\beta_{n}\right) / L-M_{R} X_{\Theta_{n}}(-L)\right] \\
& d_{4 n}=M f_{1}\left(\beta_{n}\right) / L+\left[I_{R_{2}}+M_{R}\left(X^{2}+L^{2}\right)\right] \Theta_{n}(-L)-M_{R} L S_{n x}(-L) \\
& d_{5 n}=\omega_{0} M_{R} X\left[L \Theta_{n}(-L)+2 S_{n x}(-L)\right] \\
& d_{E n}=M_{R}\left[X S_{n y}(-L)-Y S_{n x}(-L)\right] \\
& d_{7 n}=M_{R} \omega_{0} X Y \Theta_{n}(-L) \\
& d_{g n}=\omega_{n}^{2} Y S_{n x}(-L)
\end{aligned}
$$

(2) Direct Method's Equation (For nth mode)

$$
\begin{aligned}
d_{1 n}= & M f_{2}\left(\beta_{n}\right) / L+M_{R} L S_{n y}(-L)+M_{R} X L_{\theta_{n}}(-L)+M_{R} Y^{2} S_{n y}^{\prime}(-L) \\
& -M_{R} X Y S_{n x}^{\prime}(-L)+I_{R_{1}} S_{y n}^{\prime}(-L)
\end{aligned}
$$

$$
d_{2 n}=2 \omega_{0} M_{R} Y S_{n x}(-L)+\left(-2 \omega_{0} M \%_{R} Y^{2}-\omega_{0} I_{R_{1}}+\omega_{0} I_{R_{2}}-\omega_{0} M_{R} X^{2}-\omega_{0} I_{R_{3}}\right) \theta_{n}(-L)
$$

$$
d_{3 n}=\omega_{0}^{2}\left[M f_{2}\left(\beta_{n}\right) / L+M_{R} X Y S_{n x}^{\prime}(-L)-M_{R} Y^{2} S^{\prime}{ }_{y n}(-L)\right.
$$

$$
\left.+M_{R} L S_{n y}(-L)+I_{R_{2}} S^{\prime}{ }_{y n}-I_{R_{3}} S_{y n}^{\prime}(-L)\right]
$$

$$
d_{4 n}=M f_{1}\left(B_{n}\right) / L-M_{R} L S_{n x}(-L)+I_{R_{2}} S_{n x}^{\prime}(-L)+M_{R} Y L \otimes_{n}(-L)
$$

$$
-M_{R} X Y S_{n y}^{\prime}(-L)+M_{R} X^{2} S_{n x}^{\prime}(-L)
$$

$$
d_{5 n}=-M_{R} \omega_{0} X S_{n x}(-L)+M_{R} \omega_{0} Y L S_{n y}^{\prime}(-L)-M_{R} \omega_{0} X L S_{n x}^{\prime}(-L)+M_{R} \omega_{0} X Y \Theta_{n}(-L)
$$

$$
d_{E n}=M_{R} X S_{n y}(-L)-M_{R} Y S_{n x}(-L)+M_{R} X Y \Theta_{n}(-L)+M_{R} Y^{2} s_{n}(-L)+I_{R_{3}} \theta_{n}(-L)
$$

$$
d_{7 n}=M_{R} \omega_{0} Y^{2} S_{n y}^{\prime}(-L)+M_{R} \omega_{0} X Y S_{n x}^{\prime}(-L)-\omega_{0} I_{R_{2}} S_{n y}^{\prime}(-L)+\omega_{0} I_{R_{3}} S_{n y}^{\prime}(-L)
$$

$$
M_{R} X L \omega_{0} \Theta_{n}(-L)+\omega_{0} I_{R_{1}} S_{n y}^{\prime}(-L)
$$

$$
d_{g_{n}}=M_{R} \omega_{0}^{2} Y S_{n x}(-L)+M_{R} \omega_{0}^{2} X S_{n y}(-L)-\omega_{0}^{2} I_{R_{1}} \Theta_{n}(-L)+\omega_{0}^{2} I_{R_{2}} \Theta_{n}(-L)
$$

where

$$
\begin{aligned}
& f_{1}\left(\beta_{n}\right)=A_{1 n}\left(\frac{L \cos \beta_{n} L}{\beta_{n}}-\frac{\sin \beta_{n} L}{\beta_{n}^{2}}\right)+B_{1 n}\left(\frac{L \sin \beta_{n} L}{\beta_{n}}+\frac{\cos \beta_{n} L}{\beta_{n}^{2}}+\frac{1}{\beta_{n}^{2}}\right) \\
& +C_{1 n}\left(\frac{\sinh _{n} L}{B_{n}^{2}}-\frac{L \cosh B_{n} L}{\beta_{n}}\right)+D_{1 n}\left(\frac{L \sinh B_{n} L}{\beta_{n}}-\frac{\cosh B_{n} L}{\beta_{n}^{2}}+\frac{1}{B_{n}^{2}}\right) \\
& f_{2}\left(\beta_{n}\right)=A_{2 n}\left(\frac{L \cos \beta_{n} L}{\beta_{n}}-\frac{\sin _{n} L}{\beta_{n}^{2}}\right)+B_{2 n}\left(\frac{L \sin \beta_{n} L}{\beta_{n}}+\frac{\cos \beta_{n} L}{\beta_{n}^{2}}-\frac{1}{\beta_{n}^{2}}\right) \\
& +C_{2 n}\left(\frac{\sinh _{n} L}{B_{n}^{2}}-\frac{L \cosh \beta_{n} L}{B_{n}}\right)+D_{2 n}\left(\frac{L \sinh \beta_{n} L}{B_{n}}-\frac{\cosh \beta_{n} L}{B_{n}^{2}}+\frac{1}{B_{n}^{2}}\right) \\
& S_{n x}^{\prime}(-L)=\beta_{n}\left[A_{1 n} \operatorname{cosi}_{n} L+B_{1 n} \sin _{n} L+C_{1 n} \cosh _{n} L+D_{1 n} \sinh \beta_{n} L\right] \\
& S_{n y}^{\prime}(-L)=B_{n}\left[A_{2 n} \operatorname{coss}_{n} L+B_{2 n} \sin _{n} L+C_{2 n} \cosh \beta_{4} L+D_{2 n} \sinh _{n} L\right]
\end{aligned}
$$

3. System Parameters
(1) Inertial Moment

| $I_{s_{1}}=905,443$ | slug- $f t^{2}$ |
| :--- | :--- |
| $I_{s_{2}}=6,789,100$ | slug-f. $t^{2}$ |
| $I_{s_{3}}=7,086,601$ | slug- $f t^{2}$ |
| $I_{s_{4}}=145,393$ | slug- $f t^{2}$ |
| $I_{R_{1}}=4,969$ | slug- $f t^{2}$ |
| $I_{R_{2}}=4,969$ | slug-ft |
| $I_{R_{3}}=9,938$ | slug-ft |

(2) First Four Modal Coefficients

| Mode No. (n) | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| $\beta_{n}$ | 1.19 | 1.29 | 1.97 | 2.54 |
| $B^{\prime}$ | 0.033 | 0.039 | 0.092 | 0.152 |
| $\omega_{n}$ | 0.274 | 0.322 | 0.748 | 1.24 |
| $\mathrm{A}_{1 \mathrm{n}}$ | 0.161 | 0.072 | 0.022 | 0.068 |
| $\mathrm{B}_{1 \text { n }}$ | -0.196 | -0.084 | -0.059 | -0.063 |
| $C_{1 n}$ | -0.168 | -0.075 | -0.023 | -0.068 |
| $\mathrm{D}_{1 /}$ | 0.195 | 0.084 | 0.059 | 0.063 |
| $\mathrm{A}_{2 \mathrm{n}}$ | -0.039 | 0.125 | 0.025 | -0.105 |
| $\mathrm{B}_{2 \mathrm{n}}$ | 0.069 | -0.196 | 0.003 | 0.094 |
| $\mathrm{C}_{2 n}$ | 0.058 | -0.167 | -0.025 | 0.107 |
| $\mathrm{D}_{2 \mathrm{n}}$ | -0.069 | 0.196 | -0.003 | -0.093 |
| $\mathrm{A}_{3 \mathrm{n}}$ | -0.032 | 0.003 | 0.072 | 0.011 |
| $\mathrm{B}_{3 n}$ | 0.158 E | -0.109E | -0.131E | -0.123 |

(3) Other Values

| $\omega_{0}=7.27 \mathrm{E}-5$ | $\mathrm{rad} / \mathrm{sec}$ |
| :--- | ---: |
| $M=12.42$ | slug |
| $M_{R}=12.42$ | slug |
| $X=18.75$ | ft |
| $Y=-32.5$ | ft |
| $L=-130$ | ft |

## ORIGINAL PAGE IS <br> OF POOR QUALITY

IV. Control structure Interaction - Preliminary study of the Effect of Actuator Mass on the Design
of Control Laws

The dynamics of large space structures are described using the finite element method as ${ }^{1}$ :
$M \ddot{x}+c \dot{x}+k x=E リ$
where
$x=n \times 1$ vector representing degrees of freedom
$M=n \times n$ mass matrix
$c=n \times n$ damping matrix
$k=n \times n$ stiffness matrix
$B=n \times m$ control influence matrix
$11=m \times 1$ control vector
Using modal analysis ${ }^{2}$ and modern control theory ${ }^{3}$, state variable feedback control laws of the form

$$
\begin{equation*}
u=-F_{r} \dot{x}-F_{p} x \tag{2}
\end{equation*}
$$

where
$F_{r}$ and $F_{p}$ are rate and position control gain matrices of appropriate dimensions are designed. To implement the control law given by equation (2) physical actuators are needed. These physical actuators have finite mass and, thus, change the mass and stiffness of the structure to be controlled. This mass can be as much as fifteen percent of the uncontrolled structure. ${ }^{4}$ Thus the control 1 aws designed without taking this mass into consideration have to be reevaluated tor their stability and performance degradation. tssumingaM and $\Delta K$ are the changes in the mass and stiffness
matrices due to actustors the dynamics of the controlled system can be written as:

$$
\begin{equation*}
(M+\Delta M) \ddot{x}+\left(C+B F_{r}\right) \ddot{x}+\left(K+\Delta K+8 F_{p}\right) x=0 \tag{3}
\end{equation*}
$$

since the control law is designed for the stability of the controlled system, the matrices $M, C+B F r$, and $K$ are positive difinite matrices. 5 If the changes in the mass matrix and stiffness matrix, $\Delta M$ and $\Delta k$, are also assumed to be positive definite then the matrices $(M+\Delta M),\left(C+E F_{r}\right)$, and $\left(K+\Delta K+B F_{p}\right)$ are also positive definite. Thus, equation (3) is stable, though performance degradation can not be commented on. The assumption that. $\Delta M$ and $\Delta K$ are positive is a valid assumption, as the dynamics of the oscillatory motion of the structure with he added actustor masses can be desaribed besed on the finite element method or energy conservation techniques, and thus, $(M+\Delta M)$ and ( $K+\Delta K$ ) must be positive definite. $A s(M+\Delta M),(K+\Delta K)$ are positive definite and $(M+1$ $\Delta M),\left(K+E F_{p}\right)$ are positive defirite, the matrices $(M+\Delta M)$, $K+$ $\left.\Delta K+B F_{p}\right)$ are also positive definite. Thus, $a s M$ and $K$ are positive definite and ( $M+\Delta M$ ) and ( $K+\Delta K$ ) are positive definite, $\Delta k$ and $\Delta k$ are positive definite. The effect of the actuator mass on the structural damping is not considered here.

In this anaiysis the modal truncation $i \equiv$ not taken into account znd, thus, the control spill-over problem will not arise. The performance degradation is analysed using a two mass; two spring, two sctuator system.

Numerical Example:

The two-mass two-spring system is shown in figure 1 and its equations of motion are written as:

$$
\left[\begin{array}{ll}
m_{1} & 0  \tag{4}\\
0 & m_{2}
\end{array}\right]\left[\begin{array}{l}
\ddot{x}_{1} \\
\ddot{x}_{2}
\end{array}\right]+\left[\begin{array}{cc}
k_{1}+k_{2} & -k_{2} \\
-k_{2} & k_{2}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]
$$



## Numerical Example:

The two-mass two-spring system is shown in figure 1 and its equations of motion are written as:

$$
\left[\begin{array}{ll}
m_{1} & 0  \tag{4}\\
0 & m_{2}
\end{array}\right]\left[\begin{array}{l}
\ddot{x}_{1} \\
\ddot{x}_{2}
\end{array}\right]+\left[\begin{array}{cc}
k_{1}+k_{2} & -k_{2} \\
-k_{2} & k_{2}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{i} \\
u_{2}
\end{array}\right]
$$

The control law of the form

$$
\left[\begin{array}{l}
u_{1}  \tag{5}\\
u_{2}
\end{array}\right]=-\left[\begin{array}{ll}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{array}\right]\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2}
\end{array}\right]
$$

is designed with the following numerical values for the mass, stiffiness and control gain matrices.

$$
\begin{aligned}
& m_{1}=2, \quad m_{2}=1, k_{1}=4, k_{2}=1 \\
& f_{11}=1, f_{12}=f_{21}=0, f_{22}=1
\end{aligned}
$$

The numerical simulation is conducted varying the masses and stiffnesses, ons at atime, and the closed-100p eigenvalues are tabulated in Table 1. From Table 1, it can be observed that the change in mass, m2. has a maximum effect on the cagradation of the closed-1000 eigenvalues. A $15 \%$ change in $m_{2}$ pushed the leftmost eigenvalus to the right by around $11 \%$ while the second zajenvalue moved to the right by around $0 \%$. A $15 \%$ change in mo mon the efgenvalue oloeest to the
 axamia and numerical simulation demonatratas that the

Bctuator masees attect the derformance of the control iaw that is designed without taking these masees into consideration. It is also worthwhile to observe that a change in the stiftness moves the eigenvalues to the right as well as to the left and can be explained as the effect of the increase in the stiftness on one mass or the other.

To understand the fertormance degradation due to actuators an exhaustive simulation st the clised-1oop controlied eystem has to be done with the tollowing oonsiderations:

1. The masses needed to implement specirio control Torces have to be evaluated.
2. The change in stiffness due to chanje in mias haz to je determined.
3. Simulation has to be conducted with changes in the total mase and atitriess metrious rather than individual masaez as is done in this study.

A controi aystem dezth to acoommodate the efiect ot the actuator masess has to be done in iteratiy tiashion inodrporating the ohenge in mass and stiffneri into the dynamio moasl until asatistantory control iaw a arrived at.

$$
F E H=r=n c z a
$$





3. Kwakernaak, 」. and Sivan, R.. Linear optimal control Systems, John wiley \& Sons. New york. 1970.
4. Earth pointing satellite (EFS) Structure Description, NAEA Internal Document, Jan.. 1989.
5. Bellman, F. , Introduction to Matrix Analysis. McGraw Hill Book Company Inc., New York, 1960.


Figure 1: Two Mass-Two Spring System

| Percentage change in | Closed loop eigenvalues |
| :--- | :--- |
| masses and stiffnesses | (complex conjugate pairs) |


| $\mathrm{m}_{1}$ | $\mathrm{m}_{2}$ | $\mathrm{k}_{1}$ | $\mathrm{k}_{2}$ | 1 |  | 2 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | -0.472 | $\pm j 0.710$ | -. 277 | j0.163 |
| 15 | 0 | 0 | 0 | -0.464 | $\pm j 0.709$ | -. 254 | j0.154 |
| 0 | 15 | 0 | 0 | -0.420 | $\pm \mathrm{j} 0.684$ | -. 268 | j0.163 |
| 0 | 0 | 15 | 0 | -0.477 | $\pm j 0.729$ | -. 272 | j0.171 |
| 0 | 0 | 0 | 15 | -0.465 | $\pm j 0.756$ | - . 284 | j0.168 |

Table 1 : Closed-loop Eigenvalues due to Changes in Mass and Stiffness Values.

## V. CONCLUSIONE AND RECOMMENGATIONE

The maximum principle of pontryagin has been applied to the rapid maneuvering problem of the planar, flexible orbiting sCOLE. The resulting two-point boundary value problem is solved by applying the quasilinearization technique, and the near-minimum time is obtained by shortening the maneuvering time in a sequential manner until the controls are near the bang-bang type. The results indicate that responses of the nonlinear system for the flexible modal amplitudes may be significantly different from those of the corresponding linearized system for rapid slewing maneuvers. This research is currently being extended to the three dimensional slewing of the flexible sCOLE system.

From an analysis and simulation of the sCOLE stationkeeping dynamics it is found that the flexible vibrations of the mast are not greatly excited during typical stationkeeping operations. Eystem responsee are highly dependent on the force actuator locations and the numerical values of the state and control penalty matrices included in the LQR control law design. Force actuators mounted at $1 / 3$ and $2 / 3$ of the mast length along the mast are effective in supplessing the flexible mast vibrations.

A preliminary examination of the etfect of actuator mass on the design of control laws for large flexible space gyatems demonstrates that actuator maseez can intluance the
performance of the closed-loop system where the control law has been designed without taking these masses into consideration. To understand better the possible degradation in performance due to the presence of actuator masses additional studies are required to accurately evaluate the changes in the stiffness matrix due to specific actuator masses, and simulations must be performed incorporating changes in the total mass and stiffness matrices, rather than individual masses as was done here.

Finally, the current (1989-90) grant work has been redirected so as to lend greater support to the new controls/Etructures Interaction (CSI) program and focusing on specific SSI evolutionary configurations, in addition to the treatment of the scole $3-\mathrm{D}$ slewing problem.

