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GUIDANCE ANALYSIS OF THE AEROGlide  
PLANE CHANGE MANEUVER AS A TURNING  
POINT PROBLEM

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## SUMMARY

The problem of developing guidance information for changing the orbit of a vehicle by using its aerodynamic lifting capability is considered. Aerodynamic maneuvers reduce propulsive control system requirements to achieve a range of orbit transfers of practical interest. As a consequence, rocket fuel, and hence, the tremendous cost of transporting additional fuel mass to orbit are saved. In order to achieve these savings, guidance laws must be developed for the aerodynamic portion of the maneuver sequence. When the atmospheric maneuver is accomplished entirely aerodynamically, it is termed an aeroglide maneuver. The approach taken for developing the aeroglide guidance law is to analytically approximate the solutions to those control problems which optimize the final path angles for a given energy loss. A detailed analysis of the optimal heading angle problem is provided, but the methods used are equally applicable to the coplanar, orbit transfer problem wherein the final flight path angle is optimized.

The optimal heading angle problem for a given energy loss is equivalent to the minimum energy loss problem that accomplishes a desired change in the orbital inclination of the vehicle. Analytic expressions for the optimal controls (bank angle and lift) are developed as solutions to the approximate state/Euler system of differential equations for the optimal heading angle problem. The optimal control solutions are characterized by three approximations, valid in separate regions of the flight, which are derived using asymptotic theory of linear differential equations containing a small parameter. The perturbation parameter depends on the scale height of the atmosphere, assumed to be exponentially varying with altitude, and the planet's radius. The optimal control solutions are a composite of two slowly varying (outer) solutions, that are valid in the region of flight near the boundaries, and a rapidly varying (inner) solution that is valid in the region where the minimum altitude for the flight occurs. Numerical analyses, required to determine the matching conditions that continue the two outer solutions through the inner, transition zone are not included. Nevertheless, the analytic formulas for the optimal controls form the basis for an on-board guidance algorithm and are also useful for developing engineering insight into the optimal steering policy.

Aerodynamic heating constraints, neglected in this analysis, need to be imposed to obtain useful guidance laws. Additionally, guidance corrections for off-nominal atmospheres must be made for on-board use.

## LIST OF SYMBOLS

A,B	constants in general solution for $w$
a,b	coefficients in algebraic solution for $\lambda$
$a_1, a_2$	constants in general solution for $\lambda_{\gamma_0}^l$
$b_1, b_2$	constants in general solution for $\lambda_{\gamma_0}^r$
C	coefficient in solution for $\lambda_{\gamma_u}$
$C_D$	drag coefficient
$C_{D_0}$	drag coefficient at zero lift

$C_L$  lift coefficient  
 $c$  integration constant  
 $c_1, c_2$  constants in general solution of parabolic cylinder equation  
 $D$  drag force on vehicle  
 $D_0$  parabolic cylinder function  
 $E$  specific energy of vehicle  
 $E$   $\exp\left(\int_{x_e}^{x_f} (q/\gamma) dx\right)$   
 $\epsilon$  nondimensional energy  
 $E^*$  maximum lift to drag ratio  
 $F$   $\exp\left(\int_x^{x_f} (q/\gamma) dx\right)$   
 $g_c$  conversion factor for expressing slugs in pounds-mass  
 $H$  Hamiltonian  
 $H$  geometric altitude of vehicle above Earth's surface  
 $h$  nondimensional altitude  
 $i$  inclination of orbital plane with respect to Earth's polar axis  
 $J$  performance index  
 $K$  induced drag factor  
 $L$  lift force on vehicle  
 $M$  inner expansion for bank angle  
 $m$  mass of vehicle  
 $n_t$  "reduced" vertical load factor at the turning point  
 $O$  order symbol  
 $Q$  variable displacement coefficient in turning-point-problem equation  
 $q$  variable proportional to gravitational less centripetal acceleration  
 $R$  radius of Earth  
 $r$  radial distance from Earth's center to vehicle  
 $S$  vehicle reference area  
 $s$  nondimensional arc length of trajectory  
 $t$  elapsed time  
 $u$  state variable replacing energy in approximate vehicle model  
 $V$  speed of vehicle

$v$	nondimensional speed of vehicle
$W$	weight of vehicle
$W_0$	zeroth-order inner, dependent variable for turning point problem
$w$	outer, dependent variable for turning point problem
$X$	stretched independent variable for turning point problem
$x$	nondimensional arc length relative to the turning point
$Z$	nondimensional lift loading
$\alpha$	value of flight path angle derivative at turning point
$\beta$	reciprocal scale height of Earth's atmosphere
$\gamma$	flight path angle
$\Gamma$	gamma function
$\Delta V$	velocity impulse approximation for fuel usage
$\epsilon$	small parameter
$\theta$	down range angle of vehicle
$\Lambda_\gamma$	inner expansion for $\lambda_\gamma$
$\lambda$	nondimensional lift
$\lambda(\rho)$	Lagrange multiplier variable for state ( $\rho$ )
$\mu$	bank angle of lift vector, measured positive clockwise from vertical, when viewed from vehicle's nose
$\bar{\mu}$	gravitational constant for Earth
$\nu$	index of parabolic cylinder function
$\xi$	"dummy" variable in recurrence relation
$\rho$	air density
$\tau$	"dummy" variable of integration
$\phi$	cross range angle of vehicle
$\psi$	heading angle of vehicle

#### Subscripts

$e$	initial value (at vehicle entry into atmosphere)
$f$	final value (at vehicle exit from atmosphere)
$\min$	denotes minimum value
$o$	denotes zeroth-order approximation
$r$	denotes reference value for exponential atmosphere approximation
$t$	denotes value at turning point
$u$	denotes uniform approximation

### Superscripts

- l denotes solution to left of turning point
- r denotes solution to right of turning point
- \*
- ( )' denotes differentiation with respect to nondimensional arc length

### INTRODUCTION

Aerodynamically assisted, orbit transfers are known to be more fuel efficient than all propulsive (Hohmann) transfers (ref. 1.). One application for aero-assisted maneuvers is a high-Earth-orbit to low-Earth-orbit transfer for returning vehicles, that service satellites, to the Shuttle. Another application is for the Mars mission where aero-assisted maneuvers can be used for rendezvous operations with other vehicles. NASA soon plans to launch an aero-assist flight experiment to make measurements of the environment surrounding the vehicle during sustained high altitude, hypersonic flight.

Recent numerical studies have appeared in the literature to determine atmospheric trajectories that will minimize the amount of fuel needed for an orbit transfer. In reference 2 optimal atmospheric trajectories are presented for a coplanar transfer in which the apogee of the orbit is lowered. In reference 3 optimal trajectories are computed for changing the orbital inclination of a vehicle. Implementation of optimal control policies on-board a vehicle, to take advantage of the aero-assist maneuver's fuel efficiency, requires either the storage of optimal controls and trajectories for a large number of boundary conditions or rapid on-board computations of optimal trajectories for specific boundary conditions corresponding to the desired mission. Thus, either large storage requirements or fast, on-board computational capabilities are necessary to implement optimal maneuvers. An alternative to either of these methods of implementing optimal steering is to approximate the optimal control problem so that it may be solved analytically. If sufficiently accurate guidance solutions can be found, the demands upon on-board computational capability are reduced.

In references 3, 4, and 5, different approximations have been made to determine guidance laws for the orbit plane change problem. In reference 3, the term appearing in the differential equation for the flight path angle, known as Loh's "constant," is assumed to be constant on an optimal trajectory, which makes possible the derivation of an optimal guidance approximation. In reference 4, a regular perturbation approach is used to derive optimal guidance approximations by first neglecting Loh's "constant" and then correcting the solution by allowing variation in Loh's constant. In reference 5, a singular perturbation approximation is derived which simplifies the optimal control problem by reducing the order of the state/Euler system, resulting in a relatively simple guidance solution. The approach taken herein is similar to those cited above in that the equations of motion are written in terms of a small parameter so that asymptotic theory for differential equations can be applied to arrive at a solution to the guidance problem. This research was undertaken to improve the accuracy of the approximations cited above.

## MODELING AND PROBLEM STATEMENT

The problem of developing a guidance scheme for transferring a vehicle from one orbital plane above the Earth to another, using a minimum of fuel, is considered. If heating constraints on the vehicle are not imposed, the maneuver sequence consists of exoatmospheric portions, during which the vehicle maneuvers using its rocket motor, and an atmospheric portion, during which the vehicle maneuvers aerodynamically to change heading by modulating angle of attack and bank angle. The imposition of heating constraints on the vehicle during atmospheric flight may lead to additional thrusting arcs to avoid overheating, reference 6, but such constraints are not considered in the present formulation. The maneuver sequence begins with a deorbit impulse to cause the vehicle to descend toward the Earth's atmosphere. After entering the atmosphere, the vehicle maneuvers to change its heading to the desired orbital inclination by modulating angle of attack and bank angle. A boost phase follows the aerodynamic maneuver to increase altitude of the vehicle to a desired apogee. Finally, a rocket burn is used to circularize the orbit. Because the exoatmospheric portions of the flight can be calculated in closed form using Kepler's equations and  $\Delta V$  impulses to approximate fuel usage, interest centers on the atmospheric phase of flight for obtaining analytical expressions that can be used for guiding the vehicle. The atmospheric maneuver problem can be stated in terms of the  $\Delta V$  impulses of the minimum fuel problem, resulting in a one-dimensional parameterization of the atmospheric guidance problem in terms of the deorbit  $\Delta V$  impulse and a performance index which minimizes the energy lost during the aerodynamic turn, reference 3. The vehicle's motion is described in terms of six state variables: specific energy,  $E = V^2/2 - \bar{\mu}/r$ ; flight path and heading angles,  $\gamma$  and  $\psi$ ; down range and cross range angles,  $\theta$  and  $\phi$ ; and the radial distance,  $r$  of the vehicle from the Earth's center. The controls for the vehicle are lift,  $L$ ; and bank angle,  $\mu$ . The velocity of the vehicle is denoted by  $V$ ,  $\bar{\mu}$  is the universal gravitational constant for the Earth,  $m$  is the vehicle's mass, and  $D$  is the drag force on the vehicle, which is assumed to be parabolic with respect to lift. Time,  $t$ , is replaced by nondimensional arc length,

$$s = \int (V/r) dt \quad (1)$$

as the independent variable. The point-mass equations of motion of the vehicle may be written with respect to a vehicle centered reference frame that rotates above a spherical Earth, inertial frame as depicted in figure 1.

$$dr/ds = r \sin \gamma \quad (2)$$

$$d\theta/ds = \cos \gamma \cos \psi / \cos \phi \quad (3)$$

$$d\phi/ds = \cos \gamma \sin \psi \quad (4)$$

$$dE/ds = -Dr/m \quad (5)$$

$$d\gamma/ds = Lr \cos \mu / (mV^2) - [\bar{\mu}/(rV^2) - 1] \cos \gamma \quad (6)$$

$$d\psi/ds = Lr \sin \mu / (mV^2 \cos \gamma) - \cos \psi \cos \gamma \tan \phi \quad (7)$$

The cross range angle,  $\phi$ , is small near a node so that the last term on the right-hand side of equation (7) may be neglected if the turn is made in the

vicinity of a node. If the boundary conditions on  $\phi$  and  $\theta$  are not prescribed, these state variables are ignorable since neither couples into the remaining equations. Thus, the order of the system is reduced by two. Since  $\phi$  is small, the equation for orbit inclination,

$$\cos i = \cos \phi \cos \psi \cong \cos \psi \quad (8)$$

and the heading angle approximates the inclination angle near a node.

$$r' = r \sin \gamma \quad (9)$$

$$E' = -Dr/m \quad (10)$$

$$\psi' = Lr \sin \mu / (mV^2 \cos \gamma) \quad (11)$$

$$\gamma' = Lr \cos \mu / mV^2 - [\bar{\mu}/(rV^2) - 1] \cos \gamma \quad (12)$$

The following nondimensional variables are introduced to determine if any further simplifications can be made. Let

$$Z = \rho_r S C_L^* r_r e^{-h/2m} \quad (13)$$

$$h = \beta(H - H_r) \quad (14)$$

where  $H$  is the altitude of the vehicle above the Earth's surface,  $H_r$  is the reference altitude for the exponentially varying atmosphere,  $\rho_r$  is the air density at  $H_r$ ,  $S$  is the vehicle's reference area, and

$$r_r = R + H_r \quad (15)$$

where,  $R$  is the radius of the Earth and

$$C_L^* = C_L \text{ at max } (L/D) \quad (16)$$

The drag coefficient is assumed to be parabolic

$$C_D = C_{D_0} + KC_L^2 = C_{D_0}(1+\lambda^2) = C_L^* (1+\lambda^2)/2E^* \quad (17)$$

where,  $E^* = (L/D)_{\max}$  and  $\lambda = C_L/C_L^*$ . Also let

$$v = V \sqrt{r_r/\bar{\mu}} \quad (18)$$

$$\epsilon = Er_r/\bar{\mu} = v^2/2 - (1 + \epsilon h)^{-1} \quad (19)$$

$$r/r_r = (1 + \epsilon h) \quad (20)$$



be the nondimensional speed, energy, and radius variables, respectively, where  $\epsilon = (\beta r_r)^{-1}$  is a small parameter (of order  $10^{-3}$  for the Earth). Substituting equations (13) through (20) into equations (9) through (12), one may rewrite the governing equations of motion for the vehicle as

$$h' = \epsilon^{-1}(1+\epsilon h) \sin \gamma \quad (21)$$

$$\epsilon' = -Z_r e^{-h(1+\lambda^2)} (\epsilon + 1 + \epsilon h \epsilon) / E^* \quad (22)$$

$$\psi' = Z_r e^{-h(1+\epsilon h)} \lambda \sin \mu / \cos \gamma \quad (23)$$

$$\gamma' = Z_r e^{-h} \lambda \cos \mu (1+\epsilon h) - (1+\epsilon[h/(\epsilon+1)-h]-1) / [2(\epsilon+1)] \cos \gamma \quad (24)$$

to  $O(\epsilon^2)$ , where the binomial theorem has been used to expand  $r/r_r$ . After making a final change of variables

$$u = \ln(\epsilon+1) \quad (25)$$

and substituting equation (25) into the system, (21) through (24), one obtains to  $O(\epsilon)$  the following approximate model, describing the vehicle's motion:

$$u' = -Z_r e^{-h(1+\lambda^2)} / E^* \quad (26)$$

$$\psi' = Z_r e^{-h} \lambda \sin \mu / \cos \gamma \quad (27)$$

$$\gamma' = Z_r e^{-h} \lambda \cos \mu - q \cos \gamma \quad (28)$$

$$h' = \epsilon^{-1}(1+\epsilon h) \sin \gamma \quad (29)$$

where,

$$q = e^{-u/2} - 1. \quad (30)$$

The approximate model of the vehicle, (26) through (29), retains the term,  $q$ , which is a factor in Loh's "constant" but does not account for variations in gravitational acceleration due to altitude. Additionally, the model assumes that the effect of altitude changes on the motion are dominated by changes in air density rather than changes in potential energy. Thus, potential energy changes are neglected on the right-hand sides of equations (26) through (29).

Finally, variations with Mach number of the aerodynamic coefficients,  $C_L^*$  and  $E^*$ , have been neglected at this order of approximation so that  $Z_r$  is regarded as constant.

#### OPTIMAL CONTROL PROBLEM

If the final path angles are specified, a minimum energy loss trajectory to a specified final altitude will result in minimum  $\Delta V$  fuel burns to achieve a desired final orbit. By restating this problem in terms of a specified energy loss, one may instead, optimize the final path angles. The purpose of the restatement is to simplify the final boundary conditions on the optimal vertical lift component, which equals zero in this instance. Only the plane

change problem is discussed in detail herein. The plane change problem is equivalent to extremizing the final heading angle for a specified energy loss, so that the payoff is

$$J = \psi_f = \int \psi' ds. \quad (31)$$

The Hamiltonian for this problem is

$$H = \lambda_0 \psi' + \lambda_u u' + \lambda_h h' + \lambda_\gamma \gamma'. \quad (32)$$

where,

$$\lambda_0 = \begin{cases} +1 \Rightarrow \max \psi_f \Rightarrow \max i \\ -1 \Rightarrow \min \psi_f \Rightarrow \min i \end{cases} \quad (33)$$

By choosing  $\lambda_0 = +1$ , one seeks the optimal bank angle and lift controls to maximize the final heading angle with starting conditions,  $u_e, h_e, \gamma_e$ , and final conditions  $u_f, h_f$ , so that the final path angles are optimized. Substituting the state equations, (26) through (29), into equation (32), one obtains

$$H = Z_{re}^{-h} \lambda \sin \mu / \cos \gamma - \lambda_u Z_{re}^{-h} (1 + \lambda^2) / E^* + \lambda_h \epsilon^{-1} (1 + \epsilon h) \sin \gamma + \lambda_\gamma (Z_{re}^{-h} \lambda \cos \mu - q \cos \gamma) \quad (34)$$

Since the final arc length is not specified,  $H=0$  on an extremal. The optimal controls are given by

$$\partial H / \partial \lambda = -2 \lambda_u \lambda / E^* + \sin \mu + \lambda_\gamma \cos \mu = 0 \quad (35)$$

and

$$\partial H / \partial \mu = \cos \mu - \lambda_\gamma \sin \mu = 0, \quad (36)$$

where second-order terms in  $\gamma$  have been neglected.

Solving equations (35) and (36) simultaneously, one obtains

$$\lambda = E^* \sqrt{1 + \lambda_\gamma^2} / (2 \lambda_u) \text{ with } \lambda_u > 0 \quad (37)$$

$$\cot \mu = \lambda_\gamma \quad (38)$$

The Lagrange multipliers satisfy

$$\lambda_u' = -\partial H / \partial u = -\lambda_\gamma \epsilon^{-u/2} \quad (39)$$

$$\lambda_\gamma' = -\partial H / \partial \gamma = -\epsilon^{-1} \lambda_h (1 + \epsilon h) \cos \gamma - (\lambda_\gamma q + Z \lambda \sin \mu \sec^2 \gamma) \sin \gamma \quad (40)$$

$$\begin{aligned} \lambda_h' &= -\partial H / \partial h = H - \lambda_h \epsilon^{-1} (1 + \epsilon h) \sin \gamma + q \lambda_\gamma \cos \gamma - \lambda_h \sin \gamma \\ &= -\lambda_h \epsilon^{-1} [1 + \epsilon (h+1)] \sin \gamma + q \lambda_\gamma \cos \gamma \end{aligned} \quad (41)$$

since  $H = 0$ . Differentiating equation (40), one obtains

$$\epsilon \lambda_\gamma'' = -\sin \gamma \lambda_\gamma' - q \lambda_\gamma + 0(\epsilon) + 0(\sin^2 \gamma) \quad (42)$$

The solution to equation (42), rewritten as

$$\epsilon \lambda_\gamma'' + \gamma \lambda_\gamma' + q \lambda_\gamma = 0 \quad (43)$$

for small  $\gamma$ , gives the cotangent of the optimal bank angle control. Equation (43) is regarded as a linear equation with coefficients depending on the independent variable,  $s$ . The behavior of the solution depends on the coefficient of the damping term, which is the path angle,  $\gamma$ . According to reference 7, if  $\gamma(s_e) < 0$ , there will be no boundary layer for the solution of  $\lambda_\gamma$  at  $s_e$ , and that if  $\gamma(s_f) > 0$ , there can be no boundary layer for  $\lambda_\gamma$  at  $s_f$ . In the present problem  $\gamma(s_e) < 0$  for entry into the atmosphere and  $\gamma(s_f) > 0$  for exit. Thus, no boundary layers are expected in the solution of (43). Since  $\gamma(s_e) < 0$  and  $\gamma(s_f) > 0$ ,  $\gamma(s_t) = 0$  for some  $s_t$ , such that  $s_e < s_t < s_f$ . The simplest possible situation is that  $\gamma$  vanishes only once on the interval so that

$$\begin{array}{lll} \gamma < 0 & \text{for} & s_0 \leq s < s_t \\ \gamma > 0 & \text{for} & s_t < s \leq s_f \\ \gamma(s_t) = 0. & & \end{array} \quad (44)$$

The solution to equation (43) can be sought in two steps with the aid of assumptions (44). The first step is to find the solutions for  $s$  away from  $s_t$ , which break down at  $s = s_t$ . The second step is to find a solution in a neighborhood of  $s = s_t$  which is asymptotic to the two solutions found in the first step. Problems that contain internal transition zones, such as this, are known as turning point problems, reference 8, with the turning point, in this case, occurring at  $s = s_t$  when  $\gamma = 0$ .

#### ANALYSIS OF THE TURNING POINT PROBLEM

The optimal bank angle is found from the solution to equation (43), which can be placed in a standard form whose solution is known. For convenience later, the independent variable is translated by letting

$$x = (s - s_t) \quad (45)$$

so that  $x(0) = -s_t$  and  $x(s_t) = 0$ . The dependent variable is transformed from  $\lambda_\gamma$  to  $w$  by letting

$$\lambda_\gamma = w \exp\left(-\int_0^x (\gamma/2\epsilon) d\tau\right). \quad (46)$$

Substituting equation (46) into equation (43) and factoring off exponential terms, one obtains

$$w'' - (2\epsilon)^{-2} Q(x)w = 0 \quad (47)$$

where.

$$Q(x) = \gamma^2 + 2\epsilon\gamma' - 4\epsilon q. \quad (48)$$

Equation (47) is the standard form for analyzing turning point problems with a large parameter,  $(2\epsilon)^{-2}$ , reference 8.

### Outer Solutions

The general solution to equation (47) is given by

$$w = |Q(x)|^{-1/4} \left[ A \exp\left(\int_c^x (\sqrt{Q}/2\epsilon) dt\right) + B \exp\left(-\int_c^x (\sqrt{Q}/2\epsilon) dt\right) \right] + O(\epsilon) \quad (49)$$

where A, B, and c are constants to be determined. The fractional powers of Q(x) in (49) are determined using the binomial expansion, assuming  $\gamma(x)$  is away from zero, outside the transition zone. From equation (48)

$$\sqrt{Q} = \gamma + \epsilon\gamma'/\gamma - 2\epsilon q/\gamma \quad (50)$$

$$|Q|^{1/4} = \sqrt{|\gamma|} + O(\epsilon). \quad (51)$$

Substituting equation (46) into equation (49), together with (50) and (51), one obtains for the zeroth-order approximation to  $\lambda_\gamma$  to the left of the turning point

$$\lambda_{\gamma_0}^l = a_1 E^{-1} + a_2 e^{-(h-h_{\min})} E/\gamma; \quad x < 0 \quad (52)$$

where,

$$E = \exp\left(\int_{x_e}^x (q/\gamma) dt\right). \quad (53)$$

To the right of the turning point,

$$\lambda_{\gamma_0}^r = b_1 F^{-1} + b_2 e^{-(h-h_{\min})} F/\gamma; \quad x > 0 \quad (54)$$

where,

$$F = \exp\left(-\int_x^{x_f} (q/\gamma) dt\right). \quad (55)$$

The zeroth-order solutions for  $\lambda_\gamma$  given by equations (52) and (54) describe the cotangent of the bank angle control outside the transition zone, i.e., for  $|x| > 0$ .

The solutions may be regarded as a mixture of terms  $(a_1 E^{-1}, a_2 E/\gamma, b_1 F^{-1}, b_2 F/\gamma)$  which are functions of the slow scale x and of the term,  $e^{-(h-h_{\min})}$ .

which is a function of the fast scale,  $x/\epsilon$ , since  $h-h_{\min} = \int(\gamma/\epsilon)dx$ . The solutions (52) and (54) break down in the transition zone where  $\gamma=0$ . When  $x$  is small in the transition zone,  $\gamma$  is assumed to have the Taylor series expansion

$$\gamma = \gamma'|_{x=0}x + \dots \equiv \alpha x + \dots \quad (56)$$

where,  $\alpha > 0$ . Likewise

$$\gamma' \rightarrow \alpha \text{ and } q \rightarrow q(0) \equiv q_t \quad (57)$$

as  $x \rightarrow 0^\pm$ . Under assumptions (56) and (57), the differential equation for  $w$ , in (47), becomes

$$W_0'' - (2\epsilon)^{-2}[(\alpha x)^2 + 2\epsilon\alpha - 4\epsilon q_t]W_0 = 0 \quad (58)$$

where,  $W_0$  denotes the zeroth-order inner variable for the solution to equation (47) when  $x$  is small. Likewise, under the small  $x$  assumptions, the outer solutions are, from equations (52) and (54):

$$\lambda_{\gamma_0}^l = a_1 \exp\left(-\int_{x_e}^x (q_t/\alpha\tau)d\tau\right) + a_2(\alpha x)^{-1} e^{-\alpha x^2/2\epsilon} \exp\left(\int_{x_e}^x (q_t/\alpha\tau)d\tau\right) \quad (59)$$

$$\lambda_{\gamma_0}^r = b_1 \exp\left(\int_x^{x_f} (q_t/\alpha\tau)d\tau\right) + b_2(\alpha x)^{-1} e^{-\alpha x^2/2\epsilon} \exp\left(-\int_x^{x_f} q_t/\alpha\tau d\tau\right). \quad (60)$$

The inner solution to equation (58) can be expressed in terms of the stretched dependent variable,  $X = \sqrt{\alpha/\epsilon} x$ . This solution leads to an inner expansion

for  $\lambda_\gamma$ , denoted by  $\Lambda_{\gamma_0}$ . The inner solution,  $\Lambda_{\gamma_0}$ , for large  $X$  can be matched with the outer solutions, equations (59) and (60), for small  $x$ .

### Inner Solution

Equation (58) for the inner variable  $W_0$  can be rewritten in terms of the stretched dependent variable  $X = \sqrt{\alpha/\epsilon} x$  as

$$d^2W_0/dX^2 - (X^2/4 + 1/2 - q_t/\alpha)W_0 = 0. \quad (61)$$

Equation (61) is the parabolic cylinder equation whose solution is known in terms of the parabolic cylinder functions,  $D_\nu(X)$  and  $D_\nu(-X)$ , when  $\nu$  is not a nonnegative integer. The solution to equation (61) is given in reference 9 as

$$W_0(X) = c_1 D_\nu(X) + c_2 D_\nu(-X) \quad (62)$$

where,  $\nu = q_t/\alpha - 1$ , so that the solution for  $\Lambda_{\gamma_0}$  is given by

$$\Lambda_{\gamma_0}(X) = e^{-X^2/4} [c_1 D_{\nu}(X) + c_2 D_{\nu}(-X)]. \quad (63)$$

The asymptotic properties of the parabolic cylinder functions for large  $X$  are used to match with the outer solutions for small  $x$ . The asymptotic properties of the parabolic cylinder functions are given by

$$D_{\nu}(X) \sim (X)^{\nu} e^{-X^2/4} \quad \text{as } X \rightarrow +\infty \quad (64)$$

$$D_{\nu}(-X) \sim (X)^{-\nu-1} e^{X^2/4} \sqrt{2\pi}/\Gamma(-\nu) \quad \text{as } X \rightarrow +\infty \quad (65)$$

where  $\Gamma$  denotes the gamma function.

### Matching and the Formation of a Uniform Expansion for $\lambda_{\gamma}$ (ref. 10)

The inner expansion, equation (63), is first expressed in terms of the asymptotic limits of the parabolic cylinder functions as  $X \rightarrow \pm\infty$  and then matched with the outer expansions for small  $x$  as  $x \rightarrow 0^{\pm}$ . From equations (63) through (65) as  $X \rightarrow +\infty$ ,

$$\Lambda_{\gamma_0}(X) = c_1(X)^{q_t/\alpha-1} e^{-X^2/2} + c_2(X)^{-q_t/\alpha} \sqrt{2\pi}/\Gamma(1-q_t/\alpha). \quad (66)$$

Since the first term on the right-hand side of (66) is exponentially small compared with the second, the first term can be neglected in comparison with the second. Next, the inner limit may be expressed in terms of the outer dependent variable,  $x$ , in preparation for matching, as

$$\Lambda_{\gamma_0} \sim c_2 C^{-1} x^{-q_t/\alpha}, \quad \text{as } X \rightarrow +\infty \quad (67)$$

where,

$$C = (\sqrt{\alpha/\epsilon})^{q_t/\alpha} [\Gamma(1-q_t/\alpha)]/\sqrt{2\pi}. \quad (68)$$

Similarly, to the left of the turning point the inner limit is given by

$$\Lambda_{\gamma_0} \sim c_1 C^{-1} (-x)^{-q_t/\alpha}, \quad \text{as } X \rightarrow -\infty. \quad (69)$$

Equation (67) is matched with  $\lambda_{\gamma_0}^r$  as  $x \rightarrow 0^+$ , and equation (69) is matched

with  $\lambda_{\gamma_0}^l$  as  $x \rightarrow 0^-$ . The second term in equation (60) for  $\lambda_{\gamma_0}^r$  is, for small

$\epsilon$ , exponentially small compared to the first so that the second term may be neglected in comparison with the first. Thus,

$$\lambda_{\gamma_0}^r \sim b_1 e^{-(q_t/\alpha) \ln|x/x_f|}, \quad \text{as } x \rightarrow 0^+. \quad (70)$$

Similarly, from equation (59)

$$\lambda_{\gamma_0}^l \sim a_1 e^{-(q_t/\alpha) \ln[-x/(-x_e)]}, \text{ as } x \rightarrow 0^- \quad (71)$$

A digression to discuss the boundary condition on  $\lambda_{\gamma_0}^r$  is necessary before proceeding with the matching process. The boundary condition on  $\lambda_{\gamma_0}^r$  is zero since the final value of  $\gamma$  is unspecified. This boundary condition can only be satisfied asymptotically for large  $x$  (since  $b_1 \neq 0$ ). This is possible since  $q > 0$  and  $\gamma > 0$  to the right of the turning point so that the outer solution,  $\lambda_{\gamma_0}^r$ , from equation (54) decays from the turning point. Thus,  $b_1$  is simply a scale factor for  $\lambda_{\gamma_0}^r$ . Thus, the upper limits on the integrals in equation (60) are set equal to  $+\infty$  to indicate that the final boundary condition on  $\lambda_{\gamma_0}^r$  is satisfied asymptotically, i.e.,  $\lambda_{\gamma_0}^r(x) \rightarrow 0$  as  $x \rightarrow +\infty$ , and  $b_1 \equiv 1$ . Returning to the matching procedure, one matches equation (54) with equation (67), using (70), by equating the two limits as  $x \rightarrow +\infty$  and  $x \rightarrow 0^+$ :

$$c_2 \rightarrow C \exp \int_x^{+\infty} (q/\gamma - q_t/\alpha\tau) d\tau \quad (72)$$

so that

$$c_2 = C \exp \int_0^{+\infty} (q/\gamma - q_t/\alpha\tau) d\tau \quad (73)$$

Similarly, to the left of the turning point, equation (52) is matched with equation (69), using (71), by equating the two limits as  $x \rightarrow -\infty$  and  $x \rightarrow 0^-$ :

$$c_1 \rightarrow a_1 (-x_e)^{q_t/\alpha} C \exp \int_x^{+\infty} (q/\gamma - q_t/\alpha\tau) d\tau \quad (74)$$

so that

$$c_1 = a_1 (-x_e)^{q_t/\alpha} C \exp \int_0^{+\infty} (q/\gamma - q_t/\alpha\tau) d\tau \quad (75)$$

Equations (73) and (75) provide the connections between the coefficients,  $c_1$  and  $c_2$ , of the inner solution and the coefficients of the outer solutions,  $a_1$  and  $b_1$ , that are determined by the boundary conditions. Thus, the two outer solutions have been continued through the transition zone. The matching procedure showed that exponentially small terms were neglected in forming a leading order uniform approximation. Thus,  $a_2 = b_2 = 0$  for the leading order

approximation. These terms enter into the solution at the next higher approximation (ref. 10). The leading order uniform approximation for  $\lambda_\gamma$  is constructed by combining equation (66) with the connection formulas, (72) and (74):

Uniform Approximation for  $\lambda_\gamma$ :

$$\lambda_{\gamma u} = C e^{-\alpha x^2/4\epsilon} \{A(x)D_v(\sqrt{\alpha/\epsilon}x) + B(x)D_v(\sqrt{\alpha/\epsilon}x)\} \quad (76)$$

where,

$$A(x) = a_1 (-x_e)^{q_t/\alpha} \exp \int_x^{x_e} (q/\gamma - q_t/\alpha\tau) d\tau \quad (77)$$

$$B(x) = \exp \int_x^{+\infty} (q/\gamma - q_t/\alpha\tau) d\tau \quad (78)$$

$$C = (\sqrt{\alpha/\epsilon})^{q_t/\alpha} [\Gamma(1 - q_t/\alpha)] / \sqrt{2\pi} \quad (79)$$

$$v = q_t/\alpha - 1 = -(n_t - 2) / (n_t - 1) \quad (80)$$

$$n_t = (L \cos\mu / (W - mV^2/r_r))_x = 0. \quad (81)$$

In equation (81),  $n_t$  is the "reduced" vertical load factor, evaluated at the turning point; that is,  $n_t$  is the ratio of the vertical lift force at the turning point to the weight of the vehicle, reduced by the centrifugal force at the turning point.

The results of this section are summarized by equations (76) through (81) and the following equations for the outer and inner solutions:

Outer Solutions:

$$\lambda_{\gamma_o}^r = \exp \int_x^{+\infty} (q/\gamma) d\tau, \quad x \gg 0 \quad (82)$$

$$\lambda_{\gamma_o}^l = \lambda_\gamma(x_e) \exp \left( -\int_{x_e}^x (q/\gamma) d\tau \right), \quad x \ll 0 \quad (83)$$

Inner Solution:

$$\lambda_{\gamma_o} = e^{-\alpha x^2/4\epsilon} \{c_1 D_v(\sqrt{\alpha/\epsilon}x) + c_2 D_v(-\sqrt{\alpha/\epsilon}x)\} \quad (84)$$



$$c_1 = C A (0) \quad (85)$$

$$c_2 = C B (0) \quad (86)$$

Equations (76) through (81) give the leading order uniform approximation for the cotangent of the optimal bank angle control over the entire interval,  $x_0 \leq x \leq +\infty$ . In equation (79), C is a constant for a fixed atmosphere, "reduced" load factor at the turning point and desired minimum altitude for the maneuver. The coefficients A(x) and B(x) are functions of the slow dependent variable, x, while the parabolic cylinder functions and the exponential factor are functions of the fast dependent variable,  $X = \sqrt{\alpha/\epsilon} x$ , in the transition zone. The uniform expansion, for large |X|, is asymptotic to the outer solutions, which are functions of the slow variable, x, and satisfy the boundary conditions on  $\lambda_\gamma$  at the two end points. The inner solution, equation (84), depends on the minimum altitude for the maneuver and the "reduced" load factor at the turning point. With the uniform approximation for the bank angle control in hand, the reader's attention is focused on determining the optimal, normalized lift,  $\lambda$ .

#### Determination of Normalized Lift

The approximation for the multiplier,  $\lambda_\gamma$ , has been derived in terms of the state and dependent variables, so that  $\lambda_\gamma$  can be determined, also, in terms of the state and dependent variables. The Hamiltonian can thus be expressed in terms of the state and dependent variables and the unknown multiplier,  $\lambda_u$ , needed to determine the optimal normalized lift,  $\lambda$ .

The Hamiltonian can be rewritten in terms of  $\lambda'_\gamma$  and  $(2\lambda_u/E^*)$  as

$$H = (2\lambda_u/E^*)^2 + 2b(2\lambda_u/E^*) - (1 + \lambda_{\gamma u}^2) = 0 \quad (87)$$

using equations (34), (37), (38), and (40). In equation (87)

$$b = Z^{-1} (\gamma \lambda'_\gamma + \lambda_{\gamma u} q) \quad (88)$$

$$Z = \rho_r S C_L^* r_r e^{-h/2m} \quad (\text{eq. 13}) \quad (89)$$

$$\lambda_{\gamma u} = C e^{-\alpha x^2/4} \epsilon \{ A(x) D_v(\sqrt{\alpha/\epsilon} x) + B(x) D_v(-\sqrt{\alpha/\epsilon} x) \} \quad (\text{eq. 80}) \quad (90)$$

Thus,  $(2\lambda_u/E^*)$  is determined as the positive root of equation (87),

$$(2\lambda_u/E^*) = -b + \sqrt{b^2 + (1 + \lambda_{\gamma u}^2)} \quad (91)$$

Thus, the approximation for the optimal normalized lift is, from equation (37).

$$\lambda = a + \sqrt{a^2 + 1} \quad (92)$$

where,

$$a = b / \sqrt{1 + \lambda_{\gamma_u}^2} \quad (93)$$

The variable, b, in equation (93) depends on  $\lambda'_{\gamma_u}$ , from equation (88). Differentiating equation (90), one obtains

$$\begin{aligned} \lambda'_{\gamma_u} = & C e^{-\alpha x^2/4\epsilon} \left( -(q/\gamma - q_t/\alpha x) A(x) D_{\nu}(X) - (q/\gamma - q_t/\alpha x) B(x) D_{\nu}(-X) \right) \\ & + A(x) d[e^{-\alpha x^2/4\epsilon} D_{\nu}(X)]/dx + B(x) d[e^{-\alpha x^2/4\epsilon} D_{\nu}(-X)]/dx \end{aligned} \quad (94)$$

The derivatives in (94) can be found from the recurrence relation (refs. 11 and 12)

$$d[e^{-\xi^2/4} D_{\nu-1}(\xi)]/d\xi = -e^{-\xi^2/4} D_{\nu}(\xi), \quad (95)$$

so that equation (94) may be simplified as

$$\lambda'_{\gamma_u} = - (q/\gamma - q_t/\alpha x) \lambda_{\gamma_u} + C e^{-\alpha x^2/4\epsilon} \sqrt{\alpha/\epsilon} [-A(x) D_{q_t/\alpha}(X) + B(x) D_{q_t/\alpha}(-X)] \quad (96)$$

From equation (88)

$$b = Z^{-1} \gamma \left\{ (q_t/\alpha x) \lambda_{\gamma_u} + C e^{-\alpha x^2/4\epsilon} \sqrt{\alpha/\epsilon} [-A(x) D_{q_t/\alpha}(X) + B(x) D_{q_t/\alpha}(-X)] \right\} \quad (97)$$

so that the coefficient of  $(2\lambda_u/E^*)$  in equation (87) can be determined, and hence, the value of  $\lambda$  can be determined from equations (92) and (93).

The value of  $\lambda$  for large  $|X|$  can be determined from the asymptotic properties of  $D_{q_t/\alpha}(X)$ :

$$\begin{aligned} D_{q_t/\alpha}(X) & \sim (X)^{-q_t/\alpha-1} \sqrt{2\pi} e^{X^2/4} / \Gamma(-q_t/\alpha) \\ & = \Gamma(1-q_t/\alpha) D_{q_t/\alpha-1}(X) / \Gamma(-q_t/\alpha) \\ & = (-q_t/\alpha) D_{q_t/\alpha-1}(X), \text{ as } X \rightarrow +\infty \end{aligned} \quad (98)$$

Thus, for large positive X, equation (97) has the asymptotic property,

$$b \sim Z^{-1} \gamma \left\{ (q_t/\alpha x) \lambda_{\gamma_u} - (q_t/\alpha x) \lambda_{\gamma_u} \right\} = 0, \quad (99)$$

where exponentially small terms have been neglected. Thus, outside the transition zone,  $b \sim 0$ , so that

$$\lambda \sim 1 \quad \text{as } X \rightarrow +\infty \quad (100)$$

which implies

$$C_L \sim C_L^* \quad \text{as } X \rightarrow +\infty. \quad (101)$$

Similarly,

$$C_L \sim C_L^* \quad \text{as } X \rightarrow -\infty \quad (102)$$

also.

Formulas have been derived for an approximation to the optimal normalized lift coefficient,  $\lambda$ , by solving a quadratic algebraic equation for the unknown multiplier,  $2\lambda_u/E^*$ , from which  $\lambda$  is obtained from the extremal control formula in terms of  $2\lambda_u/E^*$ . The value of  $\lambda$  was shown to be asymptotic to one, outside the transition zone. This implies that the lift coefficient is asymptotic to its value at  $\max(L/D)$ , outside the transition zone.

The next section contains a summary of the formulas needed to synthesize the optimal guidance commands.

#### Summary of Optimal Guidance Commands

Formulas have been derived in the previous two sections for the optimal bank angle and lift controls for the problem of maximizing heading angle for a given energy loss. The formulas assume that full, reduced-order state and dependant variable estimates are available from the vehicle's navigation computer. For preliminary simulation studies one may assume that the reduced-order model, derived herein, supplies these estimates perfectly (i.e., uncorrupted by noise and other uncertainties).

The following is a summary of the optimal control formulas:

#### Bank Angle Control

$$\mu = \text{arccot}(\lambda\gamma_u) \quad (103)$$

where,

$$\lambda\gamma_u = C e^{-\alpha x^2/4\epsilon} [A(x) D_v(\sqrt{\alpha/\epsilon} x) + B(x) D_v(-\sqrt{\alpha/\epsilon} x)] \quad (104)$$

$$A(x) = \lambda\gamma(x_e) (-x_e) q_t / \alpha \exp \int_x^x e^{(q/\gamma - q_t/\alpha)\tau} d\tau \quad (105)$$

$$B(x) = \exp \int_x^{+\infty} (q/\gamma - q_t/\alpha) d\tau \quad (106)$$

$$C = (\sqrt{\alpha/\epsilon}) q_t / \alpha [\Gamma(1 - q_t/\alpha)] / \sqrt{2\pi} \quad (107)$$

$$v = q_t / \alpha - 1 = -(n_t - 2) / (n_t - 1) \quad (108)$$

$$n_t = [L \cos \mu / (W - mV^2 / r_r)]_{x=0} \quad (109)$$

$$\epsilon = 1 / (\beta r_r) \quad (110)$$

$D_v(x)$ ,  $D_v(-x)$  are given in tabular form (ref. 13)

$$x = s_t + \int_0^s (V/r) d\tau ; x_e = -s_t \quad (111)$$

$$q = e^{-u/2} - 1 = (\bar{\mu} V^2 r_r - 1) ; q_t = q(0) \quad (112)$$

$$\gamma = \int \gamma dx \quad (113)$$

$$\gamma' = Z \lambda \cos \mu - q ; \alpha = \gamma|_{x=0} \quad (114)$$

$$z = z_r e^{-h} ; h = \beta(H - H_r) \quad (115)$$

$$z_r = \rho_r S C_L^* r_r / 2m \quad (116)$$

### Lift Control

$$\lambda = a + \sqrt{a^2 + 1} \quad (117)$$

where,

$$a = b / \sqrt{1 + \lambda \gamma_u^2} \quad (118)$$

$$b = Z^{-1} \gamma \{ (q_t / \alpha x) \lambda \gamma_u + C_e \alpha x^2 / 4 \epsilon \sqrt{\alpha / \epsilon} [-A(x) D_{q_t} / \alpha (\sqrt{\alpha / \epsilon} x) + B(x) D_{q_t} / \alpha (-\sqrt{\alpha / \epsilon} x)] \} \quad (119)$$

### Values of the State and Related Variables

$$h = \int h' dx ; u = \int u' dx ; \psi = \int \psi' dx \quad (120)$$

$$h' = \epsilon^{-1} (1 + \epsilon h) \gamma \quad (121)$$

$$u' = -Z (1 + \lambda^2) / E^* \quad (122)$$

$$\psi' = Z \lambda \sin \mu \quad (123)$$

$$E = \bar{\mu} (e^u - 1) / r_r \quad (124)$$

$$V = \sqrt{2 [E + \bar{\mu} / (r_r + H)]} \quad (125)$$

In order to implement the feedback formulas for the optimal guidance commands, one must write a numerical simulation to determine the integrals for the state equations as well as the integrals needed for matching. While this has not been completed, herein, an analytical example of the procedure for conducting numerical studies is provided in the next section.

### ANALYTICAL EXAMPLE AND DISCUSSION

Although numerical simulations are necessary to complete the matching procedure and to determine the accuracy of the optimal guidance approximation, the behavior of the bank angle control may be examined in the transition zone where the guidance law reduces to an analytical formula. Data from reference 3, listed in Table 1, are used for the vehicle model in this example and tabulated values of the parabolic cylinder and gamma functions, given in reference 13, are used to calculate the guidance commands.

Table 2 contains estimates of the guidance parameters as a function of minimum altitude for the maneuver. The constants,  $c_1 = -.0542$  and  $c_2 = .2748$ , were selected to form the inner approximation,  $\Lambda_{\gamma_0}$ , as a linear combination of parabolic cylinder functions multiplied by  $e^{-X^2/4}$ . The index,  $\nu = -.5$ , for the parabolic cylinder functions was selected for this example, which corresponds approximately to a 150,000 foot altitude trajectory with a bank angle at the turning point of  $75^\circ$  and a normalized lift coefficient,  $\lambda = 1$ .

The index for the parabolic cylinder functions,  $\nu$ , depends on the vertical, "reduced" load factor,  $n_t$ , at the turning point.  $n_t$  is a function of  $H_{min}$ ,  $V$ , and  $\lambda \cos \mu$  at the turning point, so that care must be exercised in estimating  $\nu$ . The values for  $\nu$  given in Table 2 lie in the range,  $-1 < \nu < 0$ , which corresponds to a range of reduced load factors,  $n_t > 2$ . For  $1 \leq n_t \leq 2$ ,  $\nu$  is non-negative, and when  $\nu$  is a non-negative integer, the guidance formulas are not valid since  $D_\nu(X)$  and  $D_\nu(-X)$  are linearly dependent in this case. Although the guidance solutions can be formulated more generally in terms of other linearly independent solutions of the parabolic cylinder equation, one would expect, for most applications, that the reduced load factor exceeds a value of two. In this event  $\nu$  is a negative number greater than  $-1$ .

Data for  $\Lambda_{\gamma_0}$  and the corresponding bank angle,  $M_0$ , are tabulated in Table 3 and plotted in figure 2. Values for the two outer solutions,  $\lambda_{\gamma_0}^R$  and  $\lambda_{\gamma_0}^L$ , are tabulated in table 4 as functions of the inner variable,  $X$ . The outer solution approximations for small  $x$ :

$$\mu_0^R = \cot^{-1}[c_2(X/2)^{-1/2}] ; X > 0 \quad (126)$$

$$\mu_0^L = \cot^{-1}[c_1(-X/2)^{-1/2}] ; X < 0 \quad (127)$$

are also plotted in figure 2 so that the behavior of the inner and outer solutions can be observed in the overlapping regions, where these solutions match.

For purposes of comparison the optimal bank angle control, given in reference 3, is superimposed on figure 2. The agreement between the inner solution guidance approximation,  $M_0$ , and the optimal bank angle control,  $\mu_{opt}$ , is excellent in the neighborhood of the turning point,  $X = 0$ . The agreement between the optimal bank angle,  $\mu_{opt}$ , and its approximation,  $\alpha x$ , near the turning point is also good. Away from the turning point the outer solution approximations for small  $|x|$ ,  $\mu_0^r$  and  $\mu_0^l$ , are asymptotic the inner solution  $M_0$ , for large  $|X|$ . The outer solutions for large  $X$  are not in numerical agreement with the optimal solution for the bank angle, indicating the need to adjust the initial condition on  $\mu_0$  by adjusting  $\lambda_{\gamma_0}^l(x_e)$ . Matching the inner and outer solutions for specified initial conditions requires a numerical simulation, which is not included in this report.

For numerical studies one could begin by integrating trajectories from the turning point to the boundaries using the estimates for the state given in Table 2. The integration to the right of the turning point could start at  $x = 0^+$ , by extrapolating the state from  $x = 0$  to  $x = 0^+$  using a Taylor expansion. Using the guidance formulas as feedback controls, the reduced state equations would be integrated to determine the change in the state from the turning point. The flight path angle must remain positive to the right in order to satisfy the boundary condition on  $\lambda_{\gamma_0}^r$ . At the terminal point the energy lost and heading angle change from the turning point would be available. These values would be added to the values of "delta" heading angle and energy loss for a similar state-equation integration to the left of the turning point. The integration to the left, however, will undoubtedly not match the initial conditions on the state. In this case the initial condition on  $\lambda_{\gamma_0}^l$  can be adjusted to change the outer solution to the left of the turning point. An iterative process ensues because  $\lambda_{\gamma_0}^l(x_e)$  is proportional to  $c_1$  of the inner expansion,  $\Lambda_{\gamma_0}$ . The numerical analysis would proceed by adjusting not only  $c_1$  but also some of the parameters in Table 2. New values for the index of the parabolic cylinder functions and a new estimate for  $\Lambda_{\gamma_0}$  would be calculated. This procedure would be repeated until the initial conditions are satisfied sufficiently accurately and the final condition,  $\lambda_{\gamma_0}^r \sim 0$  as  $x \rightarrow +\infty$  with  $\gamma > 0$ , holds. A numerical procedure such as that outlined above should produce bank angle commands that are qualitatively similar to those computed with a numerical optimization computer code.

To complete the numerical analysis, one would use the uniform guidance law, obtained by matching with the reduced model, as a feedback law for the full system, equations (2) through (7), to compare optimally guided trajectories with trajectories computed using a numerical optimization procedure.

This analytical example has shown that optimal trajectories for the minimum fuel/plane change problem can be parameterized by the minimum altitude for the maneuver in the following sense. Each minimum altitude corresponds to a trajectory that satisfies initial conditions and results in a loss of energy

and change in heading angle to the left of the turning point. The energy loss and "reduced" vertical load factor at the turning point determine the index of the parabolic cylinder functions. The loss of energy and change in heading angle to the right of the turning point, when added to the corresponding values to the left of the turning point, result in an energy loss and maximum (minimum) heading angle change for the entire maneuver, parameterized by the minimum altitude of the maneuver.

#### CONCLUDING REMARKS

Formulas have been derived for computing approximations to extremal controls, normalized lift and bank angle, for the problem of minimizing energy loss to change the orbital inclination of a vehicle. Asymptotic analysis was used to derive formulas for the bank angle control that were valid in three different regions of the trajectory. Two outer solutions were found to approximate the bank angle control near the upper limits of the Earth's atmosphere. These solutions break down at the minimum altitude of the maneuver, when the flight path angle is zero. In the transition zone, near  $\gamma = 0$ , the optimal bank angle control was described in terms of parabolic cylinder functions. The two outer solutions were connected through the transition zone by forming a uniform solution, in terms of the parabolic cylinder functions, to approximate the optimal bank angle control over the entire trajectory. The approximate, optimal lift coefficient was subsequently determined by solving a quadratic algebraic equation. The lift coefficient was determined to be asymptotic to its value at max (L/D) when the trajectory is far from the turning point. The guidance formulas are in terms of state and dependent variables and parameters that define the index and argument of the parabolic cylinder function. A numerical example was given to illustrate the utility of the method used to construct the guidance formulas. Construction of the inner solution showed that optimal trajectories will be parameterized by the minimum altitude selected for the flight and the "reduced" vertical load factor at the turning point, which determines the index of the parabolic cylinder function. Numerical studies, however, must be conducted to obtain matching conditions and to verify the accuracy of the guidance law. The guidance formulas require feedback of energy, altitude, and flight path angle from the vehicle's navigation computer and are suitable for forming the basis for an on-board optimal guidance system.

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TABLE 1

VEHICLE/MODEL PARAMETERS	
$S = 126 \text{ ft}^2$	$\bar{\mu} = 1.409 \times 10^{16} \text{ ft}^3\text{-sec}^{-2}$
$m = 332 \text{ slugs}$	$\epsilon = 1.29 \times 10^{-3}$
$C_L^* = .151$	$H_r = 180,000 \text{ ft}$
$\beta = (27000 \text{ ft})^{-1}$	$\rho_r = .356 \times 10^{-4} \text{ lbm/ft}^3$
$r_r = 20,900,000 \text{ ft}$	$g_c = 32.2 \text{ lbm/slug}$
$Z_r = C_L^* \rho_r S r_r / 2mg_c = .662 \text{ (eq. 116)}$	$E^* = 2.36$

TABLE 2

GUIDANCE SYSTEM PARAMETER ESTIMATES (Ref. 3)							
$H_{\min} \text{ (Est)}$ (ftx1000)	$Z(H_{\min})$ (eq 115)	$V_t \text{ (Est)}$ (ftx1000)	$q_t$ (eq 112)	$n_t (\mu_t = 75^\circ)$ (eq 109)	$v$ (eq 108)	$\alpha$ (eq 114)	$\sqrt{\alpha/\epsilon}$
175	.797	25	.0787	2.62	-.383	.128	10.0
150	2.01	24	.170	3.06	-.515	.350	16.5
125	5.08	23	.274	4.80	-.737	1.04	28.5

TABLE 3

Inner Solution

$$\Lambda_{\gamma_0}(X) = [-.0542 D^{-1/2}(X) + .2748 D^{-1/2}(-X)]e^{-X^2/4}$$

$$M_0(X) = \cot^{-1}(\Lambda_{\gamma_0})$$

X	Multipliers		Bank Angles (deg)	
	$\Lambda_{\gamma}(X)$	$\Lambda_{\gamma}(-X)$	M(X)	M(-X)
0.0	.2688	.2688	75.0	75.0
0.5	.3420	.1620	71.1	80.8
1.0	.3640	.0621	70.0	86.4
1.5	.3433	-.0050	71.1	90.3
2.0	.3045	-.0368	73.1	92.1
2.5	.2668	-.0459	75.1	92.6
3.0	.2375	-.0455	76.6	92.6
3.5	.2159	-.0425	77.8	92.4
4.0	.1998	-.0396	78.7	92.3
4.5	.1871	-.0371	79.4	92.1
5.0	.1767	-.0350	80.0	92.0

TABLE 4

Outer Solutions					
$\lambda_{\gamma_0}^r = .2748 (X/2)^{-1/2} ; \mu_0^r = \cot^{-1}(\lambda_{\gamma_0}^r)$ $\lambda_{\gamma_0}^l = -.0542 (-X/2)^{-1/2} ; \mu_0^l = \cot^{-1}(\lambda_{\gamma_0}^l)$					
Multipliers			Bank Angles (deg)		
X	$\lambda_{\gamma_0}^r$	$\lambda_{\gamma_0}^l$	$\mu_0^r$	$\mu_0^l$	
.25	.7773	-.1533	52.1	98.7	
.50	.5496	-.1084	61.2	96.2	
1.0	.3886	-.0767	68.8	94.4	
2.0	.2748	-.0542	74.6	93.1	
3.0	.2248	-.0443	77.4	92.5	
4.0	.1943	-.0383	79.0	92.2	
5.0	.1738	-.0343	80.1	92.0	

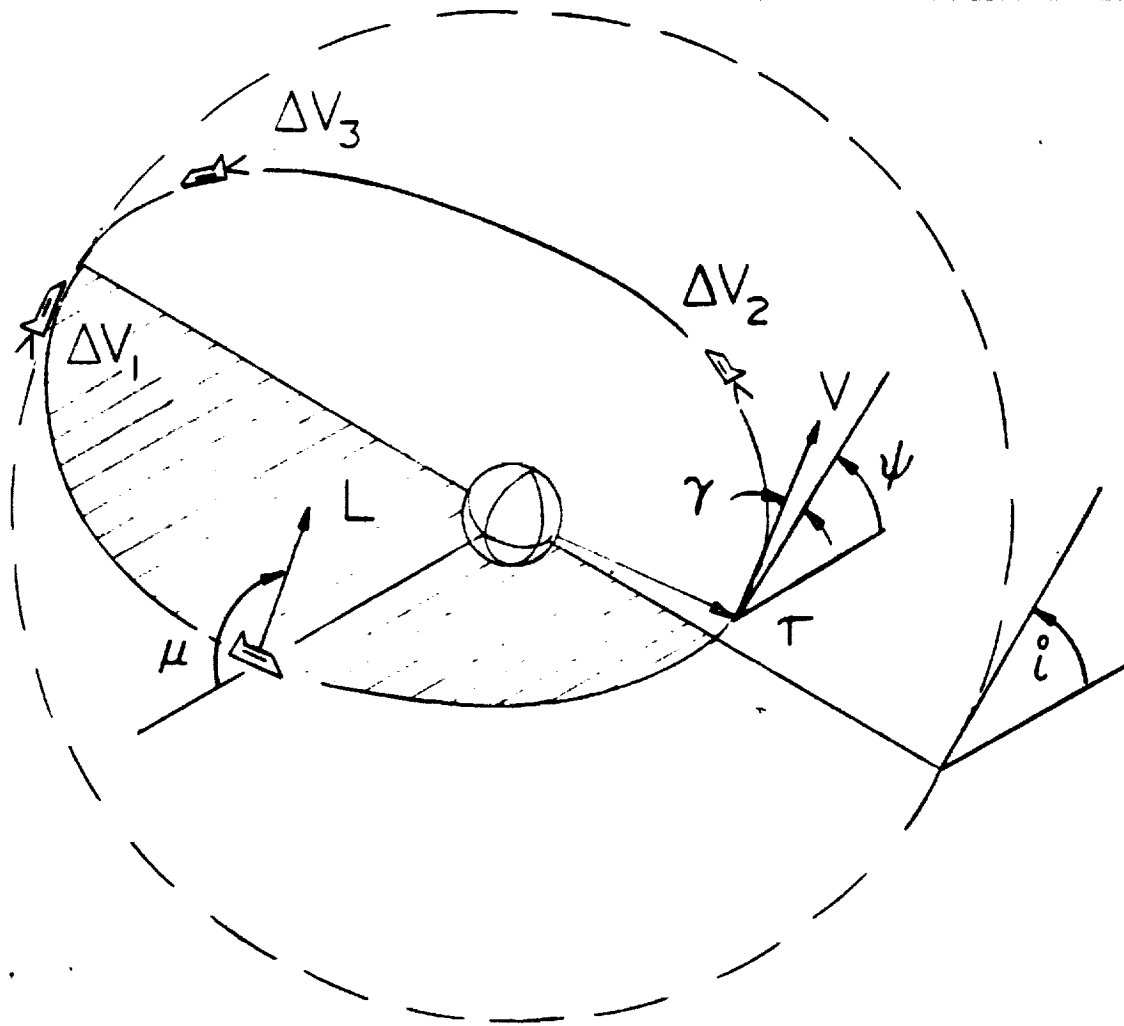


FIGURE 1. MANEUVER SEQUENCE

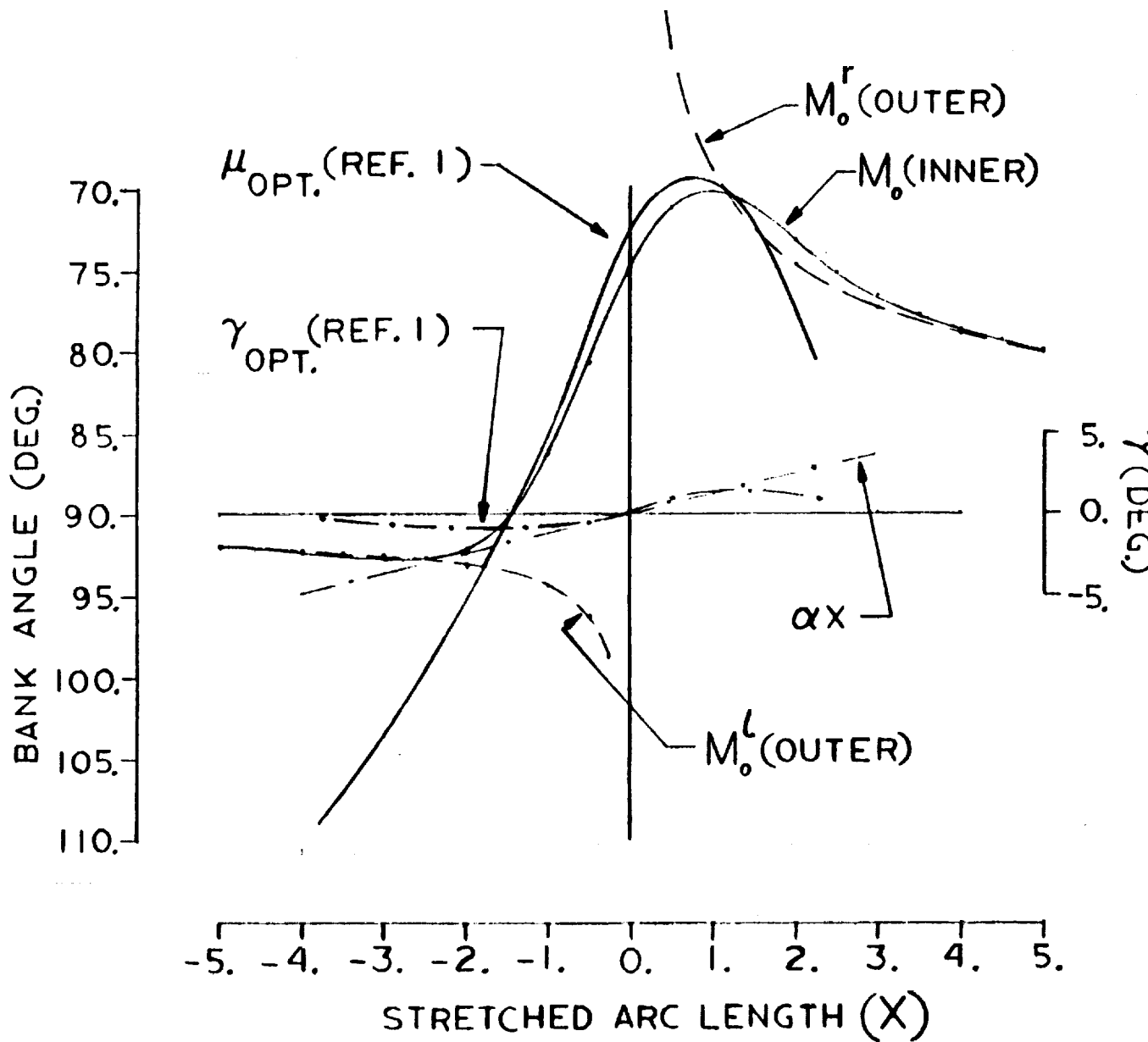


FIGURE 2. BANK AND FLIGHT PATH ANGLE COMPARISONS



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16. Abstract <p>The development of guidance approximations for the atmospheric (aeroglide) portion of the minimum fuel, orbital plane change, trajectory optimization problem is described. Asymptotic methods are used to reduce the two point, boundary value, optimization problem to a turning point problem for the bank angle control. The turning point problem solution, which yields an approximate optimal control policy, is given in terms of parabolic cylinder functions, which are tabulated, and integral expressions, which must be numerically computed. Comparisons of the former, over their region of validity, with optimal control solutions show good qualitative agreement. Additional work and analysis is needed to compute the guidance approximation work.</p>					
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