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Group-Kinetic Theory and Modeling of Atmospheric Turbulence

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FINAL REPORT FOR CONTRACT NAS8-36153
"GROUP-KINETIC THEORY AND MODELING OF ATMOSPHERIC TURBULENCE"

Attached is the Final Report prepared by the Research Specialist, Dr. C. M. Tchen, who performed the research sponsored under Contract NAS8-36153. In this Report, Dr. Tchen documents the results of his three-year effort, as called for in Attachment J-2 of the Contract.

Also attached are two additional documents which describe Tchen's analysis method in detail and apply it to Rossby Wave Turbulence.

Respectfully submitted,

M. H. Davis, Ph.D.

Program Director for USRA

August 8, 1989

FINAL REPORT

GROUP-KINETIC THEORY AND MODELING OF ATMOSPHERIC
TURBULENCE

By C. M. TCHEN

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GROUP-KINETIC THEORY AND MODELING OF TURBULENCE

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1. Research objectives and method of treatment

A group-kinetic method is developed for analysing eddy transport properties and relaxation to equilibrium. The purpose is to derive the spectral structure of turbulence in incompressible and compressible media. Of particular interest are: direct and inverse cascade, boundary layer turbulence, Rossby wave turbulence, two-phase turbulence; compressible turbulence, and soliton turbulence. Soliton turbulence can be found in large-scale turbulence, turbulence connected with surface gravity waves and nonlinear propagation of acoustical and optical waves.

Since the group-kinetic method is basic to our statistical theory of turbulence, two new Technical Reports are enclosed here:

#1. Group-kinetic method of turbulence

#2. Group-kinetic theory of Rossby wave turbulence.

2. Group-kinetic theory of turbulence

The most appealing method in the last three decades for treating turbulence statistically has been the "renormalization perturbation expansion" and the "kinetic method of turbulence". The former has enabled Kraichnan (1959, 1977) to develop his DIA method and its generalization. A systematic method of developing the expansion in a general form was given by Martin, Siggia and Rose (1972). By the difficulty of convergent summation and by the indefinite complings

1

among modes without explicit representation of physical processes, the perturbation method has not succeeded in analytically deriving the eddy coefficients (eddy viscosity and eddy amplification) that can discriminate between the direct and inverse cascades. These difficulties become more evident when turbulence is characterized by more than one parameter such as shear turbulence, convective turbulence and Rossby wave turbulence.

In fact, the molecular viscosity as a transport property is known to be of kinetic origin such that it has to be calculated from the Boltzmann equation (Chapman and Cowling, 1970). If by analogy, the eddy viscosity has to be derived from the "collision operator", the kinetic equation of turbulence has first to be formulated. Any attempt that leads to many-point distributions (Bogoliubov, 1962; Struminskii, 1985) has not shown progress.

The determination of the spectral solution of the Navier-Stokes equation by an analytic statistical method is important not only for reproducing the Kolmogoroff spectrum and evaluating the Kolmogoroff constant, but also for understanding the explicit transport processes and transport coefficients. This examination will help in assessing the applicability of the statistical method to other problems of turbulence.

For the description of the microdynamic state of turbulence, we let the pressure fluctuations govern the elementary interaction between fluid elements, by writing the Navier-Stokes equation in the form

$$(\partial_t + \hat{u} \cdot \nabla - \nu \nabla^2)\hat{u} = \hat{E}, \quad \nabla \cdot \hat{u} = 0 \tag{2.1}$$

The kinetic equivalent is called the master equation

$$(\partial_t + \hat{L})\hat{f}(t,\underset{\sim}{x},\underset{\sim}{v}) = 0, \qquad (2.2)$$

by $\hat{f}(t,\underset{\sim}{x},\underset{\sim}{v}) = \delta[\underset{\sim}{v} - \hat{\underset{\sim}{u}}(t,\underset{\sim}{x})]$. Here $\hat{\underset{\sim}{u}}$ is fluid velocity, $\hat{\underset{\sim}{E}} = -\frac{1}{\rho}\nabla\hat{p}$ is the gradient of pressure $\hat{p}$ per unit mass-density ρ, and ν is molecular viscosity. The velocity distribution is $\hat{f}$ in the phase-space $t,\underset{\sim}{x},\underset{\sim}{v}$. The differential operator is $\hat{L} = \underset{\sim}{v}\cdot\nabla - \nu\nabla^2 + \hat{\underset{\sim}{E}}\cdot\underset{\sim}{\partial}, \quad \underset{\sim}{\partial} \equiv \partial/\partial\underset{\sim}{v}$. An instantaneously fluctuating function is denoted by $\hat{(\ )} = \bar{(\ )} + \tilde{(\ )}$, and can be decomposed into an ensemble average $\bar{(\ )} \equiv <\ >$ and a fluctuation $\tilde{(\ )}$. The scaling operator $\bar{A}$ and $\tilde{A}$ can be used.

We develop a group-kinetic method that incorporates the transport processes of spectral evolution, eddy viscosity and relaxation as macrogroup, microgroup and subgroup into the multi-scale distribution functions f^0, f', $f'' \equiv f^{(2)} + f^{(3)} + \cdots$. The scaling operators A^0, A', A'' can be used. The subgroup distribution f'' forms a cluster of many high-order distributions. By random encounter, these distributions are homogenized, lose their identity in v-dependence and their role of memory-transmission. This random behavior leads to a closure of the hierarchy.

By a macrogroup dynamics, the transport equation for f' is integrated to form the "eddy collision" so as to derive the kinetic equation for the evolution of the macro-distribution f^0 in the form

$$(\partial_t + A^0\hat{L})f^0 = -L^0\bar{f} + \underset{\sim}{\partial}\cdot\underset{=}{D}'\cdot\underset{\sim}{\partial}\{f^0(t-\tau)\} \qquad . \qquad (2.3)$$

The "collision operator"

$$C'\{\} \equiv \underset{\sim}{\partial}\cdot\underset{=}{D}'\cdot\underset{\sim}{\partial}\{\} \qquad (2.4)$$

3

depends on the "diffusion operator" $\underset{\approx}{D}{}'$ due to $\underset{\sim}{E}{}'$-fluctuations.

By taking the moment, the kinetic equation is transformed into a "renormalized Navier-Stokes equation"

$$(\partial_t + \hat{\underset{\sim}{u}} \cdot \underset{\sim}{\nabla} - \nu\nabla^2)\underset{\sim}{u}{}^0 = \underset{\sim}{E}{}^0 + \underset{\sim u}{J}{}^0 , \tag{2.5}$$

i.e. in a form identical to the original Navier-Stokes equation augmented by the "eddy damping"

$$\underset{\sim u}{J}{}^0 = \int d\underset{\sim}{v}\ \underset{\sim}{v}\ C'\{f^0(t-\tau)\} \quad . \tag{2.6}$$

The eddy damping represents the coupling between the macrogroup f^0 and the microgroup fluctuations that organize the collision operator. At the same time the equation of spectral evolution is obtained as

$$\tfrac{1}{2}\,\partial_t < \underset{\sim}{u}{}^{02} > = - T_u^0 \quad . \tag{2.7}$$

This equation involves the transfer function

$$T_u^0 = - <\underset{\sim}{u}{}^0 \cdot \underset{\sim u}{J}{}^0 > = \begin{vmatrix} K'(\underset{\sim}{\nabla}\underset{\sim}{u}{}^0)^2 & \text{direct cascade} \\[2mm] -\lambda'< \underset{\sim}{u}{}^{02} > & \text{inverse cascade} \end{vmatrix} \tag{2.8}$$

that governs the cascade transfer by direct and inverse cascades across the spectrum. By a microgroup dynamics, the eddy viscosity K' and the coefficient of eddy amplification λ' are determined from the collision operator of the kinetic equation. The change of signs indicates the change of eddy damping into eddy amplification so that $-K'\{\ \}$ may be interpreted as an operator due to "negative

viscosity".

The approach of the transport properties K' and λ' to equilibrium is regulated by a relaxation process as described by the path perturbation $\ell''(\tau)$ in a time interval τ along the trajectory that is perturbed by turbulence. The path perturbation enters into the orbit function $< \exp - ik'' \cdot \ell''(\tau) >$ and is related to the path diffusivity K_ℓ''.

In order to replace the customary mixing-length hypothesis needed in modelings of turbulence (Monin and Obukhov, 1953; Mellor and Yamada, 1974, 1982), we formulate a subdynamics of relaxation. It consists of a Fokker-Planck equation of path transition and a Langevin equation of turbulence. The former determines the orbit function in terms of the path diffusivity K_ℓ''. The latter derives the relationship between the path diffusivity K_ℓ'' and the field diffusivity D''. Since K' has been previously derived from the D'-dependent collision operator C' in the microgroup dynamics, we obtain a system of integral equations for K_ℓ'' and K'. Their solutions determine the eddy viscosity K' that approaches equilibrium by relaxation from K_ℓ''.

The eddy viscosity is derived as

$$K' = c_K R^{0\frac{1}{2}} k^{-2} \tag{2.9}$$

with $R^0 = < (\nabla u^0)^2 >$, $c_K = (2/9)^{\frac{1}{2}} P_t = 0.471$, for $P_t = 0.545$.

The spectral functions of energy, temperature variance, and pressure variance are found as follows:

$$F(k) = A\epsilon^{2/3} k^{-5/3}, \quad F_\theta(k) = B\epsilon_\theta \epsilon^{-1/3} k^{-5/3}, \quad F_p(k) = C\epsilon^2 \epsilon^{4/3} k^{-7/3} \tag{2.10}$$

with A = 1.650, B = 0.899 and C = 0.907. The rate of dissipations are ε and ε_{ij}.

It is to be noted that a singlet distribution f^0, that is governed by the kinetic equation of turbulence with the diffusivity-dependent collision operator, suffices for the determination of the spectrum by our group-kinetic kinetic method, while a system of two kinetic equations for the one-point distribution $F_1(t,\underset{\sim}{x},\underset{\sim}{v})$ and two-point distribution $F_{12}(t_1,\underset{\sim}{x}_1,\underset{\sim}{v}_1;t_2,\underset{\sim}{x}_2,\underset{\sim}{v}_2)$ is necessary by the Bogoliubov method.

The detailed description of the group-kinetic theory is presented in Technical Report #1.

By the group-kinetic method, the Navier-Stokes equation is transformed into an equation with "eddy damping" or "eddy amplification". Such an equation is called the renormalized Navier-Stokes equation. Since the eddy coefficients aré derived analytically, the renormalized equation forms a logical basis for the modeling of turbulence. Such a modeling is superior to the customary modeling by the artificial damping with the differential ∇^n of n-th order or by an empirical second-order closure.

It is to be remarked that the determination of the transfer function from the collision operator in the group-kinetic theory involves the quadruple correlation $< \underset{\sim}{E}'(t)\underset{\sim}{E}'(t-\tau) > < f^0(t)f^0(t-\tau)>$ in the form of a product of two binary correlations, and not in the form of the quadruple correlation $< \underset{\sim}{\tilde{E}}(t)\,\underset{\sim}{\tilde{E}}(t-\tau)\underset{\sim}{\tilde{u}}{}^0(t)\underset{\sim}{\tilde{u}}{}^0(t-\tau) >$ that can only be factorized by the hypothesis of quasi-normality in the perturbation expansion theory.

6

The group-kinetic decomposition into transport processes yields the property of "nearest-neighbor group-interaction". This property leads to many simplifications in the development of the theory.

3. Group-kinetic theory of Rossby wave turbulence

To include the Coriolis force in the Navier-Stokes equation (2.1), we write $\hat{E} = \hat{E}_p + \hat{E}_C$ as consisting of a pressure field $\hat{E}_p = - \frac{1}{\rho} \nabla \hat{p}$ and a Coriolis force $\hat{E}_C = \omega_c \hat{u} \times \hat{z} = \hat{E}_c + \hat{E}_\beta$, that is the sum of $\hat{E}_c = \omega_0 \hat{u} \times \hat{z}$ and $\hat{E}_\beta = \beta x_2 \hat{u} \times \hat{z}$. The Coriolis parameter $\omega_c = \omega_0 + \beta x_2$ has a uniform rotation ω_0 and a differential rotation β.

The curl differentiation $\hat{\zeta} = \nabla \times \hat{u}$ of the Navier-Stokes equation yields the well-known vorticity equation for a drift wave

$$(\partial_t + \tilde{u} \cdot \nabla - \nu \nabla^2)\tilde{\zeta} = - \beta \tilde{u}_2 \qquad (3.1)$$

in two dimensions $x = (x_1, x_2)$.

By dimensional arguments based upon the parameter β, the energy spectrum

$$F_{22}(k) \approx \beta^2 k^{-5} \qquad (3.2)$$

from $\tilde{u}_2$-fluctuations in the drift-range has been proposed by Rhines (1975).

In the inertial-range that is not controlled by β, Eq. (2.1) is reduced to the form

$$(\partial_t + \hat{u} \cdot \nabla - \nu \nabla^2) \tilde{\zeta} = 0 \qquad (3.3)$$

that identically governs quasi-geostrophic turbulence. Quasi-geostrophic turbulence has been treated by Kraichnan (1967), Herring (1980), and Tchen (1982, 1983), finding the spectrum

$$F(k) = C \varepsilon_\zeta^{2/3} k^{-3} \tag{3.4}$$

It has been tacitly assumed that the uniform rotation ω_0 plays the role of a circular translation and will not influence the eddy transport, so that Rossby wave turbulence without drift becomes reduced to the quasi-geostrophic turbulence. However, evidence indicates that ω_0 controls the eddy transport coefficients. Hence the loss of ω_0-dependence in the vorticity equation (3.1) is a weakness of the current vortex dynamics because the Navier-Stokes equation has been prematurely differentiated.

It is true that the large-scale structure is drifted by the differential rotation β as indicated by Eq. (3.1), but for the maintenance of the large-scale structure, the small-scale transients must organize a ω_0-dependent eddy amplification in order to perform the inverse cascade. Without this transfer, the drift cannot find the necessary balance to produce the spectrum (3.2). On the other hand, the transients can organize an eddy damping by the ω_0-dependent eddy viscosity to perform the direct cascade in the inertial range. Finally the change of the direct cascade into the inverse cascade defines a transition range in the spectral distribution. The three spectral ranges (drift-range, inertia-range and transition-range) cannot be formulated from the traditional vorticity equation (3.1). Thus a new vortex dynamics is necessary.

By the group-kinetic method, we derive the renormalized vorticity equation

$$(\partial_\tau + \underline{u}^0 \cdot \underline{\nabla} - \nu\nabla^2 - C')\zeta^0 = -\beta\, u_2^0 + J_\zeta^0 \tag{3.5a}$$

It has all the terms of the traditional vorticity equation (3.1), augmented with the eddy damping

$$J_\zeta^0 = \underline{\nabla} \times \underline{J}_u^0 \,, \quad \underline{J}_u^0 = \int d\underline{v}\; \underline{v}\, C'\{f^0(t-\tau)\} \tag{3.5b}$$

Upon multiplying Eq. (3.5a) by ζ^0 and averaging, we find the equation of evolution for the enstrophy

$$\frac{1}{2}\partial_\tau \langle \zeta^{02} \rangle = \kappa^0 - T_\zeta^0 \tag{3.6}$$

to be governed by the drift function

$$\kappa^0 = \beta^2\, K_d^0 \tag{3.7a}$$

and the transfer function

$$T_\zeta^0 = \begin{vmatrix} K' \langle (\nabla\zeta^0)^2 \rangle & \text{direct cascade} \\[2mm] -\lambda' \langle \zeta^{02} \rangle & \text{inverse cascade} \end{vmatrix} \tag{3.7b}$$

The eddy coefficients are found as

$$K' = \frac{3}{2}\omega_0^{-1} \int_k^\infty dk''\, F(k'')$$

$$\lambda' = \frac{3}{4}\omega_0^{-1} R'\,, \quad R' \equiv 2\int_k^\infty dk''\, k''^2\, F(k'')$$

$$K_d^0 = 2\,\omega_0^{-1}\int_0^k dk'\, F_{22}(k') \tag{3.8}$$

9

and are proportional to ω_0^{-1} . The eddy viscosity $\kappa' \approx \omega_0^{-1} <\underline{u}'^2>$ has been suggested by Blackadar (1962) empirically.

The energy spectra and the eddy coefficients are found:

(i)

$$F_{22}(k) = \frac{8}{3} \beta^2 k^{-5} \tag{3.9a}$$

$$\lambda' = \omega_0^{-1} \beta^2 k^{-2} \tag{3.9b}$$

(ii) <u>inertial-range</u>

$$F(k) = (8/3)^{\frac{1}{2}} (\omega_0 \epsilon_\zeta)^{\frac{1}{2}} k^{-3} \tag{3.10a}$$

$$\kappa' = (3/2)^{\frac{1}{2}} (\epsilon_\zeta/\omega_0)^{\frac{1}{2}} k^{-2} \tag{3.10b}$$

(iii) <u>transition-range</u>

$$F(k) = 2(\omega_0 \epsilon_\zeta/\beta^2)k^{-1} \tag{3.11}$$

Here ϵ_ζ is the rate of enstrophy dissipation.

By comparing (3.10a) with (3.4), it is seen that the inertial-ranges of Rossby wave turbulence and quasi-geostrophic turbulence share the same -3 power law but differ in intensities by the ratio

$$(\omega_0^3/\epsilon_\zeta)^{1/6} > 1 \tag{3.12}$$

in strong rotation.

The detailed theory is presented in Technical Report #2.

10

4. Renormalized vortex dynamics

By taking the curl differentiation of the Navier-Stokes equation we obtain the following equations for the evolution of the vorticity $\zeta = \nabla \times \hat{u}$:

(i) <u>Vorticity equation for Rossby wave turbulence in three dimensions</u>

$$(\partial_t + \hat{u}\cdot\nabla - \nu\nabla^2)\hat{\zeta} = (\hat{\zeta}\cdot\nabla)\hat{u} - \beta\hat{u}_2\hat{z} + \omega_0\nabla_3\hat{u}, \quad \nabla\cdot\hat{u} = 0. \tag{4.1}$$

(ii) <u>Vorticity equation for quasi-geostrophic turbulence in three dimensions</u>

$$(\partial_t + \hat{u}\cdot\nabla - \nu\nabla^2)\hat{\zeta} = (\hat{\zeta}\cdot\nabla)\hat{u}, \quad \nabla\cdot\hat{u} = 0. \tag{4.2}$$

These equations have been considered by Lundgren (1982) and Saffman (1985).

In two dimensions, Eq. (4.1) and (4.2) are reduced to:

(iii) <u>Rossby wave turbulence</u>

$$(\partial_t + \hat{u}\cdot\nabla - \nu\nabla^2)\hat{\zeta} = -\beta\hat{u}_2, \quad \nabla\cdot\hat{u} = 0 \tag{4.3}$$

(iv) <u>quasi-geostrophic turbulence</u>

$$(\partial_t + \hat{u}\cdot\nabla - \nu\nabla^2)\hat{\zeta} = 0. \tag{4.4}$$

Equations (4.3) and (4.4) are the traditional equations for Rossby wave turbulence and quasi-geostrophic turbulence. The losses of pressure-gradient $\frac{1}{\rho}\nabla\hat{p}$ and uniform rotation ω_0 have been seen as expedient simplifications for analysis and numerical computations.

The advantage is however illusory. The losses of ω_0 and of $\overline{\nabla u}$ as entailed from $\hat{\nabla p}$ have prevented the formulation of the eddy coefficients that depend on those parameters.

With regard to (4.4), the omission of $\hat{\nabla p}$ changes the numerical coefficient but not the power law k^{-3}. But the loss of the eddy coefficients that depend on $\overline{\nabla u}$ misses the analysis of baroclinicity.

In order to avoid these difficulties, we develop a renormalized vortex dynamics and exploit further the group-kinetic method. This program will be investigated in future research.

5. Surface boundary layer

By the group-kinetic method, we have derived the spectral structure in an atmospheric surface layer. In the stable layer, the spectrum takes the form $k^{-5/3}$, k^{-1}, gap, k^{-3}, due to isotropic turbulence, shear turbulence, stable convection, and Coriolis rotation, respectively. In the unstable layer the spectrum takes the form $k^{-5/3}$, k^{-1} and k^{-3} due to isotropic turbulence, shear turbulence and Coriolis rotation. The gap disappears in unstable convection. These results are verified by experiments. The detailed theory is presented in Paper #3.

6. Two-phase turbulence

Consider a particulate phase of number-density $\hat{N}$ velocity $\hat{u}_2$, pressure P_2 suspended in a fluid phase of velocity $\hat{u}_1$ pressure $\hat{P}_1$ and constant density ρ. The two phases are coupled by the constant friction γ. The governing equations are:

$$\hat{N}^{-1}\rho[(\partial_t + \hat{u}_1 \cdot \nabla)\hat{u}_1 + \frac{1}{\rho}\nabla\hat{P}_1] = \gamma(\hat{u}_2 - \hat{u}_1) \tag{6.1}$$

$$\nabla \cdot \hat{u}_1 = 0 \tag{6.2}$$

$$m_s(\partial_t + \hat{u}_2 \cdot \nabla)\hat{u}_2 + \nabla\hat{P}_2 = -\gamma(\hat{u}_2 - \hat{u}_1) \tag{6.3}$$

$$\partial_t \hat{N} + \nabla \cdot (\hat{N}\hat{u}_2) = 0 . \tag{6.4}$$

For $\hat{P}_2$ a constitutive equation is added.

The two species are denoted by $(\ )_a$ with $a = 1, 2$ for fluid phase and particulate phase, respectively. An instantaneously fluctuating function is denoted by $\hat{(\ )} = \bar{(\ )} + \tilde{(\ )}$ and can be decomposed into an ensemble average $\bar{(\ )} \equiv <\ >$ and a fluctuation $\tilde{(\ )}$.

The variable number density $\hat{N}$ does not enter in the momentum equation for the particulate phase, but does enter in the momentum equation (6.1) for the fluid phase. This equation resembles the one that governs convective turbulence, where the Boussinesq approximation is acceptable if the phase-coupling can be compared with the convection. With this approximation the number density in the left side can be assumed to be a constant N_0, simplifying the system of equations (6.1) - (6.4) into

$$N_0^{-1}[\rho(\partial_t + \hat{u}_1 \cdot \nabla)\hat{u}_1 + \nabla p_1] = \gamma(\hat{u}_2 - \hat{u}_1) \qquad (6.5)$$

$$\nabla \cdot \hat{u}_1 = 0 \qquad (6.6)$$

$$m_s(\partial_t + \hat{u}_2 \cdot \nabla)\hat{u}_2 + \nabla \hat{p}_2 = -\gamma(\hat{u}_2 - \hat{u}_1) \qquad (6.7)$$

$$\nabla \cdot \hat{u}_2 = 0 \qquad (6.8)$$

If $\nabla \hat{p}_2 \ll \gamma(\hat{u}_2 - \hat{u}_1)$, the system of equations (6.1) – (6.4) is reduced to

$$N^{-1}[\rho(\partial_t + \hat{u}_1 \cdot \nabla)\hat{u}_1 + \nabla \hat{p}_1] = \gamma(\hat{u}_2 - \hat{u}_1) \qquad (6.9)$$

$$\nabla \cdot \hat{u}_1 = 0 \qquad (6.10)$$

$$m_s(\partial_t + \hat{u} \cdot \nabla)\hat{u}_2 = -\gamma(\hat{u}_2 - \hat{u}_1) \qquad (6.11)$$

$$\partial_t \hat{N} + \nabla \cdot (\hat{N} \hat{u}_2) = 0 \qquad (6.12)$$

for dilute suspension, called "dusty gas" by Saffman (1962).

The treatment of two-phase turbulence from the system of equations (6.5) – (6.8) has been our first attempt. The spectral results are:

14

a. In the inertia-range

$$F_a(k) = A_a \, \epsilon_a^{2/3} \, k^{-5/3} \tag{6.13}$$

with $\epsilon_a = \nu_a \langle (\vec{\nabla} \tilde{u}_a)^2 \rangle + \alpha_a \langle \tilde{v}_b^2 \rangle$, $\alpha_1 = n_0 \gamma / \rho$, $\alpha_2 = \gamma / m_s$, where $(\ )_b$ is the other species.

b. In the drag-range

$$F_1(k) = B_1 \, u_{*1}^2 \, k^{-1}, \quad F_2(k) = B_2 \, u_{*2}^2 \, k^{-1} \tag{6.14}$$

with $u_{*a} = (\epsilon_a / \alpha_a)^{\frac{1}{2}}$. The numerical coefficients A_a, B_a are evaluated. The detailed description is presented in Tech. Rep. #7.

The treatment of the dusty gas model (6.9) - (6.12), as well as the compressible model (6.1) - (6.4) will be done in future research.

7. Soliton turbulence

The soliton turbulence is governed by the Schrödinger equation

$$(i \, \partial_t + \nu_n \nabla^2 - \frac{1}{2} \omega_n \, \tilde{A} \, |\tilde{E}|^2) \tilde{E} = \tilde{X} \ .$$

Here $\tilde{E}$ is the envelope of field fluctuations, such that the density is $\tilde{N} \approx \tilde{A} |\tilde{E}|^2$, $\tilde{X}$ is the driving force, ν_n and ω_n are constants, and $\tilde{A} = 1 - \bar{A}$ is a fluctuating operator and gives the deviation from the average $\bar{A} \equiv \langle \ \rangle$. This equation applies to nonlinear problems in hydrodynamics (gravity waves and surface waves, shallow water waves) optics (nonlinear propagation, self-focusing, saturation of scintillation) and biological systems. Thus $\tilde{E}$ depends on the nonlinear

15

problem considered. The transformation of the hydrodynamic equations by taking the envelope of the wave packets is made (Tchen, 1986a). We derive the spectral structure $F_N \sim k^{-1}$ and $F_N \sim k^{-5}$ for density fluctuations, and

$$F_E \sim k^{-1} , \qquad F_E \sim k^{-3}$$

for field fluctuations. The results agree with plasma experiments (Truc, 1984). The detailed description is presented in papers #8, #9.

8. Compressible turbulence

The following models may be used for the description of the microdynamic state of turbulence.

(i) The Navier-Stokes equations are

$$\partial_t \hat{\rho} \hat{u}_i + \nabla_j \hat{\rho} \hat{u}_j \hat{u}_i = \hat{\rho} \hat{E}_{pi} , \quad \text{with} \quad \hat{\rho}\hat{E}_p \equiv -\nabla \hat{p} \tag{8.1a}$$

$$\partial_t \hat{\rho} + \nabla \cdot \hat{\rho}\hat{u} = 0 \tag{8.1b}$$

The equation of momentum may be written as

$$(\partial_t + \hat{u} \cdot \nabla)\hat{u} = \hat{E}_p \tag{8.2}$$

A constitutive relation $\hat{p} = \hat{\rho} R \hat{T}$ for ideal gas or $(\hat{p}/p_0) = (\hat{\rho}/\rho_0)^\gamma$ for adiabatic gas can be used. Here $\hat{\rho}, \hat{u}, \hat{E}, R, \gamma$ are pressure, density, velocity, pressure gradient, gas constant and ratio of specific heats, respectively.

(ii) The equations of sound propagation in turbulence are

16

$$(\partial_t + \hat{\underset{\sim}{u}} \cdot \nabla)\hat{u} = \hat{\underset{\sim}{E}}_p \tag{8.3a}$$

$$(\partial_t^2 - \underset{\sim}{\nabla} \cdot \hat{c}^2 \ \nabla)\hat{\rho} = \hat{r} \tag{8.3b}$$

with a source $\hat{r} \equiv \underset{\sim}{\nabla}\,\underset{\sim}{\nabla} : \hat{\rho}\,\hat{\underset{\sim}{u}}\,\hat{\underset{\sim}{u}}$. The equation of propagation (8.3b) with variable speed $\hat{c} = (d\hat{p}/d\hat{\rho})^{\frac{1}{2}}$ is obtained by a cross differentiation of (8.1a) and (8.1b) with respect to x_j and t.

(iii) For an adiabatic gas with

$$\frac{\hat{p}}{p_0} = \left(\frac{\hat{\rho}}{\rho_0}\right)^{\gamma}$$

and by introducing the speed of sound

$$\hat{c} = (\gamma\,\hat{p}/\hat{\rho})^{\frac{1}{2}}$$

and an auxiliary function

$$\hat{w} = \frac{2}{\gamma-1}\,\hat{c}$$

We transform Eqs. (8.1a) and (8.1b) into the following system of symmetric equations

$$(\partial_t + \hat{u} \cdot \nabla)\hat{u} = -\frac{\gamma-1}{2}\,\hat{w} \cdot \nabla\hat{w} \tag{8.4a}$$

$$(\partial_t + \hat{u} \cdot \nabla)\hat{w} = \frac{\gamma-1}{2}\,\hat{w}\,\nabla \cdot \hat{u} \tag{8.4b}$$

This system resembles the Riemann equations for rarefaction waves. Note that

$$\hat{\phi} = \frac{\gamma-1}{4}\,\hat{w}^2 = \frac{1}{\gamma-1}\,\hat{c}^2 \tag{8.5}$$

gives the potential of the driving force

17

$$\hat{E}_p = -\nabla\hat{\phi}_p \tag{8.6}$$

(iv) For interstellar gas, the hydrodynamic equations are

$$\partial_t \hat{\rho}\hat{u}_i + \nabla_j \hat{\rho}\hat{u}_j\hat{u}_i = \hat{E}_i \tag{8.7a}$$

$$\partial_t \hat{\rho} + \nabla \cdot \hat{\rho}\hat{u} = 0 \tag{8.7b}$$

The driving force

$$\hat{E} = \hat{E}_p + \hat{E}_g \tag{8.8}$$

consists of a pressure gradient

$$\hat{E}_p = -\frac{1}{\hat{\rho}}\nabla\hat{p} \tag{8.9a}$$

and a gravitational force

$$\hat{E}_g = -\nabla\hat{\phi}_g \tag{8.9b}$$

where the potential $\hat{\phi}_g$ satisfies the Poisson equation

$$\nabla^2\hat{\phi}_g = 4\pi G\hat{\rho} \tag{8.9c}$$

For the problems of compressible turbulence above, the group-kinetic method is best suitable. By raising to higher dimensionality by

$$\hat{f}(t,x,v) = \hat{\rho}\,\delta[v - \hat{u}(t,x)] \ , \tag{8.10}$$

Eqs. (8.1a) and (8.1b) is transformed into the master equation

$$(\partial_t + v\cdot\nabla + \hat{E}\cdot\partial)\,\hat{f}(t,x,v) = 0 \ . \tag{8.11}$$

18

It is not difficult to show by moments $\int d\underline{v}\, \hat{f} = \hat{\rho}$, $\int d\underline{v}\, \underline{v}\, \hat{f} = \hat{\rho}\, \hat{\underline{u}}$
that the master equation can reproduce the Navier-Stokes
equations. An instantaneously fluctuating function as denoted by
$\hat{(\ )} = \bar{(\ )} + \tilde{(\ )}$ can be decomposed into an ensemble average $\bar{(\ )} \equiv < >$
and a fluctuation $\tilde{(\ )}$. It is interesting to see that turbulence,
whether incompressible or compressible, is represented by the same
master equation for the description of the microdynamic state of
turbulence.

We have investigated the Navier-Stokes equations (8.1a) and
(8.1b) for compressible turbulence by the group-kinetic method.
The following spectral intensities are derived:

$$< u'^2 > \, \approx\, k^{-2}, \quad < E'^2 > \, \approx\, k^{-2}, \quad < \rho'^2 > \, \approx\, k^{-4}\,. \tag{8.12}$$

The details of derivation are given in Technical Report #13.

9. Conclusions

Past analytic efforts in the literature have not succeeded in
deriving explicit expressions of cascades (direct and inverse
cascades) and of eddy coefficients (eddy viscosity, eddy coef-
ficient of amplification, and eddy diffusivity of drift), that con-
trol the spectral evolution of turbulence. Since the spectral
evolution, the transport coefficients and the relaxation for the
approach of the eddy coefficients to equilibrium are the three tran-
sport processes of turbulence, we represent them by a macro-kinetic
group, a micro-kinetic group and a subgroup, respectively.

By group-kinetic considerations, we formulate a kinetic equation

19

of turbulence, and derive the eddy coefficients analytically. The approach to equilibrium is analysed by a subdynamics that consists of a Fokker Plank equation of path perturbations and a system of Langevin equations of turbulence. We derive the "renormalized Navier-Stokes equations" and the "renormalized vorticity equations". They contain explicit cascades through the analytically derived eddy coefficients.

By the group-kinetic method we have investigated some aspects of Navier-Stokes turbulence with the Kolmogoroff spectrum, Rossby wave turbulence, boundary layer turbulence, two-phase turbulence, soliton turbulence, and compressible turbulence. Other aspects will be investigated in future research.

REFERENCES

BOGOLIUBOV, N. N. 1962 Problems of a dynamical theory in statistical physics, in: Studies in Statistical Mechanics, I ed. J. de Boer and G. E. Uhlenbeck North-Holland, Amsterdam.

CHAPMAN, S. & COWLING, T. G. 1939 The Mathematical Theory of Non-Uniform Gases, Cambridge University Press.

HERRING, J. R. 1980 Statistical theory of quasi-geostrophic turbulence. J. Atmos. Sci. 37, 969-977.

HERRING, J. R. 1988 The inverse cascade range of quasi-geostrophic turbulence. Meteorol. Atmos. Phys. 38, 106-115.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. J. Fluid Mech., 5, 497-513.

KRAICHNAN, R. H. 1967 Inertial range in two-dimensional turbulence. Phys. Fluid, 10, 1417-1423.

KRAICHNAN, R. H. 1977 Eulerian and Lagrangian renormalization in turbulent theory. J. Fluid Mech., 83, 349-374.

LUNDGREN, T. S. 1982 Strained spiral vortex model for turbulent
    fine structure. Phys. Fluids, 25, 2193-2203.

MARTIN, P. C. SIGGIA, E. D. & ROSE, H. A. 1973 Statistical dynamics
    of classical systems. Phys. Rev., A8, 423-437.

MELLOR, G. L. & T. YAMADA 1974 A hierarchy of turbulence closure
    models for planetary boundary layers. J. Atmos. Sci., 31,
    1791-1806.

MELLOR, G. L., & T. YAMADA 1982 Development of a turbulent
    closure model for geophysical fluid problems. Rev. of Geophys.
    and Space Phys., 20, 851-875.

MONIN, A. S. & A. M. OBUKHOV 1953 Dimensionless characteristics of
    turbulence in the atmospheric surface layer. Doklady Akad.
    Nauk, SSSR, 93, 223-226.

RHINES, P. B. 1975 Waves and turbulence on a ,-plane. J. Fluid
    Mech., 69, 417-443.

SAFFMAN, P. G. 1962 On the stability of laminar flow of a dusty gas.
    J. Fluid Mech., 13, 120.

SAFFMAN, P. G. 1985 Perspectives of vortex dynamics.   Physico-
    Chemical Hydrodynamics, 6, 711-726.

STRUMINSKII, V. V. 1985 Theoretical principles of turbulent flows.
    Sov. Phys. Dokl., 30, 1-3.

TCHEN, C. M. 1982 Group-scaling theory for the enstrophy transfer
    in two-dimensional turbulence. In: Theory and Modeling of
    Atmospheric Turbulence, Part I by C. M. Tchen. NASA Contractor
    Report 3787, pp C1-31.

TCHEN, C. M. 1983 Group-kinetic theory of two-dimensional
    geostrophic turbulence. In: Theory and Modeling of Atmospheric
    Turbulence, Part II, by C. M. Tchen. NASA Contractor Report
    3817, pp. 2-1 to 2-30.

TCHEN, C. M. 1986a Statistical foundation of the fluid analogue of
    the soliton formalism. Physica Scripta, 34, 289-294.

TCHEN, C. M. 1986b Transport by negative eddy viscosity in soliton
    turbulence. Physica Scripta, 34, 295303.

PUBLICATIONS AND TECHNICAL REPORTS COMPLETED
DURING CONTRACT PERIOD BY C. M. TCHEN

1. Group-kinetic method of turbulence.

2. Group-kinetic theory of Rossby wave turbulence.

3. Large-scale spectral structure of turbulence with a gap in the stably stratified atmosphere. Physica Scripta, 31, 616-620 (1985).

4. Group-kinetic treatment of atmospheric turbulence and diffusion.

5. Group-kinetic theory of rotating turbulence.

6. Statistical theory of Rossby wave turbulence.

7. Group-kinetic theory of two-phase turbulence. NASA Contractor Report 3896.

8. Statistical foundation of the fluid analogue of the soliton formalism. Physica Scripta, 34, 289-294 (1986).

9. Transport by negative eddy viscosity in soliton turbulence. Physica Scripta, 34, 295-303 (1986).

10. Kinetic approach to soliton turbulence. NASA Contractor Report 3969.

11. Fluid analogue of the soliton turbulence. NASA Contractor Report 3969.

12. Group-kinetic theory of dispersive soliton turbulence. NASA Contractor Report 3894.

13. Excitation of turbulence by density waves. NASA Contractor Report 3905.
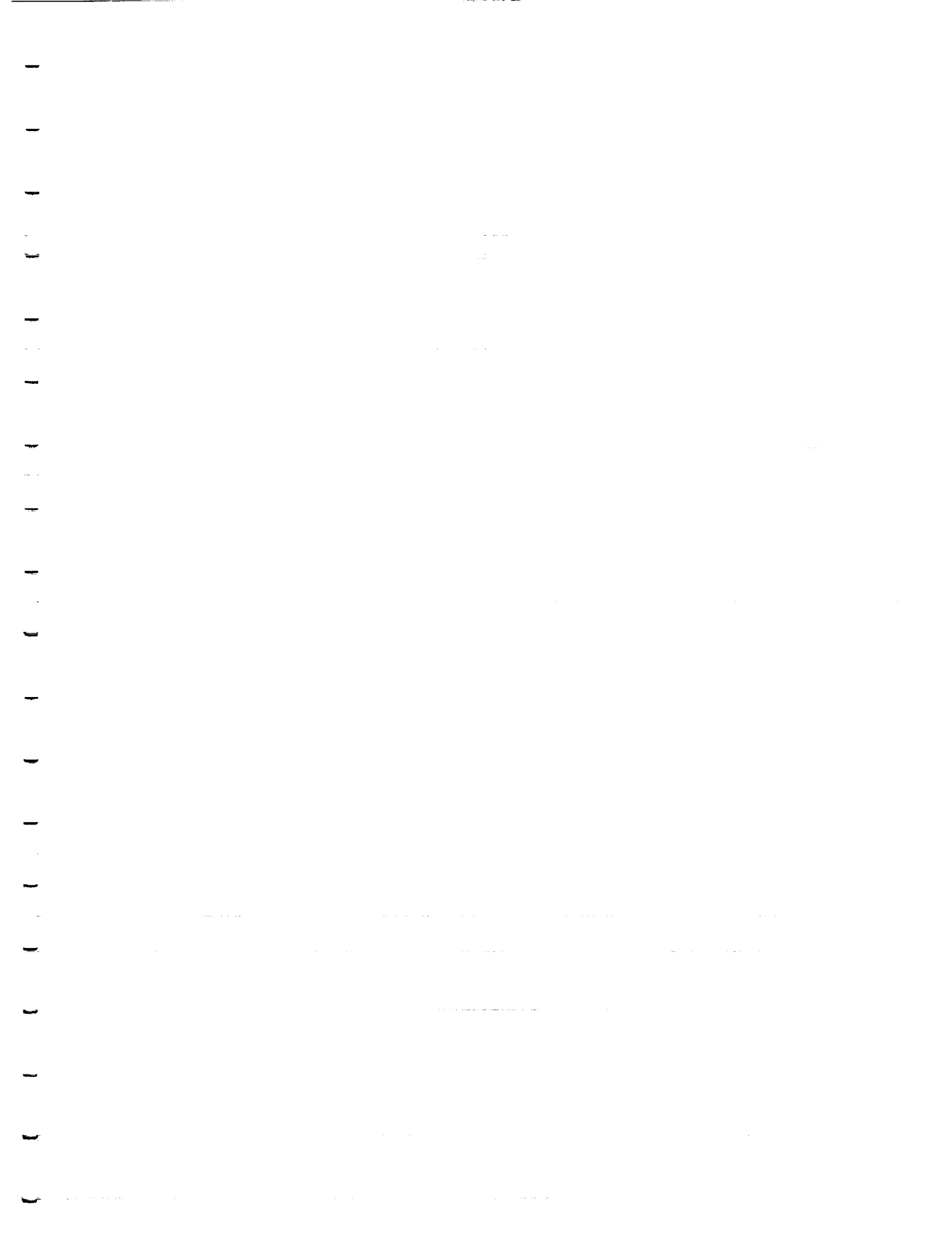
GROUP-KINETIC THEORY OF ROSSBY WAVE TURBULENCE

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The Rossby wave turbulence is formed by large-scale two-dimensional vortices in the atmosphere or ocean on a rotating planet. It has long been thought that the standard vortex model $(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla} - \nu\nabla^2)\hat{\zeta} = -\beta\hat{u}_2$ is the basic governing equation. Here $\hat{\underline{u}}$ is fluid velocity, $\hat{\zeta} = \underline{\nabla} \times \hat{\underline{u}}$ is vorticity, ν is molecular viscosity, and β is differential rotation. The present work demon&strates that this model would fail in analysing the spectral composition of the Rossby wave turbulence, because it had canceled the uniform rotation ω_0 upon which the eddy transport coefficients must depend. A renormalized vortex dynamics is developed by the use of the group-kinetic method. It describes the evolution of the large-scale vortices under drift and, in addition, their ω_0-dependent interactions with small-scale transients. The velocity distribution is decomposed into three groups representing three transport processes (evolution, eddy transport coefficients and relaxation). The kinetic equation of turbulence is derived. The eddy coefficients are calculated from the "collision operator". A subdynamics determines the relaxation to equilibrium, and consists of a Fokker-Planck equation of transition and a system of two Langevin equations. The closure is found by memory-loss and not by the hypothesis of quasi-normality.

The spectral structure is divided into an inertial-range by direct cascade, a transition-range from direct cascade to inverse cascade, and a drift-range by inverse cascade, in the order of decreasing wavenumbers. The corresponding spectral laws are found to be: $F(k) = A_1(\omega_0 \epsilon_\zeta)^{\frac{1}{2}} k^{-3}$, $F(k) = A_2(\omega_0 \epsilon_\zeta/\beta^2)k^{-1}$, $F_{22}(k) = A_3\beta^2 k^{-5}$.

The numerical coefficients are evaluated: $A_1 = (8/3)^{\frac{1}{2}}$, $A_2 = 2$ and $A_3 = 8/3$.

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1. Introduction

The Rossby wave turbulence is described by the Navier-Stokes equation of motion with a Coriolis force that has a uniform rotation ω_0 and a drift from the differential rotation β. The works on large-scale turbulence are well documented (Charney, 1971; Baer 1972; Salmon, Holloway and Hendershott, 1976; Salmon, 1978; Lambert, 1981) and have shown a spectrum k^{-3} in the inertial-range. The change to the spectrum k^{-5} in the drift range was predicted by Rhines (1975). Observations have shown a spectrum k^{-1} in the transition range. It will be desirable to distinguish between the Rossby wave turbulence and the two-dimensional geostrophic turbulence in their inertial-ranges where the same power law -3 is shared.

For the description of the Rossby waves that are large-scale planetary vortices in the rotating atmosphere and ocean, a vortex dynamic is required. The curl differentiation of the Navier-Stokes equation

2

$$(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla} - \nu \nabla^2)\hat{\underline{u}} = \hat{\underline{E}} ,\tag{1.1}$$

with

$$\hat{\underline{E}} = \hat{\underline{E}}_p + \hat{\underline{E}}_c , \quad \hat{\underline{E}}_p = -\frac{1}{\rho} \underline{\nabla} \hat{p} , \quad \hat{\underline{E}}_C = \omega_c \hat{\underline{u}} \times \hat{\underline{z}} = \hat{\underline{E}}_c + \hat{\underline{E}}_\beta \tag{1.2a}$$

$$\hat{\underline{E}}_c = \omega_0 \hat{\underline{u}} \times \hat{\underline{z}}, \quad \hat{\underline{E}}_\beta = \beta x_2 \hat{\underline{u}} \times \hat{\underline{z}}, \quad \omega_c = \omega_0 + \beta x_2 , \quad \hat{\underline{z}} = (0,0,1), \tag{1.2b}$$

yields the vortex model

$$(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla} - \nu \nabla^2)\hat{\zeta} = -\beta \hat{u}_2 \tag{1.3}$$

for the evolution of the vorticity $\hat{\zeta} = \underline{\nabla} \times \hat{\underline{u}}$ in two dimensions with a drift by β. The fluid velocity satisfies the condition of incompressibility $\underline{\nabla} \cdot \hat{\underline{u}} = 0$. Here ν is molecular viscosity, $\hat{p}$ is pressure, ρ is density. The Coriolis parameter ω_c has a uniform rotation ω_0 and a differential rotation β. A fluctuating function, as denoted by $(\hat{\ }) = (\bar{\ }) + (\tilde{\ })$ is the superposition of the ensemble average $(\bar{\ }) \equiv < >$ and the fluctuation $(\tilde{\ })$.

In the inertial-range of the Rossby wave turbulence, the drift is negligible, so that Eq. (1.3) is reduced to the homogeneous form

$$(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla} - \nu \nabla^2)\hat{\zeta} = 0 \tag{1.4}$$

that is identical to the equation for two-dimensional quasi-geostrophic turbulence. This simplification has appealed to theoretical studies and observational interpretations, but is deceptive, especially since ω_0 should control the eddy coefficients and governs the spectral structure of the Rossby wave turbulence.

We believe that a proper vortex dynamics should describe the evolution of the large-scale structure by the curl differentiation

3

of the renormalized Navier-Stokes equation that includes the interaction with the small-scale transients for organizing the ω_0-dependent eddy damping. In this respect, recall that a ω_0-dependent eddy viscosity has been suggested by Blackadar (1962) empirically.

A particularly appealing method for treating turbulence has been the "renormalization perturbation expansion". It has enabled Kraichnan (1959, 1977) to develop his DIA method and generalizations. A systematic method of developing the expansion was given in a general form by Martin, Siggia and Rose (1972) and by Dubois and Espedal (1978). Without a convergent summation, the method of perturbation expansion cannot easily derive the eddy coefficients which may be of kinetic origin. The couplings among modes in expansion are too complicate and cannot clearly describe the physical processes they represent. For this reason, we develop a group-kinetic method for Rossby wave turbulence, not from the vortex model (1.3) but from a "master equation". The master equation is based upon the Navier-Stokes equation and describes the microdynamic state of turbulence. It is only after having determined the ω_0-dependent eddy transport coefficients and derived the "renormalized Navier-Stokes equation", i.e. a Navier-Stokes equation with "augmented damping", that we pass to the "renormalized vorticity equation". Our vortex dynamics will have the same form as Eq. (1.3) but is augmented with an "eddy damping" that depends on ω^0 and is analysed from the "eddy collision" of the kinetic equation of turbulence.

By group-kinetic scaling, the three transport processes of spec-

tral evolution, eddy viscosity and relaxation are represented by three groups of distributions (Section 2). In the macrogroup dynamics, we transform the Navier-Stokes equation into a master equation and derive the kinetic equation of turbulence (Section 3). The "diffusion operator" with orbit functions enters into the "collision operator" (Section 4). The kinetic equation is reverted to the renormalized Navier-Stokes equation by moment, and subsequently to the renormalized vorticity equation by curl differentiation. The evolution of enstrophy and the governing transport functions are found (Section 5).

In particular we are interested in a microgroup dynamics that derives the eddy transport coefficients from the collision operator (Section 6).

A subdynamics of relaxation is developed in Section 6 for the approach of the eddy viscosity to equilibrium, and consists of a Fokker-Planck equation of path transition and a system of two Langevin equations of turbulence. The subdynamics determines the orbit functions and the path diffusivity. Finally, the transport coefficients (eddy viscosity, coefficient of amplification, and drift diffusivity) for the direct cascade, the inverse cascade, and the drift are investigated in Section 7. The spectral structure is found in Section 8. A summary with discussions is presented in Section 9.

5

2. Group-kinetic method

Being a nonhomogeneous partial differential equation, the Navier-Stokes equation is not convenient for the formulation of a theory of transport in the gradient form or in the form of an eddy damping (or amplification). We raise its dimensionality and transform it into a homogeneous equation

$$(\partial_t + \hat{L})\hat{f} = 0, \qquad\qquad (2.1)$$

called the "master equation"

by writing $\hat{f}(t,\underline{x},\underline{v}) = \delta[\underline{v} - \underline{\hat{u}}(t,\underline{x})]$. Here $\hat{L} = v \cdot \nabla - \nu\nabla^2 + \underline{\hat{E}} \cdot \underline{\partial}$, $\partial_t = \partial/\partial t$, $\underline{\partial} = \partial/\partial\underline{v}$. Since $\underline{v}$ is an independent variable the master equation has lesser nonlinearity. It is not difficult to verify by taking the moments that the master equation reproduces the Navier-Stokes equation and the equation of continuity.

From the statistical viewpoint, the master equation describes the microdynamical state of turbulence where the elementary interaction among fluid elements is represented by $\hat{\underline{E}}$-fluctuations. A kinetic theory can be developed, either by the reduction of the N-point distributions into F_1, F_2, F_3 according to the method of Bogoliubov (1962), or by the decompositions $\hat{f} = f^0 + f'$, $f' = f^1 + f''$, $f'' = f^{(2)} + f^{(3)} + \cdots$ from the group-kinetic method (Tchen, 1978, 1984, 1986). The three groups f^0, f', f'' called macrogroup, microgroup and subgroup represent the transport processes of spectral evolution, eddy viscosity, and relaxation, respectively, with durations of correlation $\tau_c^0 > \tau_c' > \tau_c''$ in the order of increasing randomness. The scaling operators A^0, A', A'' can be used.

6

We choose the group-kinetic method in view of its benefits over Bogoliubov's method and the method of perturbation expansion. For the derivation of spectrum, the singlet distribution f^0 suffices in our group method, while two distributions F_1, F_2 are needed in the Bogoliubov method. The eddy viscosity being of kinetic origin cannot be easily derived by a continuum theory. Although the Fourier modes in groups interact indefinitely and may overlap in wavenumbers, their statistical properties are well separated by the transport processes they represent. Consequently, the groups interact statistically as <u>nearest-neighbours</u>. Hence a quadruple correlation takes the form of a product of two binary correlations as

$$< f^0(t)f^0(t-\tau) >_{\upsilon} \quad < \underset{\sim}{E}'(t) \, \underset{\sim}{E}'(t-\tau) > \qquad (2.2)$$

and not in the form $< \underset{\sim}{\overset{\curvearrowright}{u}}(t)\underset{\sim}{\overset{\curvearrowright}{u}}(t-\tau)\underset{\sim}{\overset{\curvearrowright}{E}}(t)\underset{\sim}{\overset{\curvearrowright}{E}}(t-\tau) >$ that requires the hypothesis of normality for factorization, as was in perturbation expansion theory. The subgroup $f'' = f^{(2)} + f^{(3)} + \ldots$ forms a cluster of many high-order distributions. By random encounters, they are "homogenenized" and lose their identity in v-dependence. As a result, they can be simulated by an effective medium of friction coefficient $C''(k)$. Since it is by the v-dependence that the memory is transmitted, the loss of v-dependence causes the loss of memory, leading to irreversibility and closure, and providing a mechanism of relaxation for the approach of the eddy viscosity to equilibrium.

The correlation functions of macrogroup, microgroup and subgroup have the limits $(0,k)$, (k,∞) and (k'',∞) in integrations $\int_0^k dk' \ldots$, $\int_k^\infty dk'' \ldots$, $\int_{k''}^\infty dk''' \ldots$. These limits may be regarded as the statistical demarcations of the groups.

3. Kinetic equations of turbulence

By the use of A^0 and A^1 we transform the master Eq. (2.1) into the following transport equations

$$(\partial_t + A^0 \hat{L}) f^0 = - L^0 \bar{f} + C^1 \{f^0\}, \quad \text{with } C^1 \{f^0\} \equiv - A^0 L^1 f^1 \tag{3.1}$$

$$(\partial_t + A^1 \hat{L} - C'') f^1 = - L^1 f^0 . \tag{3.2}$$

The auxiliary operator A^1 gives $A^1 \hat{f} = f^1$. Note that f^1 evolves in a medium that offers a friction $C'' f^1$. The integration of Eq. (3.2) gives f^1 and a multiplication by L^1 yields the eddy collision in the form

$$C^1 \{f^0\} \equiv - A^0 L^1 f^1 = \underset{\sim}{\partial} \cdot \underset{\sim}{D^1} \cdot \underset{\sim}{\partial} \{ f^0(t-\tau) \} \tag{3.3}$$

or

$$C' \{f^0\} = - A^0 L' f' = \underset{\sim}{\partial} \cdot \underset{\sim}{D'} \cdot \underset{\sim}{\partial} \{ f^0(t-\tau) \} . \tag{3.4}$$

The approximation $C^1 \overset{\sim}{=} C'$ is a consequence of the nearest-neighbour group-interaction. The diffusion operator is

$$\underset{\approx}{D'} \{ \ \} = \int_0^{t \to \infty} d\tau < \underset{\sim}{E'}(t) \ \underset{\sim}{E'}(t - \tau) > \{ \ \} \tag{3.5}$$

and the collision operator is $C'\{ \} \equiv \underset{-}{\partial} \cdot \underset{=}{D'} \cdot \partial \{\}$. Hence we derive the kinetic equation of turbulence

$$(\partial_t + A^0 \hat{L}) f^0 = - L^0 \bar{f} + C' \{f^0\} . \tag{3.6}$$

8

4. Orbit functions in strong rotation

The diffusion operator (3.5) is obtained by the time integration of the Lagrangian correlation of field-fluctuations. In view of the separation of transport processes, the diffusion operator is adiabatic, _i.e._ $t \rightarrow \infty$. In strong rotation, _i.e._ $\hat{E} \simeq \tilde{E}_c$, the Lagrangian correlation in the operator description

$$< \underline{E}'_c(t)\underline{E}'_c(t-\tau) > = < \underline{E}'_c(t)A'U(t,t-\tau)\underline{E}'_c(t-\tau) > , \qquad (4.1a)$$

or, equivalently, in the orbital description

$$< \underline{E}'_c(t)\underline{E}'_c(t-\tau) > = < \underline{E}'_c(t)\underline{E}'_c[t-\tau,\hat{\underline{x}}(t-\tau)] > , \qquad (4.1b)$$

can be calculated from the evolution operator $U(t,t-\tau)$ that is governed by the equation

$$(\partial_t + A^1L^1 - C'')U(t,t') = 0, \qquad \text{with} \quad U(t,t) = 1, \quad t' \leqslant t, \qquad (4.2)$$

or from the perturbed trajectory

$$\hat{\underline{x}}(t-\tau) = \underline{x} - \int_0^\tau d\tau' \ \tilde{\underline{v}}_c(\tau') - \tilde{\underline{\ell}}(\tau)$$

$$= \underline{x} - m\nu \times \hat{\underline{z}} - \tilde{\underline{\ell}}(\tau) \qquad (4.3)$$

with

$$\tilde{\underline{v}}_c(\tau) = \tfrac{1}{2} \omega_0 \tau \ \underline{v} \times \hat{\underline{z}}, \qquad m = \tfrac{1}{4} \omega_0 \tau^2 . \qquad (4.4)$$

We choose the orbital description.

By Fourier transformation, the Lagrangian correlations are

$$\langle u^0(t) f^0(\tau-\tau) \rangle = \int dk' \, \chi \langle u^0(k') f^0(-k', v) \rangle \, h_c(\tau, k', v) h_\ell(\tau, k') \tag{4.5a}$$

$$\langle E'_c(t) E'_c(\tau-\tau) \rangle = \int dk'' \, \chi \langle E'_c(k'') E'_c(-k'') \rangle \, h_c(\tau, k', v) h_\ell(\tau, k'') \tag{4.5b}$$

$$\langle E''_c(t) E''_c(\tau-\tau) \rangle = \int dk''' \, \chi \langle E''_c(k''') E''_c(-k''') \rangle h''_c(\tau, k'''), \tag{4.5c}$$

the diffusivities are

$$D'(t, x, v) = \int_0^\infty d\tau \int dk'' \, \chi \langle E'_c(k'') E'_c(-k'') \rangle h_c(\tau, k'', v) h_\ell(\tau, k'') \tag{4.6a}$$

$$D''(t, x) = \int_0^\infty d\tau \int dk'' \, \mathrm{tr}\, \chi \langle E''_c(k''') E''_c(-k''') \rangle \, h''_\ell(\tau, k'''), \tag{4.6b}$$

and the orbit functions are

$$h_c(\tau, k'', v) = \exp[-imk'' \cdot (v \times \hat{z})] \, J \tag{4.7a}$$

$$h_\ell(\tau, k'') = [\exp C''(k'') \tau] \langle \exp - ik'' \cdot \ell''(\tau) \rangle, \quad h''_\ell(\tau, k'') = \langle \exp - ik \cdot \ell''(\tau) \rangle \tag{4.7b}$$

for streaming and relaxation. The path fluctuation ℓ'' organizes a path diffusivity

$$K''_\ell = \tfrac{1}{2} \lim \frac{d}{d\tau} \, \mathrm{tr} \langle \ell''(\tau) \ell''(\tau) \rangle$$

$$= \int_0^\infty d\tau' \, \mathrm{tr} \langle u'(0) u'(\tau) \rangle . \tag{4.8}$$

The meandering caused by ℓ^0 and ℓ^1 is the lowest-order perturbation expansion in quasilinear theory and is negliglible in strong turbulence.

The Fourier decomposition is truncated in a region within which the function is quasi-homogeneous, and the truncation factor is χ. The limits of the volume integrations in (4.5a) – (4.5b) and in the following are understood to extend from $-\infty$ to $+\infty$.

5. Renormalized equations, transport functions and eddy coefficients

5.1. Renormalized equations

By taking the moment of the kinetic equation (3.6) we get the renormalized Navier-Stokes equation

$$(\partial_t + \underline{u}^0 \cdot \nabla - \nu \nabla^2)\underline{u}^0 = \underline{E}^0 + \underline{J}^0_u \tag{5.1}$$

with the "augmented damping" or "eddy dampings"

$$\underline{J}^0_u = \int d\underline{v} \ \underline{v} \ C'\{f^0(t-\tau)\} \ . \tag{5.2}$$

By a curl differentiation $\zeta^0 = \nabla \times \underline{u}^0$ we transform Eq. (5.1) into the renormalized vorticity equation in two dimensions

$$(\partial_t + \underline{u}^0 \cdot \nabla - \nu \nabla^2)\zeta^0 = -\beta u^0_2 + J^0_\zeta \ , \quad \text{with} \quad J^0_\zeta = \nabla \times \underline{J}^0_u \ . \tag{5.3}$$

The eddy dampings represent the coupling between the macro-distribution f^0 and the collision operator C' that depends on $\nabla P'$ and $\omega_0 \underline{u}' \times \hat{z}$. Thus the pressure gradient and the uniform rotation which were missing in the traditional vorticity model (1.3) are recovered in the renormalized vorticity equation (5.3).

Upon multiplying Eq. (5.3) by ζ^0 and averaging, we find the equation of evolution

11

$$\tfrac{1}{2} \partial_t < \zeta^{0\,2} > \; = \; - \beta < u_2^0 \, \zeta^0 > \; - \nu < (\underline{\nabla} \zeta^0)^2 > \; - T_\zeta^0 \qquad (5.4)$$

for the enstrophy $\tfrac{1}{2} < \zeta^{0\,2} >$. The transport functions are: the drift function

$$W^0 \; = \; - \beta < u_2^0 \, \zeta^0 > \qquad (5.5)$$

the transfer function

$$T_\zeta^0 \; \equiv \; - < \zeta^0 \, J_\zeta^0 > \; = \; - < \zeta^0 \underline{\nabla} \times \underline{J}_u^0 > \qquad (5.6)$$

and the dissipation function $\nu < (\nabla \zeta^0)^2 > \,$.

5.2. Drift function

The flux $- < u_2^0 \, \zeta^0 >$ is calculated by integrating Eq. (5.3) for the evolution of ζ^0 rewritten as

$$(\partial_t + \underline{u}^0 \cdot \underline{\nabla} - \nu \nabla^2 - C')\zeta^0 \; = \; -\beta \, u_2^0 \qquad (5.7)$$

with a friction constant $C'(t,\underline{x})$. An integration with respect to time along the perturbed trajectory, a multiplication by $\underline{u}^0$ and an ensemble average yield the flux

$$- < u_2^0 \, \zeta^0 > \; = \; \beta \, K_d^0 \quad , \qquad (5.8)$$

transforming (5.5) into the drift function

$$W^0 \; = \; \beta^2 \, K_d^0 \quad . \qquad (5.9)$$

The drift diffusivity

$$K_d^0 = \int_0^\infty d\tau < u_2^0(t) u_2^0(t-\tau) >$$

$$= \int_0^\infty d\tau \int d\underline{k}' \; X < u_2^0(\underline{k}') u_2^0(-\underline{k}') > h_\ell(\tau, k') \quad . \tag{5.10}$$

is controlled by the orbit function

$$h_\ell(\tau, \underline{k}') = [\exp C'(\underline{k}')] < \exp -i\underline{k}' \cdot \underline{\ell}'(\tau) > \tag{5.11}$$

for relaxation.

5.3. <u>Transfer function</u>

The transfer function (5.6) can be written in the form

$$T_\zeta^0 = - \underline{\nabla} \times \int d\underline{v} \; \underline{v} \; (\underline{\partial} \cdot \underline{\underline{D}}' \cdot \underline{\partial}) \{ < \zeta^0(t) f^0(t-\tau) > \}$$

$$= - \int d\underline{v} \; \underline{v} \iint d\underline{k}'' \; d\underline{k}' \; X < \underline{E}_c'(\underline{k}'') \underline{E}_c'(-\underline{k}'') > h_\ell(\tau, \underline{k}'') h_\ell(\tau, \underline{k}')$$

$$\int_0^\infty d\tau \; M^0(\tau, \underline{k}'', \underline{k}', \underline{v}) \{ X < \zeta^0(t, \underline{k}') f^0(t, -\underline{k}', \underline{v}) > \} \quad ,$$
$$\tag{5.12}$$

by (5.2), (3.4), (4.6a) and by Fourier transformation. The memory operator is

$$M^0\{ \} = \partial_j h_c(\tau, \underline{k}'', \underline{v}) \partial_j h_c(\tau, \underline{k}', \underline{v}) \{ \} \quad . \tag{5.13}$$

In the differentiation $\partial_j \, \partial_j$ there are terms which cancel by integrations with respect to $d\underline{v} \, d\underline{k}'' d\underline{k}'$. The remaining terms give the following memory operators:

(i) <u>for direct cascade</u>

$$M^0\{ \} = h_c(\tau, \underline{k}'', \underline{v}) \partial^2 h_c(\tau, \underline{k}', \underline{v})\{ \}, \text{ with } \partial^2 h_c(\tau, \underline{k}', \underline{v}) = -m^2 k'^2 h_c(\tau, \underline{k}' \underline{v}) \tag{5.14a}$$

13

$$\lim_{y=0} M^0\{\} = - m^2 k'^2, \quad m = \tfrac{1}{4}\omega^0 \tau^2 ,$$

(5.14b)

by (4.7a);

(ii) <u>for inverse cascade</u>

$$M^0\{\} = [\, \partial^2 h_c(\tau,\underset{\sim}{k}'',v)\,] h_c(\tau,\underset{\sim}{k}',v)$$

(5.15a)

$$\lim_{v=0} M^0\{\} = m^2 k''^2 .$$

(5.15b)

With the loss of memory by omitting the v-dependence, we tranform the memory operator into (5.14b) and (5.15b), and reduce the transfer function (5.12) into

$$T_\zeta^0 = - \int_0^\infty d\tau \iint d\underset{\sim}{k}'' \, d\underset{\sim}{k}' \; \text{tr} X < E_c'(\underset{\sim}{k}'') \, E_c'(-\underset{\sim}{k}'') > h_\ell(\tau,\underset{\sim}{k}'')$$

$$\times m^2 \begin{pmatrix} -k'^2 \\ k''^2 \end{pmatrix} X < \zeta^0(\underset{\sim}{k}') \, \zeta^0(-\underset{\sim}{k}') >$$

$$= - \int d\underset{\sim}{k}' \; D'\{ \, m^2 \begin{pmatrix} -k'^2 \\ k''^2 \end{pmatrix} X < \zeta^0(\underset{\sim}{k}') \zeta^0(-\underset{\sim}{k}') > \}$$

$$= \begin{vmatrix} D'\{m^2\} \; R_\zeta^0 \,, & \text{with } R_\zeta^0 = < (\underset{\sim}{\nabla} \zeta^0)^2 > \\[2mm] -D'\{m^2 k''^2\} < \zeta^{02} > \,, & \text{with } < \zeta^{02} > = \int d\underset{\sim}{k}' \; X < \zeta^0(\underset{\sim}{k}') \zeta^0(-\underset{\sim}{k}') > \end{vmatrix}$$

$$= \begin{vmatrix} K' R_\zeta^0 \,, & \text{direct cascade} \\[4mm] - \lambda' < \zeta^{02} > \,, & \text{inverse cascade} . \end{vmatrix}$$

(5.16a)

(5.16b)

Here

$$\underset{\approx}{D'} = \int_0^\infty d\tau \int d\underset{\sim}{k}'' \; \text{tr} X < E_c'(\underset{\sim}{k}'') \, E_c'(-\underset{\sim}{k}'') > h_\ell(\tau,\underset{\sim}{k}'')$$

(5.17a)

14

$$K' \equiv D'\{m^2\} = \int dk'' \; \mathrm{tr} \, \chi \; < E'_c(k'') \; E'_c(-k'') > \int_0^\infty d\tau \; m^2(\tau) h_\ell(\tau, k'') \qquad (5.17b)$$

$$\lambda' = K'\{k''^2\} = \int dk'' \; k''^2 \; \mathrm{tr} \, \chi < E'_c(k'') E'_c(-k'') > \int_0^\infty d\tau \; m^2(\tau) h_\ell(\tau, k'') \qquad (5.17c)$$

are called path diffusivity, eddy viscosity, and coefficient of am-
plification, respectively.

The results (3.4), (5.2) and (5.16a) indicate an equivalence

$$C'(t, x, v) = C'(t=\infty, x) = K' \nabla^2 \qquad (5.18a)$$

between the integral operator $C'(t, x, v)\{ \; \}$ and the differential
operator $C'(t=\infty, x)$, provided the eddy viscosity is computed from
the eddy collision of the kinetic equation. In an analogous way, we
have

$$C''(x) = K'' \nabla^2 \qquad \text{and} \qquad C''(k) = -k^2 K'' \; . \qquad (5.18b)$$

6. Subdynamics of relaxation

The principal aim of the subdynamics is to determine the path
diffusivity K''_ℓ and the orbit function $h''_\ell(\tau, k'')$ to find a relaxa-
tion for the eddy coefficients to approach equilibrium. To this
end, we consider the Fokker-Planck equation of transition and a
system of two Langevin equations of turbulence.

15

6.1. Fokker-Planck equation of path transition

The probability for the path to be in the interval ℓ and $\ell + d\ell$ in a time interval τ is $p(t,x;t-\tau, x-\ell)d\ell$, or $p(\tau,\ell)d\ell$ by abbreviation. The probability density $p(\tau,\ell)$ satisfies the condition of normalization and is governed by the Fokker-Planck equation (Tchen, 1944).

$$\frac{\partial}{\partial \tau} p(\tau,\underset{\sim}{\ell}) = K_\ell'' \frac{\partial^2}{\partial \ell^2} p(\tau,\underset{\sim}{\ell}) \quad . \tag{6.1}$$

The path diffusivity K_ℓ'' as defined by (4.8) is adiabatic, _i.e._ independent of τ and $\underset{\sim}{\ell}$. The Fourier form is

$$\frac{\partial}{\partial \tau} p(\tau,\underset{\sim}{k''}) = - k''^2 K_\ell'' \; p(\tau,\underset{\sim}{k''}) \tag{6.2}$$

The solution

$$p(\tau,\underset{\sim}{k''}) = \frac{1}{(2\pi)^d} \exp(-k''^2 K_\ell'' \tau) \tag{6.3}$$

in $d = 2$ dimensions determines the orbit function

$$h''(\tau,\underset{\sim}{k''}) \equiv \; < \exp -i\underset{\sim}{k''}\cdot\underset{\sim}{\ell}(\tau) > \; = \int d\underset{\sim}{\ell} \; e^{-i\underset{\sim}{k''}\cdot\underset{\sim}{\ell}} p(\tau,\underset{\sim}{\ell}) = (2\pi)^d p(\tau,\underset{\sim}{k''})$$

$$= (2\pi)^d p(\tau,\underset{\sim}{k''}) = \exp(-k''^2 K_\ell'' \tau) \tag{6.4}$$

6.2. Langevin equations of turbulence

The orbit function $h_\ell(\tau,\underset{\sim}{k''})$ depends on the path diffusivity K_ℓ'' and governs the relaxation for the approach of the ω_0-dependent eddy viscosity to equilibrium. Note that β governs the drift in the macrogroup transport (5.9) and does not enter into the subdynamics.

We consider a system of two Langevin equations

$$\frac{du''_G}{dt} + \gamma_G \, u''_G = G''$$

(6.5a)

$$\frac{du''}{dt} + \gamma_G \, u'' = E''_{cG} \, .$$

(6.5b)

The first Langevin equation has a random noise G'' with an auto-correlation

$$\text{tr} < G''(0)G''(\tau) > = \pi^{-1}D''\delta(\tau)$$

(6.6)

that depends on the field diffusivity D''. It determines the friction constant γ_G for the second Langevin equation to analyse k''_ℓ and calculate h''_ℓ. In the second Langevin equation, the field is

$$E''_{cG} = \omega_0 \, u''_G \times \hat{z}_v \, ,$$

(6.7)

by (1.2b). The system is immersed in a quasi-geostrophic turbulence, labeled by the suffix $(\ )_G$ as the background medium with a spectrum and an intensity of the form

$$F(k) \Big|_G = C_G \, \varepsilon_\zeta^{2/3} k^{-3} \quad \text{and} \quad < u''^2_G > = C_G \, \varepsilon_\zeta^{2/3} k^{-2} \, .$$

(6.8)

Here $\varepsilon_\zeta = \nu <(\nabla\zeta)^2>$ is the rate of dissipation and C_G is a numerical constant.

6.3. Determination of the friction constant γ_G from the first Langevin equation

Upon multiplying the first Langevin equation (6.5a) throughout by the path length ℓ_G'' and averaging, we have

$$\frac{1}{2}\frac{d^2}{dt^2}<\ell_G''^2> - \left(\frac{d\ell_G''}{dt}\right)^2 + \gamma_G <u_G''\cdot\ell_G''> = 0$$

or

$$- <u_G''^2> + 2\,\gamma_G\,K_{\ell G}'' = 0 \quad,$$

and obtain the relationship between the path diffusivity $K_{\ell G}''$ and the friction constant γ_G in the form

$$K_{\ell G}'' = \tfrac{1}{2}\,\gamma_G^{-1} <u_G''^2>$$

$$= \tfrac{1}{2}\,C_G\,\epsilon_\zeta^{2/3}\,\gamma_G^{-1}k^{-2} \quad, \tag{6.9}$$

by (6.8). We have written $K_{\ell G}'' = \mathrm{tr}<u_G''\,\ell_G''>$. The term $\tfrac{1}{2}\,d^2<\ell_G''^2>/dt^2$ is negligible in statistically steady turbulence.

On the other hand, we calculate

$$K_{\ell G}'' = \int_0^\infty d\tau \int dk''\,\mathrm{tr}\,X <u_G''(k'')u_G''(-k')> h_{\ell G}(\tau,k'')$$

$$= \int dk''\,\mathrm{tr}\,X <u_G''(k'')u_G''(-k'')> \int_0^\infty d\tau\,\exp(-k''^2\,K_{\ell G}''\tau)$$

$$= \int dk''\,\mathrm{tr}\,X <u_G''(k'')u_G''(-k'')> (k''^2 K_{\ell G}'')^{-1} \quad, \tag{6.10a}$$

by (4.8) and (4.7b). The integral equation gives the solution

$$K_{\ell G}^{''} = [\, 2 \int_k^\infty dk^{''} \; k^{''-2} \; F(k^{''}) \,]_G^{\frac{1}{2}}$$

$$= (\tfrac{1}{2} C_G \epsilon_G^{2/3})^{\frac{1}{2}} k^{-2} \quad , \tag{6.10b}$$

by (6.8).

By a comparison between (6.9) and (6.10b) the friction constant is determined as

$$\gamma_G = (\tfrac{1}{2} C_G \; \epsilon_G^{2/3})^{\frac{1}{2}} \quad . \tag{6.11}$$

6.4. <u>Determination of the relationship between $K_{\ell}^{''}$ and $D^{''}$ from the second Langevin equation</u>

From the second Langevin equation (6.5b), we derive the relationship

$$K_\ell^{''} = \gamma_G^{-2} \; D^{''} \tag{6.12}$$

between the path diffusivity $K_\ell^{''}$ and the field diffusivity

$$D^{''} = \int dk^{''} \; \mathrm{tr} \; \chi < E_{cG}^{''}(k^{''}) \; E_{cG}^{''}(-k^{''}) > \int_0^\infty d\tau \; h_\ell(\tau, k^{''})$$

$$= \int_k^\infty dk^{''} \; F_c(k^{''}) \Big|_G (k^{''2} \; K_\ell^{''})^{-1} \quad , \tag{6.13}$$

by (6.4). The detail of calculations has been omitted. The spectrum $F_c(k)\big|_G$ has been introduced such that

$$\int dk^{''} \; \mathrm{tr} \; \chi < E_{cG}^{''}(k^{''}) E_{cG}^{''}(-k^{''}) > = \int_k^\infty dk^{''} \; F_c(k^{''}) \Big|_G \quad .$$

The system of equations (6.12) and (6.13) are solved, determining

$$D^{''} = [\, 2 \gamma_G^2 \int_k^\infty dk^{''} \; k^{''-2} F_c(k^{''}) \,]_G^{\frac{1}{2}}$$

$$\tag{6.14a}$$

19

$$K''_\ell = \gamma_G^{-1} [\, 2 \int_k^\infty dk'' \; k''^{-2} \; F_c(k'') \,]_G^{\frac{1}{2}} \quad . \tag{6.14b}$$

By writing $F_c(k)\big|_G = \omega_0^2 \; F(k)\big|_G$ from (6.7) and (6.8), we calculate (6.14) and find

$$D'' \cong \gamma_G^2 \; \omega_0 \; k^{-2} \tag{6.15a}$$

$$K''_\ell \cong \omega_0 \; k^{-2} \quad . \tag{6.15b}$$

It follows

$$h''_\ell = \exp(-\omega_0 \tau), \tag{6.16a}$$

$$\int_0^\infty d\tau \; h''_\ell(\tau) = \omega_0^{-1}, \qquad \int_0^\infty d\tau \; \tau^4 \; h''_\ell(\tau) = 24 \; \omega_0^{-5} \;, \tag{6.16b}$$

by (6.4).

7. Eddy coefficients

The eddy viscosity K' and the eddy coefficient λ' from (5.17b) and (5.17c), determine the transfer function in direct cascade (5.16a) and inverse cascade (5.16b), while the eddy diffusivity K_d^0 in (5.10) determines the drift function (5.9). These eddy coefficients approach their equilibrium by relaxation through the orbit functions $h_\ell(\tau, k'')$ and $h''_\ell(\tau, k'')$ which differ by $C''(k'') = -k''^2 K''$, from (5.11) and (5.18b). However, an inspection of (5.17b) and (5.14b) reveals that K'' is inversely proportional to ω_0 while K''_ℓ is proportional to ω_0 by (6.15b), so that the contribution of C'' becomes negligible, approximating

$$h_\ell \cong h''_\ell \tag{7.1}$$

7.1. Field diffusivity

The field diffusivity $\underset{\approx}{D}'(t,x)$ is defined by (4.6). By writing the field intensity $tr\,X < \underset{\sim}{E}'_c(\underset{\sim}{k}')\,\underset{\sim}{E}'_c(-\underset{\sim}{k}')> = \omega_0^2\,tr<\underset{\sim}{u}'(\underset{\sim}{k}')\underset{\sim}{u}'(-\underset{\sim}{k}')>$ and by the use of (6.16b) we get

$$D'(t,x)\{\} = \omega_0 \int dk''\; tr\,X < \underset{\sim}{u}'(\underset{\sim}{k}'')\underset{\sim}{u}'(-\underset{\sim}{k}'')>\{\}$$

$$= \omega_0 \int_k^\infty dk''\; F(k'')\{\}\;. \tag{7.2}$$

7.2. Eddy viscosity K'

The eddy viscosity as defined by (5.17b) is $K' = D'(t,x)\{m^2\}$, with $m^2 = (\tfrac{1}{4}\,\omega_0\,\tau^2)^2$, by (5.14b), or

$$K' = (\tfrac{1}{4}\,\omega_0)^2\; D'\{\tau^4\}$$

$$= (\tfrac{1}{4}\,\omega_0)^2 \int_k^\infty dk''\; F(k'') \int_0^\infty d\tau\; \tau^4 h''_\ell(\tau,k'')$$

$$= \tfrac{3}{2}\,\omega_0^{-1} \int_k^\infty dk''\; F(k'')\;, \tag{7.3}$$

by (7.2) and (6.16 b).

7.3. Coefficient of amplification λ'

The eddy coefficient λ' as defined by (5.17c) is calculated from the integral operator $K'\{\;\}$. Thus we find

$$\lambda' = \tfrac{3}{2}\,\omega_0^{-1} \int_k^\infty dk''\; k''^2\, F(k'') = \tfrac{3}{4}\,\omega_0^{-1} R'(k)\,, \tag{7.4}$$

21

from (7.3), with the vorticity function $R' = 2 \int_k^\infty dk'' \, k''^2 F(k'')$.

7.4. Drift diffusivity κ_d^0

The drift diffusivity κ_d^0 is defined by (5.10) and can be calculated in the same manner as for (7.3). We find

$$\kappa_d^0 = 2 \, \omega_0^{-1} \int_0^k dk' \, F_{22}(k') , \tag{7.5}$$

where $F_{22}(k)$ is the spectrum such that $< u_2^0 u_2^0 > = 2 \int_0^k dk' \, F_{22}(k')$.

The expressions (7.3) – (7.5) show that the eddy coefficients κ', λ' and κ_d^0 are proportional to ω_0^{-1} .

8. Spectral structure of Rossby wave turbulence

We consider three ranges of Rossby wave turbulence: (i) inertial-range by direct cascade, (ii) drift-range by inverse cascade, and (iii) transition-range from direct cascade to inverse cascade in the direction of decreasing k.

8.1. Inertial-range by direct cascade

In the inertial-range, the transfer occurs at the constant rate of enstrophy dissipation ϵ_ζ and is governed by

$$\kappa' R_\zeta^0 = \epsilon_\zeta \tag{8.1}$$

by (5.16a), where κ' is given by (7.3) and $R_\zeta^0 = \int_0^k dk' k'^4 F(k')$.
In terms of $F(k)$, Eq. (8.1) is

$$\frac{3}{2}\,\omega_0^{-1}\int_k^\infty dk''\,F(k'')\int_0^k dk'\,k'^4\,F(k') = \epsilon_\zeta\,.\tag{8.2}$$

Upon dividing both members by any one of the integrals and differentiating, we solve the integral equation and find the spectrum

$$F(k) = (8/3)^{\frac{1}{2}}\,(\omega_0\,\epsilon_\zeta)^{\frac{1}{2}}\,k^{-3}\tag{8.3}$$

and the intensity

$$< \underline{u}'^2 > = (8/3)^{\frac{1}{2}}\,(\omega_0\,\epsilon_\zeta)^{\frac{1}{2}}\,k^{-2}\,.\tag{8.4}$$

The eddy viscosity is

$$K' = (3/2)^{\frac{1}{2}}\,(\epsilon_\zeta/\omega_0)^{\frac{1}{2}}\,k^{-2}\,,\tag{8.5}$$

by (7.3). The spectrum (8.3) should not be confounded with the spectrum of quasi-geostrophic turbulence (6.8) although the two spectra share the same power law.

8.2. Drift-range

The large-scale structure formulates a drift k^0, while the small-scale transients perform an inverse cascade to amplify the enstrophy at the rate λ'. The maintenance of the large-scale structure is described by the balance between the drift and the cascade, a phenomenon called "blocking". The balance in the differential form is

$$\frac{dk^0}{dk} - \frac{dT_\zeta^0}{d(-k)} = 0\,.\tag{8.6}$$

Here d/dk and $d/d(-k)$ are differentiations in the direction of increasing and decreasing k, respectively.

23

We write the drift function

$$W^0 \equiv \beta^2 K_d^0 = 2 \beta^2 \omega_0^{-1} \int_0^k dk' \, F_{22}(k') \quad , \tag{8.7}$$

by (5.9) with K_d^0 given by (7.5), and the transfer function

$$T_\zeta^0 = - \lambda' \langle \zeta^{02} \rangle = - \frac{3}{8} \omega_0^{-1} R'(k') R^0(k') \, , \tag{8.8}$$

by (5.16b) with λ' given by (7.4). Here $\langle \zeta^{02} \rangle = \frac{1}{2} R^0$.

Note

$$dW^0/dk = 2 \beta^2 \omega_0^{-1} F_{22}(k) \tag{8.9a}$$

and

$$dT_\zeta^0/d(-k) = \frac{3}{4} \omega_0^{-1} k^2 F(k)[R'(k) - R^0(k)]$$

$$\simeq \frac{3}{4} \omega_0^{-1} k^2 F(k) R'(k) \quad , \tag{8.9b}$$

because $R^0 \ll R'$ in the region of small k where the blocking occurs.

If we confine the transfer to the spectral component $F_{22}(k)$ along which the drift operates, the spectral balance (8.6) takes the form

$$R'(k) \equiv \langle (\nabla_j u_2')^2 \rangle = \frac{8}{3} \beta^2 k^{-2} \quad , \tag{8.10}$$

after simplification by (8.9a) and (8.9b). The integral equation is solved, giving the spectrum

$$F_{22}(k) = \frac{8}{3} \beta^2 k^{-5} \tag{8.11}$$

and the intensity

$$\langle u_2'^2 \rangle = \frac{4}{3} \beta^2 k^{-4} \quad . \tag{8.12}$$

24

The eddy coefficient is

$$\lambda' = \omega_0^{-1} \beta^2 k^{-2} \quad , \tag{8.13}$$

by (7.4).

Formula (8.12) indicates that the drift turbulence is limited to a

length $\frac{3}{4} \pi^{-1} (<u_2'^2> /\beta^2)^{\frac{1}{4}}$.

8.3. Transition-range

A spectral gap lies between the inertial-range and the drift-range. It describes the transition from the direct cascade at rate ϵ_ζ to the inverse cascade $\lambda'<\zeta^{02}>$, and is governed by the balance between the two cascades in the form

$$\lambda'<\zeta^{02}> = \epsilon_\zeta \quad . \tag{8.14}$$

This equation governs the enstrophy transfer in a background medium having a transport property λ' by (8.13). By writing $<\zeta^{02}> = \frac{1}{2}R^0 = \int_0^k dk' k'^2 F(k')$, we solve Eq. (8.14) to obtain

$$F(k) = 2(\omega_0 \epsilon_\zeta/\beta^2) k^{-1} \quad . \tag{8.15}$$

By a comparison between the two spectra (8.11) and (8.15), the critical wavenumber that separates the transition-range from the drift-range is found to be

$$k_c = (4/3)^{\frac{1}{4}} \beta(\omega_0 \epsilon_\zeta)^{-\frac{1}{4}} \quad . \tag{8.16}$$

The critical wavenumber also determines the passage from isotropy to anisotropy.

25

9. Summary and discussions

The Navier-Stokes equation with the Coriolis force is trans-
formed into a master equation to describe the microdynamic state of
turbulence. The group-kinetic method describes the three groups of
transport processes (spectral evolution, eddy coefficients and
relaxation). In the macrogroup dynamics, we derive the kinetic equa-
tion of turbulence (3.6), the equations of evolution for the vor-
ticity (5.3) and the enstrophy (5.4), and their governing transport
functions of transfer T_ζ^0 and drift W^0 by (5.16) and (5.9). In the
microgroup dynamics, we derive the eddy coefficients consisting of
the eddy viscosity K' for direct cascade, the eddy coefficient of
amplification λ' for inverse cascade, and the diffusivity K_d^0 for
drift turbulence. These coefficients are calculated from the colli-
sion integral of the kinetic equation. The subdynamics treats the
mechanism of relaxation by which the eddy coefficients approach
their equilibrium. The closure of turbulence is obtained by the
loss of memory. The subdynamics consists of a Fokker-Planck equa-
tion (6.1) for path transition and a system of two Langevin equa-
tions of turbulence (6.5a) and (6.5b).

All the eddy coefficients K', λ' and K_d^0 are found in (7.3) –
(7.5) to be inversely proportional to rotation ω^0. The dependence
of K' on ω_0^{-1} has been suggested by Blackadar (1962) empirically.

The spectral structure of Rossby wave turbulence is investigated
for the inertial-range and the drift-range as characterized by a
direct cascade and an inverse cascade, respectively. The spectral
gap as formed by the transition between the two cascades is also
examined. The corresponding spectral laws k^{-3}, k^{-5} and k^{-1} are found

26

in (8.3), (8.11) and (8.15).

The spectrum k^{-3} in (8.3) and the spectrum k^{-3} of geostrophic turbulence (6.8) have separate parameters and should not be confounded. This difference cannot be revealed by a theory from the vortex model (1.4).

It is to be remarked that by the cancellation of ω^0 and the subsequent constraint to quasi-geostrophic turbulence in the traditional vortex model (1.4), the small eddies cannot organize the ω^0-dependent cascades to analytically treat the inertial-range and the transition range. The drift function cannot find a cascade for balance to derive the drift-range. This explains why no theoretical treatment has succeeded in attacking the Rossby wave turbulence.

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REFERENCES

BAER, F. 1972 An alternate scale representation of atmospheric energy spectra. J. Atmos. Sci., 29, 649-664.

BLACKADAR, A. K. 1962 The vertical distribution of wind and turbulent exchange in a neutral atmosphere. J. Geophys. Res. 67, 3095-3102.

BOGOLIUBOV, N. N. 1962 Problems of a dynamical theory in statistical physics. In: Studies in Statistical Mechanics. J. H. de Boer and G. E. Uhlenbeck, eds., Vol. 1, North-Holland, Amsterdam.

CHAPMAN, S. & COWLING, T. G. 1970 The Mathematical Theory of Non-Uniform Gases. Cambridge University Press, 110-131.

CHARNEY, J. G. 1971 Geostrophic turbulence. J. Atmos. Sci. 28, 1087-1095.

DUBOIS, D. F. & ESPEDA, M. 1978 Direct interaction approximation and plasma turbulent theory. Plasma Phys. 20, 1209-1239.

HOLLOWAY, G. & HENDERSHOTT, M. C. 1977 Stochastic closure of nonlinear Rossby wave. J. Fluid Mech. 82, 747-765.

KRAICHNAN, R. H. 1959 The structure of isotrophic turbulence at very high Reynolds numbers. J. Fluid Mech. 5, 497-513.

KRAICHNAN, R. H. 1977 Eulerian and Lagrangian renormalization in turbulent theory. J. Fluid Mech. 83, 349-374.

LAMBERT, S. J. 1981 A diagnostic study of global energy and enstrophy fluxes and spectra. Tellus 33, 411-414.

MARTIN, P. C., SIGGIA, E. D. & ROSE, H. A. 1973 Statistical dynamics of classical systems. Phys. Rev. A8, 423-437.

RHINES, P. B. 1975 Waves and turbulence on a β-plane. J. Fluid Mech. 69, 417-443.

SALMON, R. 1978 Two-layer quasi-geostrophic turbulence in a simple case. Geosphys. Astrophys. Fluid Dyn. 10, 25-51.

SALMON, R., HOLLOWAY. G. & HENDERSHOTT, M. S. 1976 The equilibrium statistical mechanics of simple quasi-geostrophic models. J. Fluid Mech. 75, 691-703.

TCHEN, C. M. 1944 Enige wiskundige betrekkingen welke een rol spelen in diffuse-problemen (Some mathematical relations which play a role in diffusion problems). Versl. Ned. Akad. Wetensch. Afd. Natuurkunde, 53, 400-410.

TCHEN, C. M. 1978 Modéle cinétique de turbulence dans un fluide incompressible. C. R. Acad. Sc. Paris, B 287, 175-178.

TCHEN, C. M. 1984 Theory and Modeling of Atmospheric Turbulence, Parts I and II. NASA Contractor Report 3787 and 3817.

TCHEN, C. M. 1986 Transport by negative eddy viscosity in soliton turbulence. Physica Scripta, 34, 295-303.
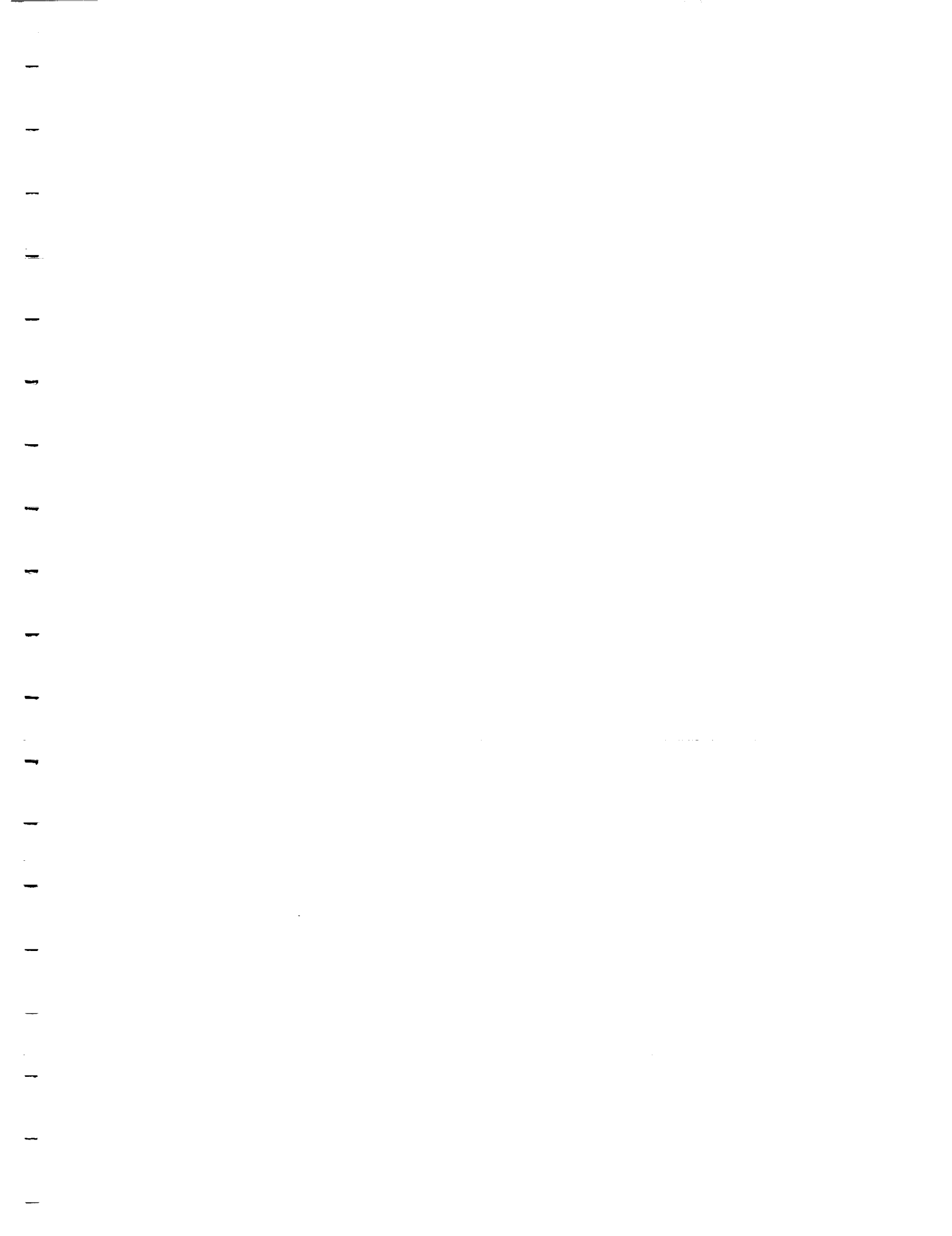
GROUP-KINETIC METHOD OF TURBULENCE

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The inhomogeneous Navier-Stokes equation is transformed into a master equation in the phase space for the description of the microdynamical state of turbulence. The elementary interaction among the fluid elements is represented by the pressure-gradient. For the three transport processes of spectral evolution, eddy viscosity and relaxation, a macrogroup dynamics, a microgroup dynamics and a subdynamics are developed. The kinetic equation of turbulence is derived. The eddy viscosity is calculated from the "eddy collision". The simulation of the subcluster of many high-order distributions by an effective fluid medium causes a loss of memory. This mechanism of relaxation for the approach of eddy viscosity to equilibrium solves the closure problem. The subdynamics consists of a Fokker-Planck equation of path transition and a Langevin equation of turbulence for the determination of the orbit function and the path diffusivity. Although the Fourier modes interact indefinitely, their statistical quantities interact by nearest neighbour-groups. By this property the high-order correlations enter in the form of a product of pair correlations without invoking the hypothesis of normality or relying on perturbation expansion and subsequent summation.

The spectral results are found as follows: (i) In the inertial range of the Navier-Stokes turbulence, the energy spectrum is

1

$F(k)=1.650 \ \varepsilon^{2/3}k^{-5/3}$, and the pressure variance spectrum is $S(k)=0.907\rho^2\varepsilon^{4/3}k^{-7/3}$, where ε is the energy dissipation rate, and ρ is the fluid density. (ii) In the convective-inertial range of heat diffusion, the temperature variance spectrum is $F_\theta(k)=0.899\varepsilon_\theta\varepsilon^{-1/3}k^{-5/3}$, where ε_θ is the rate of dissipation of temperature variance.

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1. Introduction

The derivation of the energy spectrum has attracted the attention of many authors. Kolmogoroff (1941) presented a dimensional argument, and Heisenberg (1948) proposed an energy transfer from the large eddies that present a macro-gradient toward the small eddies that organize an eddy viscosity. A particularly appealing method has been the renormalization. The group-renormalization transformation has succeeded in finding a correction to the 5/3 spectrum from intermittencies of turbulence but not in deriving the 5/3 spectrum itself (Grossmann & Schnedler, 1977). This correction is nevertheless too small to be verified by direct measurements. The method of renormalization perturbation expansion has enabled Kraichnan (1959, 1977) to develop his DIA method and generalizations. A systematic method of developing the expansion was given in a general form by Martin, Siggia & Rose (1972), and by Dubois & Espedal (1978). However, the following difficuties remain: The necessity of the molecular viscosity and an artificial driving force as the basis of perturbation expansion, the uncertainty of convergence, the validity of the lowest-order expansion which amounts to a quasilinear theory, and the hypothesis of normality for factorizing

a quadruple correlation.

In order to alleviate these difficulties and clarifying the physical roles played by the mathematical operators and functions (e.g. evolution operator, collision operator, diffision operator and orbit functions), we develop a kinetic method based upon the transformation of the Navier-Stokes equation into the kinetic equation of turbulence. We calculate the eddy viscosity from the eddy collision-integral, not by perturbation as was done in the Chapman-Enskog method for the derivation of the molecular viscosity from the Boltzmann equation (Chapman & Cowling, 1939). With the knowledge of the eddy viscosity, we derive the spectral structure.

As the scope of research, we let the pressure-gradient define the elementary interaction between fluid elements. The inhomogeneous partial differential equation of Navier-Stokes is then transformed into a homogeneous master equation in the phase space. The master equation has lesser nonlinearity.

The decomposition into a macro-distribution f^0, a micro-distribution f' and a sub-distribution f'' represents the three transport processes of spectral evolution, eddy viscosity and relaxation (Section 2). This decomposition is superior to the reduction of the N-body distribution function into singlet-,binary-, and triplet-distribution functions in many-body statistical mechanics (Bogoliubov, 1962), because the singlet-distribution f^0 in the group form suffices for describing the spectral function, and the groups interact as nearest-neighbours statistically,

For analysing the three transport processes, we develop a macro-

3

group dynamics for f^0 a microgroup dynamics for f' and a sub-dynamics of relaxation. The kinetic equation of turbulence for f^0 is derived in Section 3, and the eddy viscosity is found in Section 4. The eddy viscosity approaches its equilibrium by relaxation for which a subdynamics is developed and consists of a Fokker-Planck equation of transition and a Langevin equation of turbulence. The former equation governs the path fluctuations and determines the orbit function, and the latter finds a path diffusivity (Section 5). In sum, the macrogroup dynamics determines the spectral evolution from the kinetic equation, the microgroup dynamics and the subdynamics produce a system of integral equations for the eddy viscosity and the path diffusivity. The conversion of pressure field fluctuations into velocity fluctuations is made in Section 6. In Section 7, we derive the energy spectrum $k^{-5/3}$, the temperature variance spectrum $k^{-5/3}$, and the pressure variance spectrum $k^{-7/3}$. A summary and a discussion are presented in Section 8.

2. Group-kinetic method

Fluid turbulence is described by the Navier-Stokes equation

$$(\partial_t + \hat{u} \cdot \nabla - \nu \nabla^2)\hat{u} = \hat{E}, \qquad \partial_t \equiv \partial/\partial t \tag{2.1}$$

for the fluid velocity $\hat{u}$ in a driving field $\hat{E} = -\frac{1}{\rho}\nabla\hat{p}$. Here $\hat{p}$ is the pressure, ρ is the density, and ν is the molecular viscosity. The incompressible fluid satisfied the equation of continuity $\nabla \cdot \hat{u} = 0$. An instantaneously fluctuating function is denoted by $\hat{(\ )}$. For raising the dimensionality we introduce $\hat{f}(t,x,v) = \delta[v - \hat{u}(t,x)]$ and transform the inhomogeneous Navier-Stokes equation into the

4

homogeneous master equation

$$(\partial_t + \hat{L})\,\hat{f} = 0. \tag{2.2}$$

The normalization is $\int dv\,\hat{f} = 1,$ and the differential operator is $\hat{L} = v\cdot\nabla - \nu\nabla^2 + \hat{E}\cdot\partial$ with $\partial \equiv \partial/\partial v$ The master equation describes the microdynamic state of turbulence. A scaling is necessary for statistical treatment. We decompose the instantaneously fluctuating function $\hat{f} = \bar{f} + \tilde{f}$ into an ensemble average $\bar{f} \equiv \langle\hat{f}\rangle$, and a fluctuation $\tilde{f}$. We represent the three transport processes of spectral evolution, eddy viscosity and relaxation by a macrogroup f^0, a microgroup $f' = f^1 + f''$ and a subgroup $f'' = f^{(2)} + f^{(3)} + \cdots$, respectively. The operators $\bar{A}$, $\tilde{A}$, A^0, A', A'' can be used. The groups are in the order of decreasing coherence with durations of correlation $t > \tau_c^0 > \tau_c' > \tau_c''$.

Since the spectrum evolves in a medium of eddy viscosity, and the latter approaches its equilibrium by relaxation, the three transport processes, together with the three groups that represent them, interact by nearest-neighbours. This property in association with separated coherence constitutes the "nearest neighbour group-interaction". Consequently, a quadruple correlation appears in the form of a product of two binary correlations as

$$\langle E^1(t)E^1(t-\tau)\rangle \langle f^0(t)f^0(t-\tau)\rangle = \langle E'(t)E'(t-\tau)\rangle \langle f^0(t)f^0(t-\tau)\rangle , \tag{2.3}$$

and not in the form $\langle \bar{E}(t)\bar{E}(t-\tau)\tilde{f}(t)\tilde{f}(t-\tau)\rangle$ that requires the hypothesis of normality for factorization.

There is a certain analogy between the decomposition of the distribution $\tilde{f}$ into groups f^0, f', f'' and the reduction of the

5

many-point distribution $F_{12...N}$ into the singlet-distribution F_1, the pair-distribution F_2, and the triplet-distribution F_3 (Bogoliubov, 1962). Both the decomposition and the reduction represent the three transport processes of evolution, transport property and relaxation that interact as nearest neighbours. However, a difference exists in the determination of the spectrum: a kinetic equation for f^0 suffices in the first case, and a system of two kinetic equations for F_1 and F_2 is required in the second case.

The instantaneously fluctuating groups can be transformed into Fourier modes with overlapping wavenumbers, but the statistical groups are separated by adjacent wavenumber limits $0 \leqslant k' \leqslant k$, $k \leqslant k'' \leqslant \infty$, $k^\nu \leqslant k''' \leqslant \infty$ as demarcation. Thus

$$< \underset{\sim}{E^0} \underset{\sim}{E^0}> = \int d\underset{\sim}{k}' \; \chi <\underset{\sim}{E^0}(\underset{\sim}{k}') \; \underset{\sim}{E^0}(-\underset{\sim}{k}')>$$

$$< \underset{\sim}{E}' \underset{\sim}{E}'> \{\} = \int d\underset{\sim}{k}'' \chi <\underset{\sim}{E}'(\underset{\sim}{k}'') \; \underset{\sim}{E}'(-\underset{\sim}{k}'')> \{\}$$

$$< \underset{\sim}{E}'' \underset{\sim}{E}''> \{\} = \int d\underset{\sim}{k}''' \; \chi <\underset{\sim}{E}''(\underset{\sim}{k}''') \underset{\sim}{E}''(-\underset{\sim}{k}''')> \{\}$$

It is understood that the volume integrals with respect to $d\underset{\sim}{k}'$, $d\underset{\sim}{k}''$, $d\underset{\sim}{k}'''$ extend from $-\infty$ to $+\infty$ and that k separates the macro-intensity and the micro-intensity, while other wavenumbers are dummy variables of integration. The Fourier decomposition is truncated with a truncation factor χ .

3. Kinetic equation of turbulence

The master equation is scaled by A^0 and A^1 to give the transport equations

$$(\partial_t + A^0\hat{L})f^0 = -L^0\bar{f} + C^1\{f^0\}, \quad C^1\{f^0\} \equiv -A^0L^1f^1 \tag{3.1}$$

$$(\partial_t + A^1\hat{L})f^1 = -L^0(\bar{f} + f^0) + C''\{f^1\}, \quad C''\{f^1\} \equiv -A^1L''F''. \tag{3.2}$$

The coefficients of "eddy collision" $C^1(t,x,v)\{\}$, $C''(t,x,v)\{\}$ are integral operators. By random encounters, the many distributions in the cluster $f'' = f^{(2)} + f^{(3)} + \cdots$ become homogenized and lose their individuality in v-dependence. As a result, the cluster can be simulated by a frictional medium so that the coefficient of collision $C''(t,x,v)\{\} \cong C''(t,x)$ ceases to be an integral operator. By (2.3) we can write $C^1(t,x,v) \cong C'(t,x,v)$ and $L^1(\bar{f} + f^0) \cong L'f^0$, reducing (3.1) and (3.2) in the closed form:

$$(\partial_t + A^0\hat{L})f^0 = -L^0\bar{f} + C'\{f^0\}, \quad C'\{f^0\} = -A^0L'f' \tag{3.3}$$

$$[\partial_t + A'\hat{L}(t,x,v) - C''(t,x)]f' = -L'f^0 \tag{3.4}$$

The homogeneous equation

$$(\partial_t + A'\hat{L} - C'')U(t,t') = 0, \quad \text{with} \quad U(t,t) = 1 \tag{3.5}$$

governs the evolution operator $U(t,t')$ by means of which we in-

tegrate (3.6) to get

$$f' = - \int_0^t d\tau \; A'U(t,t-\tau) \; L'(t-\tau)f^0(t-\tau) \tag{3.6}$$

and find the eddy collision

$$C'\{f^0(t-\tau)\} = A^0 \int_0^t d\tau \; L'(t,t-\tau)A'U(t,t-\tau)L'(t-\tau)f^0(t-\tau)$$

$$= \partial \cdot D' \cdot \partial \{f^0(t-\tau)\}. \tag{3.7}$$

The diffusivity

$$D' = \int_0^{t=\infty} d\tau \; <E'(t) \; A'U(t,t-\tau)E'(t-\tau) > \tag{3.8}$$

is adiabatic by the large-time diffusion on account of the separation of groups E' and f^0. By the knowledge of the collision operator (3.7), Eq. (3.3) is called the kinetic equation of turbulence.

By taking the moment of (3.3) we obtain the Navier-Stokes equation

$$(\partial_t + A^0 \hat{u} \cdot \nabla - \nu \nabla^2)u^0 = E^0 + J^0 \tag{3.9}$$

with the added "eddy damping"

$$J^0 = \int dv \; v \; C^1(t,x,v)\{f^0(t-\tau)\}. \tag{3.10}$$

By multiphying by u^0 and averaging, we find the equation for the spectral evolution in quasi-homogeneous turbulence

$$\frac{1}{2} \partial_t < u^{0\,2} > = - \nu <(\nabla_j u_i^0)^2> - T^0 \tag{3.11}$$

8

with the transfer function

$$T^0 = < \underline{u}^0 \cdot \int d\underline{v} \, \underline{v} \, C'(t,\underline{x},\underline{v}) \{ f^0(t-\tau) \} > \; .$$ (3.12)

4. Kinetic origin of the eddy viscosity

4.1. Transformation of the evolution operator into orbit functions

At the instant of time $t-\tau$ the Lagrangian field $A'U(t,t-\tau)E'(t-\tau) \equiv A'\underline{E}'[t-\tau,\hat{\underline{x}}(t-\tau)]$ takes the position $\hat{\underline{x}}(t-\tau) = \underline{x} - \underline{v}\tau - \hat{\underline{\ell}}(\tau)$ along the perturbed trajectory that passes through the phase point $\underline{x}, \underline{v}$ at the time t. Here $\hat{\ell}(\tau)$ is the path length traveled in the interval of time τ. Thus the Lagrangian correlations are

$$< \underline{u}^0(t) f^0(t-\tau) > = \int d\underline{k}' \; X < \underline{u}^0(\underline{k}') f^0(-\underline{k}',\underline{v}) > h_v(\tau,\underline{k}',\underline{v}) h_\ell(\tau,\underline{k}')$$ (4.1)

$$< \underline{E}'(t)\underline{E}'(t-\tau) > = \int d\underline{k}'' \; X < \underline{E}'(\underline{k}'')\underline{E}'(-\underline{k}'') > h_v(\tau,\underline{k}'',\underline{v}) h_\ell(\tau,\underline{k}'')$$ (4.2)

$$< \underline{E}''(t)\underline{E}''(t-\tau) > = \int d\underline{k}''' X < \underline{E}''(\underline{k}''')\underline{E}'''(-\underline{k}''') > h_\ell''(\tau,\underline{k}'') .$$ (4.3)

The field correlations can be integrated to form the adiabatic diffusivities:

$$\underline{\underline{D}}'(t,\underline{x},\underline{v}) = \int_0^\infty d\tau \int d\underline{k}'' \; X < \underline{E}'(\underline{k}'') \, \underline{E}'(-\underline{k}'') > h_v(\tau,\underline{k}'',\underline{v}) h_\ell(\tau,\underline{k}'')$$ (4.4)

9

$$\underset{\sim}{D}''(t,\underset{\sim}{x}) = \int_0^\infty d\tau \int d\underset{\sim}{k}'' \; X < \underset{\sim}{E}''(\underset{\sim}{k}'') \; \underset{\sim}{E}''(-\underset{\sim}{k}'') > h_\ell''(\tau,\underset{\sim}{k}'').$$ (4.5)

The orbit functions are:

$$h_v(\tau,\underset{\sim}{k}',\underset{\sim}{v}) = \exp(-i\underset{\sim}{k}'' \cdot \underset{\sim}{v} \tau)$$ (4.6)

$$h_\ell(\tau,\underset{\sim}{k}'') = [\exp C''(k'')\tau] < \exp - i\underset{\sim}{k}'' \cdot \underset{\sim}{\ell}''(\tau) >, \quad h_\ell''(\tau,\underset{\sim}{k}'') = < \exp - i\underset{\sim}{k}'' \cdot \underset{\sim}{\ell}''(\tau) >$$ (4.7a,b)

$$h_m(\tau,\underset{\sim}{k}'') = < \exp - i\underset{\sim}{k}'' \cdot (\underset{\sim}{\ell}^0 + \underset{\sim}{\ell}^1) > = \exp\left(-\frac{1}{2} k''^2 u_0^2 \tau^2\right)$$ (4.8)

Here h_v governs the streaming, h_ℓ and h_ℓ'' govern the relaxation by $\underset{\sim}{\ell}''$, and h_m represents a meandering by $\underset{\sim}{\ell}^0$ and $\underset{\sim}{\ell}^1$ with an effective velocity $\underset{\sim}{u}^0$. The meandering without relaxation appears in the lowest-order perturbation expansion in quasilinear theory. The strong turbulence is predominantly governed by streaming and relaxation.

4.2. Transfer function

By substituting (4.1) and (4.4) in (3.12), we get the transfer function

$$T^0 = - \int d\underset{\sim}{v} \; v_i \iint d\underset{\sim}{k}'' d\underset{\sim}{k}' \; X < \underset{\sim}{E}'(\underset{\sim}{k}'')\underset{\sim}{E}'(-\underset{\sim}{k}'') > h_\ell(\tau,\underset{\sim}{k}'')h_\ell(\tau,\underset{\sim}{k}')$$

$$X \int_0^\infty d\tau \; M^0(\tau,\underset{\sim}{k}'',\underset{\sim}{k}',v) \; \{X < u_i^0(\underset{\sim}{k}')f(-\underset{\sim}{k}',\underset{\sim}{v}) > \}$$ (4.9)

with the memory operator

$$M^0\{ \; \} = \partial_j h_v(\tau,\underset{\sim}{k}'',v) \; \partial_j h_v(\tau,\underset{\sim}{k}',\underset{\sim}{v})$$ (4.10)

10

In differentiations $\partial_j \partial_j$ we find terms that cancel upon integrations with respect to $dv\ dk''\ dk'$. The remaining terms reduce the memory operator to:

(i) <u>transport in a macro-gradient</u>

$$M^0\{\} = h_v(\tau,k'',v)\ \partial^2 h_v(\tau,k',v)\{\}, \quad \lim_{v=0} M^0\{\} = -k'^2\tau^2 \tag{4.11}$$

(ii) <u>transport without gradient</u>

$$M^0\{\} = -\partial^2 h_v(\tau,k'',v)\ h_v(\tau,k'v)\{\}, \quad \lim_{v=0} M^0\{\} = k''^2\tau^2 \ . \tag{4.12}$$

Note that (4.11) and (4.12) govern the memory transmission by v-dependence through h_v . The loss of memory at the limit $v=0$ cancels the role of operator and reduces the transfer function into

$$
T^0 = \begin{cases} K'\ R^0 & \text{for direct cascade} & (4.13a) \\[2em] -\lambda' < u^{02} > & \text{for inverse cascade} & (4.13b) \end{cases}
$$

by (4.11) and (4.12), respectively. In the direct cascade, energy is transferred from the large eddies that present a macro-velocity gradient $\nabla_j u_i^0$ such that

$$R^0 \equiv <(\nabla_j u_i^0)^2> = 2 \int_0^k dk'\ k'^2\ F(k') \tag{4.14}$$

towards the small eddies that organize an eddy viscosity

$$K' = tr \underset{\sim}{K'} = \int dk' tr \chi <\underset{\sim}{E'}(\underset{\sim}{k'})\; \underset{\sim}{E'}(-\underset{\sim}{k'})> \int_0^\infty d\tau\; \tau^2\; h_\ell(\tau,\underset{\sim}{k'})\;. \qquad (4.15)$$

Here $F(k)$ is the energy spectrum. In the inverse cascade, the small eddies organize an eddy amplification at a rate

$$\lambda' = \int dk'' \; k''^2 tr \chi < \underset{\sim}{E'}(\underset{\sim}{k''})\underset{\sim}{E'}(-\underset{\sim}{k''}) > \int_0^\infty d\tau\; \tau^2\; h_\ell(\tau,\underset{\sim}{k''}) \qquad (4.16)$$

and feed energy into the large eddies.

The approximation $h_\ell(\tau,\underset{\sim}{k''})h_\ell(\tau,\underset{\sim}{k'}) \overset{\sim}{=} h_\ell(\tau,\underset{\sim}{k''})$ by $k' \ll k''$ has been made. The moment $\int d\underset{\sim}{v} \; \underset{\sim}{v} \; f^0 = \underset{\sim}{u}^0$ has been taken, and other details of calculation have been omitted. In the following we confine ourselves to the transport by K' in a direct cascade only.

For the sake of abbreviation, we write

$$\int_0^\infty d\tau\; \tau^2\; h_\ell(\tau,\underset{\sim}{k'}) = 2\gamma'^{-3}, \qquad \gamma'(k') = C'(k') + \gamma'_\ell(k') \qquad (4.17a,b)$$

$$\int_0^\infty d\tau\; h''_\ell(\tau,\underset{\sim}{k''}) = \int_0^\infty d\tau\; <exp-i\underset{\sim}{k''}\cdot\underset{\sim}{\ell}''(\tau)>\; \gamma''^{-1}_\ell, \qquad \gamma''_\ell(k'') = k''^2 K''_\ell\;. \qquad (4.18a,b)$$

The subdynamics for the analytical derivation of (4.18b) is given in Section 5.

The relaxation times γ'^{-1} and γ''^{-1}_ℓ govern the approach to equilibrium for the eddy viscosity

$$\underset{\sim}{K'} = 2< \underset{\sim}{E'}\underset{\sim}{E'}> \{\gamma'^{-3}\} \qquad (4.19)$$

and the path diffusivity

12

$$K_{\ell}'' \equiv \frac{1}{2} \lim_{\tau=\infty} \frac{d}{d\tau} <\ell''(\tau)\, \ell''(\tau)> = < E''E''> \{\gamma_{\ell}''^{-3}\}, \qquad (4.20)$$

respectively.

4.3 Eddy coefficient

Recall that upon simulating f'' by an effective fluid medium, we have reduced the collision integral $C''(t,x,v)\{\} \overset{\sim}{=} C''(t,x)$ into a collision coefficient $C''(t,x)$. If the same reduction is made for $C'(t,x,v) \overset{\sim}{=} C'(t,x)$, we reduce the eddy damping (3.10) into $J^0 = C'(t,x)\, u^0(t,x)$ and the transfer function (3.12) into

$$T^0 = - < u^0 \cdot C'(t,x)\, u^0 > . \qquad (4.21)$$

By comparing with (4.13a) we identify

$$C'(t,x) = K'\nabla^2 \qquad (4.22)$$

and similarly

$$C''(t,x) = K''\nabla^2, \qquad \text{or} \qquad C''(k) = -k^2 K'' . \qquad (4.23)$$

Hence we find

$$\gamma'(k') = k'^2(K' + K_{\ell}') = (P_t + 1)\gamma_{\ell}', \qquad (4.24)$$

from (4.17b), and transform (4.19) into

$$K' = 2(P_t+1)^{-3}\, tr<E'E'> \{\gamma_{\ell}''^{-3}\}, \qquad P_t = K'/K_{\ell}' . \qquad (4.25a,b)$$

The path diffusivity governs the orbit functions in (4.17a) and (4.18a), and controls the approach of the eddy viscosity (4.25a) to

13

equilibrium. By a subdynamics in Section 5, we shall analyse this transport coefficient.

5. Subdynamics of relaxation

The orbit function $< \exp -ik'' \cdot \ell''(\tau) >$ and the path diffusivity K_ℓ'' govern the relaxation. For their determination we develop a subdynamics consisting of a Fokker-Planck equation of transition and a Langevin equation of turbulence. These equations are in the configuration space by definition of relaxation.

5.1. <u>Fokker-Planck equation of transition</u>

The probability that a path-length is made between ℓ and $\ell + d\ell$ in a time interval τ along a trajectory that passes through the point x at the time instant t is

$$p(t,x;t-\tau, x-\ell)d\ell \qquad \text{or briefly} \qquad p(\tau,\ell)d\ell$$

in quasi-stationary processes. The probability density of retrograde transition is governed by the Fokker-Planck equation (Tchen, 1944)

$$\frac{\partial}{\partial \tau} p(\tau,\ell) = K_\ell'' \frac{\partial^2}{\partial \ell \, \partial \ell} p(\tau,\ell) \tag{5.1}$$

and satisfies the condition of normalization

$$\int d\ell \; p(\tau,\ell) = 1 . \tag{5.2}$$

The streaming is absent in relaxation. The path diffusivity K_ℓ'' as defined by (4.20) is adiabatic, i.e. independent of τ and ℓ.

By Fourier decomposition, we transform Eq. (5.1) into

$$\frac{\partial}{\partial \tau} p\, (\tau, \underset{\sim}{k}'') = -\, \underset{\sim}{k}''\, \underset{\sim}{k}'' : \underset{\approx}{K}_\ell''\, p(\tau, \underset{\sim}{k}'') \tag{5.3}$$

with the solution

$$p(\tau, \underset{\sim}{k}'') = \frac{1}{(2\pi)^3} \exp[-\underset{\sim}{k}''\, \underset{\sim}{k}'' : \underset{\approx}{K}_\ell''\, \tau]\, . \tag{5.4}$$

The coefficient of integration is determined by the condition (5.2). Hence we obtain

$$< \exp -i\underset{\sim}{k}'' \cdot \underset{\sim}{\ell}''(\tau) > = \int d\underset{\sim}{\ell}\; e^{-i\underset{\sim}{k}'' \cdot \underset{\sim}{\ell}}\, p(\tau, \underset{\sim}{\ell})$$

$$= \exp(-\gamma_\ell''\, \tau), \quad \text{and} \quad \gamma_\ell'' = k''^2\, K_\ell''\, . \tag{5.5a,b}$$

The path diffusivity and the relaxation time as determined by (5.5a,b) have been introduced in connection with (4.17) and (4.18).

5.2. Langevin equation of turbulence

The Langevin equation of turbulence

$$\frac{du''(\underset{\sim}{k}'')}{dt} + \gamma_\ell''\, \underset{\sim}{u}''(\underset{\sim}{k}'') = \underset{\sim}{E}''(\underset{\sim}{k}'') \tag{5.6}$$

governs the relaxation in $\underset{\sim}{k}''$-space. It has a certain analogy with the Langevin equation for Brownian motion.

By two successive integration we get

$$\underline{\ell}'' = \int_0^t dt' \int_0^t dt'' \exp[-\gamma_\ell''(t'-t'')] \; \underline{E}''(t'')$$

$$= \int_0^t dt'' \int_{t''}^t dt' \exp[-\gamma_\ell''(t'-t'')] \; \underline{E}''(t'')$$

$$= \gamma_\ell''^{-1} \int_0^t dt'' \; \underline{E}''(t'') (1 - \exp[-\gamma_\ell''(t-t'')]) \tag{5.7}$$

An interchange of order of integrations has been made. In calculating K_ℓ'' by definition (4.20), we find four double integrals, one of which is

$$< |\underline{\ell}''\underline{\ell}''| > = \gamma_\ell''^{-1} \int_0^t dt' \int_0^t dt'' < |\underline{E}''(t')\underline{E}''(t'')| >$$

$$= 2\gamma_\ell''^{-2} \int_0^t dt' \int_0^{t'} d\tau < |\underline{E}''(0)\underline{E}''(\tau)| >$$

$$= 2\gamma_\ell''^{-2} \int_0^t d\tau (t-\tau) < |\underline{E}''(0)\underline{E}''(\tau)| > \; . \tag{5.8}$$

An interchange of order of integrations leads to

$$\frac{1}{2} \lim_{t \to \infty} \frac{d}{dt} < |\underline{\ell}''(t)\underline{\ell}''(t)| > = \gamma_\ell''^{-2} \int_0^{t \to \infty} d\tau < |\underline{E}''(0)\underline{E}''(\tau)| > \tag{5.9}$$

or

$$K_\ell'' = D''\{\gamma_\ell''^{-2}\} \; , \tag{5.10}$$

by (4.20). Note that the remaining three double integrals decay exponentially and do not contribute. Also note that (5.9) is written in k''-space, by (5.6), i.e.

$$< |\underline{\ell}''(t)\underline{\ell}''(t)| > = < \underline{\ell}''(t,\underline{k}'') \; \underline{\ell}''(t,-\underline{k}'') >, \quad < |\underline{E}''(0)\underline{E}''(\tau)| > = < \underline{E}''(t,\underline{k}'')\underline{E}''(t-\tau,-\underline{k}'') >$$

16

It is to be remarked that a relationship analogous to (5.10) exists in Brownian motion and is called Einstein's formula of fluctuation dissipation.

By substituting for D'' from (4.5), we obtain

$$K''_\ell \equiv \text{tr } \underset{\approx}{K''_\ell} = \text{tr } <\underset{\sim}{E''} \underset{\sim}{E''}> \{\gamma''^{-3}_\ell\} \qquad (5.11)$$

and confirm (4.20b).

A comparison between (5.11) and (4.25a) yields the quartic equation

$$P_t(P_t+1)^3 = 2 \cdot \qquad (5.12)$$

The solution determines the turbulent Prandtl number $P_t = 0.545$.

6. Eddy coefficients as functions of energy spectrum

6.1. Equation of state of turbulence

By the condition of incompressibility $\nabla \cdot \underset{\sim}{u} = 0$ the Navier-Stokes equation yields the equation

$$\nabla \cdot \underset{\sim}{E''} = r'', \qquad \text{or} \qquad \underset{\sim}{E''}(\underset{\sim}{k''}) = i \, \underset{\sim}{k''} \, k''^{-2} \, r''(\underset{\sim}{k''}) \qquad (6.1)$$

with $r'' = A'' \underset{\sim}{\nabla} \underset{\sim}{\nabla} : \underset{\sim}{u} \underset{\sim}{u}$. It applies to relaxation in equilibrium. Upon neglecting the non-stationary effects by meanderning, we decompose $\nabla \underset{\sim}{u}(x) = \nabla u^0 + \nabla u''(x)$ into a quasi-homogeneous background ∇u^0 and a variable sub-gradient $\nabla u''(x)$, and write

$$r'' = A''(\nabla_i u_j^0 + \nabla_i u_j'')\ (\nabla_j u_i^0 + \nabla_j u_i'')\quad . \tag{6.2}$$

By selecting the subgroup, and assuming isotropy, we get

$$<r''^2> = \frac{2}{9}\ R^0\ R''\,,\quad \text{with}\quad R^0 = <(\nabla_j u_i^0)^2>,\quad R'' = <(\nabla_j u_i'')^2> \tag{6.3}$$

and derive

$$<\underline{E}''^2> = \frac{2}{9}\ R^0 <\underline{u}''^2>, \tag{6.4}$$

by (6.1).

6.2. Eddy coefficients

By writing (6.4) in the operator form $<\underline{E}''\,\underline{E}''>\{\} = \frac{2}{9}\ R^0 <\underline{u}''\,\underline{u}'' > \{\}$, we derive

$$K_\ell'' = \frac{2}{9}\ R^0\ \mathrm{tr} <\underline{u}''\,\underline{u}''> \{\gamma_\ell''^{-3}\} \tag{6.5}$$

by (5.11), or

$$K_\ell''(k') = \frac{2}{9}\ R^0(k)\ K_\ell''(k')\{[\gamma_\ell''(k'')]^{-2}\}\quad . \tag{6.6}$$

by $K_\ell'' = \mathrm{tr} <\underline{u}''\underline{u}''> \{\gamma_\ell''^{-1}\}$ from definition (4.20).

The integral equation (6.6) can be solved by a differentiation with respect to k', yielding

$$K_\ell''(k) = (2/9)^{\frac{1}{2}}\ R^{0\frac{1}{2}}\ k^{-2}\quad . \tag{6.7}$$

18

It follows

$$K' = c_K R^{0^{\frac{1}{2}}} k^{-2} \tag{6.8}$$

from (4.25b) and (6.7). The numerical coefficient is

$$c_K = (2/9)^{\frac{1}{2}} P_t = 0.471, \quad \text{for } P_t = 0.545, \tag{6.9}$$

by (5.12).

7. Spectral structure

7.1. Energy spectrum

Formula (6.8) for the eddy viscosity K' helps to determine the energy transfer across the spectrum. In the inertial-range, the transfer at a constant rate of dissipation ε is

$$K' R^0 = \varepsilon, \quad \text{or} \quad (2/9)^{\frac{1}{2}} P_t R^{0\,3/2} k^{-2} = \varepsilon, \tag{7.1}$$

from (3.8) and (4.13a). The vorticity function R^0 is defined by (4.14). By isolating R^0 and differentiating, we obtain the Kolmogoroff law

$$F(k) = A \, \varepsilon^{2/3} \, k^{-5/3} . \tag{7.2}$$

The numerical coefficient is evaluated to be $A = \frac{2}{3} c_K^{-2/3} = 1.650$, by (6.9).

7.2. Pressure fluctuations

The pressure fluctuation $\rho^{-1} p(k) = -i k^{-2} k . E(k)$ can be cal-

19

culated from (6.1), giving

$$\mathcal{X} < |p''(\underset{\sim}{k})|^2 > = \rho^2 \; k^{-2} \; \mathcal{X} < |\underline{E}''(\underset{\sim}{k})|^2 >$$

$$= \frac{2}{9} \rho^2 \; R^0(k) \; k^{-2} \; \mathcal{X} < |\underline{u}''(\underset{\sim}{k})|^2 >, \qquad (7.3)$$

by (6.4).

By introducing the spectrum $S(k)$ of pressure fluctuations such

that $2 \int_k^\infty dk'' \; S(k'') = \int dk'' \; \mathcal{X} < |p''(\underset{\sim}{k''})|^2 >$ we can write (7.3) in the spectral form

$$S(k) = \frac{2}{9} \rho^2 \; R^0 \; k^{-2} \; F(k)$$

$$= C \rho^2 \; \varepsilon^{4/3} \; k^{-7/3}, \qquad (7.4)$$

with $\quad C = \frac{1}{3} A^2 = 0.907$. The use of (7.2) has been made.

7.3. <u>Thermal spectrum</u>

The evolution of a passive scalar, <u>e.g.</u> the temperature fluctuation $\overset{\sim}{\theta}$, is governed by the homogeneous diffusion equation

$$(\partial_t + \tilde{A} \; \overset{\wedge}{\underset{\sim}{u}} \cdot \underset{\sim}{\nabla} - \kappa \nabla^2)\overset{\sim}{\theta} = 0 \qquad (7.5)$$

in isotropic turbulence. The heat transfer in the convective-inertial range is described by

$$K_\ell' \; R_\theta^0 = \varepsilon_\theta \quad \text{or} \quad (2/9)^{\frac{1}{2}} \; R^{0\frac{1}{2}} \; R_\theta^0 = \varepsilon_\theta \quad , \qquad (7.6)$$

20

where ε_θ is the dissipation rate by the molecular diffusivity κ,

and $R_\theta^0 \equiv \langle (\nabla \theta^0)^2 \rangle = 2 \int_0^k dk'\, k'^2\, F(k')$ is the variance of the temperature

gradient.

By the same procedure as was used for deriving (7.2) from (7.1), we obtain the solution of (7.6) as

$$F_\theta(k) = B\, \varepsilon_\theta\, \varepsilon^{-1/3} k^{-5/3} . \tag{7.7}$$

The numerical coefficient is found to be $B = 0.899$ in consistency with the condition $B = P_t A$.

8. Summary and discussion

By letting the pressure gradient represent the elementary interaction among fluid elements and by raising the Navier-Stokes equation to higher dimensionality, we obtain the master equation for the description of the microdynamical state of turbulence. Being a homogeneous equation, the master equation is suitable for the explicit formulation of the eddy transport in the gradient form (direct cascade) and in the form of an eddy damping (inverse cascade). By the macrogroup dynamics and the microgroup dynamics, the kinetic equation is derived, and the eddy viscosity is found to depend on the orbit functions of streaming and relaxation, in the roles of memory-transmission and memory-loss, respectively. The loss of memory leads to closure and irreversibility.

The subdynamics consists of a Fokker-Planck equation of transition and a Langevin equation of turbulence, and governs the path

21

perturbation along an accelerated trajectory and the path diffusivity.

The eddy viscosity and the path diffusivity are found to be governed by a system of integral equations. The solutions determine the energy spectrum $F(k) = 1.650 \, \varepsilon^{2/3} \, k^{-5/3}$, the pressure variance spectrum $S(k) = 0.907 \rho^2 \varepsilon^{4/3} k^{-7/3}$ in the inertial range, and the temperature variance spectrum $F_\theta(k) = 0.899 \, \varepsilon_\theta \, \varepsilon^{-1/3} \, k^{-5/3}$ in the convection-inertial range.

If the orbit function is degenerated into a non-adiabatic diffusion as the lowest order perturbation expansion in the quasilinear theory, a meandering at an effective velocity u_0 is found. The parameter $\varepsilon \, u_0'$ yields a spectrum $F(k) \approx (\varepsilon \, u_0)^{1/2} \, k^{-3/2}$ reproducing the result of DIA.

Since the groups represent the transport processes of spectral evolution, eddy viscosity and relaxation, and are analysed by a macrogroup dynamics, a microgroup dynamics and a subdynamics separately, the high-order correlations appear in the form of group-interactions $\langle \underline{u}' \underline{u}' \rangle \langle f^0 f^0 \rangle$ by our group-kinetic theory and not in the form of a quadruple correlation $\langle \tilde{\underline{u}} \tilde{\underline{u}} \tilde{f} \tilde{f} \rangle$ as would be by a perturbation theory. The quadruple correlation requires the hypothesis of normality for factorization and closure.

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22

10. REFERENCES

BOGOLIUBOV, N. N. 1962 Problems of a dynamical theory in statistical physics, in: _Studies in Statistical Mechanics_, I ed. J de Boer & G. E. Uhlenbeck, North-Holland, Amsterdam.

CHAPMAN, S. & COWLING, T. G. 1939 _The Mathematical Theory of Non-Uniform Gases_, Cambridge University Press.

DUBOIS, D. F. & ESPEDAL, M. 1978 Direct interaction approximation and plasma turbulent theory. _Plasma Phys._ 20, 1209-1239.

GROSSMAN, S. & SCHNEDLER, E. 1977 Fluctuation correlation of the turbulent spectrum. _Z. Physik_ B 26, 307-317.

HEISENBERG, W. 1948 Zur Statistichen Theorie der Turbulenz, _Zeit. Physik_ 124, 628-657.

KOLMOGOROFF, A. N. 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. _C. R. Acad. Sci._ URSS, 30, 299-303.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. _J. Fluid Mech._, 5, 497-513.

KRAICHNAN, R. H. 1977 Eulerian and Lagrangian renormalization in turbulent theory. _J. Fluid Mech._, 83, 349-374.

MARTIN, P. C., SIGGIA, E. D. & ROSE, H. A. 1973 Statistical dynamics of classical systems. _Phys. Rev._, A8, 423-437.

TCHEN, C. M. 1944 Enige wiskundige betrekkingen welke een rol spelen in diffusie-problemen (Some meathematical relations which play a role in diffusion problems). _Versl. Ned. Akad. Wetensch. Afd. Natuurkunde_, 53, 400-410.