ON LINEAR STRUCTURE AND PHASE ROTATION INVARIANT PROPERTIES OF BLOCK 2^l -PSK MODULATION CODES

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ABSTRACT

In this correspondence, we investigate two important structural properties of block 2^ℓ-ary PSK modulation codes, namely: linear structure and phase symmetry. For an AWGN channel, the error performance of a modulation code depends on its squared Euclidean distance distribution. Linear structure of a code makes the error performance analysis much easier. Phase symmetry of a code is important in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization. It is desirable for a modulation code to have as many phase symmetries as possible. In this paper, we first represent a 2^ℓ-ary modulation code as a code with symbols from the integer group, $S_{2^\ell\text{-PSK}} = \{0, 1, 2, \ldots, 2^\ell - 1\}$, under the modulo-$2^\ell$ addition. Then we define the linear structure of block 2^ℓ-ary PSK modulation codes over $S_{2^\ell\text{-PSK}}$ with respect to the modulo-2^ℓ vector addition, and derive conditions under which a block 2^ℓ-ary PSK modulation code is linear. Once the linear structure is developed, we study phase symmetry of a block 2^ℓ-ary PSK modulation code. In particular, we derive a necessary and sufficient condition for a block 2^ℓ-ary PSK modulation code, which is linear as a binary code, to be invariant under $180°/2^{\ell-h}$ phase rotation, for $1 \leq h \leq \ell$. Finally, a list of short 8-PSK and 16-PSK modulation codes is given together with their linear structure and the smallest phase rotation for which a code is invariant.

ON LINEAR STRUCTURE AND PHASE ROTATION INVARIANT PROPERTIES OF BLOCK 2^ℓ-PSK MODULATION CODES

1. Introduction

As the application of coded modulation in bandwidth-efficient communications grows, there is a need of better understanding of the structural properties of modulation codes, especially those properties which are useful in: error performance analysis, implementation of optimum (or suboptimum) decoders, efficient resolution of carrier-phase ambiguity, and construction of better codes. In this paper, we investigate two important structural properties of block 2^ℓ-ary PSK modulation codes, namely: linear structure and phase symmetry. For an AWGN channel, the error performance of a modulation code depends on its squared Euclidean distance distribution [1-4]. Linear structure of a code makes the error performance analysis much easier [2, 4]. Furthermore, it may lead to a simpler implementation of encoder and decoder. Phase symmetry of a code is important in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization [1, 5-8]. It is desirable for a modulation code to have as many phase symmetries as possible.

Suppose the integer group $\{0, 1, 2, \ldots, 2^\ell - 1\}$ under the modulo-2^ℓ addition, denoted $S_{2^\ell\text{-PSK}}$, is chosen to represent a two-dimensional 2^ℓ-PSK signal set. Then a block 2^ℓ-ary PSK modulation code C of length n may be regarded as a block code of length n over the integer group $S_{2^\ell\text{-PSK}}$, and a codeword in C is simply an n-tuple over $S_{2^\ell\text{-PSK}}$. If each integer in $S_{2^\ell\text{-PSK}}$ is represented by its binary expression of ℓ bits, then a block code of length n over $S_{2^\ell\text{-PSK}}$ can be considered as a binary block code of length ℓn. The resultant binary code is linear if it is closed under the component-wise modulo-2 addition. Most of the known block 2^ℓ-ary PSK modulation codes are linear as binary codes. A linear code in this sense is not necessarily closed under the component-wise modulo-2^ℓ addition. For two integers s and s' in $S_{2^\ell\text{-PSK}}$, the squared Euclidean distance between two signal points represented by s and s' respectively depends only on $s - s'$ (modulo 2^ℓ), but is not always determined by the Hamming distance between the binary expressions of s and s'. For an additive white Gaussian noise (AWGN) channel, error performance of a modulation code is determined by its squared

Euclidean distance distribution. If a code C over $S_{2^\ell\text{-PSK}}$ is either closed under the component-wise modulo-2^ℓ addition or a union of relatively small number of cosets of a subcode which is closed under the component-wise modulo-2^ℓ addition, then the error performance analysis of C is much easier than a code without such a property [2, 4]. In this paper, we present a condition for a code over $S_{2^\ell\text{-PSK}}$, which is linear as a binary code, to be closed under the component-wise modulo-2^ℓ addition. In particular, we present a necessary and sufficient condition for a basic multilevel block code over $S_{2^\ell\text{-PSK}}$, which is linear as a binary code, to be closed under the component-wise modulo-2^ℓ addition.

An important issue in coded modulation is the resolution of carrier-phase ambiguity. Several methods have been proposed to resolve the carrier-phase ambiguity for coded PSK modulations [6, 8, 9]. In these methods, the phase-rotation invariant property of a code over $S_{2^\ell\text{-PSK}}$ plays the central role. Tanner [8] has proposed a simple phase ambiguity resolution method for 2^ℓ-ary PSK modulation codes which are invariant under $360°/2^\ell$ phase shift. In this paper, we present a necessary and sufficient condition for a code over $S_{2^\ell\text{-PSK}}$, which is linear as a binary code, to be invariant under $180°/2^{\ell-h}$ phase shift with $1 \leq h \leq \ell$.

Finally, we give a list of short block 8-PSK and 16-PSK modulation codes together with their closure (or linear) properties under the component-wise modulo-2^ℓ addition, the smallest phase shifts for which these codes are invariant, and other parameters.

2. Linear Block 2^ℓ-PSK Modulation Codes

Let ℓ be a positive integer. Suppose the integer group $\{0, 1, 2, \ldots, 2^\ell - 1\}$ under the modulo-2^ℓ addition, denoted $S_{2^\ell\text{-PSK}}$, is used to represent a two-dimensional 2^ℓ-PSK signal set. We define the distance between two integers s and s' in $S_{2^\ell\text{-PSK}}$, denoted $d(s, s')$, as the squared Euclidean distance between the two 2^ℓ-PSK signal points represented by s and s' respectively. Then $d(s, s')$ is given below:

$$d(s, s') = 4\sin^2\left(2^{-\ell}\pi(s - s')\right). \tag{2.1}$$

Let d_i denote $d(2^{i-1}, 0)$. From (2.1), we see that

$$d_i = 4\sin^2(2^{i-\ell-1}\pi).$$

For a positive integer n, let $S^n_{2^\ell-PSK}$ denote the set of all n-tuples over $S_{2^\ell\text{-PSK}}$. Define the distance between two n-tuples $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ and $\bar{\mathbf{v}}' = (v'_1, v'_2, \ldots, v'_n)$ over $S_{2^\ell\text{-PSK}}$, denoted $d(\bar{\mathbf{v}}, \bar{\mathbf{v}}')$, as follows:

$$d(\bar{\mathbf{v}}, \bar{\mathbf{v}}') \triangleq \sum_{j=1}^{n} d(v_j, v'_j) \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$d(\bar{\mathbf{v}}, \bar{\mathbf{v}}') = d(\bar{\mathbf{v}} - \bar{\mathbf{v}}', \bar{0}) \tag{2.3}$$

where "$-$" denotes the component-wise modulo-2^ℓ subtraction and $\bar{0}$ denotes the all-zero n-tuple over $S_{2^\ell\text{-PSK}}$. For an n-tuple $\bar{\mathbf{v}}$ over $S_{2^\ell\text{-PSK}}$, define $|\bar{\mathbf{v}}|_d$ as follows:

$$|\bar{\mathbf{v}}|_d \triangleq d(\bar{\mathbf{v}}, \bar{0}). \tag{2.4}$$

We may regard that $|\bar{\mathbf{v}}|_d$ is the squared Euclidean weight of $\bar{\mathbf{v}}$.

Consider a block code C of length n over $S_{2^\ell\text{-PSK}}$. The minimum distance of C, denoted $D[C]$, with respect to the distance measure $d(\cdot,\cdot)$ given by (2.2) is defined as follows:

$$D[C] \triangleq \min\left\{d(\bar{\mathbf{v}}, \bar{\mathbf{v}}') : \bar{\mathbf{v}}, \bar{\mathbf{v}}' \in C \text{ and } \bar{\mathbf{v}} \neq \bar{\mathbf{v}}'\right\}. \tag{2.5}$$

If each component of a codeword $\bar{\mathbf{v}}$ in C is mapped into the corresponding signal point in the two-dimensional 2^ℓ-PSK signal set, we obtain a block 2^ℓ-PSK modulation code with minimum squared Euclidean distance $D[C]$. The effective rate of this code is given by

$$R[C] = \frac{1}{2n}\log_2|C|, \tag{2.6}$$

which is simply the average number of information bits transmitted per dimension.

Let $\bar{\mathbf{u}} = (u_1, u_2, \ldots, u_n)$ and $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ be two n-tuples over $S_{2^\ell\text{-PSK}}$. Let $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ denote the following n-tuple over $S_{2^\ell\text{-PSK}}$:

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} \triangleq (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n),$$

where $u_i + v_i$ is carried out in modulo-2^ℓ addition. A code over the integer group $S_{2^\ell\text{-PSK}}$ is said to be linear with respect to (w.r.t.) "$+$", if C is closed under the component-wise modulo-2^ℓ addition, i.e., for any $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ in C, $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ is also in C. It follows from (2.3) to (2.5) that, for a linear code C w.r.t. $+$, we have

$$D[C] = \min\left\{|\bar{\mathbf{v}}|_d : \bar{\mathbf{v}} \in C - \{\bar{0}\}\right\}. \tag{2.7}$$

As a result, for a linear code C over $S_{2^\ell\text{-PSK}}$ w.r.t. $+$, the error performance analysis of C based on the distance measure $d(\cdot,\cdot)$ is reduced to that of C in terms of the weight measure $|\cdot|_d$. This simplifies the error performance analysis and computation of code C [2, 4].

Let $(b_1, b_2, \ldots, b_\ell)$ be the binary representation of an integer s in $S_{2^\ell\text{-PSK}}$, where b_1 and b_ℓ be the least and most significant bits respectively. Then $s = \sum_{i=1}^{\ell} b_i 2^{i-1}$. Let $\bar{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ be an n-tuple over $S_{2^\ell\text{-PSK}}$ with $v_j = \sum_{i=1}^{\ell} v_{ij} 2^{i-1}$ and $v_{ij} \in \{0, 1\}$ for $1 \leq i \leq \ell$ and $1 \leq i \leq n$. Then $\bar{\mathbf{v}}$ can be expressed as the following sum:

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}^{(1)} + 2\bar{\mathbf{v}}^{(2)} + \cdots + 2^{\ell-1}\bar{\mathbf{v}}^{(\ell)}, \tag{2.8}$$

where $\bar{\mathbf{v}}^{(i)} = (v_{1i}, v_{2i}, \ldots, v_{ni})$ is a binary n-tuple, for $1 \leq i \leq \ell$. We call $\bar{\mathbf{v}}^{(i)}$ the i-th binary component n-tuple of $\bar{\mathbf{v}}$. The sum of (2.8) may be regarded as the binary expansion of the n-tuple $\bar{\mathbf{v}}$. For $1 \leq i \leq \ell$, let C_i be a binary (n, k_i) code with minimum Hamming distance δ_i. Define the following block code C over $S_{2^\ell\text{-PSK}}$,

$$\begin{aligned} C &\triangleq C_1 + 2C_2 + \cdots + 2^{\ell-1}C_\ell \\ &\triangleq \left\{\bar{\mathbf{v}}^{(1)} + 2\bar{\mathbf{v}}^{(2)} + \cdots + 2^{\ell-1}\bar{\mathbf{v}}^{(\ell)} \ : \ \bar{\mathbf{v}}^{(i)} \in C_i \text{ for } 1 \leq i \leq \ell\right\}. \end{aligned} \tag{2.9}$$

The code C defined by (2.9) is called a basic multi-level code. Basic multilevel codes were first introduced by Imai and Hirakawa [10] and then studied by other [3, 11, 12]. For $1 \leq i \leq \ell$, C_i is called the i-th binary component code of C. The minimum distance of C is

$$D[C] = \min_{1 \leq i \leq \ell} \delta_i d_i, \tag{2.10}$$

where $d_i = d(2^{i-1}, 0)$. If every component of a codeword in C is mapped into a signal point in a two-dimensional 2^ℓ-PSK signal constellation, then C is a basic multi-level 2^ℓ-PSK modulation code with a minimum squared Euclidean distance,

$$D[C] = \min_{1 \leq i \leq \ell}\{4\delta_i \ \sin^2(2^{i-\ell-1}\pi)\}.$$

For n-tuples $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ over $S_{2^\ell\text{-PSK}}$, let $\bar{\mathbf{u}} \oplus \bar{\mathbf{v}}$ denote the n-tuple over $S_{2^\ell\text{-PSK}}$, such that the i-th binary component n-tuple of $\bar{\mathbf{u}} \oplus \bar{\mathbf{v}}$ is the modulo-2 vector sum of the i-th binary component n-tuple of $\bar{\mathbf{u}}$ and the i-th binary component n-tuple of $\bar{\mathbf{v}}$. A code C over $S_{2^\ell\text{-PSK}}$ is said to be linear w.r.t. $\oplus$, if C is closed under addition $\oplus$. Most of the known block codes for

2^ℓ-PSK modulation are linear w.r.t. $\oplus$. A linear code w.r.t. $\oplus$ is not necessarily linear w.r.t. $+$. In the following, we will derive a condition for a linear code w.r.t. $\oplus$ to be linear w.r.t. $+$.

Let $\bar{u}$ and $\bar{v}$ be two n-tuples over $S_{2^\ell\text{-PSK}}$, and let $\bar{w}$ denote $\bar{u}+\bar{v}$. For $1 \leq i \leq \ell$, let the i-th binary component n-tuples of $\bar{u}$, $\bar{v}$ and $\bar{w}$ be represented as $\bar{u}^{(i)} = (u_{1i}, u_{2i}, \ldots, u_{ni})$, $\bar{v}^{(i)} = (v_{1i}, v_{2i}, \ldots, v_{ni})$, and $\bar{w}^{(i)} = (w_{1i}, w_{2i}, \ldots, w_{ni})$, respectively. Then the following recursive equations hold [13]:

$$w_{ji} = u_{ji} \oplus v_{ji} \oplus x_{ji}, \quad \text{for } 1 \leq i \leq \ell, \tag{2.11}$$

$$x_{ji} = u_{ji-1}v_{ji-1} \oplus (u_{ji-1} \oplus v_{ji-1})x_{ji-1}, \quad \text{for } 1 < i \leq \ell, \tag{2.12}$$

$$x_{j1} = 0. \tag{2.13}$$

For $1 \leq i \leq \ell$, let $c^{(i)}(\bar{u}, \bar{v})$ be defined as

$$c^{(i)}(\bar{u}, \bar{v}) \triangleq (x_{1i}, x_{2i}, \ldots, x_{ni}). \tag{2.14}$$

For two binary n-tuples, $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$, let $\bar{a} \cdot \bar{b}$ be defined as

$$\bar{a} \cdot \bar{b} \triangleq (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n),$$

where $a_j \cdot b_j$ denotes the logical product of a_j and b_j.

It follows from (2.11) to (2.14) that for $1 \leq i < \ell$,

$$c^{(i+1)}(\bar{u}, \bar{v}) = \bar{u}^{(i)} \cdot \bar{v}^{(i)} \oplus \left(\bar{u}^{(i)} \oplus \bar{v}^{(i)}\right) \cdot c^{(i)}(\bar{u}, \bar{v}). \tag{2.15}$$

Let $c(\bar{u}, \bar{v})$ be defined as

$$c(\bar{u}, \bar{v}) \triangleq c^{(1)}(\bar{u}, \bar{v}) + 2c^{(2)}(\bar{u}, \bar{v}) + \cdots + 2^{\ell-1}c^{(\ell)}(\bar{u}, \bar{v}). \tag{2.16}$$

Then,

$$\bar{u} + \bar{v} = \bar{u} \oplus \bar{v} \oplus c(\bar{u}, \bar{v}). \tag{2.17}$$

Now consider a block code C over $S_{2^\ell\text{-PSK}}$ which is linear w.r.t. $\oplus$. Let $\bar{u}$ and $\bar{v}$ be two codewords in C. Then it follows from (2.17) that $\bar{u} + \bar{v} \in C$ if and only if

$$c(\bar{u}, \bar{v}) \in C. \tag{2.18}$$

For $1 \leq i \leq \ell$, let $C^{(i)}$ and C_i be defined as

$$C^{(i)} \triangleq \left\{\bar{v}^{(i)} : \bar{v}^{(1)} + \cdots + 2^{i-1}\bar{v}^{(i)} + \cdots + 2^{\ell-1}\bar{v}^{(\ell)} \in C\right\}, \tag{2.19}$$

$$C_i \triangleq \left\{\bar{v}^{(i)} : 2^{i-1}\bar{v}^{(i)} \in C\right\}. \tag{2.20}$$

By definition

$$C_i \subseteq C^{(i)}. \tag{2.21}$$

Since C is linear w.r.t. $\oplus$, $C^{(i)}$ and C_i are also linear w.r.t. $\oplus$ and

$$C_1 + 2C_2 + \cdots + 2^{\ell-1}C_\ell \subseteq C, \tag{2.22}$$

where the equality holds if C is a basic multilevel code. For binary codes C and C' of the same length, let $C \cdot C'$ be defined as

$$C \cdot C' \triangleq \left\{ \bar{u} \cdot \bar{v} \ : \ \bar{u} \in C \ \text{ and } \ \bar{v} \in C' \right\}.$$

Now we present two lemmas regarding to the closure property of a 2^ℓ-PSK code.

Lemma 1: Suppose that C is a linear code over $S_{2^\ell\text{-PSK}}$ w.r.t. $\oplus$ and for $1 \le i \le \ell$,

$$C^{(i)} \cdot C^{(i)} \subseteq C_{i+1}. \tag{2.23}$$

Then C is closed under the component-wise modulo-2^ℓ addition, and hence is linear w.r.t. $+$.

Proof: By induction, we show that for $1 \le i \le \ell$

$$c^{(i)}(\bar{u}, \bar{v}) \in C_i. \tag{2.24}$$

Since $c^{(1)}(\bar{u}, \bar{v}) = \bar{0}, c^{(1)}(\bar{u}, \bar{v}) \in C_1$. Suppose that $c^{(j)}(\bar{u}, \bar{v}) \in C_j$ for $1 \le j \le i < \ell$. Since $C^{(i)}$ and C_{i+1} are linear w.r.t. $\oplus$, it follows from (2.15), (2.21) and (2.23) that $c^{(i+1)}(\bar{u}, \bar{v}) \in C_{i+1}$. Consequently (2.18) follows from (2.16), (2.22) and (2.24), and this lemma holds. $\Delta\Delta$

Lemma 2: Suppose that C is a linear basic multilevel code over $S_{2^\ell\text{-PSK}}$ w.r.t. $\oplus$. Then $C(= C_1 + 2C_2 + \cdots + 2^{\ell-1}C_\ell)$ is closed under the component-wise modulo-2^ℓ addition, if and only if

$$C_i \cdot C_i \subseteq C_{i+1}, \quad \text{for } 1 \le i < \ell. \tag{2.25}$$

Proof: Only if part: Let $\bar{u}$ (or $\bar{v}$) denote the n-tuple over $S_{2^\ell\text{-PSK}}$ whose i-th binary component n-tuple is $\bar{u}^{(i)} \in C_i$ (or $\bar{v}^{(i)} \in C_i$) and whose other binary component n-tuples are the all-zero n-tuple $\bar{0}$. Assume that $\bar{u} + \bar{v} \in C$. It follows from (2.11) to (2.13) that for these specific $\bar{u}$ and $\bar{v}$,

$$x_{j\,i+1} = u_{ji} v_{ji}, \quad \text{for } \ 1 \le i \le \ell. \tag{2.26}$$

From (2.14),(2.18) and (2.26), we see that

$$c^{(i+1)}(\bar{u}, \bar{v}) = \bar{u}^{(i)} \cdot \bar{v}^{(i)} \in C_{i+1}.$$

That is, $C_i \cdot C_i \subseteq C_{i+1}$.

If part: Since C is a basic multilevel code, $C_i = C^{(i)}$ for $1 \leq i \leq \ell$. Then if part follows from Lemma 1. $\Delta\Delta$

3. A Necessary and Sufficient Condition for a 2^ℓ-PSK Modulation Code to be Invariant Under $180°/2^{\ell-h}$ Phase Shift with $1 \leq h \leq \ell$

Now we consider the phase symmetry of a block 2^ℓ-ary PSK modulation code. To determine the phase symmetry of a code, we need to know the smallest rotation under which the code is invariant.

For $1 \leq h \leq \ell$, let $2^{h-1}\bar{1}$ denote the n-tuple over $S_{2^\ell\text{-PSK}}$ whose h-th binary component n-tuple is the all-one n-tuple and whose other binary component n-tuples are the all-zero n-tuple. A code C of length n over $S_{2^\ell\text{-PSK}}$ is said to be invariant under $180°/2^{\ell-h}$ phase shift if for any codeword $\bar{v}$ in C,

$$\bar{v} + 2^{h-1}\bar{1} \in C. \tag{3.1}$$

By letting $\bar{u} = 2^{h-1}\bar{1}$ in (2.11) to (2.16), we obtain the following equations:

(1)

$$w_{ji} = v_{ji} \oplus x_{ji}, \qquad \text{for} \quad 1 \leq i \leq \ell. \tag{3.2}$$

(2) If $h < \ell$, then

$$x_{ji} = v_{ji-1} x_{ji-1}, \qquad \text{for} \quad h < i \leq \ell. \tag{3.3}$$

(3)

$$x_{jh} = 1. \tag{3.4}$$

(4) If $1 < h$, then

$$x_{ji} = 0, \qquad \text{for} \quad 1 \leq i < h. \tag{3.5}$$

It follows from (3.2) to (3.5) that we have Lemma 3.

Lemma 3: For $1 \leq h \leq \ell$, a linear code C over $S_{2^\ell\text{-PSK}}$ w.r.t. $\oplus$ is invariant under $180°/2^{\ell-h}$ phase shift if and only if for any codeword $\bar{v}^{(1)} + 2\bar{v}^{(2)} + \cdots + 2^{\ell-1}\bar{v}^{(\ell)}$ in C,

$$2^{h-1}\bar{1} + 2^h\bar{v}^{(h)} + 2^{h+1}\left(\bar{v}^{(h)} \cdot \bar{v}^{(h+1)}\right) + \cdots + 2^{\ell-1}\left(\bar{v}^{(h)} \cdot \bar{v}^{(h+1)} \cdot \cdots \cdot \bar{v}^{(\ell-1)}\right) \in C, \qquad (3.6)$$

where $\bar{1}$ denotes the all-one n-tuple.

$\Delta\Delta$

If C is a linear basic ℓ-level code w.r.t. $\oplus$, denoted $C_1 + 2C_2 + \cdots + 2^{\ell-1}C_\ell$, then the necessary and sufficient condition (3.6) is expressed as follows:

(1)

$$\bar{1} \in C_h, \qquad \text{and} \qquad (3.7)$$

(2)

$$\text{if } h < \ell, \text{ then } C_h \cdot C_{h+1} \cdot \cdots \cdot C_{j-1} \subseteq C_j, \text{ for } h+1 < j \leq \ell. \qquad (3.8)$$

Obviously, a linear code C over $S_{2^\ell\text{-PSK}}$ w.r.t. $+$ is invariant under $180°/2^{\ell-h}$ phase shift, if and only if $\bar{1}_h \in C$.

4. Code Examples

In Table 1, seven basic multilevel block codes [3] and four nonbasic block codes for 8-PSK and 16-PSK modulations are given. The number of states of a trellis diagram for each basic multilevel block code is computed based on the numbers of states of trellis diagrams for its binary component codes [14]. Among four nonbasic codes, two zero-tail Ungerboeck trellis codes for 8-PSK modulation [1] are shown. In Table 1, V_n, P_n, $P_n^\perp$, $RM_{i,j}$, $s\text{-}RM_{i,j}$ and ex-Golay denote the set of all the binary n-tuples, the set of all even weight binary n-tuples, the dual code of P_n which consists of the all-zero and all-one n-tuples, the j-th order Reed-Muller code of length 2^i, a shortened j-th order Reed-Muller code of original length 2^i, and the extended (24,12) code of binary Golay code. F_1 and F_2 denote two codes over $\{0,1,2,3\}$ which are defined as following [4]. Let $p(x_1, x_2, \cdots, x_h)$ be a boolean polynomial which is used to represent the binary 2^h-tuple whose i-th bit is given by $p(i_1, i_2, \cdots, i_h)$ where $(i_1, i_2, \cdots, i_h)$ is the binary representation of the integer $i-1$, i.e. $i - 1 = \sum_{j=1}^{h} i_j 2^{j-1}$. Let $\bar{g}_{h,i}$ denote the

Next we consider the phase rotation invariant property of codes given in Table 1. Since codes $C[1], C[4], C[5], C[6]$ and $C[11]$ are linear w.r.t. $+$ and $\bar{1}$ is contained in $P_n^{\perp}$, $RM_{i,j}$ or ex-Golay, there codes are invariant under $180^\circ/2^{\ell-1}$ phase shift. It follows from the properties (i) and (ii) of Reed-Muller codes that codes $C[8]$, $C[9]$ with $n \equiv 0 \bmod 4$ and $C[10]$ are readily shown to meet the conditions given by (3.7) and (3.8) with $h = 1$. Code $C[2]$ is shown to contain $2\bar{1}$, and therefore is invariant under 90° phase shift. Code $C[3]$ contains $2^2\bar{1}$ only and is invariant only under 180° phase shift, and code $C[7]$ does not contain even $2^2\bar{1}$.

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Table 1: Some Short 8-PSK, 16-PSK Codes

modulation	definition	n	$R[C]$	$D[C]$	The number of states of a trellis diagram	linearity w.r.t. +	phase shift invariancy
8-PSK	$C[1] \triangleq P_8^\perp + 2P_8 + 4V_8$	8	1	4	2^2	Yes	45°
	$C[2] \triangleq F_1 + 4V_8$	8	1	4	2^2	Yes	90°
	$C[3] \triangleq$ zero-tail Ungerboeck code	n	$\frac{n-1}{n}$	4	2^2	No	180°
	$C[4] \triangleq RM_{4,1} + 2P_{16} + 4V_{16}$	16	$\frac{9}{8}$	4	2^4	Yes	45°
	$C[5] \triangleq F_2 + 4V_{16}$	16	9/8	4	2^4	Yes	45°
	$C[6] \triangleq$ ex-Golay+$2P_{24} + 4V_{24}$	24	$\frac{59}{48}$	4	2^7	Yes	45°
	$C[7] \triangleq$ zero-tail Ungerboeck code	n	$\frac{2n-3}{2n}$	4.586	2^3	No	360°
	$C[8] \triangleq P_{16}^\perp + 2RM_{4,2} + 4P_{16}$	16	$\frac{27}{32}$	8	2^5	No	45°
	$C[9] \triangleq P_n^\perp + 2s\text{-}RM_{5,3} + 4P_n$	$16 < n \leq 32$	$\frac{n-3}{n}$	8	2^6	No	45° for $n \equiv 0 \pmod 4$
	$C[10] \triangleq RM_{5,1} + 2RM_{5,3} + 4P_{32}$	32	$\frac{63}{64}$	8	2^9	No	45°
16-PSK	$C[11] \triangleq P_{32}^\perp + 2RM_{5,2} + 4P_{32} + 8V_{32}$	32	$\frac{5}{4}$	4	2^8	Yes	22.5°