DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
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GUIDANCE AND CONTROL STRATEGIES FOR AEROSPACE VEHICLES

Dr. J. L. Hibey, Principal Investigator,
Dr. D. S. Naidu, Co-Principal Investigator, and
Mr. C. D. Charalambous, Research Assistant

Progress Report
For the period 1 July 1989 to 31 December 1989

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This progress report consists of the following two   parts:

I. Fuel-Optimal Trajectories for Aeroassisted Coplanar Orbital Tranfer

   Vehicles in the Presence of Uncertainties due to Modelling Inaccuracies

II. Neighboring Optimal Guidance for Aeroassisted Noncoplanar Orbital
    Transfer.

**Fuel-Optimal Trajectories for
Aeroassisted Coplanar Orbital Tranfer Vehicles in the
Presence of Uncertainties due to Modelling Inaccuracies**

Dr. J. L. Hibey, Principal Investigator
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ABSTRACT

We intend to devise a neighboring optimal guidance scheme for a nonlinear dynamic system with stochastic inputs and perfect measurements as applicable to fuel optimal control of an aeroassisted orbital transfer vehicle. For the deterministic nonlinear dynamic system describing the atmospheric maneuver, a nominal trajectory is determined. Then, a neighboring, optimal guidance scheme is obtained for open loop and closed loop control configurations. Taking modelling uncertainties into account, a linear, stochastic, neighboring optimal guidance scheme is devised. Finally, the optimal trajectory is approximated as the sum of the deterministic nominal trajectory and the stochastic neighboring optimal solution. Numerical results are presented for a typical vehicle.

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NOMENCLATURE

$A \quad = S \, \rho_R/2m$

$A1 \quad = C_{DO} \, S \, \rho_R \, Ha/2m$

$A2 \quad = C_{LR} \, S \, \rho_R \, Ha/2m$

$a_c \quad = Rc/Ra$

$a_d \quad = Rd/Ra$

$b \quad = Ra/Ha$

$C_D \quad = C_{DO} + K \, C_L^2$

$C \quad = C_L/C_{LR}$

$Em \quad = (L/D) \, max$

$C_{LR} = \sqrt{C_{DO}/K}$

$h \quad = H/Ha$

C_{DO} = zero-lift drag coefficient

C_L = Lift coefficient

$g \quad$ = gravitational acceleration

$k \quad$ = induced drag factor

$m \quad$ = vehicle mass

$R \quad$ = radius of earth center

Ra = radius of atmosphere

Re = radius of earth

$S \quad$ = aerodynamic reference area

$H \quad$ = altitude

$V \quad$ = velocity

$\gamma \quad$ = flight path angle

$\rho \quad$ = density

$t \quad$ = time

P   - costate variable

n̲   - Lagrange multiplier

Δv - characteristic velocity

J   - performance index

J̄   - augmented performance index

I. INTRODUCTION

The derivation of the optimal control law resulting from a minimum fuel criterion is addressed for the coplanar orbital transfer maneuver [6,7,11]. This maneuver involves a transfer from high earth orbit to a low earth orbit with the control law minimizing the fuel consumption. $(fig \#1)$ The performance index is expressed in terms of the characteristic velocities for deorbit and reorbit [10]. A TPBVP is solved by applying the Pontryagin minimum principle with given initial and final constraints, and a fixed terminal time. The open loop control arising from the minimum principle requires a new computation of the control when the initial and/or final condition is subject to small changes; this introduces errors in the solution. Errors are also introduced by the inexact knowledge of the true model.

To be more specific, after solving the second variation problem [3,4] that results from linearizing around a nominal solution, a control is computed that describes the correction to the original nonlinear control problem. This correction is due to small perturbations in the initial and/or terminal conditions. Furthermore, the perturbed control law is used as a control law that will drive the stochastic dynamic equation when perfect knowledge of the state is assumed (see figure #2 and figure #3). The stochastic differential equations are subject to uncertainties due to lack of knowing the exact model associated with the nonlinear TPBVP.

This report represents a continuation of the original effort reported in [10,11]. There, the coplanar and noncoplanar problem is solved by neglecting the contraints imposed on the control, whereas now these constraints are fully addressed. The outline of the procedure used for the open loop feedback control and the closed loop feedback control are shown below.

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Open Loop Feedback Control:

Step #1. Solve the TPBVP for the deterministic problem $\dot{\underline{x}} - f(\underline{x},\underline{u},t)$.

Step #2. Save the nominal state $\underline{x}_0(t)$ and nominal control $\underline{u}_0(t)$.

Step #3. Solve the neighboring TPBVP along the nominal solution of Step #2.

Step #4   Save the gain matrices K_1 and K_2 that correspond to the control feedback law generated in step #3,

$\Delta\underline{u} - k_1\Delta\underline{x} + k_2 \Delta\underline{P}$

where $\Delta\underline{P}$ are the adjoint variables.

Step #5   Linearize $\dot{\underline{x}} - f(\underline{x},\underline{u},t)$ along $\underline{x}_0$, $\underline{u}_0$ and add noise $\underline{n}$:

$$\Delta\dot{\underline{x}} - f_x\bigg|_{\underline{x}_0,\underline{u}_0}\Delta\underline{x} + f_u\bigg|_{\underline{x}_0,\underline{u}_0}\Delta\underline{u} + \underline{n}, \quad \Delta\underline{x}(t_0) - \text{given}$$

Step #6. Simulate the stochastic equation of Step #5 by using the control law described in Step #4.

Closed Loop Feedback Control:

Step #1. Solve the TPBVP for the deterministic problem $\dot{\underline{x}} - f(\underline{x},\underline{u},t)$.

Step #2. Save the nominal state $\underline{x}_0(t)$ and nominal control $\underline{u}_0(t)$.

Step #3. Solve the matrix Riccati equation backward in time.

Step #4. Save the gain matrix k that corresponds to the control feedback law
$\Delta\underline{u} - k \Delta\underline{x}$.

Step #5. Linearize $\dot{\underline{x}} - f(\underline{x}, \underline{u}, t)$ along $\underline{x}_0$, $\underline{u}_0$ and add noise $\underline{n}$:

$$\Delta\dot{\underline{x}} - f_x\bigg|_{\underline{x}_0,\underline{u}_0}\Delta\underline{x} + f_u\bigg|_{\underline{x}_0,\underline{u}_0}\Delta\underline{u} + \underline{n}$$

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Step #6.   Simulate the stochastic equation of Step #5 by using the control

law given in Step #4.

II.   DETERMINISTIC RESPONSE

The transfer of the space vehicle from a HEO at radius R_d to a LEO at R_c
is due to the velocity reduction caused by atmospheric drag forces. Given a
tangential propulsive burn, having characteristic velocity ΔVd, the vehicle
travels through an elliptical orbit (figure #1). At point E, the spacecraft
enters the atmosphere with flight path angle γ_e and velocity V_e. During the
atmospheric flight the vehicle experiences a reduction of velocity. At point
F, the spacecraft leaves the atmosphere with flight path angle γ_f and velocity
V_f. As a final stage, the maneuver ends with a circularizing or reorbit burn
having a characteristic velocity ΔVc to make the spacecraft enter into the low
earth orbit. A mathematical model representing this scenario is now proposed.

1.   Deterministic Modeling

To study the effects of aerodynamic forces on an orbital transfer reentry
vehicle, the planetary atmosphere in which the flight takes place should be
modeled. It is convenient to treat the atmosphere as a uniform gas of non-
varying composition. The effect of the atmosphere on the vehicle is considered
in terms of the density. There are some important assumptions that have been
made in order to model the reentry vehicle so that the complexity of the
problem becomes tractable [1, 2].

(a) Assumption of Spherical Symmetry

A great simplification in the atmospheric modeling is obtained by assuming
that the atmosphere has a spherical symmetry. However, the true model of the
system should be represented by oblateness.

(b) Assumption of Non-rotating System

The earth atmosphere which the space vehicle encounters is considered to be stationary with zero rotational speed. The rotational effects would depend on the latitude of the vehicle at all times. Furthermore, the rotational effects would also depend on the inclination of the vehicle's trajectory to the equator. However, the simplification of the model is more important than including the rotational effects, so these are excluded.

(c) Assumption of Stationary Atmosphere

The atmosphere is assumed to be stationary, that is, no winds are expressed in the model; also, the time dependence of the parameters is neglected. As a result of the above assumptions, the atmospheric density is assumed to be exponential and is given by

$$\rho = \rho_0 \, e^{-\beta H}$$

where β is a scale factor.

With these assumptions in mind, the re-entry model for an aeroassisted coplanar orbital transfer vehicle is given by:

$$\dot{H} = V\sin\gamma \tag{II.1}$$

$$\dot{V} = - \, g\sin\gamma - \frac{D}{m} \tag{II.2}$$

$$\dot{\gamma} = \left(\frac{v}{r} - \frac{\mu}{r^2 v} \right) \cos\gamma + \frac{L}{mv} \tag{II.3}$$

where

$$D = \frac{\rho S \, C_D \, v^2}{2} \tag{II.4}$$

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$$L - \frac{\rho S \, C_L \, V^2}{2} \qquad\qquad \text{II.5}$$

and g is the Newtonian gravitational attraction that is given by

$$g - \mu/r^2.$$

Furthermore, it is assumed that the drag polar is parabolic, given by

$$C_D - C_{DO} + K \, C_L^2 . \qquad\qquad \text{II.6}$$

Equation II.6 is valid if we assume no extreme hypersonic conditions, thereby allowing independence of the aerodynamic coefficient on the Mach number and the Reynolds number [6]. Using normalized variables the equations of motion can be rewritten as

$$\frac{dh}{d\tau} - b\nu\sin\gamma \qquad\qquad \text{II.7}$$

$$\frac{d\nu}{d\tau} - - A_1 b(1 + c^2)\delta\nu^2 - \frac{b^2\sin\gamma}{(b-1+h)^2} \qquad\qquad \text{II.8}$$

$$\frac{d\gamma}{d\tau} - A_2 b \, c \, \delta \, \nu + \frac{b \, \nu \, \cos \, \gamma}{(b-1+h)} - \frac{b^2\cos \, \gamma}{(b-1+h)^2\nu} \qquad\qquad \text{II.9}$$

where

$h - H/Ha$ 　　　　　　　　　　　$b - Ra/Ha$

$\nu - V/\sqrt{\mu/Ra}$ 　　　　　　　$A_1 - C_{DO}S \, \rho_R \, Ha/2m$

$\tau - t/Ra\sqrt{Ra/\tau}$ 　　　　　$A_2 - C_{LR}S \, \rho_R Ha/2m$

$\delta - \rho/\rho_R$ 　　　　　　　　　$C_{LR} - \sqrt{C_{DO}/K}$

$c - C_L/C_{LR}$ 　　　　　　　　$C_{DR} - 2 \, C_{DO}$

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2. Optimization Problem

For minimum fuel consumption, the performance index is given by

$$J = \Delta\nu = \Delta\nu_d + \Delta\nu_c = \phi[\ \underline{x}(t_f)\ ] \qquad\qquad \text{II.10}$$

$$\Delta\nu_d = \sqrt{1/a_d} - (\nu_o/a_d)\cos(\ - \gamma_o) \qquad\qquad \text{II.11}$$

$$\Delta\nu_c = \sqrt{1/a_c} - (\nu_f/a_c)\cos(\gamma_f) \qquad\qquad \text{II.12}$$

The atmosphere entry and exit conditions are given by the relationships

$$\left(2 - \nu_o^2\right)a_d^2 - 2ad + \nu_o^2\cos^2\nu_o = 0 \qquad\qquad \text{II.13}$$

$$\left(2 - \nu_f^2\right)a_c^2 - 2ac + \nu_f^2\cos^2\gamma_f = 0 \qquad\qquad \text{II.14}$$

which are the result of applying the principle of conservation of energy and angular momentum at the HEO-entry point and exit point-LEO.

Also, at the entry and exit point we have:

$$h(\tau_o) = 1 \qquad \text{entry} \qquad\qquad \text{II.15}$$

$$h(\tau_f) = 1 \qquad \text{exit.} \qquad\qquad \text{11.16}$$

The formulation of the problem falls into the category of a continuous system with functions of the state variables prescribed at a terminal time.

The two constraining functions are:

$$\psi_1 = h(\tau_f) - 1 \qquad\qquad \text{II.17}$$

$$\psi_2 = \left(2 - \nu_f^2\right)a_c^2 - 2a_c + \nu_f^2\cos\gamma_f\ . \qquad\qquad \text{II.18}$$

Incorporating these constraints, the performance index becomes

$$\bar{J} = \Delta\nu_c + \Delta\nu_d + n_1\psi_1 + n_2\psi_2 = \bar{\phi} \qquad\qquad \text{II.19}$$

which is referred to as the augmented performance index.

The Hamiltonian function is given by

$$H = P_h \left[b\nu \sin\gamma \right] + P_\nu \left[-A_1 b(1+C^2)\,\delta\nu^2 - \frac{b^2 \sin\gamma}{(b-1+h)^2} \right]$$

$$+ P_\gamma \left[A_2 bc\delta\nu + \frac{b\nu \cos\gamma}{(b-1+h)} - \frac{b^2 \cos\gamma}{(b-1+h)^2 \nu} \right] \qquad \text{II.20}$$

where P_h, P_ν, P_γ are the adjoint variables.

The unconstrained optimal control is obtained from the necessary condition

$$\frac{\partial H}{\partial c} = 0 \qquad \text{II.21}$$

and the adjoint variables are obtained from

$$\dot{P}_h = -\frac{\partial H}{\partial h} \qquad \text{II.22}$$

$$\dot{P}_\nu = -\frac{\partial H}{\partial \nu} \qquad \text{II.23}$$

$$\dot{P}_\gamma = -\frac{\partial H}{\partial \gamma} \qquad \text{II.24}$$

From equation II.21 the control is given by

$$C = \frac{A_2 P_\gamma}{2A_1 P_\nu V} \qquad \text{II.25}$$

However, realistically, the control is bounded. The constraint inequality is given by

$$-C_{max} \leq C \leq C_{max} \qquad \text{II.26}$$

where the value of C_{max} is determined by the aerodynamic characteristics of the space vehicle under consideration. The Hamiltonian given by equation II.20 is a quadratic function of the control whose second derivative H_{uu} possesses a positive definite condition; therefore the strengthened Legendre-Clebsch condition

$$H_{uu} > 0 \qquad \qquad \text{11.27}$$

is satisfied. This allows us to get the following explicit form of the optimal control law [7]:

$$C = \begin{cases} C_{max} & \text{when } C \geq C_{max} \\ C & \text{when } |C| < C_{max} \\ -C_{max} & \text{when } C \leq -C_{max} \end{cases} \qquad \text{II.28}$$

Boundary Conditions:

The boundary conditions due to the nature of the problem are:

$$h(r_o) = 1 \qquad \qquad \text{II.29}$$

$$\nu(r_o) = \text{specified} \qquad \qquad \text{II.30}$$

$$\gamma(r_o) = \text{known} \qquad \qquad \text{II.31}$$

$$h(r_f) = 1. \qquad \qquad \text{II.32}$$

Also there are two terminal constraints given by equations II.17 and II.18. From the generalized boundary conditions arising from the calculus of variation we have [4]

$$\underline{P}(t_f) = \left(\frac{\partial \phi}{\partial \underline{x}} + \underline{n}^T \frac{\partial \psi}{\partial \underline{x}} \right) \Big|_{r = r_f} \qquad \text{II.33}$$

where $\quad \underline{x} = [\ h,\ \nu,\ \gamma\ ]^T$.

$\qquad\qquad\quad \underline{n} = [\ n_1,\ n_2\ ]^T$.

The six differential equations II.7 - II.9 and II.22 - II.24 along with the three initial conditions II.29 - II.31 and three final conditions given by II.33, form a two-point boundary value problem with two parameters n_1 and n_2 to be found in equation II.33 so that the two constrained algebraic equations II.17, II.18 are satisfied.

3. Deterministic Perturbation Control

The perturbation control problem arises when small changes in initial and/or terminal conditions require a new computation of the entire control. Also, since the actual model state vector $\underline{x}$ and control $\underline{u}$ need to be near

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the nominal path $\underline{x}_0$ and $\underline{u}_0$, the perturbation variable will preserve the optimality [9]. Applying the second variation technique a linear feedback control scheme can be developed [4].

(a) Linear-Quadratic Neighboring-Optimal Control.

The generation of the perturbation variables $\Delta\underline{x}$, $\Delta\underline{u}$ arise when the nonlinear model of part I is linearized along a nominal path by considering small perturbations in the initial and/or final conditions. The equations associated with the linearized model are:

state differential equations:

$$\Delta\dot{\underline{x}} = f_x \Delta\underline{x} + f_u \Delta\underline{u} \qquad\qquad \text{II.34}$$

adjoint variable differential equations:

$$\Delta\dot{\underline{P}} = -H_{xx}\Delta\underline{x} - f_x^T\Delta\underline{P} - H_{xu} \Delta\underline{u} \qquad\qquad \text{II.35}$$

$$H_{xu} \overset{\Delta}{=} \frac{\partial}{\partial\underline{u}} (H_x)^T .$$

The perturbed feedback control is:

$$\Delta\underline{u} = -H_{uu}^{-1} (H_{ux} \Delta\underline{x} - f_u^T \Delta\underline{P}) \qquad\qquad \text{II.36}$$

provided that H_{uu} is not singular in the interval $\tau_0 \le \tau \le \tau_f$. The control given by equation II.36 is only valid in the interval where the control constraint is not saturated. When the constraint is saturated the state and adjoint differential equations are reduced to

$$\Delta\dot{\underline{x}} = f_x\Delta\underline{x} \qquad\qquad \text{II.37}$$

$$\Delta\dot{\underline{P}} = -H_{xx} \Delta\underline{x} - f_x^T \Delta\underline{P} \qquad\qquad \text{II.38}$$

where $\qquad\qquad \Delta\underline{u} = \underline{0} \qquad\qquad \text{II.39}$

The performance criterion is expanded to second order about the nominal trajectory. This causes the first-order terms to vanish, that is, the first variation ΔJ vanishes on the optimal (nominal) path. The remaining second-order terms constitute what is called the second variation ($\Delta^2 J$), given as

$$\Delta^2 J = \frac{1}{2} \left[\Delta\underline{x}^T \left(\phi_{xx} + (\underline{n}^T \psi_x) \right) \Delta\underline{x} \right] \Bigg|_{\tau = \tau_f}$$

$$+ \frac{1}{2} \int_{\tau_o}^{\tau_f} [\Delta\underline{x}^T \; \Delta\underline{u}^T] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \Delta\underline{x} \\ \Delta\underline{u} \end{bmatrix} d\tau \qquad \text{II.40}$$

The boundary conditions associated with the neighboring perturbed optimal formulation are given by

$$\Delta\underline{x}(\tau_i) = \text{specified} \qquad\qquad \text{II.41}$$

$$\Delta\underline{P}(\tau_x) = \left[\left(\phi_{xx} + (\underline{n}^T \psi_x)_x \right) \Delta\underline{x} + \psi_x^T \Delta\underline{n} \right] \Bigg|_{\tau = \tau_f} \qquad \text{II.42}$$

From the terminal constraints on the final states we have

$$\Delta\underline{\psi} = [\psi_x \; \Delta\underline{x}] \Big|_{\tau = \tau_f} \qquad\qquad \text{II.43}$$

The deterministic model does not explicitly take into account errors associated with disturbances inputs acting upon the vehicle (i.e., wind gust acting upon the vehicle in the reentry phase) [9]. Furthermore, nothing has been said about nonstationary components such as a time-varying wind model associated with the plant dynamics. The variational problem needs to be solved to determine the perturbed control law $\Delta\underline{u}(t)$. The perturbed control $\Delta\underline{u}(t)$ yields the feedback control law that will drive the stochastic differential system.

III. STOCHASTIC FORMULATION

The deterministic model is described by equations II.7 through II.9. In deriving these equations certain assumptions and approximations have been made to simplify the model. The addition of white noise to the deterministic model implies that there are additional stochastic disturbances that drive the system. Also white noise would accommodate for oversimplifications and approximations made during deterministic modeling. The stochastic model is given by:

$$\frac{dh}{d\tau} = b\nu\sin\gamma + W_{r1} \qquad\qquad III.1$$

$$\frac{d\nu}{d\tau} = - A_1 b(1 + C^2)\ \delta\nu^2 - \frac{b^2\sin\gamma}{(b-1+h)^2} + W_{a1} \qquad\qquad III.2$$

$$\frac{d\gamma}{d\tau} = A_2 bc\delta\nu + \frac{b\nu\cos\gamma}{(b-1+h)} - \frac{b2\cos\gamma}{(b-1+h)^2\nu} + W_{a2} \qquad\qquad III.3$$

where W_{r1}, W_{a1}, W_{a2} zero mean white noise processes. More precisely, the white noise processes are used to account for the following:

W_{r1} ~ accommodates the oblateness term omitted.

W_{a2} ~ accounts for incomplete knowledge of the aerodynamic forces, assumption of spherical symmetry, and rotational or Coriolis forces.

W_{a3} ~ same form as W_{a2}.

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The process noise also compensates for atmospheric winds acting upon the vehicle. Recall the deterministic case where the initial state of the system $h(r_o)$, $\nu(r_o)$ and $\gamma(r_o)$ was known. We, however, no longer make this assumption because the initial state vector can not be measured exactly. The modeling of the initial state is assumed to be a vector valued Gaussian random variable. its mean and covariance matrix represent a priori information about the initial condition of the plant.

1. Statistical Description

The uncertainty in the overall physical process has been modeled in two parts.

(a) Initial uncertainty

The initial state vector $[h(r_o)$, $\nu(r_o)$ and $\gamma(r_o)]^T$ is viewed as a random variable.

(b) Plant uncertainty

The dynamic equations are driven by mutually independent Gaussian white noise processes W_{r1}, W_{a1}, W_{a2}

The description of the uncertainty is given as follows. The initial state vector is Gaussian with known mean and covariance matrix Z_o:

$$E\left[h(r_o), \nu(r_o), \gamma(r_o) \right]^T - \left[\bar{h}_o, \bar{\nu}_o, \bar{\gamma}_o \right]^T \text{ (assumed known)}$$

$$E\left\{ \left[(h(r_o) - \bar{h}_o), (\nu(r_o) - \bar{\nu}_o), (\gamma(r_o) - \bar{\gamma}_o)) \right]^T \right.$$

$$\left. \cdot \left[(h(r_o) - \bar{h}_o), (\nu(r_o - \bar{\nu}_o), (\gamma(r_o) - \bar{\gamma}_o)) \right] \right\} - Z_o \text{ (assumed known)}.$$

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The plant driving noises W_{r1}, W_{a1}, W_{a2} have zero mean and covariance matrix $\theta(t) \delta(t-\tau)$:

$$E \left[W_{r1}, W_{a1}, W_{a2} \right]^T - \left[0,0,0 \right]^T \quad \forall \; t \geq t_o$$

$$E \left\{ \left[W_{r1}, W_{a1}, W_{a2} \right]^T \left[W_{r1}, W_{a1}, W_{a2} \right] \right\} - \theta(t) \delta(t - \tau) \; \text{(assumed known)}.$$

The selection of the intensity matrices $\theta(t)$ and Zo which govern the strength of the uncertainty is discussed in the simulation.

2. Linearized stochastic modeling.

The deterministic formulation was based on the nominal state vector $[h_o(\tau), \nu_o(\tau), \gamma_o(\tau)]^T$ and the nominal control $C_o(\tau)$ along with the correction vector $[\Delta h(\tau), \Delta \nu(\tau), \Delta \gamma(\tau)]^T$ and the correction control $\Delta C(\tau)$. In the stochastic modeling we considered the state vectors $[h(\tau), \nu(\tau), \gamma(\tau)]^T$, $[\Delta h(\tau), \Delta \nu(\tau), \Delta \gamma(\tau)]T$ and the controls $C(\tau)$ and $\Delta C(\tau)$ to be random processes instead of deterministic. If a Taylors series expansion is made about the nominal path by using the model of equations III.1 - III.3, the following equation is obtained:

$$\Delta \dot{\underline{x}}^*(t) - f_x \Delta \underline{x}^*(t) + f_u \Delta \underline{u}^*(t) + \underline{n}(t). \qquad \qquad \text{III.4}$$

Equation III.4 is a time varying linear state equation. We expect that the feedback scheme that generates the control $\Delta \underline{u}^*$ to be of the same form as the deterministic feedback control $\Delta \underline{u}$ obtained from the second variation. The existence of the white noise process in equation III.4 can be used to account for deterministic modeling errors associated with assumptions (a), (b), and (c) of Part II.1. For example, wind gust acting on the vehicle can be modeled in this way. White noise would also compensate for the high order terms that have been excluded from equation III.4.

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3. Open Loop Feedback Control

In part II we have investigated the deterministic perturbation control obtained from the second variation. The open loop feedback control gains K_1 and K_2 can be found by solving the linear TPBVP given by:

$$\Delta \dot{\underline{x}} = f_x \, \Delta \underline{x} + f_u \, \Delta \underline{u}$$

$$\Delta \dot{\underline{P}} = - H_{xx} \Delta \underline{x} - f_x^T \, \Delta \underline{P} - H_{xu} \, \Delta \underline{u}$$

$$\Delta \underline{u} = - H_{uu}^{-1} \, (H_{ux} \, \Delta \underline{x} - f_u^T \, \Delta \underline{P})$$

where $K_1 = H_{uu}^{-1} H_{ux}$, $K_2 = H_{uu}^{-1} f_u^T \, \Delta \underline{P}$.

The boundary conditions are:

$$\Delta \dot{x}(\tau_o) = \text{specified}$$

$$\Delta \underline{P}(\tau_f) = \left[\left(\phi_{xx} + (\underline{n}^T \psi_x)_x \right) \Delta \underline{x} + \psi_x^T \Delta \underline{n} \right] \Bigg|_{\tau = \tau_f}$$

$$\Delta \psi = \left[\psi_x \, \Delta \underline{x} \right] \Bigg| \tau = \tau_f = \text{specified.}$$

A member of the neighboring optimal correction state and control is determined by integrating the linear stochastic model

$$\begin{bmatrix} \Delta \dot{h}^*(\tau) \\ \Delta \dot{\nu}^*(\tau) \\ \Delta \dot{\gamma}^*(\tau) \end{bmatrix} = f_x \begin{bmatrix} \Delta h^*(\tau) \\ \Delta \nu^*(\tau) \\ \Delta \gamma^*(\tau) \end{bmatrix} + f_u \, \Delta C^*(\tau) + \begin{bmatrix} n_1(\tau) \\ n_2(\tau) \\ n_3(\tau) \end{bmatrix} \qquad \text{III.5}$$

along with the control law

$$\Delta C^*(\tau) = \underline{K}_1(\tau) \begin{bmatrix} \Delta h^*(\tau) \\ \Delta \nu^*(\tau) \\ \Delta \gamma^*(\tau) \end{bmatrix} + K_2(\tau) \qquad\qquad \text{III.6}$$

by using initial conditions

$$\begin{bmatrix} \Delta h^*(\tau_o) \\ \Delta \nu^*(\tau_o) \\ \Delta \gamma^*(\tau_o) \end{bmatrix} = \begin{bmatrix} \bar{h}(\tau_o) & - & ho(\tau_o) \\ \bar{\nu}(\tau_o) & - & \nu o(\tau_o) \\ \bar{\gamma}(\tau_o) & - & \gamma o(\tau_o) \end{bmatrix}. \qquad\qquad \text{III.7}$$

Finally a member of the optimal state and control history is evaluated by

$$\begin{bmatrix} h^*(\tau) \\ \nu^*(\tau) \\ \gamma^*(\tau) \\ C^*(\tau) \end{bmatrix} = \begin{bmatrix} ho(\tau) + \Delta h^*(\tau) \\ \nu o(\tau) + \Delta \nu^*(\tau) \\ \gamma o(\tau) + \Delta \gamma^*(\tau) \\ Co(\tau) + \Delta C^*(\tau) \end{bmatrix}. \qquad\qquad \text{III.8}$$

4.   Closed Loop Feedback Control

In most aerospace and navigational problems a closed loop feedback control is very convenient in controlling the plant dynamics and reducing the computational time.   Recall the linear system given by

$$\Delta \dot{\underline{x}} = f_x \, \Delta \underline{x} + f_u \, \Delta \underline{u}, \quad x(\tau_o) = \text{specified}$$

with criterion

$$\Delta^2 J = \frac{1}{2} \left[\Delta\underline{x}^T \quad \phi_{xx} + (\underline{n}^T\psi_x)_x \quad \Delta\underline{x} \right] \Bigg|_{\tau = \tau_f}$$

$$+ \frac{1}{2} \int_{\tau_o}^{\tau_f} \left[\Delta\underline{x}^T \; \Delta\underline{u}^T \right] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \Delta\underline{x} \\ \Delta\underline{u} \end{bmatrix} dt$$

and constraint

$$\Delta\underline{\psi} = \left[\psi_x \; \Delta\underline{x} \right] \Big|_{\tau = \tau_f}.$$

It is well known [1] that the above system can be replaced by the equivalent

$$\Delta\underline{\dot{x}} = f_x^{(1)} \, \Delta\underline{x} + f_u \, \Delta\underline{u}, \; x(\tau_o) = \text{specified} \qquad \text{III.9}$$

with criterion,

$$\Delta^2 J = \frac{1}{2} \left[\Delta\underline{x}^T \left(\phi_{xx} + (\underline{n}^T\psi_x)_x \right) \Delta\underline{x} \right] \Bigg|_{\tau = \tau_f}$$

$$+ \frac{1}{2} \int_{\tau_o}^{\tau_f} \left[\Delta\underline{x}^T \; \Delta\underline{u}^T \right] \begin{bmatrix} H_{xx}^{(1)} & 0 \\ 0 & H_{uu} \end{bmatrix} \begin{bmatrix} \Delta\underline{x} \\ \Delta\underline{u} \end{bmatrix} d\tau \qquad \text{III.10}$$

and constraint

$$\Delta\underline{\psi} = \left[\psi_x \; \Delta\underline{x} \right] \Big|_{t = t_f}. \qquad \text{III.11}$$

where
$$f_x^{(1)} = f_x - f_u H_{uu}^{-1} H_{ux} \qquad \text{III.12}$$

$$H_{xx}^{(1)} = H_{xx} - H_{xu} H_{uu}^{-1} H_{ux}. \qquad \text{III.13}$$

The system given by equation III.9 is found to be controllable by solving the

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matrix differential equation

$$\frac{d}{dt} W(\tau, \tau_f) = f_x^{(1)} W(\tau, \tau_f) + W(\tau, \tau_f) f_x^{(1)^T} - f_u f_u^T \qquad \text{III.14}$$

$$W(\tau_f, \tau_f) = \underline{0} \qquad \text{III.15}$$

where $W(\tau_o, \tau_f)$ is nonnegative definite for $\tau_f \geq \tau_o$ [17]. Since the system is controllable there exists a control $\Delta\underline{u}$ which drives the state of the system given by III.9 from some initial value $\underline{x}(\tau_o)$ to $x(\tau_f)$ with $\tau_f > \tau_o$. We will make use of the controllability in the later formulation.

Next, the performance criterion given by III.10 is investigated. Recall that the necessary conditions for the linear quadratic performance index are:

$$\left(\phi_{xx} + (\underline{n}^T \psi_x)_x \right) \geq 0$$

$$H_{xx}^{(1)} \geq 0$$

$$H_{uu} > 0 .$$

However, the matrix $H_{xx}^{(1)}$ in III.13 is not positive semidefinite. To circumvent this, we use Schwartz's inequality to bound certain entries of the matrix $H_{xx}^{(1)}$ and thereby increase the performance index to be minimized. The new matrix $H_{xx}^{(1)}$ thus becomes

$$\begin{bmatrix} \left| \dfrac{\partial^2 H}{\partial h^2} \right| & \cdot & \cdot \\ \cdot & \left| \dfrac{\partial^2 H}{\partial \nu^2} \right| & \cdot \\ \cdot & \cdot & \left| \dfrac{\partial^2 H}{\partial \gamma^2} \right| \end{bmatrix}$$

Recall that Schwartz's inequality is given by $\left| xy \right|^2 \leq \left| x \right| \cdot \left| y \right|$. Thus, the physical meaning of taking the absolute value implies that we are minimizing both positive and negative deviations from the nominal path. Solving the differential equation III.14 along with III.15 we see that the system is still controllable. The closed loop feedback control is given by [4]

$$\Delta u(\tau) = - H_{uu}^{-1} \left[f_u^T (S - RQ^{-1} R^T) \Delta x(\tau) + f_u^T RQ^{-1} \Delta \psi \right]$$

where

$$\dot{S} = - S f_x^{(1)} - f_x^{T(1)} S + S f_u H_{uu}^{-1} f_u^T S - H_{xx}^{(1)} \qquad \text{III.16}$$

$$S(\tau_f) = \left[\phi_{xx} + (n^T \psi_x)_x \right] \Bigg|_{\tau = t_f} \qquad \text{III.17}$$

$$\dot{R} = - (f_x^{(1)} - S f_u H_{uu}^{-1} f_u^T) R \qquad \text{III.18}$$

$$R(\tau_f) = \left[\psi_x^T \right] \Bigg|_{\tau = \tau_f} \qquad \text{III.19}$$

$$\dot{Q} = R^T f_u H_{uu}^{-1} f_u^T \qquad \text{III.20}$$

$$Q(\tau_f) = 0 . \qquad \text{III.21}$$

Since the system is controllable, we can force the terminal constraints

$$\Delta \psi - \psi_x \Delta x \Bigg|_{\tau = \tau_f} = 0.$$

From equation III.16, the closed loop feedback control law is given by

$$\Delta u(\tau) = - H_{uu}^{-1} f_u^T (S - RQ^{-1} R^T) \Delta \underline{x} \qquad \tau_o \leq \tau < \tau_f. \qquad \text{III.22}$$

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The sufficient conditions for the existence of a neighboring solution are [4]

$$H_{uu} > 0$$

$$Q(\tau) < 0$$

$$S(\tau) - R(\tau) \, Q^{-1}(\tau) \, R^T(\tau) \qquad \text{finite for } \tau_o \leq \tau < \tau_f.$$

As in the case of open loop feedback, a member of the neighboring correction state and control is determined by integrating the linear stochastic model given by equation III.5 with the control law given by equation III.22. The initial conditions are given by equation III.7. Again, a member of the optimal state and control history is evaluated by equation III.8. However, in the case of real-time control, the physical system is the one that provides the integration[3]. The neighboring optimal control is computed as a function of the difference between the measured state $\left[h_n(\tau), \; \nu_m(\tau), \; \gamma_m(\tau) \right]^T$ and the nominal optimal state $\left[h_o(\tau), \; \nu_o(\tau), \; \gamma_o(\tau) \right]^T$. The optimal control is defined as

$$\Delta C^*(\tau) = - H_{uu}^{-1} f_u^T \, (\, S - RQ^{-1} R^T \,) \left[h_o - h_m, \; \nu_o - \nu_m, \; \gamma_o - \gamma_m \right]^T .$$

Therefore, the total optimal control is

$$C^*(\tau) = Co(\tau) + \Delta C^*(\tau).$$

5. Evaluation of the Variational Cost Function

The deterministic non-linear cost function and the second variation cost are given by equations II.10 and II.40, respectively.

The stochastic cost of the variational performance index can be expressed as a trace of the expected values of the second variational cost given by

$$\Delta^2 J^* = \frac{1}{2} \text{Tr} \quad E \left[\Delta \underline{x}^{*T} \ (\phi_{xx} + (\underline{n}^T \psi_x)_x) \ \Delta \underline{x}^* \right] \Bigg|_{\tau = \tau_f}$$

$$+ \frac{1}{2} E \quad \int_{\tau_0}^{\tau_f} \left[\Delta \underline{x}^{*T} \ \Delta \underline{u}^{T*} \right] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \Delta \underline{x}^* \\ \Delta \underline{u}^* \end{bmatrix} d\tau \Bigg\} . \qquad \text{III.23}$$

If we assume that the combined optimization of the nominal trajectory and the perturbation stochastic feedback control is given by

$$\min \quad E \ (J_{nom} + \Delta^2 J^*) \quad = \left[\min J_{nom} \right] \Bigg|_{noise = 0}$$

$$+ \min \left[E \ \Delta^2 J^* \right] \qquad \text{III.24}$$

we can determine the total minimum cost required.

In the presence of noise, it is impossible to meet the terminal condition $\psi \ [\ \underline{x}(\tau_f), \ \tau_f \ ] = \underline{0}$ exactly. Hence, in the place of the requirement $\Delta \psi = \psi_x \ \Delta \underline{x} \ |_{\tau = \tau_f} = \underline{0}$ the quadratic approximation of the noisy case is augmented by $\psi_x^T \ N \ \psi_x$ where N is diagonal positive definite matrix [4]. Equation III.23 is replaced by

$$\Delta^2 J^* = \frac{1}{2} \text{Tr} \quad E \left[\Delta \underline{x}^{*T} \ (\phi_{xx} + (n_x^T \psi \ )_x + \psi_x^T \ N \ \psi_x) \ \Delta \underline{x}^* \right] \Bigg|_{\tau = \tau_f}$$

$$+ \quad E \quad \int_{\tau_0}^{\tau_f} \left[\Delta \underline{x}^{*T} \ \Delta \underline{u}^{*T} \right] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \Delta \underline{x}^* \\ \Delta \underline{u}^* \end{bmatrix} d\tau$$

In the case of the closed loop feedback control, the performance index becomes

$$\Delta^2 J^* = \frac{1}{2} \, \text{Tr} \quad E \left[\Delta\underline{x}^{*T} (\phi_{xx} + (\underline{n}^T \psi_{xx} + \psi_x^T N \psi_x) \Delta\underline{x}^* \right] \Bigg|_{\tau = \tau_f}$$

$$+ \quad E \quad \int_{\tau_o}^{\tau_f} \left[\Delta\underline{x}_*^T \quad \Delta\underline{u}^{T*} \right] \begin{bmatrix} H_{xx} & 0 \\ 0 & H_{uu} \end{bmatrix} \begin{bmatrix} \Delta\underline{x}^* \\ \Delta\underline{u}^* \end{bmatrix} d\tau \Bigg\}$$

which is simplified to

$$\Delta^2 J^* = \frac{1}{2} \, \text{Tr} \left\{ X_f \, S_f + \int_{\tau_o}^{\tau_f} \left[H_{xx} \, X + H_{uu} \, U \right] d\tau \right\}$$

where $\quad X = E \left[\Delta\underline{x}^* \, \Delta\underline{x}^{*T} \right]$

and $\quad U = E \left[\Delta\underline{u}^* \, \Delta\underline{u}_*^T \right]$.

Finally, since $\quad \Delta\underline{u}^* = - H_{uu}^{-1} \, f_u^T \, (S - RQ^{-1} R^T) \, \Delta\underline{x}^*$

we obtain $\quad\quad C = - H_{uu}^{-1} \, f_u^T \, (S - RQ^{-1} R^T)$

$$U = C \, X \, C^T .$$

IV. NUMERICAL DATA AND RESULTS

The data used for simulation are summarized below:

Spacecraft data

m/s — .3E03 kg/m^3

C_{DO} — .21   ----

K — .167E01   ----

C_{LR} — .3546   ----

CL_{max} — .9

Physical data

$\beta = \dfrac{1}{6900}$ m^{-1}

ρ_o — 1.316 kg/m^3

R_a — .6498E07   m

R_c — .6558E07   ----

R_d — .12996E08   ----

μ — .3986E15 m^3/sec^2

Normalized values

R_R — .6498E07   m

H_R — 120.0E03   ----

V_R — 7.832   m/sec

t_R — 887.16   sec

ρ_R — .3996E-2 kg/m^3

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Using the above mentioned data, simulations are carried out for two cases of nominal solutions. The nominal (optimal) solutions of the deterministic non-linear TPBVP and neighboring linear TPBVP are obtained by successful use of OPTSOL code developed by Deutsche Forschungs - and Versuchsanstalt fur Luft Raumfahrt (DFVLR) at Oberpfaffenhofen, West Germany.

1. Deterministic Solution

The nominal solution is obtained by solving a TPBVP. The time histories of altitude H, velocity V, flight path angle γ, and the control coefficient C_L for two cases are shown in Figures 3 and 4. The flight time through the atmosphere for the two cases is 480 and 550 seconds, respectively. Using the physical data given above, the following entry and exit conditions are obtained.

Case #1

Entry: H_e - 120 km; V_e - 9.025 km/s; γ_e - 6.14°

Exit: H_e - 120 km; V_e - 7.459 km/s; γ_e - -0.025°

Characteristic velocities:

Deorbit velocity, ΔV_d - 1.05387 km/s

Reorbit velocity, ΔV_c - .405493 km/s

Case #2

Entry: H_e - 120 km; V_e - 9.029 km/s; γ_e - 5.665°

Exit: H_e - 120 km; V_e - 7.666 km/s; γ_e - 0.302°

Characteristic velocities:

Deorbit velocity, ΔV_d - 1.04565 km/s

Reorbit velocity, ΔV_c - .20083 km/s

Consider the optimal solution obtained for case #1. Initially, the vehicle is in a circular orbit at HEO moving at a speed $V_d - \sqrt{\mu/R_d}$ - 5538.14 m/s. A deorbit impulse ΔV_d - 1053.87 m/s is executed to fly the vehicle

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through an elliptic orbit. The elliptic velocity at the deorbit point D is $\bar{V}_d$ - V_d - ΔV_d - 4484.27 m/s. At the atmospheric entry point E of altitude H_a - 120 km, the vehicle attains an orbital velocity Ve - 9.025 km/s. During the atmospheric maneuver, the velocity of the vehicle is reduced and the exit velocity is V_f - 7.459 km/s. In order to attain the desired circular orbit (LEO) at radius R_c - 6558 km, a reorbit impulse ΔV_c - 405.49 m/s is imparted. The vehicle is now in a circular orbit with speed V_c - $\sqrt{\mu/R_c}$ - 7796.2 m/s - $\bar{V}_c$ + ΔV_c where $\bar{V}_c$ - 7390.71 m/s which is the elliptic velocity along the exit-LEO elliptic path.

The simulation of equation III.5 is obtained for different strengths of white process noise. The criterion used is based on the assumption that the cost resulting from the deterministic optimization $minJ_{det}$ should be sufficiently larger than the optimization resulting from the stochastic contribution of the cost, $\min \left[E\Delta^2 J^* \right]$.

2. Open Loop Feedback Control Law

Here the neighboring-optimum optimization technique was applied along a nominal trajectory obtained from the nonlinear TPBVP. Then, the linear stochastic differential equation was implemented to generate a set of optimum trajectories for slightly different initial conditions and state covariance matrices. The optimum trajectories of altitude, velocity flight path angle and lift coefficients are shown for three different sets of data (set #1, Set #2, and Set #3). Note that the state vector does not converge to the nominal path as time is increased.

3. Closed Loop Feedback Control Law

Here the control scheme was applied by first solving the three Riccati type equations backward in time. Then, the linear stochastic differential equation was solved by considering different sets of initial conditions. This scheme is significantly important in terms of computational time required to solve the overall problem. Note that the control scheme works better for $\Delta\gamma(\tau_0) < 0$ than for $\Delta\gamma(\tau_0) > 0$. The gain values become very large when $\Delta\gamma(\tau_0)$ is greater than zero. Also the correction components of altitude velocity and flight path angle converge to zero as time approaches the final time. Better results are also obtained in terms of a stochastic cost which is less for $\Delta\gamma(\tau_0) < 0$ than for $\Delta\gamma(\tau_0) > 0$. The optimum trajectories of altitude, velocity, flight path angle and lift coefficient are shown for six sets of initial conditions (Set #4 - Set #9).

Due to the digital computer simulation, the small magnitude of the numbers required some scaling. The scaling coefficients of the standard white noise processes are L_1, L_2, L_3. The covariances of the noise W_{r1}, W_{a1}, W_{a2} are Q_1, Q_2, Q_3, respectively. The initial conditions used to generate the state and control trajectories are shown in Table 1, Table 2 and Table 3.

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V. CONCLUDING REMARKS

In this report, we have devised a stochastic neighboring optimal guidance scheme as applicable to an AOTV. We have addressed the minimization of fuel consumption during the atmospheric portion of an aeroassisted, coplanar, orbital transfer vehicle, and examined the effects of uncertainties due to modelling on the neighboring optimal guidance. In the first stage, a nominal solution has been obtained for the deterministic, nonlinear dynamics describing the atmospheric maneuver of a coplanar AOTV. In the next stage, a neighboring optimal guidance has been determined. In the third stage, with a linear stochastic model, a neighboring optimal solution has been obtained. Finally, the approximate solution has been obtained as the sum of the deterministic nominal solution and the stochastic neighboring solution. Numerical results have been presented for a typical coplanar AOTV.

Control trajectories are achieved by means of lift modulation. The performance index minimized is in terms of energy required for orbital transfer. It has been shown that the overall cost is increased when the deterministic and stochastic minimal problem is combined and lower and upper bounds on the control have been imposed.

The space vehicle is assumed to have a digital computer on board so that different nominal trajectories are stored. Whenever the vehicle desires to enter the atmosphere, a nominal trajectory is used along with the desired initial conditions so that the linear stochastic differential equation is implemented. The results obtained are used as a correction to the nominal trajectory. This approach reduces the computational time and the requirement of large computers on board the vehicle. It also takes into consideration the uncertainty associated with the inexact knowledge of the true model describing the flight through the atmosphere.

Table 1

	Set #1	Set #2	Set #3
L1	1 E-5	5 E-5	1 E-5
L2	1 E-4	1 E-4	1 E-4
L3	1 E-4	5 E-4	1 E-4
Q1	1	1	1
Q2	30	30	30
Q3	55	55	55
$\Delta \overset{*}{h}(\tau_o)$	12m	36m	36m
$\Delta \overset{*}{\nu}(\tau_o)$	46.99 m/s	39.6 m/s	156 m/s
$\Delta \overset{*}{\gamma}(\tau_o)$	0.06°	0.0573°	0.173°
$E\left[\underline{x}^{*T}\underline{x}^{*}\right]\Big|_{\tau = \tau_o}$	$\begin{bmatrix} 4E\text{-}8 & 0 & 0 \\ 0 & 5.4E\text{-}4 & 0 \\ 0 & 0 & 2E\text{-}5 \end{bmatrix}$	$\begin{bmatrix} 1E\text{-}7 & 0 & 0 \\ 0 & 1E\text{-}4 & 0 \\ 0 & 0 & 1E\text{-}4 \end{bmatrix}$	$\begin{bmatrix} 1E\text{-}7 & 0 & 0 \\ 0 & 1E\text{-}5 & 0 \\ 0 & 0 & 1E\text{-}5 \end{bmatrix}$
J_{det}	1459.36 m/s	1459.36 m/s	1459.36 m/s
$\Delta^2 J^*$	74.69 m/s	126.07 m/s	12.53 m/s

Note: The covariance matrix $E\left[\underline{x}^{*T}\underline{x}^{*}\right]$ is given in a normalized form.

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Table 2

	Set #4	Set #5	Set #6
L1	1 E-5	1 E-5	1 E-5
L2	1 E-4	1 E-4	5 E-4
L3	1 E-4	1 E-4	5 E-4
Q1	1	1	0.1
Q2	30	30	3
Q3	55	55	5.5
$\Delta \overset{*}{h}(\tau_o)$	12m	60m	120m
$\Delta \overset{*}{\nu}(\tau_o)$	46.99 m/s	39.16 m/s	70.49 m/s
$\Delta \overset{*}{\gamma}(\tau_o)$	0.057°	0.057°	.458°
$E \left[\underline{x}^{*T} \underline{x}^* \right] \Big|_{\tau \ - \ \tau_o}$	$\begin{bmatrix} 4E\text{-}8 & 0 & 0 \\ 0 & 1.2E\text{-}4 & 0 \\ 0 & 0 & 2E\text{-}5 \end{bmatrix}$	$\begin{bmatrix} 4E\text{-}10 & 0 & 0 \\ 0 & 7.5E\text{-}7 & 0 \\ 0 & 0 & 1E\text{-}8 \end{bmatrix}$	$\begin{bmatrix} 4E\text{-}10 & 0 & 0 \\ 0 & 7.5E\text{-}7 & 0 \\ 0 & 0 & 2E\text{-}7 \end{bmatrix}$
J_{det}	1459.36 m/s	1459.36 m/s	1459.36 m/s
$\Delta^2 J^*$	212 m/s	131.39 m/s	329.33 m/s

Table 3

	Set #7	Set #8	Set #9
L1	1 E-5	1 E-5	1 E-5
L2	1 E-4	1 E-4	5 E-4
L3	1 E-4	1 E-4	5 E-4
Q1	1	1	1 E-1
Q2	30	20	5
Q3	55	30	5
$\Delta \overset{*}{h}(\tau_o)$	1.2m	120m	96m
$\Delta \overset{*}{\nu}(\tau_o)$	0.39 m/s	39.16 m/s	7.83 m/s
$\Delta \overset{*}{\gamma}(\tau_o)$	2.86 E-3°	-0.286°	-0.516°

$$E\left[\underline{x}^{*T}\,\underline{x}^*\right]\Bigg|_{\tau\,=\,\tau_o}$$

Set #7:
$$\begin{bmatrix} 1E-8 & 0 & 0 \\ 0 & 1E-7 & 0 \\ 0 & 0 & 1.5E-7 \end{bmatrix}$$

Set #8:
$$\begin{bmatrix} 5E-8 & 0 & 0 \\ 0 & 5E-7 & 0 \\ 0 & 0 & 7.5E-7 \end{bmatrix}$$

Set #9:
$$\begin{bmatrix} 1E-10 & 0 & 0 \\ 0 & 5E-7 & 0 \\ 0 & 0 & 1.05E-6 \end{bmatrix}$$

	Set #7	Set #8	Set #9
J_{det}	1246.47 m/s	1246.47 m/s	1246.47 m/s
$\Delta^2 J^*$	91.41 m/s	51.69 m/s	214.29 m/s

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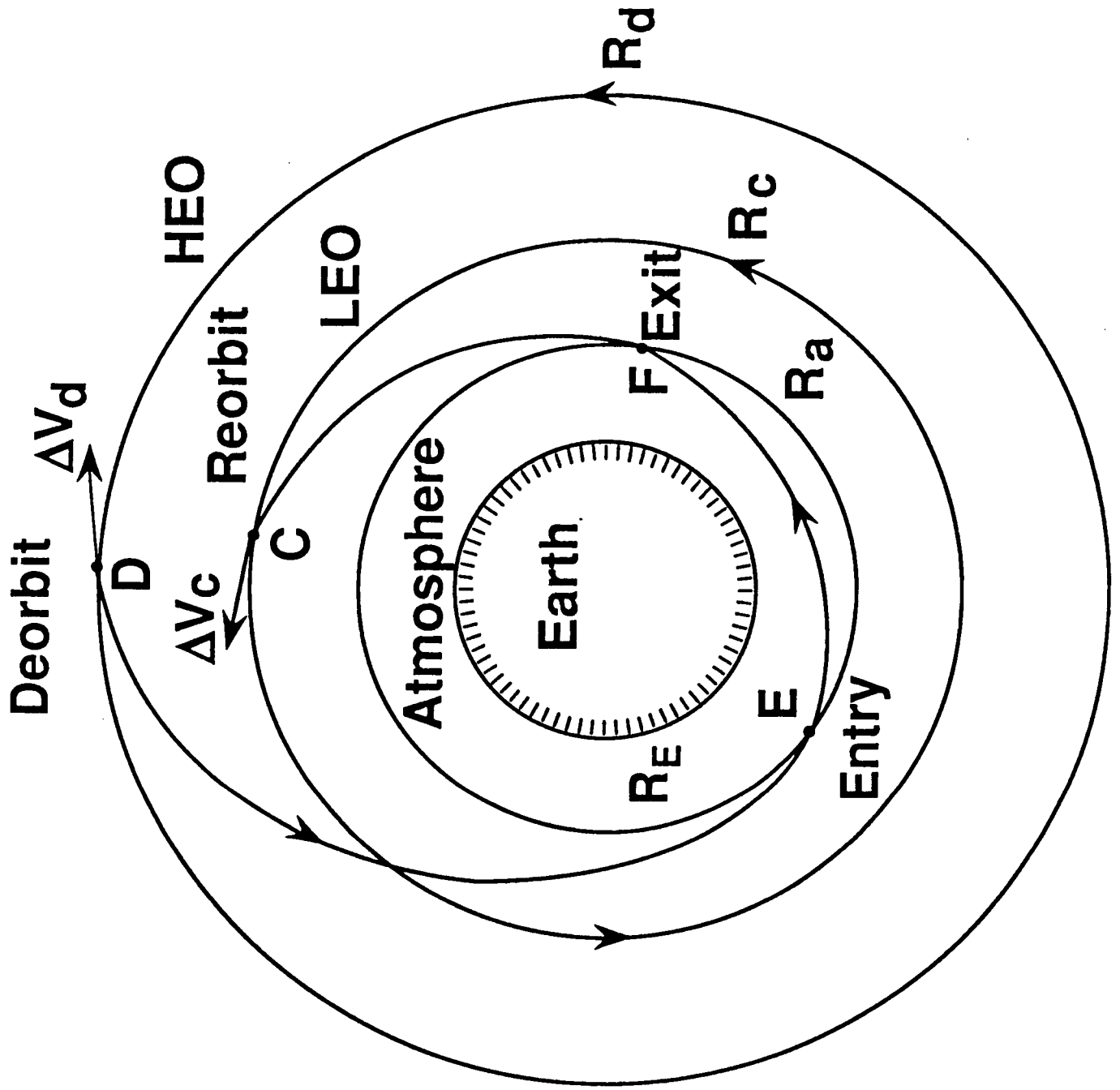
Fig. 1 Aeroassisted Coplanar Orbital Transfer
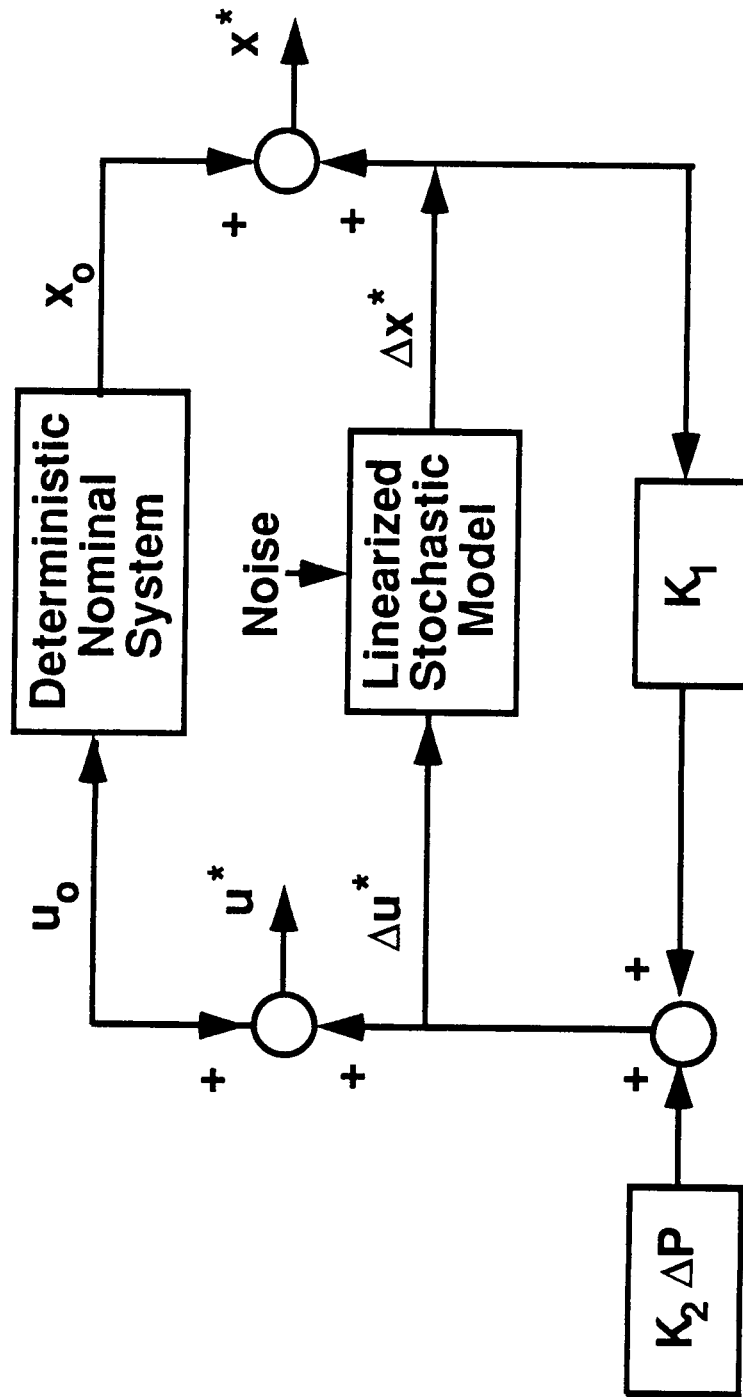
Fig. 2 Stochastic Simulation Diagram: Open Loop
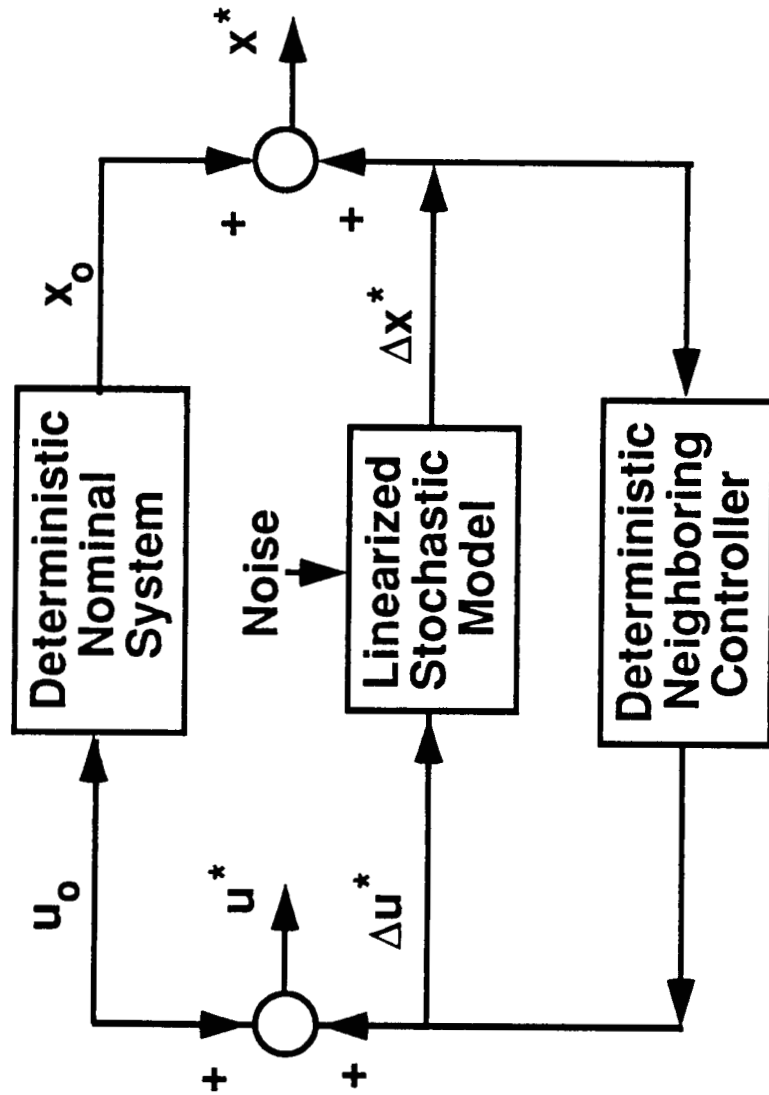
Fig. 3 Stochastic Simulation Diagram: Closed Loop

ALTITUDE
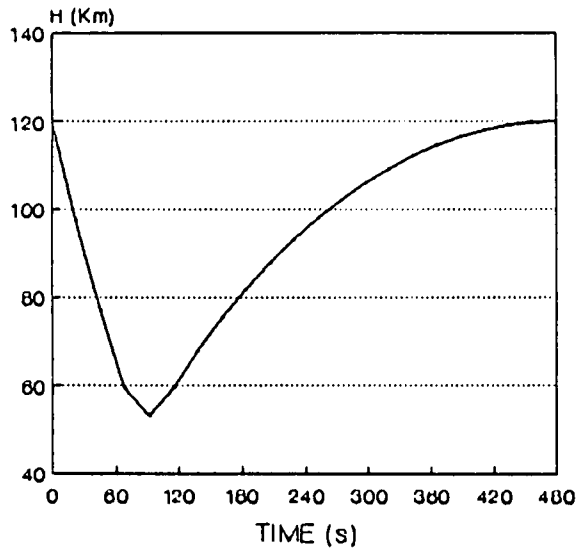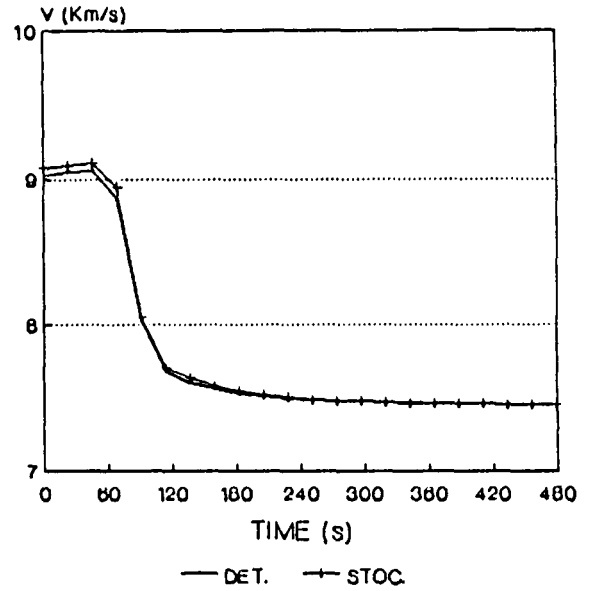

VELOCITY
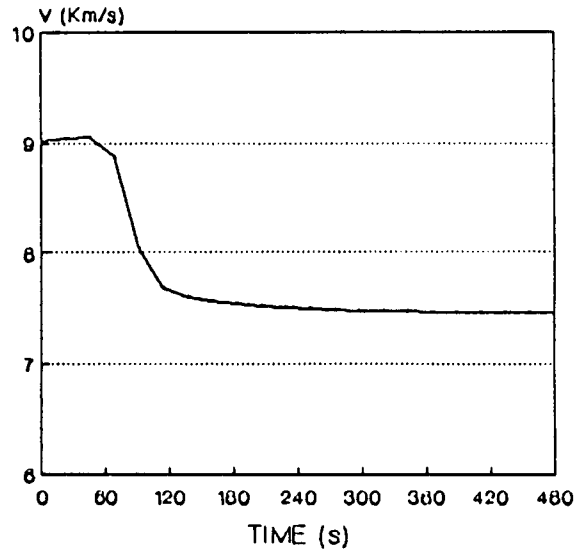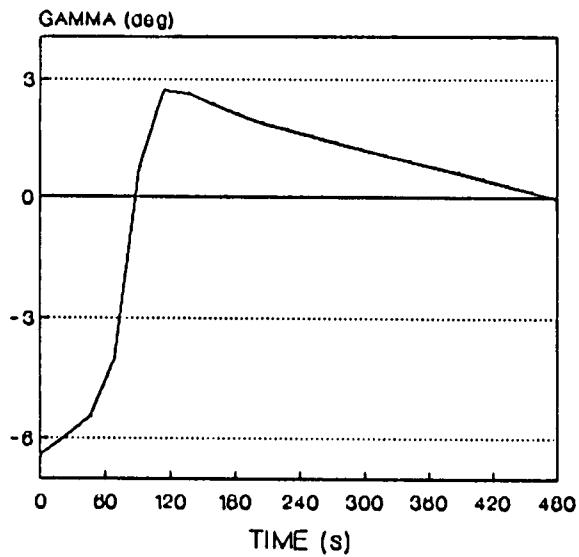
FLIGHT PATH ANGLE

LIFT COEFFICIENT
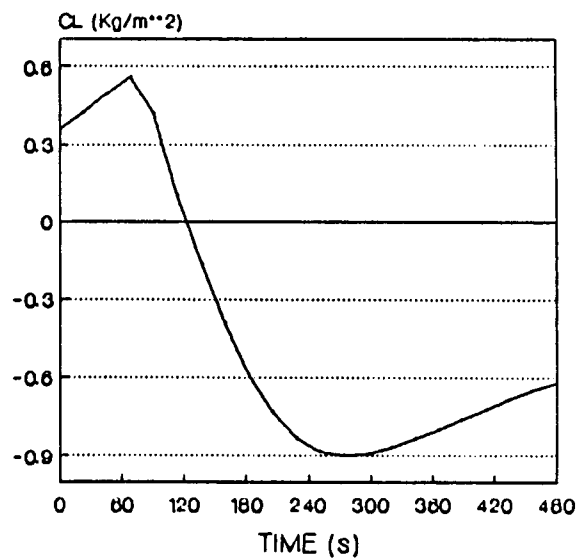
Fig. 4 Nominal Solution: Case 1

ALTITUDE

H (Km)

VELOCITY

V (Km/s)

FLIGHT PATH ANGLE

GAMMA (deg)

LIFT COEFFICIENT

CL (Kg/m**2)

TIME (s)

Fig. 5 Nominal Solution: Case 2

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Fig. 6 Stochastic Simulation: Open Loop: Set # 1

Fig. 7 Stochastic Simulation: Open Loop: Set # 2
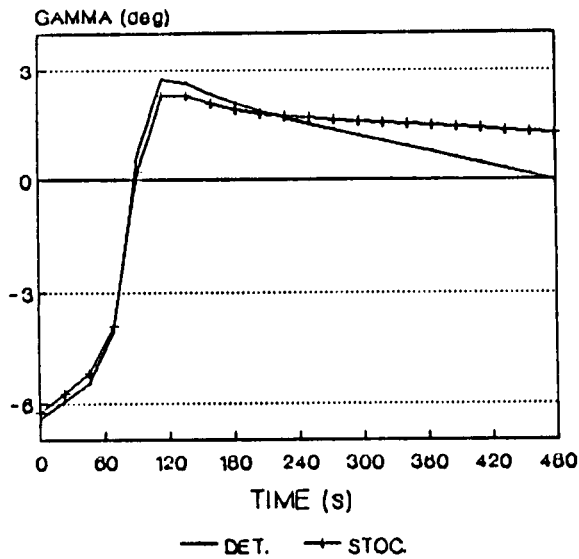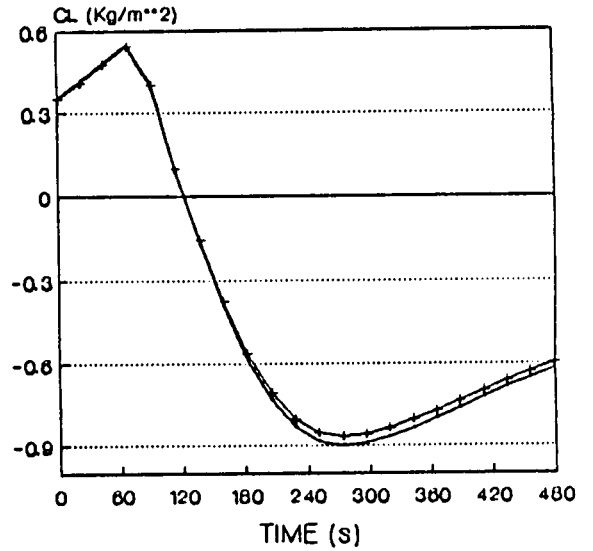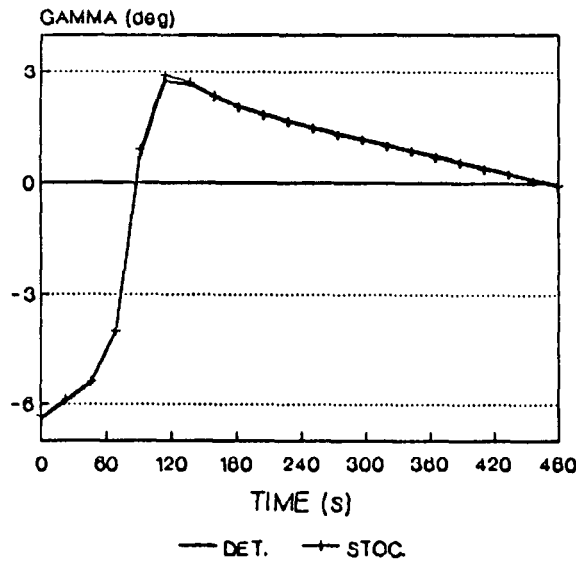
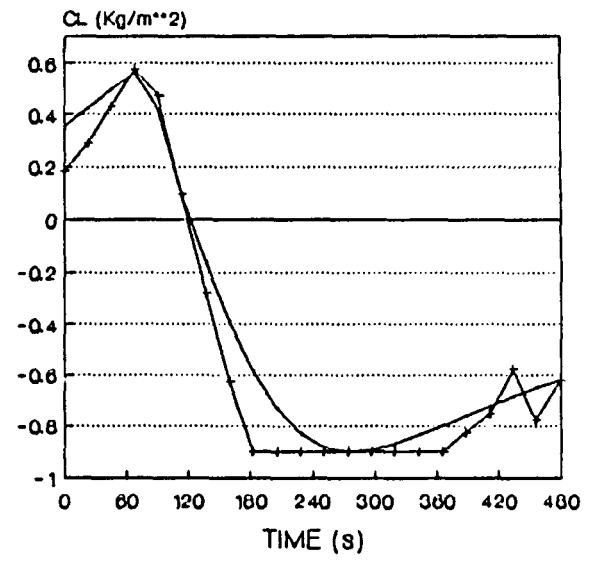Fig. 8 Stochastic Simulation: Open Loop: Set # 3

Fig. 9 Stochastic Simulation: Closed Loop: Set # 4

ALTITUDE

H (Km)

VELOCITY

V (Km)

FLIGHT PATH ANGLE

GAMMA (deg)

LIFT COEFFICIENT

CL (Kg/m**2)

—— DET.   —+— STOC.
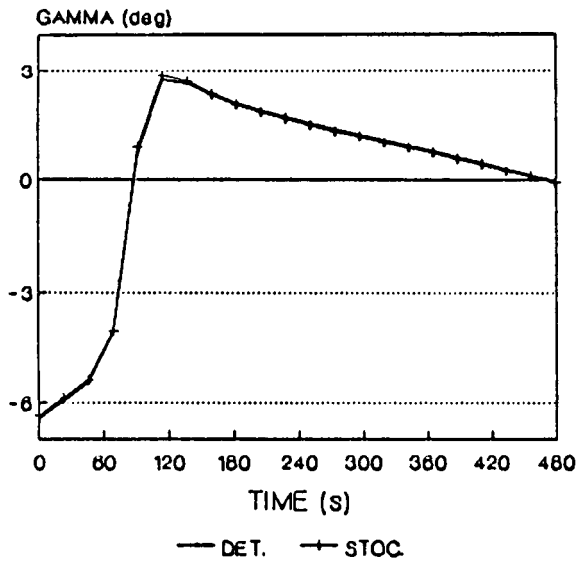
Fig. 10 Stochastic Simulation: Closed Loop: Set # 5

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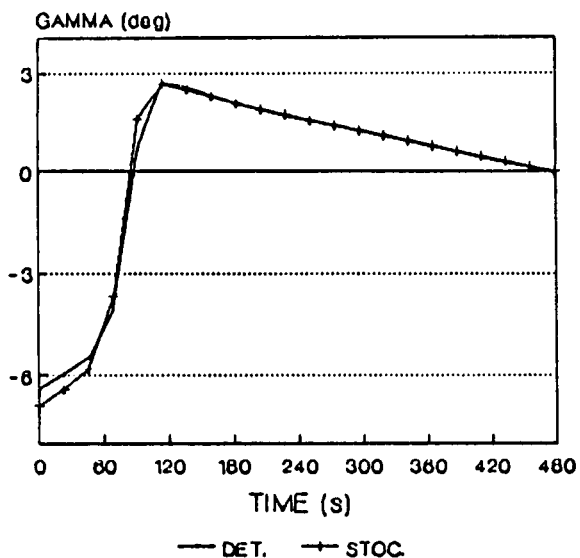
Fig. 11 Stochastic Simulation: Closed Loop: Set # 6

ALTITUDE

H (Km)

— DET. —+— STOC.

TIME (s)

VELOCITY

V (Km/s)

— DET. —+— STOC.

TIME (s)

FLIGHT PATH ANGLE

GAMMA (deg)

— DET. —+— STOC.

TIME (s)

LIFT COEFFICIENT

CL (Kg/m**2)

TIME (s)
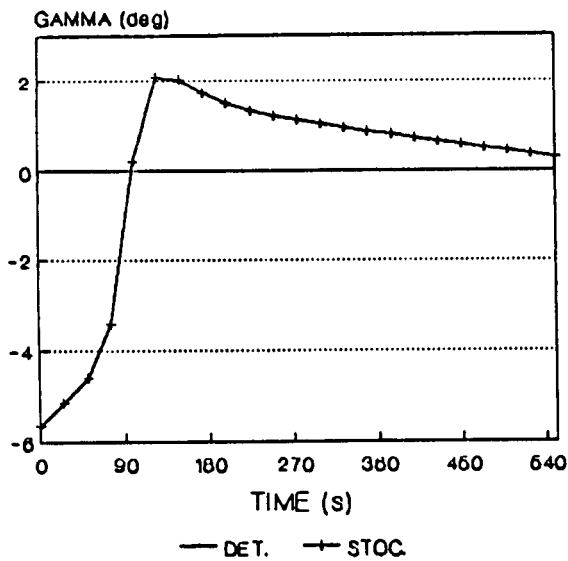
Fig. 12 Stochastic Simulation: Closed Loop: Set # 7

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ALTITUDE

H (Km)

DET.   STOC.

VELOCITY

V (Km/s)

DET.   STOC.

FLIGHT PATH ANGLE

GAMMA (deg)

DET.   STOC.

LIFT COEFFICIENT

CL (Kg/m**2)
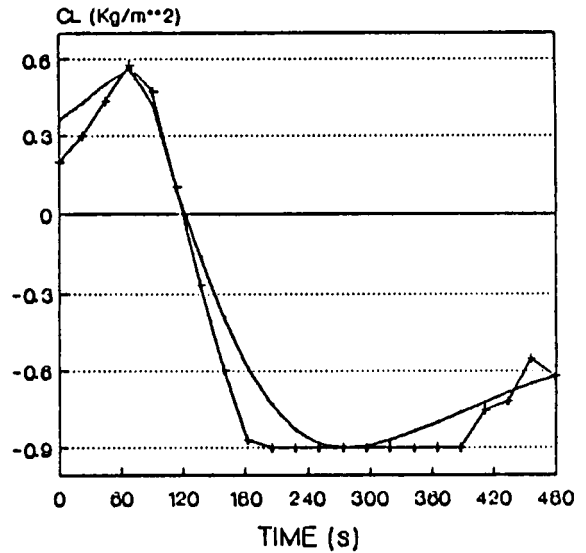
TIME (s)

Fig. 13 Stochastic Simulation: Closed Loop: Set # 8

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Fig. 14 Stochastic Simulation: Closed Loop: Set # 9

VII. APPENDIX
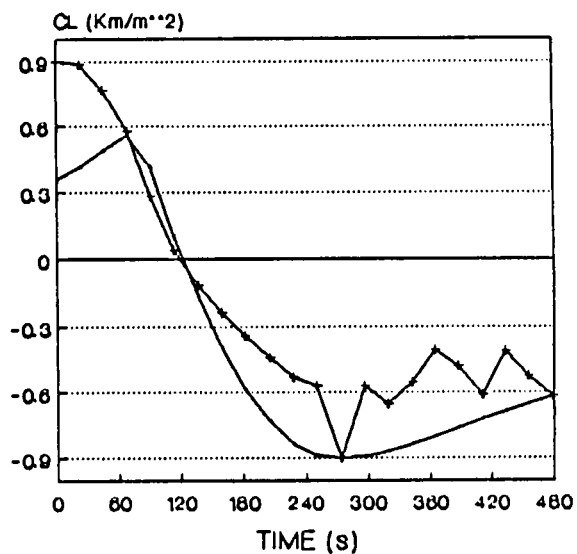
Costate variable differential equations for the nonlinear TPBVP.

$$\dot{P}_h = - P_\nu \left[\beta A_1 H_a b (1 + c^2) \nu^2 e^{-\beta h H_a} + \right.$$

$$\left. \frac{2b^2 \sin \gamma}{(b-1+h)^3} \right] + P_\gamma \left[\beta H_a A_2 b c e^{-\beta h H_a} + \frac{b\nu \cos \gamma}{(b-1+h)^2} - \right.$$

$$\left. \frac{2 b^2 \cos\gamma}{(b-1+h)^3 \nu} \right] \qquad\qquad VI.1$$

$$\dot{P}_\nu = - P_h \left[b\sin\gamma \right] + P_\nu \left[2 A_1 b (1+c^2) e^{-\beta H_a h} \nu \right]$$

$$- P_\gamma \left[A_2 bc e^{-\beta h H_a} + \frac{b \cos \gamma}{(b-1+h)} + \frac{b^2 \cos \gamma}{(b-1+h)^2 \nu^2} \right] \qquad\qquad VI.2$$

$$\dot{P}_\gamma = - P_h \left[b\nu\cos\gamma \right] + P_\nu \left[- \frac{b^2 \cos \gamma}{(b-1+h)^2} \right]$$

$$+ P_\gamma \left[\frac{b \nu \sin \gamma}{(b-1+h)} - \frac{b^2 \sin \gamma}{(b-1+h)^2 \nu} \right] \qquad\qquad VI.3$$

Equation associated with the linearized TPBVP are as follows:

State equations:

$$\Delta \dot{\underline{x}} = f_x \ \Delta \ \underline{x} + f_u \ \Delta \underline{u} \qquad\qquad VI.4$$

where
$$f_x = \begin{bmatrix} \dfrac{\partial f_h}{\partial h} & \dfrac{\partial f_h}{\partial v} & \dfrac{\partial f_h}{\partial \gamma} \\[2ex] \dfrac{\partial f_v}{\partial h} & \dfrac{\partial f_v}{\partial v} & \dfrac{\partial f_v}{\partial \gamma} \\[2ex] \dfrac{\partial f_\gamma}{\partial h} & \dfrac{\partial f_\gamma}{\partial v} & \dfrac{\partial f_\gamma}{\partial \gamma} \end{bmatrix}$$

$$f_n = \begin{bmatrix} \dfrac{\partial f_h}{\partial c} \\[2ex] \dfrac{\partial f_v}{\partial c} \\[2ex] \dfrac{\partial f_\gamma}{\partial c} \end{bmatrix}$$

$$\frac{\partial f_h}{\partial h} = 0$$

$$\frac{\partial f_h}{\partial v} = b \sin \gamma$$

$$\frac{\partial f_h}{\partial \gamma} = b\nu \cos \gamma$$

$$\frac{\partial f_\nu}{\partial h} = A_1 b \ C_{Do}^{n} \ (1+C^2) \ \beta \ Ha \ \delta \ \nu^2 + \frac{2 \ b^2 \ \sin \gamma}{(b-1+h)^3}$$

$$\frac{\partial f_\nu}{\partial \nu} = -2 \ A_1 b \ C_{Do}^{n} \ (1+C^2) \ \delta \nu$$

$$\frac{\partial f_\nu}{\partial \gamma} = - \frac{b^2 \ \cos \gamma}{(b-1+h)^2}$$

$$\frac{\partial f_\gamma}{\partial h} = - A_2 \, \beta \, H_a \, b \, C \, \delta \nu \; - \; \frac{b\nu \, \cos \gamma}{(b-1+h)^2} \quad + \quad 2 \; \frac{b^2 \, \cos \gamma}{(b-1+h)^3 \nu}$$

$$\frac{\partial f_\gamma}{\partial \nu} = A_2 \, b \, C \, \delta \; + \; \frac{b \, \cos \gamma}{(b-1+h)} \quad + \quad \frac{b^2 \, \cos \gamma}{(b-1+h)^2 \nu^2}$$

$$\frac{\partial f_\gamma}{\partial \gamma} = - \; \frac{b \, \nu \, \sin \gamma}{(b-1+h)} \; + \; \frac{b^2 \, \sin \gamma}{(b-1+h)^2 \nu}$$

$$\frac{\partial f_h}{\partial c} = 0$$

$$\frac{\partial f_\gamma}{\partial c} = -2 \, A_1 \, b \; C_{Do} \, c \, \delta \, \nu^2$$

$$\frac{\partial f_\nu}{\partial c} = A_2 \, b \, \delta \, \nu$$

Adjoint variable dif. equations:

$$\Delta \dot{\underline{P}} = - H_{xx} \, \Delta \, \underline{x} \; - \; f_x^T \, \Delta \, \underline{P} \; - \; H_{xu} \, \Delta \underline{u} \qquad\qquad\qquad \text{VI.5}$$

where
$$H_{xx} = \begin{bmatrix} \dfrac{\partial^2 H}{\partial h^2} & \dfrac{\partial^2 H}{\partial h \partial \nu} & \dfrac{\partial^2 H}{\partial h \partial \gamma} \\[2ex] \dfrac{\partial^2 H}{\partial \nu \partial h} & \dfrac{\partial^2 H}{\partial \nu^2} & \dfrac{\partial^2 H}{\partial \nu \partial \gamma} \\[2ex] \dfrac{\partial^2 H}{\partial \gamma \partial h} & \dfrac{\partial^2 H}{\partial \gamma \partial \nu} & \dfrac{\partial^2 H}{\partial \gamma^2} \end{bmatrix}$$

$$H_x = \begin{bmatrix} \dfrac{\partial H}{\partial h} & \dfrac{\partial H}{\partial \nu} & \dfrac{\partial H}{\partial \gamma} \end{bmatrix} \quad , \qquad H_{xu} = \begin{bmatrix} \dfrac{\partial^2 H}{\partial u \partial h} & \dfrac{\partial^2 H}{\partial u \partial \nu} & \dfrac{\partial^2 H}{\partial u \partial \gamma} \end{bmatrix}^T$$

$$\frac{\partial^2 H}{\partial h^2} = \left[- A_1 (\beta\, Ha)^2\, b\, C_{Do}\, (1+C^2)\, \delta\, \nu^2 - \frac{6\, b^2\, \sin\gamma}{(b-1+h)^4} \right] \cdot P_\nu$$

$$+ \left[A_2\, (\beta\, Ha)^2\, b\, c\, \delta\, \nu + \frac{2\, b\, \nu\, \cos\gamma}{(b-1+h)^3} - \frac{6\, b^2\, \sin\gamma}{(b-1+h)^4 \nu} \right] \cdot P_\gamma$$

$$\frac{\partial^2 H}{\partial h\, \partial \nu} = \left[2\, A_1\, b\, C_{Do}\, (1+C^2)\, \beta\, Ha\, \delta\nu \right] P_\nu + \left[- A_2\, b\, c\, \beta\, Ha\, \delta \right.$$

$$\left. - \frac{b\, \cos\gamma}{(b-1+h)^2} - \frac{2\, b^2\, \cos\gamma}{(b-1+h)^3 \nu^2} \right] P_\gamma$$

$$\frac{\partial^2 H}{\partial h\, \partial \gamma} = \left[\frac{2\, b^2\, \cos\gamma}{(b-1+h)^3} \right] P_\nu + \left[\frac{b\, \nu\, \sin\gamma}{(b-1+h)^2} - \frac{2\, b^2\, \sin\gamma}{(b-1+h)^3 v} \right] P_\gamma$$

$$\frac{\partial^2 H}{\partial \nu^2} = \left[- 2\, A_1\, b\, C_{Do}\, (1+C^2)\, \delta \right] P_\nu + \left[- \frac{2\, b^2\, \cos\gamma}{(b-1+h)^2 \nu^3} \right] P_\gamma$$

$$\frac{\partial^2 H}{\partial \nu\, \partial \gamma} = [b\cos\gamma]\, P_h + \left[- \frac{b\, \nu\, \sin\gamma}{(b-1+h)} - \frac{b^2\, \sin\gamma}{(b-1+h)^2 \nu^2} \right] P_\gamma$$

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$$\frac{\partial^2 H}{\partial \gamma^2} = [-b\nu\sin\gamma] \ P_h \ + \left[\frac{b^2\sin\gamma}{(b-1+h)^2} \right] P_\nu$$

$$+ \left[- \frac{b \ \nu \ \cos\gamma}{(b-1+h)} - \frac{b^2\cos\gamma}{(b-1+h)^2\nu} \right] \ P_\gamma$$

$$\frac{\partial^2 H}{\partial c \partial h} = [2 \ A_1 b \ C_{Do} \ C \ \beta \ H_a \ \delta \ \nu^2] \ P_\nu \ + [- \ A_2 b \ \beta \ H_a \ \delta \ \nu] \ P_\gamma$$

$$\frac{\partial^2 H}{\partial c \partial \nu} = [-4A_1 b \ C_{Do} \ C \ \delta \ \nu] \ P_\nu \ + [A_2 b \ \delta] \ P_\gamma$$

$$\frac{\partial^2 H}{\partial c \partial \gamma} = 0$$

The perturbed feedback control is given by

$$\Delta u = -H_{uu}^{-1} \ (H_{ux} \ \Delta \ \underline{x} + f_u^{\ T}\Delta\underline{P}) \hspace{3cm} \text{VI.6}$$

where $\hspace{2cm} H_{uu} = - 2 \ A_1 b \ C_{Do} \ \delta \ \nu^2 \ P_\nu$

$$\Delta c = \left[c \ \beta \ H_a - \frac{A_2 b \ H_a \ P_\gamma}{2 \ A_1 \ C_{Do} \ \nu \ P_\gamma} \right] \ \Delta \ P_h$$

$$+ \left[- \frac{2c}{\nu} + \frac{A_2 \ P_\gamma}{2A_1 C_{Do}\nu^2 P_\nu} \right] \cdot \Delta P_\nu$$

$$+ \left[- \frac{c}{P_\nu} \right] \Delta P_\nu \ + \left[\frac{A_2}{2A_1 C_{Do}\nu \ P_\gamma} \right] \ \Delta P_\gamma \ .$$

The boundary conditions are:

$$\Delta \underline{\psi} = [\psi_x \, \Delta \, \underline{x}] \, \Big|_{\tau \, = \, \tau_f} \qquad\qquad \text{VI.7}$$

$$\Delta h(\tau_0) = \text{specified}$$

$$\Delta \nu(\tau_0) = \text{specified}$$

$$\Delta \gamma(\tau_0) = \text{specified}$$

$$\Delta \underline{P} \, (\tau_f) = [(\phi_{xx} + (\underline{n}^T \, \psi_x)_x + \psi_x^T \, N \, \psi_x) \, \Delta \underline{x} + \psi_x^T dn] \, \Big|_{\tau \, = \, \tau_f} \qquad \text{VI.8}$$

where

$$\psi_x = \begin{bmatrix} \dfrac{\partial \psi_1}{\partial h} & \dfrac{\partial \psi_1}{\partial \nu} & \dfrac{\partial \psi_1}{\partial \gamma} \\[2mm] \dfrac{\partial \psi_2}{\partial h} & \dfrac{\partial \psi_2}{\partial \nu} & \dfrac{\partial \psi_2}{\partial \gamma} \end{bmatrix}$$

$$\psi_x = \begin{bmatrix} 1 & 0 & 0 \\[2mm] 0 & -2\nu_f(a_c^2 - \cos^2 \gamma_f) & -\nu_f^2 \sin 2 \, \gamma_f \end{bmatrix}$$

$$\Delta \psi_1 = \Delta h_f$$

$$\Delta \psi_2 = [-2 \, \nu_f \, (a_c^2 - \cos^2 \gamma_f) \, \Delta \nu_f + [- \, \nu_f^2 \sin 2 \, \gamma_f] \cdot \Delta \gamma_f$$

$$\phi_{xx} = \begin{bmatrix} 0 & 0 & 0 \\[2mm] 0 & 0 & \sin\gamma_f / ac \\[2mm] 0 & \dfrac{\sin\gamma_f}{ac} & \nu_f \cos\gamma_f / ac \end{bmatrix}$$

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$$\psi_x^T \, N \, \psi_x = \begin{bmatrix} N_1 & 0 & 0 \\ 0 & 4N_2\nu_f^2(a_c^2-\cos^2\gamma_f)^2 & 2N_2\nu_f^3(a_c^2-\cos^2\gamma_f)\sin 2\gamma_f \\ 0 & 2N_2\nu_f^3(a_c^2-\cos^2\gamma_f)\sin 2\gamma_f & N_2\nu_f^3\sin^2\nu_f \end{bmatrix}$$

The equations govern the closed loop feedback control are as follows:

$$\Delta \underline{u} = - H_{uu}^{-1} \, f_u^T \, [S - RQ^{-1} R^T] \, \Delta \underline{x} \qquad\qquad VI.9$$

where the matrices S, R, Q correspond to three Riccati equations which must be solved backwards in time.

$$\dot{S} = - S \, f_x - f_x^T \, S + S \, f_u \, H_{uu}^{-1} \, f_u^T \, S - H_{xx}$$

$$\dot{R} = - (f_x^T - S \, f_u \, H_{uu}^{-1} \, f_u^T) \, R$$

$$\dot{Q} = - R^T \, f_u \, H_{uu}^{-1} \, f_u^T \, R$$

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \quad , \quad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \end{bmatrix} \quad , \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}$$

$$\dot{S}_{11} = -2 \left[S_{11} \cdot \frac{\partial f_h}{\partial h} + S_{12} \cdot \frac{\partial f_\nu}{\partial h} + S_{13} \frac{\partial f_\gamma}{\partial h} \right]$$

$$+ \left[S_{11} \frac{\partial f_h}{\partial c} + S_{12} \frac{\partial f_\nu}{\partial c} + S_{13} \frac{\partial f_\nu}{\partial c} \right]^2 \cdot H_{uu}^{-1} - \frac{\partial^2 H}{\partial h^2}$$

$$\dot{S}_{12} = -\left[S_{11} \frac{\partial f_h}{\partial \nu} + S_{12} \frac{\partial f_\nu}{\partial \nu} + S_{13} \frac{\partial f_\gamma}{\partial \nu} \right].$$

$$-\left[S_{12} \frac{\partial f_h}{\partial h} + S_{22} \frac{\partial f_\nu}{\partial h} + S_{23} \frac{\partial f_\gamma}{\partial h} \right]$$

$$+\left[S_{11} \frac{\partial f_h}{\partial c} + S_{12} \frac{\partial f_\nu}{\partial c} + S_{13} \frac{\partial f_\nu}{\partial c} \right] \left[S_{12} \frac{\partial f_h}{\partial c} + S_{22} \frac{\partial f_\nu}{\partial c} \right.$$

$$\left. + S_{23} \frac{\partial f_\gamma}{\partial c} \right] \cdot H_{uu}^{-1} - \frac{\partial^2 H}{\partial h \partial \nu}$$

$$\dot{S}_{13} = -\left[S_{11} \frac{\partial f_h}{\partial \gamma} + S_{12} \frac{\partial f_\nu}{\partial \gamma} + S_{13} \frac{\partial f_\gamma}{\partial \gamma} \right]$$

$$-\left[S_{13} \frac{\partial f_h}{\partial h} + S_{23} \frac{\partial f_\nu}{\partial h} + S_{33} \frac{\partial f_\gamma}{\partial h} \right]$$

$$+\left[S_{11} \frac{\partial f_h}{\partial c} + S_{12} \frac{\partial f_\nu}{\partial c} + S_{13} \frac{\partial f_\nu}{\partial c} \right] \left[S_{13} \frac{\partial f_h}{\partial c} + S_{23} \frac{\partial f_\nu}{\partial c} \right.$$

$$\left. + S_{33} \frac{\partial f_\gamma}{\partial c} \right] \cdot H_{uu}^{-1} - \frac{\partial^2 H}{\partial h \partial \gamma}$$

$$\dot{S}_{22} = -\left[S_{12} \frac{\partial f_h}{\partial \nu} + S_{22} \frac{\partial f_\nu}{\partial \nu} + S_{23} \frac{\partial f_\gamma}{\partial \nu} \right]$$

$$-\left[S_{12} \frac{\partial f_h}{\partial \nu} + S_{22} \frac{\partial f_\nu}{\partial \nu} + S_{23} \frac{\partial f_\gamma}{\partial \nu} \right]$$

$$+\left[S_{12} \frac{\partial f_h}{\partial c} + S_{22} \frac{\partial f_\nu}{\partial c} + S_{23} \frac{\partial f_\gamma}{\partial c} \right]^2 \cdot H_{uu}^{-1} \cdot \frac{\partial^2 H}{\partial \nu^2}$$

$$\dot{S}_{23} = -\left[S_{12} \frac{\partial f_h}{\partial \gamma} + S_{22} \frac{\partial f_\nu}{\partial \gamma} + S_{23} \frac{\partial f_\gamma}{\partial \gamma} \right]$$

$$-\left[S_{13} \frac{\partial f_h}{\partial \nu} + S_{23} \frac{\partial f_\nu}{\partial \nu} + S_{33} \frac{\partial f_\gamma}{\partial \nu} \right]$$

$$+\left[S_{12} \frac{\partial f_h}{\partial c} + S_{22} \frac{\partial f_\nu}{\partial c} + S_{23} \frac{\partial f_\gamma}{\partial c} \right] \cdot \left[S_{13} \frac{\partial f_h}{\partial c} + \right.$$

$$\left. S_{23} \frac{\partial f_\nu}{\partial c} + S_{33} \frac{\partial f_\gamma}{\partial c} \right] \cdot H_{uu}^{-1} \cdot \frac{\partial^2 H}{\partial \nu \partial \gamma}$$

$$\dot{S}_{33} = -\left[S_{13}\ \frac{\partial f_h}{\partial \gamma} + S_{23}\ \frac{\partial f_\nu}{\partial \gamma} + S_{33}\ \frac{\partial f_\gamma}{\partial \gamma}\right]$$

$$-\left[S_{13}\ \frac{\partial f_h}{\partial \gamma} + S_{23}\ \frac{\partial f_\nu}{\partial \gamma} + S_{33}\ \frac{\partial f_\gamma}{\partial \gamma}\right]$$

$$+\left[S_{13}\ \frac{\partial f_h}{\partial c} + S_{23}\ \frac{\partial f_\nu}{\partial c} + S_{33}\ \frac{\partial f_\gamma}{\partial c}\right]^2 \cdot H_{uu}^{-1} - \frac{\partial^2 H}{\partial \gamma^2}$$

$$S_{11}\,(t_f) = N_1$$

$$S_{12}\,(t_f) = 0.0$$

$$S_{13}\,(t_f) = 0.0$$

$$S_{22}\,(t_f) = 2\,n_2\,(\cos^2\gamma_f - a_c^2) + 4N_2\,\nu_f^2\,(a_c^2 - \cos\gamma_f)\,\sin 2\gamma_f$$

$$S_{23}\,(t_f) = \frac{\sin\gamma_f}{a_c} - 2\,n_2\,\nu_f\,\sin 2\gamma_f + 2N_2\,\nu_f^3\,(a_c^2 - \cos^2\gamma_f)\,\sin 2\gamma_f$$

$$S_{33} = \frac{\nu_f \cos\gamma_f}{a_c} - 2\,n_2\,\nu_f^2\,\cos 2\,\gamma_f + N_2\,\nu_f^3\,\sin^2\gamma_f$$

$$\dot{R} = - (f_x^T - S f_u H_{uu}^{-1} f_u^T) R$$

$$\dot{R}_{11} = - \left[\frac{\partial f_h}{\partial h} \cdot R_{11} + \frac{\partial f_\nu}{\partial h} \cdot R_{21} + \frac{\partial f_\gamma}{\partial h} \cdot R_{31} \right]$$

$$+ \left[\left[S_{12} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + S_{13} \frac{\partial f_\nu}{\partial c} \cdot \frac{\partial f_\gamma}{\partial c} \right] \cdot R_{21} \right.$$

$$\left. + \left[S_{12} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + S_{13} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{31} \right] H_{uu}^{-1}$$

$$\dot{R}_{12} = - \left[\frac{\partial f_h}{\partial h} \cdot R_{12} + \frac{\partial f_\nu}{\partial h} \cdot R_{22} + \frac{\partial f_\gamma}{\partial h} \cdot R_{32} \right]$$

$$+ \left[\left[S_{12} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + S_{13} \frac{\partial f_\nu}{\partial c} \cdot \frac{\partial f_\gamma}{\partial c} \right] \cdot R_{22} \right.$$

$$\left. + \left[S_{12} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + S_{13} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] \cdot R_{32} \right] H_{uu}^{-1}$$

$$\dot{R}_{21} = - \left[\frac{\partial f_h}{\partial \nu} R_{11} + \frac{\partial f_\nu}{\partial \nu} R_{21} + \frac{\partial f_\gamma}{\partial \nu} R_{31} \right]$$

$$+ \left[\left[S_{22} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + S_{23} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{21} \right.$$

$$+ \left. \left[S_{22} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + S_{23} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{31} \right] H_{uu}^{-1}$$

$$\dot{R}_{22} = - \left[\frac{\partial f_h}{\partial \nu} R_{12} + \frac{\partial f_\nu}{\partial \nu} R_{22} + \frac{\partial f_\gamma}{\partial \nu} R_{32} \right]$$

$$+ \left[\left[S_{22} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + S_{23} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{22} \right.$$

$$+ \left. \left[S_{22} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + S_{23} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{32} \right] H_{uu}^{-1}$$

$$\dot{R}_{31} = - \left[\frac{\partial f_h}{\partial \gamma} R_{11} + \frac{\partial f_\nu}{\partial \gamma} R_{21} + \frac{\partial f_\gamma}{\partial \gamma} R_{31} \right]$$

$$+ \left[\left[S_{23} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + S_{33} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{21} \right.$$

$$\left. + \left[S_{23} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + S_{33} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{31} \right] H_{uu}^{-1}$$

$$\dot{R}_{32} = - \left[\frac{\partial f_h}{\partial \gamma} R_{12} + \frac{\partial f_\nu}{\partial \gamma} R_{22} + \frac{\partial f_\gamma}{\partial \gamma} R_{32} \right]$$

$$+ \left[\left[S_{23} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + S_{23} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{22} \right.$$

$$\left. + \left[S_{23} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + S_{33} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{32} \right] H_{uu}^{-1}$$

$$R_{11} (\tau_f) = 1 \qquad\qquad R_{12} (\tau_f) = 0$$

$$R_{21} (\tau_f) = 0 \qquad\qquad R_{22} (\tau_f) = - 2 \nu_f (a_c^2 - \cos^2\gamma_f)$$

$$R_{31} (\tau_f) = 0 \qquad\qquad R_{32} (\tau_f) = - \nu_f^2 \sin 2\gamma_f$$

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$$\dot{Q} = R^T f_u H_{uu}^{-1} f_u^T R$$

$$\dot{Q}_{11} = \left[\left[R_{21} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + R_{31} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{21} \right.$$

$$+ \left. \left[R_{21} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + R_{31} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{31} \right] H_{uu}^{-1}$$

$$\dot{Q}_{12} = \left[\left[R_{21} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + R_{31} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{22} \right.$$

$$+ \left. \left[R_{21} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + R_{31} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{32} \right] H_{uu}^{-1}$$

$$\dot{Q}_{22} = \left[\left[R_{22} \left(\frac{\partial f_\nu}{\partial c} \right)^2 + R_{32} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} \right] R_{22} \right.$$

$$+ \left. \left[R_{22} \frac{\partial f_\nu}{\partial c} \frac{\partial f_\gamma}{\partial c} + R_{32} \left(\frac{\partial f_\gamma}{\partial c} \right)^2 \right] R_{32} \right] H_{uu}^{-1}$$

$$Q_{11}(t_f) = 0 \qquad\qquad Q_{12}(t_f) = 0 \qquad\qquad Q_{13}(t_f) = 0.$$

NEIGHBORING OPTIMAL GUIDANCE FOR AEROASSISTED NONCOPLANAR ORBITAL TRANSFER

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Abstract: The fuel-optimal control problem in aeroassisted noncoplanar orbital transfer is addressed. The equations of motion for the atmospheric maneuver are nonlinear and the optimal (nominal) trajectory and control are obtained. In order to follow the nominal trajectory under actual conditions, a neighboring optimum guidance scheme is designed using linear quadratic regulator (LQR) theory for onboard real-time implementation. One of the state variables is used as the independent variable in preference to the time. The weighting matrices in the performance index are chosen by a combination of a heuristic method and an optimal modal approach. The necessary feedback control law is obtained in order to minimize the deviations from the nominal conditions.

Nomenclature

$A_1 = C_{DO} S \rho_s H_a / 2m$

$A_2 = C_{LR} S \rho_s H_a / 2m$

$b = R_a / H_a$

C_D :   drag coefficient

C_{DO}:   zero-lift drag coefficient

C_L :   lift coefficient

C_{LR}:   lift coefficient for maximum lift-to-drag ratio

D :   drag force

g :   gravitational acceleration

H :   altitude

h :   normalized altitude

K :   induced drag factor

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L : lift force

m : vehicle mass

R : distance from Earth center to vehicle center of gravity

R_a : radius of atmospheric boundary

R_E : radius of Earth

S : aerodynamic reference area

t : time

V : velocity

v : normalized velocity

β : inverse atmospheric scale height

γ : flight path angle

ψ : heading angle

σ : bank angle

θ : down range angle

ϕ : cross range angle

δ : normalized density

μ : gravitational constant of Earth

η = normalized lift coefficient

ρ : density

ρ_s : density at the sea level

τ : normalized time

I. INTRODUCTION

In space transportation system, the concept of aeroassisted orbital transfer opens new mission opportunities, especially with regard to the initiation of a permanent space station [1]. The use of aeroassisted maneuvers to affect a transfer from high Earth orbit (HEO) to low Earth orbit (LEO) has been recommended to provide high performance leverage to future space transportation systems. The aeroassisted orbital transfer vehicle (AOTV), on its return journey from HEO, dissipates orbital energy through atmospheric drag to slow down to LEO velocity. Thus, the basic idea is to employ a hybrid combination of propulsive maneuvers in space and aerodynamic maneuvers in sensible atmosphere. Within the atmosphere, the trajectory control is achieved by means of lift and bank angle modulations. Hence, this type of flight with a combination of propulsive and nonpropulsive maneuvers, is also called synergetic maneuver or space flight.

Guidance is the determination of a strategy for following a nominal flight in the presence of off-nominal conditions, wind disturbances, and navigation uncertainties [2-5]. There are two fundamentally different approaches for guidance of spacecraft through the atmosphere.

(i) **Predicted Guidance:** Here, we predict future trajectories at the required point of time by either fast computation or use of approximate closed form solutions. In prediction using fast computation, the governing equations are solved by an air-borne computer to determine all possible future trajectories. The main advantage of this type of prediction is the ability to handle any possible flight condition and accommodate large off-nominal

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conditions. The principal objection to the system is the requirement of large onboard computer. On the other hand, for the onboard prediction of trajectories, the closed form solutions are very simple and convenient for implementation. In most of the cases, the closed form solutions are obtained based on approximate techniques. Thus, the trajectories are obtained typically for constant altitude, constant deceleration, and equilibrium glide paths. Guidance using approximate closed-form solutions has the disadvantage of not having the flexibility to handle enough off-nominal conditions.

(ii) **Nominal Guidance:** In this scheme, the nominal trajectories are precomputed on ground taking into all the aspects of constraints with optimization and stored onboard. During the flight, the difference between the measured values and nominal (stored) values is used in guiding the vehicle onto the nominal trajectory (path controller) or generate a new trajectory to reach the destination (terminal controller). Here, the nominal trajectory being fixed on the ground, does not take into account any contingencies arising during the flight.

In a typical guidance scheme, the final steering command is generated as the sum of two components, an open-loop actuating (control) signal yielding the desired vehicle trajectory in the absence of external disturbances, and a linear feedback regulating signal which reduces the system sensitivity to unwanted influences on the vehicle. It is well known that the closed-loop system is stable about the nominal trajectory and has additional desirable features.

In this report, we address the fuel-optimal control problem arising in noncoplanar orbital transfer employing aeroassist technology. The maneuver

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involves the transfer from HEO to LEO with a prescribed plane change and at the same time minimization of the fuel consumption. It is known that the change in velocity, also called the characteristic velocity, is a convenient parameter to measure the fuel consumption. For minimum-fuel maneuver, the objective is then to minimize the total characteristic velocity for deorbit, boost, and reorbit (or circularization). The corresponding optimal (nominal) trajectory and control are obtained. The linearization is performed around the nominal condition and the resulting model is fitted into the framework of linear quadratic regulator (LQR) theory. Instead of using time, one of the state variables is employed as an independent variable, thus avoiding any control required due to irrelevant timing errors. Also, the elimination of time as an independent variable carries with it the advantage of order reduction for the system. The choice of weighting matrices in the performance index is made by combining a heuristic method and optimal modal control approach. The feedback control law is obtained to suppress the perturbations from the nominal condition. The results are shown for a typical AOTV.

II. FORMULATION OF THE PROBLEM

The basic equations for orbital transfer from HEO to LEO, are those for deorbit, aeroassist (or atmospheric flight), boost and circularization (or reorbit). But, for guidance and control purposes, we consider only the atmospheric flight during which phase the vehicle needs to be controlled by aerodynamic lift and bank angle to achieve the necessary velocity reduction and the plane change.

Consider a vehicle moving about a nonrotating spherical planet. The atmosphere surrounding the planet is assumed to be at rest, and the central gravitational field obeys the usual inverse square law. The equations of motion for the vehicle are given for kinematics as [6],

$$\frac{d\theta}{dt} = V\cos\gamma\cos\psi/R\cos\phi \tag{1a}$$

$$\frac{d\phi}{dt} = V\cos\gamma\sin\psi/R \tag{1b}$$

$$\frac{dR}{dt} = V\sin\gamma \tag{1c}$$

The force equations are

$$m\frac{dV}{dt} = -D - mg\sin\gamma \tag{1d}$$

$$mV\frac{d\gamma}{dt} = L\cos\sigma + m(V^2/R - g)\cos\gamma \tag{1e}$$

$$mV\frac{d\psi}{dt} = L\sin\sigma/\cos\gamma - (mV^2/R)\cos\gamma\cos\psi\tan\phi \tag{1f}$$

where,

$$L = C_L \rho S V^2/2; \quad D = C_D \rho S V^2/2; \quad C_D = C_{DO} + K C_L^2$$

$$g = \mu/R^2; \quad R = H + R_E; \quad \rho = \rho_s \exp(-H\beta)$$

Using the normalized variables,

$$\tau = t\Big/\sqrt{R_a^3/\mu}; \quad v = V/\sqrt{\mu/R_a} \tag{2}$$

and the dimensionless constants,

$$h = H/H_a; \quad b = R_a/H_a; \quad \delta = \rho/\rho_s = \exp(-h\beta H_a) \tag{3a}$$

$$\eta = C_L/C_{LR}; \quad C_{LR} = \sqrt{C_{DO}/K} \tag{3b}$$

in (1), we get the normalized form as

$$\frac{d\theta}{d\tau} = \frac{bv\cos\gamma\cos\psi}{(b-1+h)\cos\phi} \tag{4a}$$

$$\frac{d\phi}{d\tau} = \frac{bv\cos\gamma\sin\psi}{(b-1+h)} \tag{4b}$$

$$\frac{dh}{d\tau} = bv\sin\gamma \tag{4c}$$

$$\frac{dv}{d\tau} = -A_1 b(1+\eta^2)\delta v^2 - \frac{b^2\sin\gamma}{(b-1+h)^2} \tag{4d}$$

$$\frac{d\gamma}{d\tau} = A_2 b\eta\delta v\cos\sigma + \frac{bv\cos\gamma}{(b-1+h)} - \frac{b^2\cos\gamma}{(b-1+h)^2 v} \tag{4e}$$

$$\frac{d\psi}{d\tau} = \frac{A_2 b\delta\eta v\sin\sigma}{\cos\gamma} - \frac{bv\cos\gamma\cos\psi\tan\phi}{(b-1+h)} \tag{4f}$$

where, $A_1 = C_{DO} S \rho_s H_a/2m; \quad A_2 = C_{LR} S \rho_s H_a/2m$

Taking the performance index as the sum of the characteristic velocities for deorbit, boost, and recircularization, the above fuel-optimal problem is solved by using a multiple shooting method. This is the nominal (optimal) solution to be used in obtaining the guidance scheme [7].

However, under the actual flight conditions, the initial conditions may not agree with those assumed for generating the nominal trajectory and more important the vehicle may not follow the nominal trajectory due to various disturbances. Hence, rather than compute a new nominal trajectory each time some disturbance is encountered, it is proposed to implement a closed-loop control of the vehicle to compensate for the deviations from the nominal condition.

(a) Selection of Independent Variable

In a typical guidance problem, the vehicle is required to follow a specified (nominal) trajectory in three-dimensional space. Suppose, the actual vehicle at a particular time is on the nominal trajectory, but much earlier than required or reached the desired point at incorrect time. The exact time at which the vehicle reaches the various points on the trajectory may be of no concern. Thus, if the actual trajectory satisfies the spatial relationship of the desired (nominal) trajectory, but the vehicle travelled more slowly or more quickly, the regulator (designed with time as independent variable) would sense these timing errors as spatial (state) errors along the trajectory, although, in fact, these are errors of no relevance to the objective of the guidance [2-4]. Based on these spurious errors, the regulator tends to "over control" the system and thus wastes considerable amounts of control effort. Because time is used as the independent variable in the regulator design, not

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only does the regulator try to follow the desired path, but it tries to force the system to follow at a particular rate.

The LQ regulator can be made significantly more robust by the simple technique of using a trajectory (state) variable instead of time as the independent variable. Also, if one is concerned only with the relationship that should exist between the system trajectory variables on the desired path, one of the state variables could be used in place of time as the independent variable. The main appeal for this change of the independent variable lies in the fact that inherently state dependent events become time-dependent events in an indirect manner. Another obvious advantage of eliminating time as the independent variable is simply the reduction in the order of the mathematical model of the system.

In the present plane change problem, the down range θ is monotonic and is a good candidate for an independent variable. Thus, the system of equations (4) with time as independent variable is converted into the following system of equations with down range θ as the independent variable.

$$\frac{dh}{d\theta} = (b-1+h)\tan\gamma\cos\phi/\cos\psi \tag{5a}$$

$$\frac{dv}{d\theta} = -A_1(1+\eta^2)\delta v(b-1+h)\cos\phi/\cos\gamma\cos\psi \tag{5b}$$

$$\frac{d\gamma}{d\theta} = A_2\delta\eta(b-1+h)\cos\sigma\cos\phi/\cos\gamma\cos\psi \tag{5c}$$

$$\frac{d\phi}{d\theta} = \cos\phi\tan\psi \tag{5d}$$

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$$\frac{d\psi}{d\theta} = A_2 \delta\eta(b-1+h)\sin\sigma\cos\phi/(\cos\gamma)^2\cos\psi \qquad (5e)$$

where, δ is a function of h, and the controls are normalized lift coefficient η and bank angle σ. In the above equations, a valid assumption that the gravity and centrifugal forces are negligible compared to either lift force or drag force is used in order to get a simplified form of equations. Also, note that the order of the system (5) is now only five compared to the sixth order system (4).

(b) Linearization

The nonlinear equations of motion (5) are represented in a compact form as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad (6)$$

where, $\dot{\mathbf{x}} = d\mathbf{x}/d\theta$, the state vector $\mathbf{x}^T = [h, v, \gamma, \phi, \psi]$ and the control vector $\mathbf{u}^T = [\eta, \sigma]$. Let the system (4) represent the nominal (optimum) trajectory and control which we are interested in flying. The optimization results in an open-loop implementation. Unfortunately, the open-loop guidance tends to be very sensitive to external disturbances and the vehicle parameter changes. In an actual situation, the system equations may differ from reality due to various assumptions made in deriving them. Consider the perturbations as

$$\mathbf{x} = \mathbf{x}_o + \delta\mathbf{x}; \quad \mathbf{u} = \mathbf{u}_o + \delta\mathbf{u} \qquad (7)$$

where, $[\mathbf{x}_o, \mathbf{u}_o]$, and $[\mathbf{x}, \mathbf{u}]$ represent nominal and perturbed conditions respectively, and $[\delta\mathbf{x}, \delta\mathbf{u}]$ represent the deviations from the nominal condition.

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If we aim at the guidance laws that meet the performance specifications in the presence of off-nominal conditions, it is natural to seek the principles of linearity and feedback. Then, it is well known that the perturbed system satisfies the linear system [8]

$$\delta \dot{x} = A\delta x + B\delta u \tag{8}$$

where, $A = \partial f/\partial x$, and $B = \partial f/\partial u$, are evaluated along the nominal trajectory. Although the matrices A and B can be evaluated for all values of interest, it may sometimes be necessary to keep A and B constant between the intervals of updating. The errors due to this quantization procedure are assumed to be small enough so that no drastically different changes occur in the performance. Keeping A and B constant also enables us to use the time-invariant version of the Riccati equation in the regulator problem.

III. NEIGHBORING OPTIMAL GUIDANCE

The basic regulator synthesis problem is that of selecting a control policy which generates δu in such a way that $[\delta x, \delta u]$ are made as small as possible [8]. The performance index is

$$J = \frac{1}{2} \int_0^\infty [\delta x^T Q \delta x + \delta u^T R \delta u] d\theta \qquad (9)$$

where, the first and second terms under the integral represent the penalty from the nominal state and control vectors respectively. The matrix R must be positive definite, while Q must be positive semidefinite. These weighting matrices are usually symmetric and diagonal to minimize the number of parameters to be chosen.

It is shown that the omission of second order terms in getting the linearized equation (8) is justified by selecting the performance index (9) with quadratic character. Using the minimization procedure, the feedback law is obtained as

$$\delta u = - F \delta x \qquad (10)$$

Where $F = R^{-1} B^T P$ and P is the positive definite, symmetric solution of the matrix Riccati algebraic equation

$$PA + A^T P - PBR^{-1} B^T P + Q = 0 \qquad (11)$$

The linear optimal control theory results in a feedback law which computes corrections to the nominal commands as shown in Fig. 1.

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Desirable Features of LQR Guidance

(i) The closed-loop optimal control is linear.

(ii) The control is easy for implementation or mechanization.

(iii) The closed-loop system using the optimal control is asymptotically stable.

(iv) The closed-loop control is robust in the sense that, if the system contains additional white-noise terms which model turbulence, atmospheric inhomogeneities, and unmodelled high-frequency vehicle dynamics, the closed-loop control will still give the control policy which minimizes the expected value of the performance index.

(v) The closed-loop system possesses attractive sensitivity properties with respect to variations in the elements of A and B.

(vi) The guidance scheme is off-line, in the sense that the feedback coefficient matrix F (= $R^{-1}B^{T}P$) is calculated in advance and stored for implementation onboard.

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IV. SELECTION OF WEIGHTING MATRICES

Guidance requirements during the atmospheric maneuver demand good nominal path following, and small errors under a variety of perturbations and off-nominal conditions. The designer must be able to come up with a set of weighting matrices Q and R that satisfy these specifications. In general, there is no simple and systematic way of selecting these matrices.

As Q is increased, the penalty for deviating from the nominal trajectory is increased if R is unchanged. The resulting trajectory tends to be closer to the nominal trajectory, but at the expense of greater control requirements. When R is large, which means the cost of control is important, the states decrease slowly. As R decreases, the control becomes quite large.

The proper selection of weighting matrices calls for physical and engineering intuition and simulation experience with different trial runs. However, there are some techniques available for chosing the weighting matrices [9]. These are

(i) Heuristic Methods

(ii) Optimal Modal Control Methods

(iii) Asymptotic Optimal Root Loci Methods

(iv) Dynamic Weighting Methods.

(i) Heuristic Methods

One of the heuristic methods is the 'inverse-square method' based on rule-of-thumb [10]. This is still a widely used method for the selection of quadratic cost matrices. The basic idea is the normalization with respect to their expected maximum values. Also, in most practical applications, the

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matrices Q and R are selected to be diagonal so that specific components of the state and control perturbation vectors can be penalized individually. Thus,

$$
Q = \begin{bmatrix} \dfrac{1}{\delta h_{M}^{2}} & 0 & 0 & 0 & 0 \\[2ex] 0 & \dfrac{1}{\delta v_{M}^{2}} & 0 & 0 & 0 \\[2ex] 0 & 0 & \dfrac{1}{\delta \gamma_{M}^{2}} & 0 & 0 \\[2ex] 0 & 0 & 0 & \dfrac{1}{\delta \phi_{M}^{2}} & 0 \\[2ex] 0 & 0 & 0 & 0 & \dfrac{1}{\delta \psi_{M}^{2}} \end{bmatrix} \quad ; \quad R = \begin{bmatrix} \dfrac{1}{\delta \eta_{M}^{2}} & 0 \\[2ex] 0 & \dfrac{1}{\delta \sigma_{M}^{2}} \end{bmatrix} \qquad (12)
$$

(ii) Optimal Modal Control Methods

The modal control approach is to determine a state feedback law u = -Fx, so that the closed-loop system matrix $D(= A-BF)$ has a prescribed set of eigenvalues [11]. Optimal modal control is based on the conventional pole-placement technique, but, instead of choosing the feedback matrix $F(= R^{-1}B^{T}P)$ directly, the weighting matrices of the performance index in linear quadratic regulator problem, are chosen to achieve the objective of placing the eigenvalues at the desired locations. The problem is to find the state weighting matrix Q for a given control weighting matrix R that corresponds to a set of prescribed eigenvalues. The method is based on transforming the given linear system into a diagonal (decoupled) system. The approach uses explicit formulas relating the elements of the state weighting matrix, the actual and

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the desired eigenvalues.

In the modal control approach, the choice of feedback matrix F is not unique, so that there may be a number of weighting matrices giving the same closed-loop eigenvalues. However, the method is easy for computer implementation. The details of the method are omitted and only the algorithm is given below for the case of real eigenvalues [11].

Algorithm

Step 1: Initialization (i=0)

Given the system matrices A, and B and the weighting matrices Q, and R, we obtain the optimal feedback matrix F_1 using LQR theory.

Step 2: Loop

Obtain the system matrix $D_i = A - BF_i$

Step 3: Eigenvalues/eigenvectors

(i) Compute modal (eigenvector) matrix $M_i = [v_1, \ldots, v_n]$

(ii) Compute diagonal matrix $\Lambda_i = M_i^{-1} D_i M_i^T$

(iii) Compute matrix $H_i = M_i^{-1} BR^{-1}B^T M_i^{-T}$

Step 4: Update i = i+1

Step 5: Shifting of eigenvalues

The actual eigenvalue λ_{aj} is to be shifted to the desired position λ_{dj}. Compute,

$$\left[\tilde{q}_{jj} \right]_i = \frac{\lambda_{dj}^2 - \lambda_{aj}^2}{\left(h_{jj} \right)_{i-1}} \tag{13}$$

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Step 6:

$$\text{Form} \quad \tilde{Q}_1 = \text{diag}\left\{0,0,\ldots,\left[\tilde{q}_{jj}\right]_1,0,\ldots,0\right\} \tag{14}$$

Step 7: Matrix Riccati equation

Using $\tilde{Q}_1$ from step 6, solve for $\tilde{P}_1$ from the matrix Riccati algebraic equation,

$$\tilde{P}_1\Lambda_{1-1} + \Lambda_{1-1}\tilde{P}_1 - \tilde{P}_1 BR^{-1}B^T\tilde{P}_1 + \tilde{Q}_1 = 0 \tag{14}$$

Step 8: Feedback matrix

(i) Obtain the feedback matrix $\tilde{F}_1 = R^{-1}B^T\tilde{P}_1$

(ii) Compute $\bar{F}_1 = \tilde{F}_1 M_{1-1}^{-1}$

(iii) Update $F_1 = F_{1-1} + \bar{F}_1$

Step 9: Weighting matrix

(i) Compute $\quad \bar{Q}_1 = M_{1-1}^{-T}Q_1 M_{1-1}^{-1}$

(ii) Update $Q_1 = Q_{1-1} + \bar{Q}_1$

Step 10: If the required number of eigenvalues are not shifted, then change j and go to step 2.

In the present work, a combination of the inverse square method and modal control method is adapted in arriving at the required weighting matrix for state deviations.

V. RESULTS

The nominal trajectory corresponds to the fuel-optimal condition of orbital transfer with plane change [7]. At a particular point of time, the set of nominal values used is

altitude, $H = 44724.24$ m

velocity, $V = 9278.73$ m/sec

flight path angle, $\gamma = -1.092$ deg

latitude, $\phi = 0.1954$ deg

heading angle, $\psi = 9.424$ deg

lift coefficient, $C_L = 0.3231$

bank angle, $\sigma = 75.28$ deg

With this set of nominal values, the linearized model is obtained and used for LQ regulator. For a typical set of perturbations, the solutions are obtained and presented in Figs. 2 and 3. Figs. 2(a) through 2(e) show the state perturbations, while Figs. 3(a) and 3(b) present perturbed controls. The guidance scheme is tested at different points on the nominal trajectory and the is found to be good.

VI. CONCLUDING REMARKS

A neighboring optimal guidance scheme based on linear quadratic regulator has been designed for controlling an aeroassisted orbital transfer vehicle performing a plane change. The down range angle, instead of time, has been used as an independent variable. The problem has been formulated as a linear quadratic regulator with perturbed states consisting of altitude, velocity, flight path angle, latitude, and heading angle, and perturbed controls

consisting of lift coefficient, and bank angle. The weighting matrices in the performance index have been chosen using a combination of heuristic approach and optimal modal control method. The results have been presented for fuel-optimal condition of aeroassisted orbital transfer.

Acknowledgements

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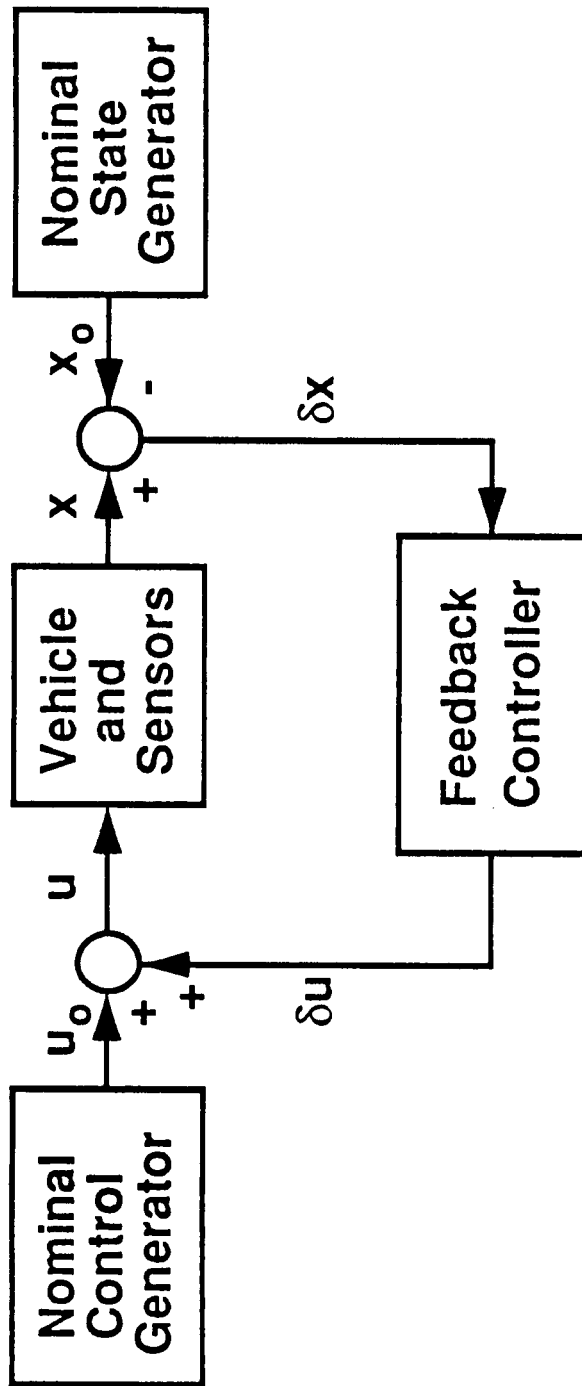
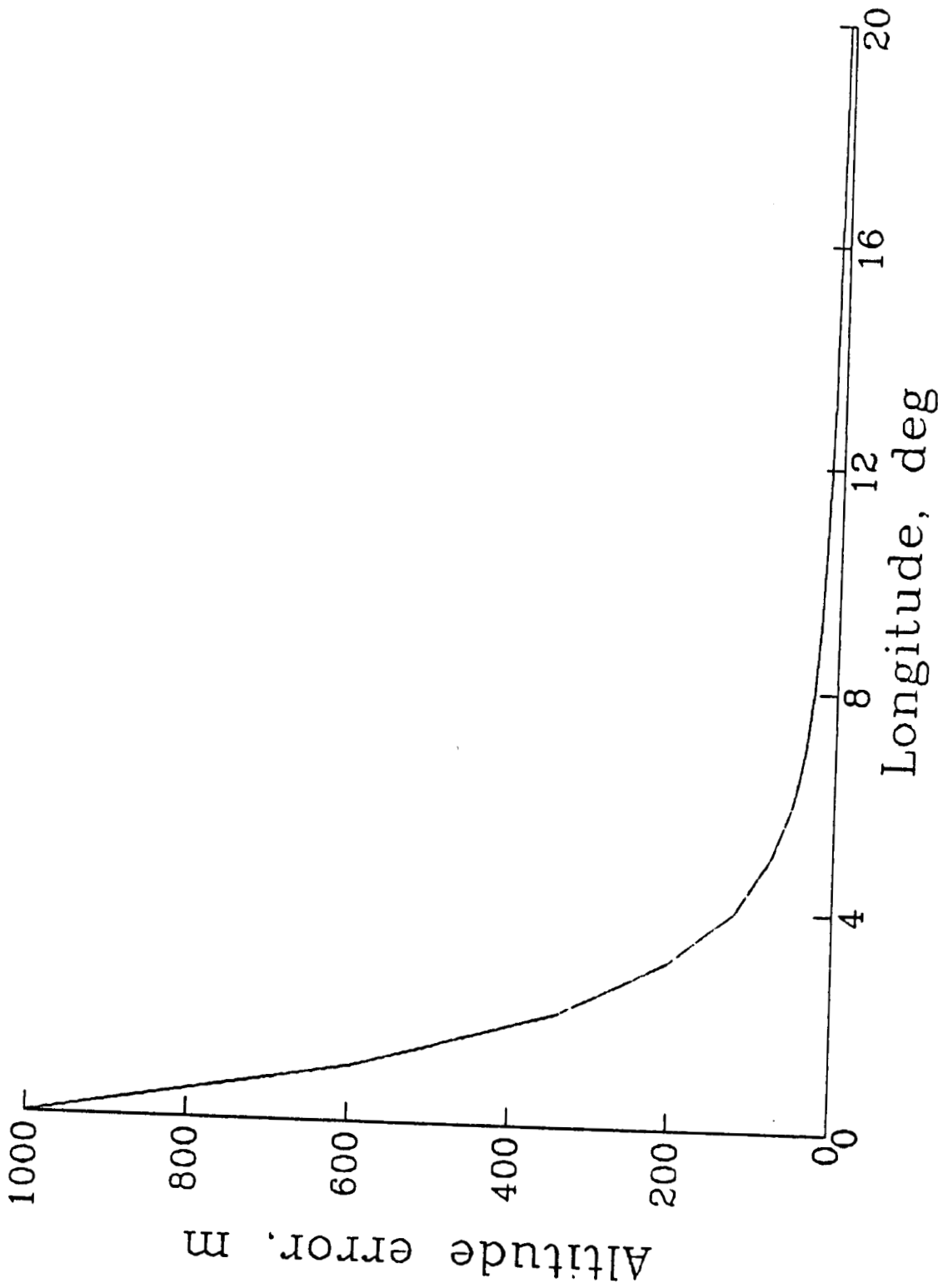**********************

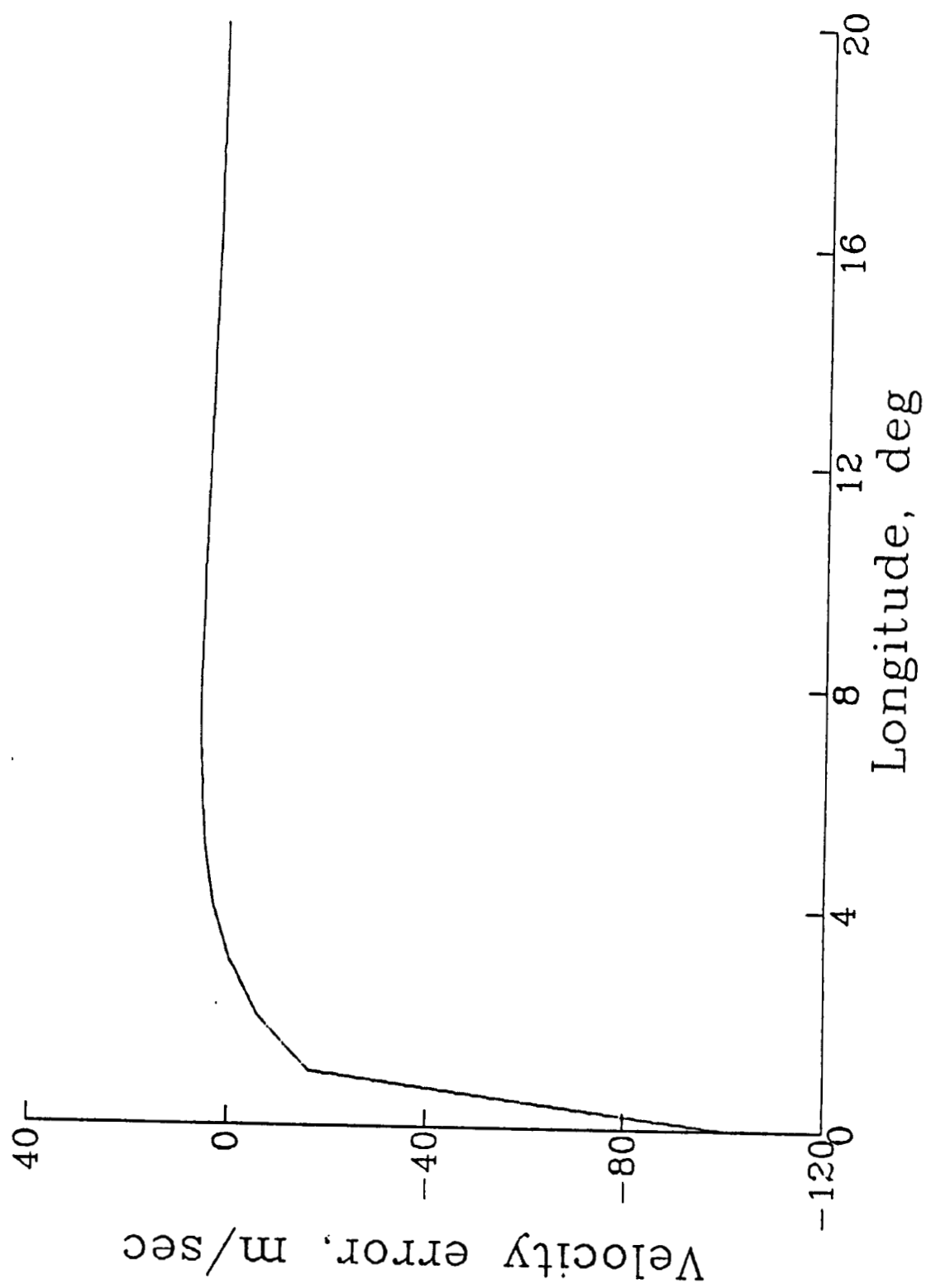FIG. 1 Neighboring Optimal Guidance

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Fig.2(a) History of altitude error

Fig.2(b) History of velocity error
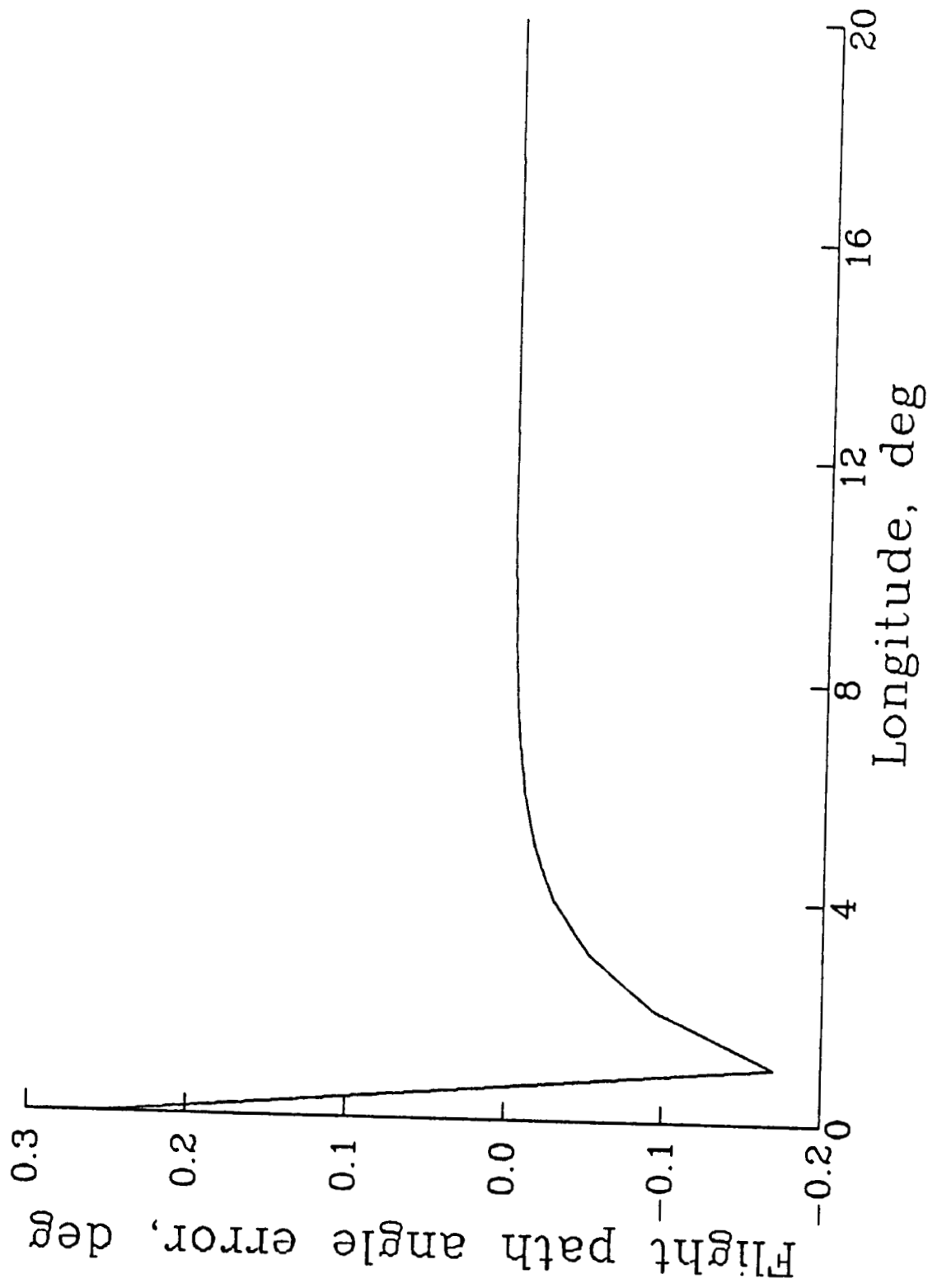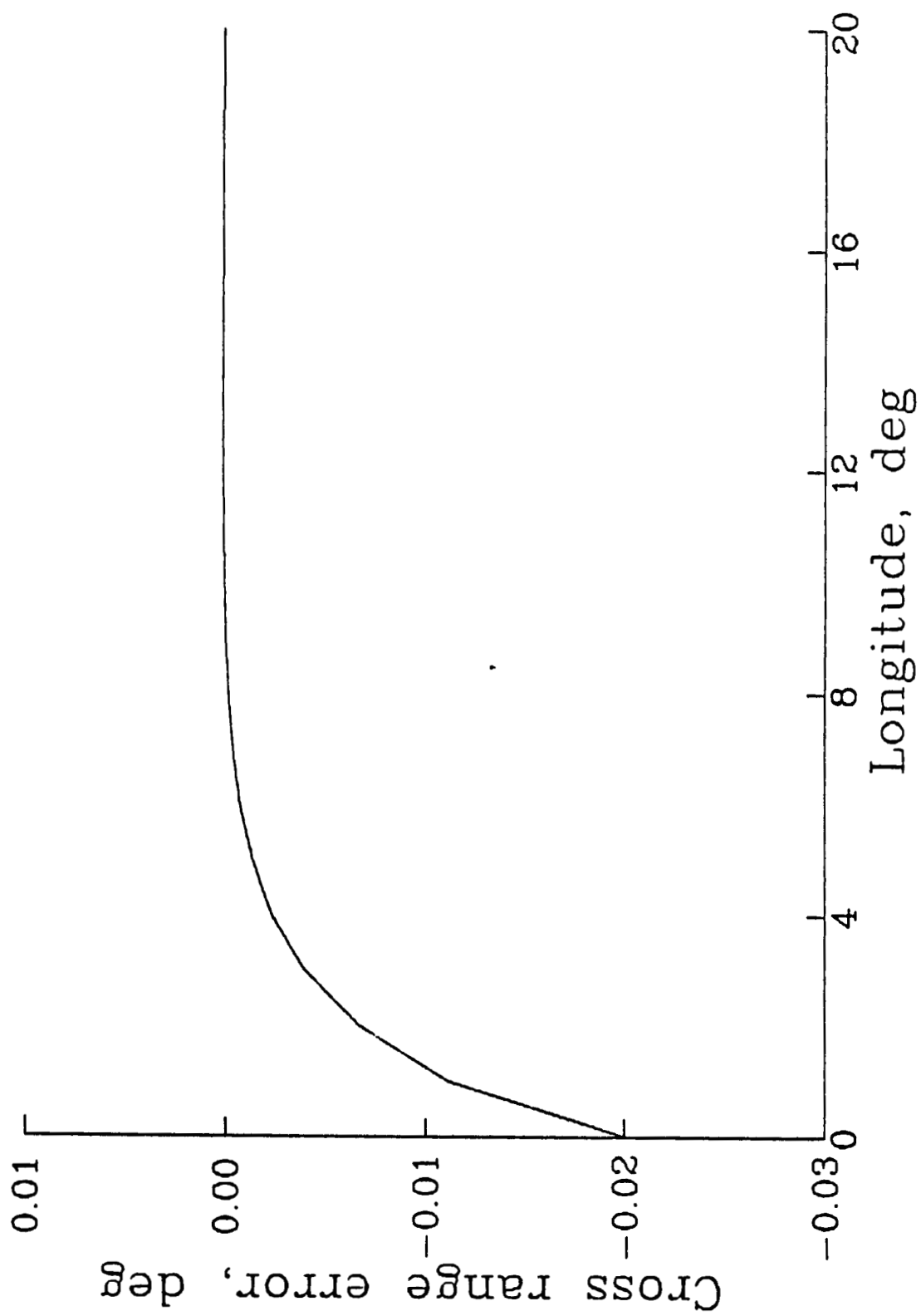
Fig.2(c) History of fligth path angle error

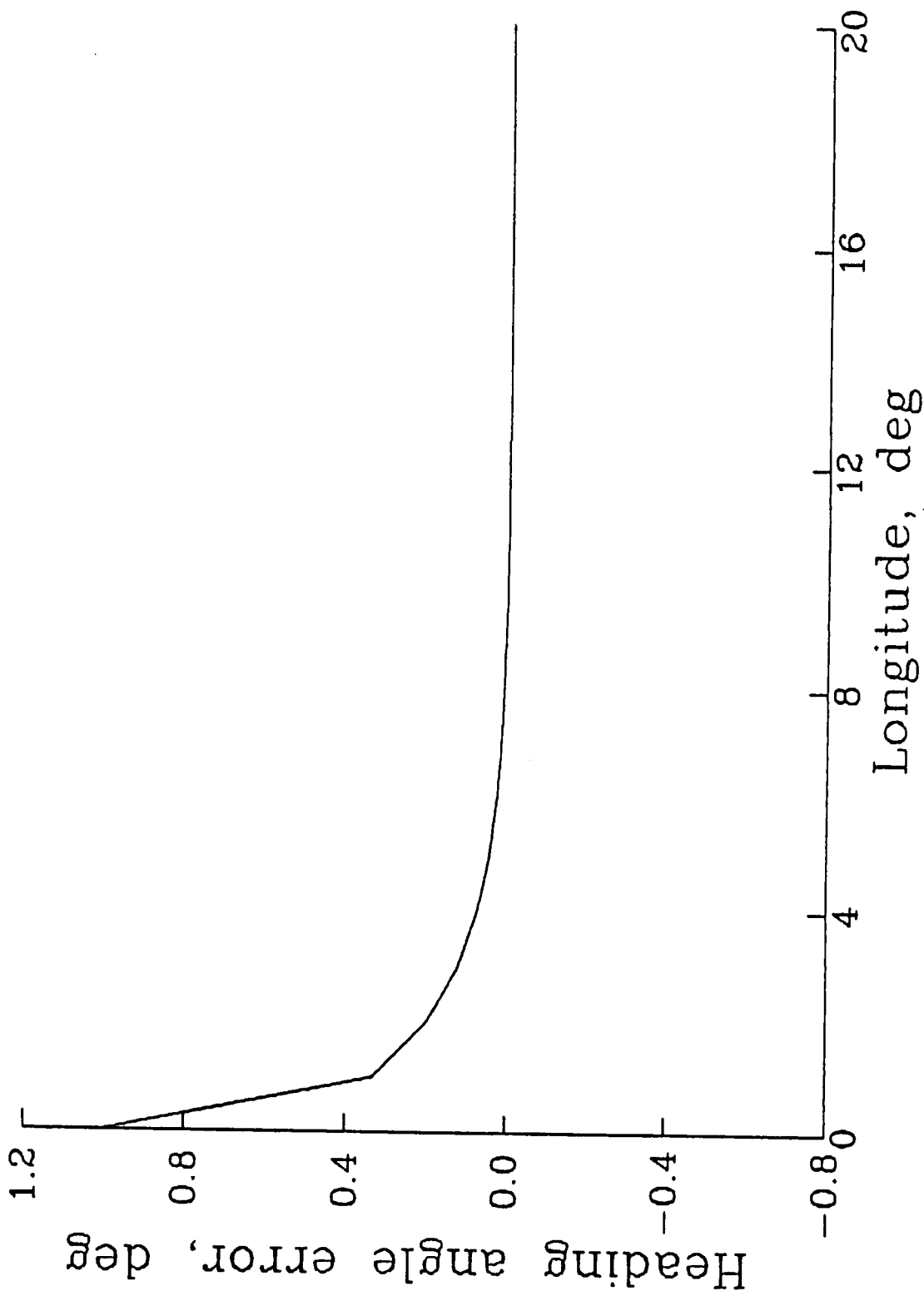Fig.2(d) History of cross range error

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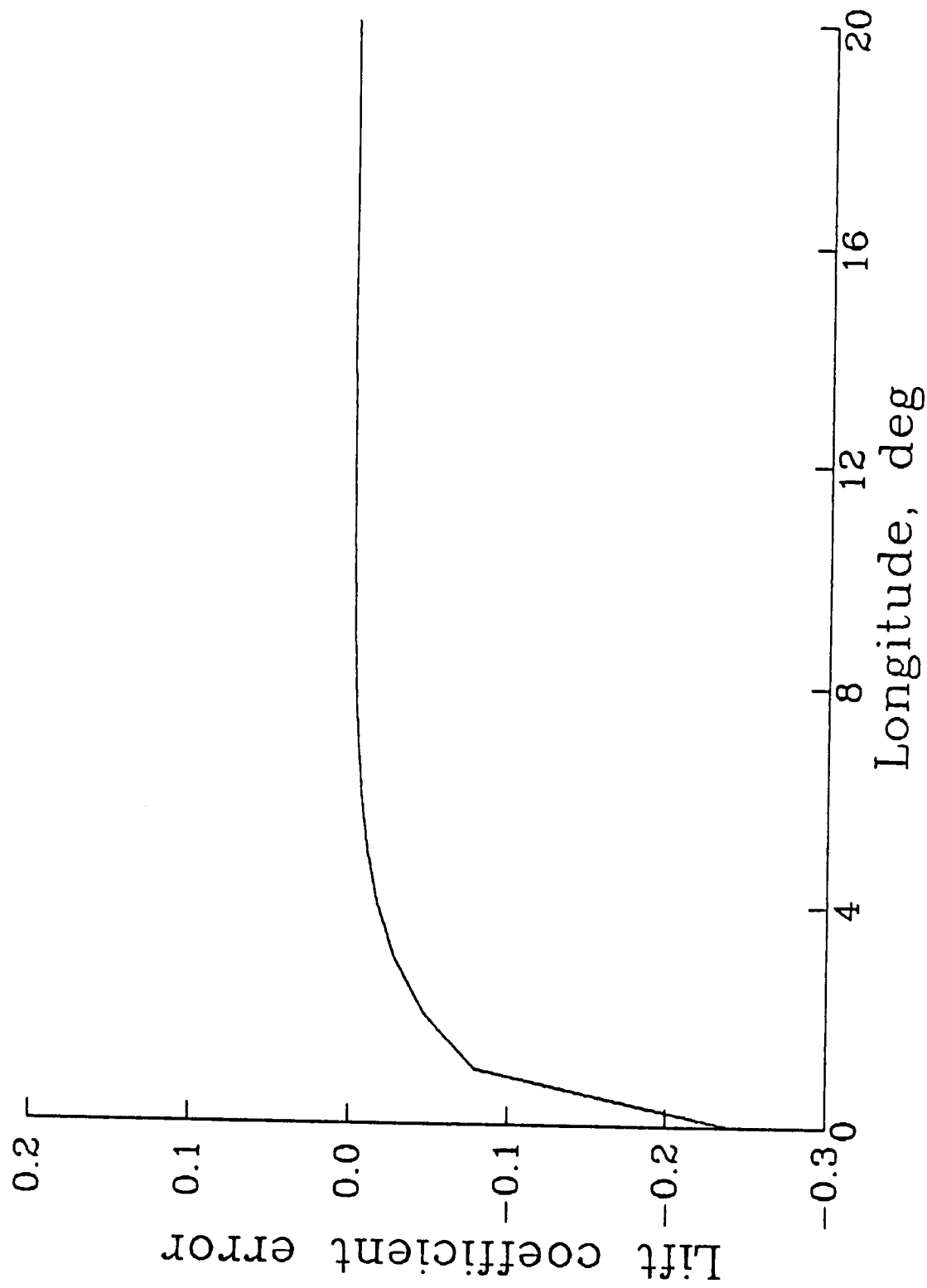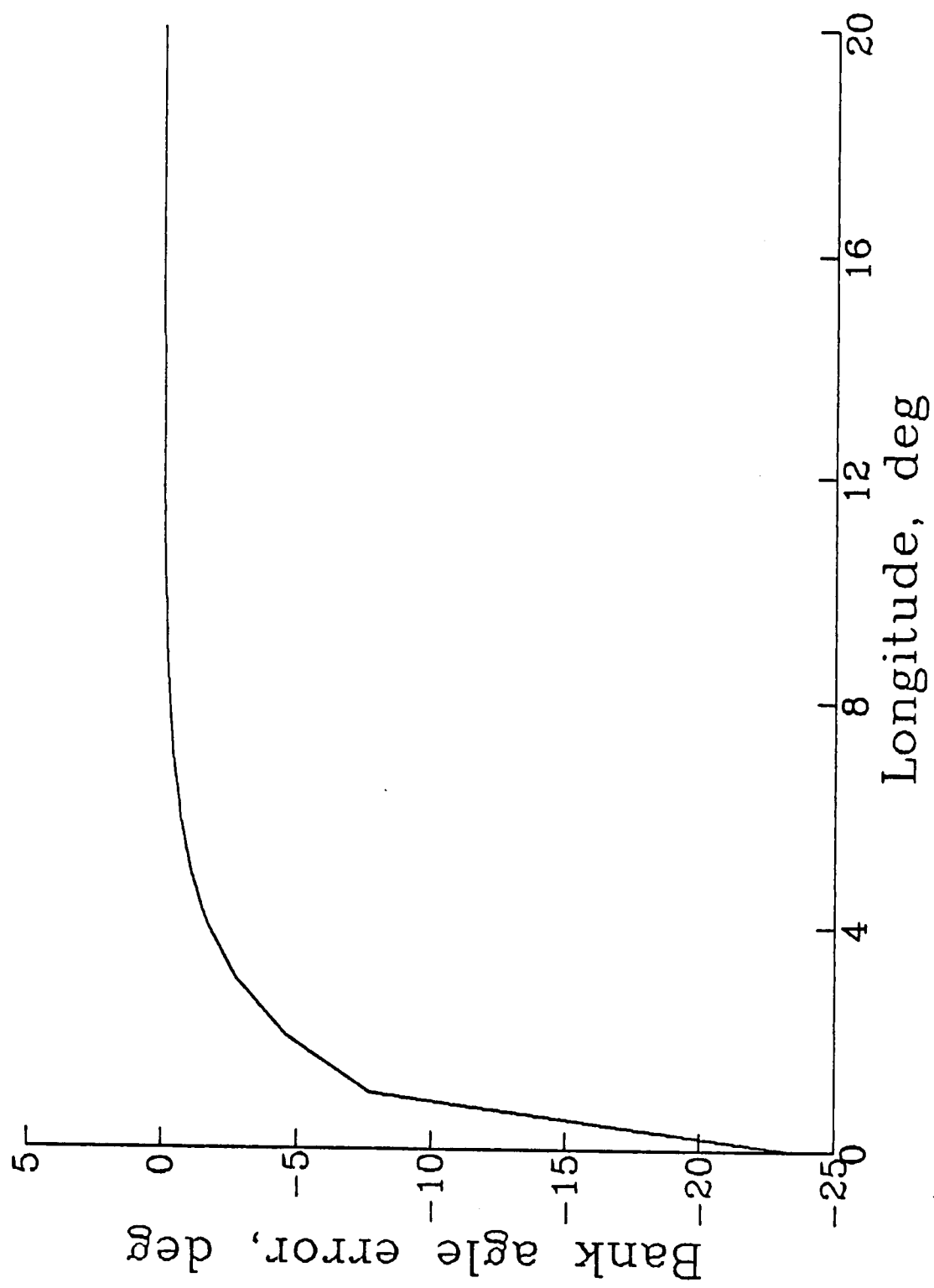
Fig.2(e) History of heading angle error

C-2

Fig.3(a) History of lift coefficient error

Fig.3(b) History of bank angle error