NASA Contractor Report 182029
ICASE Report No. 90–28

ICASE

VISCOUS SHOCK PROFILES AND PRIMITIVE FORMULATIONS

S. Karni

Contract No. NAS1–18605
March 1990

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665–5225

Operated by the Universities Space Research Association

(NASA-CR-182029) VISCOUS SHOCK PROFILES AND PRIMITIVE FORMULATIONS Final Report (ICASE) 25 p CSCL 01A

N90-20049

Unclas
G3/02 0275322

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

VISCOUS SHOCK PROFILES AND PRIMITIVE FORMULATIONS

S. Karni[1]
College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43, OAL
ENGLAND

ABSTRACT

We consider weak solutions of hyperbolic systems in primitive (non-conservation) form for which a consistent conservation form exists. We show that for primitive formulations, shock relations are not uniquely defined by the states to either side of the shock but also depend on the viscous path connecting the two. Scheme-dependent high order correction terms are derived that enforce consistent viscous shock profiles. The resulting primitive algorithm is conservative to the order of the approximation. One dimensional Euler calculations of flows containing strong shocks clearly show that conservation errors in primitive calculations are reduced to truncation levels and that both conservative and primitive flow calculations are of comparable quality.

[1]Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

1. INTRODUCTION

It is common wisdom that consistent shocked solutions can be numerically captured only if the numerical algorithm meets discrete conservation requirements. Indeed, *conservative* numerical calculations satisfyling the entropy condition converge to the correct physical solutions as the mesh size tends to zero [4]. Conversely, examples easily show that calculations which are not conservative converge to completely non-physical solutions (eg [5]). Non-conservative or *primitive* formulations, however, are strikingly simpler and are less coupled than their conservative counterparts. As such they may offer advantages, either in computational efficiency or in accuracy gains. In Fluid Dynamics, the diagonal characteristic formulation is probably the most prominent example [6]. Other formulations using the entropy function are also favoured by many. The less common choice of velocity components and pressure, often leads to accuracy gains near contact surfaces separating materials of different types [3]. In multi-dimensional setups, this choice also enables the advection of a uniform passive velocity, completely decoupled from the 1D Riemann solver in the cross direction. In high speed near vacuum conditions, the internal energy is an important but usually very small quantity, which due to numerical truncation errors may become negative. This problem may be resolved by using internal energy as a dependent variable at the cost of sacrificing conservation. Exaples of primitive forms arising in Elasticity are discussed by Colombeaux and Le Roux in [1,2].

Straight forward discretizations of primitive formulations result in both incorrect shock speed/location and wrong jump across shock transition. We show, that primitive formulations *do not* possess unique jump conditions for steady viscous shock profiles. Jump conditions depend not only on the limiting left and right states but also on the viscous path that connects them. This has also been shown in [1,2] using arguments from generalised functions theory. The secret of correct shock capturing thus lies in getting the path right. Although physically, there is only one correct such path, numerically there are many. In fact, as many as there are conservative numerical schemes. Indeed, while physical shock transition is governed by physical viscosity mechanisms, numerical shock transition is governed by numerical viscosity mechanisms, whose precise form depend on numerical truncation errors. The analysis of Le Roux and Colombeaux in [1,2] tries to enforce physical microscopic behaviour on the numerical algorithm. The physical microscopic behaviour is deduced either from a consistent conservation form or from empirical data. Ignoring numerical viscosity mechanisms, this analysis is not fully justifiable on the discrete level. In contrast, the analysis presented in this work is performed directly on the dicrete level and enforces correct numerical microscopic behaviour. We follow an idea introduced by Zwas and Roseman [10], who looked into the effect of non-linear transformations on weak solutions of conservation laws. They have considered the particular case of an original set of conservation laws which transforms into another set of conservation laws. They have looked at the *viscous form* of the equations and showed that unless the viscosity terms are included in the transformation, the latter set of conservation laws will produce inconsistent weak solutions. For a conservative system to transform into another pseudo-conservative system is, however, a very special case. More commonly, it transforms into a primitive form. Shocks obtained by primitive calculations depend inherently on getting the underlying viscous path right. We follow [10] in the case of general formulations written in primitive form, for which a consistent underlying conservation form exists. This is where we believe its great promise rests. We consider hyperbolic primitive formulations

$$\underline{w}_t + A(\underline{w})\underline{w}_x = 0$$

and derive general, scheme dependent, high order correction terms

$$\underline{w}_t + A(\underline{w})\underline{w}_x = \Delta t^p f(\underline{w}, \underline{w}_x, \underline{w}_t, \lambda)$$

where p is the order of the scheme and $\lambda = \Delta t/\Delta x$ the mesh ratio. Their inclusion on the RHS of the primitive formulation renders the viscous forms of the conservative and primitive algorithms equivalent. Though not strictly conservative, the resulting primitive algorithm is conservative to the order of the approximation. Correction terms are obtained for the first order Lax-Friedrichs and upwind schemes without reference to a particular system. Their specific form is given for the 1D Isothermal Euler equations and the complete 1D Euler equations. The effect of the correction terms is demonstrated on one dimensional Euler calculations of flows containing strong shocks. It is clearly seen that errors in weak solutions are reduced to truncation levels, and that both conservative and primitive flow calculations are of comparable quality.

2. WEAK SOLUTIONS AND VISCOSITY

Consider scalar conservation laws described by the Initial Value Problem (IVP),

$$u_t + f(u)_x = 0 \qquad (1)$$
$$u(x,0) = \varphi(x)$$

Denote by $a(u) = df/du$ the characteristic speed of the equation, then solutions to (1) can be written implicitly as

$$u(x,t) = \varphi(x-a(u)t)$$

Depending on whether $a(\varphi(x))$ is an increasing or decreasing function of x, an initially smooth solution $u(x,t)$ will either remain smooth or develope discontinuities or *shocks*. Integral conservation considerations allow the solution to be extended beyond the time of shock formation. The broader concept of *Weak Solutions* is introduced, describing piecewise smooth solutions separated by curves of discontinuity, across which the solution satisfies the *Rankine-Huganiot* jump conditions

$$s = \frac{f_R - f_L}{u_R - u_L} \qquad (2)$$

Here s is the shock speed and $(\ )_R$ and $(\ )_L$ denote the states to its immediate right and left. Weak solutions, however, are *not* unique. The criterion that rules out all but one of the solutions is known as the *Entropy Condition.* This condition can be shown [5] to select a unique solution which is the limit of solutions $u_{\epsilon}(x,t)$ of the *Viscous* problem

$$u_t + f(u)_x = \epsilon u_{xx}$$

as viscosity vanishes $\epsilon \to 0$. The concept of viscosity, thus lies in the heart of correct, entropy satisfying, shock representation. While in smooth parts of the flow, the viscosity term can be neglected on grounds of order of magnitude, in regions of rapidly varying solutions its neglect leads to ambiguities. These can only be resolved upon conceptual re-introduction of the neglected terms.

CONSERVATION FORMS, PRIMITIVE FORMS AND VISCOUS SHOCK PROFILES

The more general viscous form of the equation reads

$$u_t + f(u)_x = \epsilon(F(u)u_x)_x \tag{3}$$

for some function $F(u)$. Assume that $F(u)$ is such that stable shock profiles exists and consider a steady viscous shock profile moving at a constant speed s, $u = u\,(x-st)$, satisfying

$$u \xrightarrow[x\to -\infty]{} u_L \qquad u \xrightarrow[x\to +\infty]{} u_R \qquad (4)$$
$$u_x \xrightarrow[x\to \pm\infty]{} 0$$

Substituting (4) into (3) and integrating over $x\in(-\infty, \infty)$ gives

$$-s(u_R - u_L) + (f_R - f_L) = \epsilon\ [(F(u)u')_R - (F(u)u')_L] \tag{5}$$

By (4c), the RHS of (5) vanishes, yielding the jump conditions (2). Provided $F(u)$ is admissible, this result does not depend on its precise definition although the viscous path connecting u_L and u_R obviously does.

Let a transformation T be defined by $Tdu=dw$ and assume there exists a $g(w)$ such that $Tdf(u)=dg(w)$. Then (1) transforms into the conservation law,

$$w_t + g(w)_x = 0 \quad (6)$$

with shock solutions satisfying

$$s = \frac{g_R - g_L}{w_R - w_L}$$

which is inconsistent with (2). Any other admissible non-zero RHS in (6) of the form $\epsilon(G(w)w_x)_x$ yields the same jump relations. If, however, equation (3) is transformed together with the viscosity term in (3), then it reads

$$\begin{aligned} w_t + g(w)_x &= \epsilon T(F(u)u_x)_x \quad (7) \\ &= \epsilon(TF(u)u_x)_x - \epsilon T_x F(u)u_x \end{aligned}$$

Substituting a viscous shock profile (4) into (7) now yields

$$s(w_R - w_L) = (g_R - g_L) + \int_{-\infty}^{\infty} T_x F(u) u_x dx \quad (8)$$

and should give correct shock speed provided the viscous profile used in the integration is consistent with (3).

Note that the transformed equation (6) may not always be written in conservation form. More generally it reads

$$w_t + b(w)w_x = 0 \quad (9)$$

for some $b(w)$. Note that $b(w)$ always satisfies the conservation law

$$b(w)_t + [\tfrac{1}{2}(b(w))^2]_x = 0 \quad (10)$$

but this in itself does not make (10) more correct than (6). That will depend on whether the assumed underlying viscous form is the correct one.

Viscous conservative systems read

$$\underline{u}_t + \underline{f}(\underline{u})_x = \varepsilon(F(\underline{u})\underline{u}_x)_x$$

where $F(\underline{u})$ is now a *Viscosity Matrix*. Again, not every matrix $F(\underline{u})$ yields stable shock profiles. Under certain assumptions, the identity matrix I is admissible (see for example [7] and references cited therein).

Primitive formulations of hyperbolic systems depend crucially on the correct choice of viscous paths. Primitive systems have the form

$$\underline{w}_t + A(\underline{w})\underline{w}_x = 0$$

where $A(\underline{w})$ is not a Jacobian matrix with respect to $\underline{w}$. If the viscous primitive form is assumed to be

$$\underline{w}_t + A(\underline{w})\underline{w}_x = \epsilon \underline{w}_{xx} \tag{11}$$

steady viscous shock profiles satisfy

$$\int_{\underline{w}_L}^{\underline{w}_R} A(\underline{w})d\underline{w} = s(\underline{w}_R - \underline{w}_L) \tag{12}$$

Since $A(\underline{w})$ is not a Jacobian with respect to $\underline{w}$, the integration is path dependent and so are both the shock speed s and the jump $(\underline{w}_R - \underline{w}_L)$ (see Figure (1)). They will only be correct if the integration is along a consistent path, ie if a consistent RHS is taken in (11). If $\underline{w}$ varies linearly across shock transition, then by (12)

$$\left(\frac{1}{x_R - x_L} \int_{x_L}^{x_R} A(\underline{w})dx \right)(\underline{w}_R - \underline{w}_L) = s(\underline{w}_R - \underline{w}_L)$$

implying that the jump $(\underline{w}_R - \underline{w}_L)$ is an eigenvector of a path-dependent average of $A(\underline{w})$, and that the shock speed s is the associated eigenvalue .

4. NUMERICAL SOLUTIONS, NON-LINEAR TRANSFORMATIONS AND CORRECT SHOCK REPRESENTATION

Attempting to solve either (1) or (9) numerically immediately raises the question of consistent viscous integration paths since due to numerical viscosity, captured shocks always get smeared over a number of grid points. The precise form of shock transition depends on numerical truncation errors. Consequently, their relevance to physics is not in their precise details but in some average interpretation of shock location and in an asymptotic interpretation of the limiting states to either of its sides. While there is only one correct physical shock transition, there are many correct numerical shock transitions. Indeed, let a numerical grid be defined by the partition parameters $(\Delta x, \Delta t)$ and let $u_j^n \cong u(j\Delta x, n\Delta t)$. Consider the system of conservation laws,

$$\underline{u}_t + \underline{f}(\underline{u})_x = 0 \tag{13}$$

Then any conservative numerical scheme

$$\underline{u}_j^{n+1} = \underline{u}_j^n - (\underline{h}_{j+1/2}^n - \underline{h}_{j-1/2}^n)$$

that consistently approximates (13) produces shock transitions which are correct in that average sense. Here $\underline{h}_{j+1/2}^n = \underline{h}(\underline{u}_{j-\ell+1}, \ldots, \underline{u}_{j+r})$ is the numerical flux function at the $j+1/2$ cell interface, with ℓ and r denoting the numerical stencil width, and consistency implies

$$\underline{h}_{j+1/2}^n(\underline{u}, \underline{u}, \ldots, \underline{u}) = \underline{f}(\underline{u})$$

Other considerations dictate which shock representation is more acceptable. The role of the viscous path is revealed in a more concrete way by writing the viscous form associated with a given numerical scheme [10]. Keeping the leading order terms in the numerical truncation error this reads

$$\underline{u}_t + \underline{f}(\underline{u})_x = \Delta t^p \cdot F(\underline{u}, \underline{u}_x, \underline{u}_t, \lambda)$$

$$\underline{u}_j^{n+1} = \frac{1}{2} (\underline{u}_{j-1}^n + \underline{u}_{j+1}^n) - \frac{\lambda}{2} (\underline{f}_{j+1}^n - \underline{f}_{j-1}^n) \tag{14}$$

with the numerical flux function

$$\underline{h}_{j+1/2}^n = \frac{1}{2} (\underline{f}_{j+1}^n + \underline{f}_j^n) - \frac{1}{2\lambda} (\underline{u}_{j+1}^n - \underline{u}_j^n))$$

the viscous form reads

$$\underline{u}_t + \underline{f}(\underline{u})_x = \frac{\Delta t}{2} (\underline{u}_{xx}/\lambda^2 - \underline{u}_{tt}) \tag{15}$$

and since (14) is conservative, the resulting viscous path is consistent, though not unique. Let $\underline{w}$ be a different set of dependent variables, let T the Jacobian of the transformation $T\underline{u} = d\underline{w}$. Premultiplying (13) by T yields the primitive system,

$$\underline{w}_t + A(\underline{w})\underline{w}_x = 0 \tag{16}$$

Let (16) be approximated by a 'LxF- type' approximation

$$\underline{w}_j^{n+1} = \frac{1}{2} (\underline{w}_{j-1}^n + \underline{w}_{j+1}^n) - \frac{\lambda}{4} (A_{j+1} + A_{j-1})(\underline{w}_{j+1}^n - \underline{w}_{j-1}^n) \tag{17}$$

Then its viscous form reads

$$\underline{w}_t + A(\underline{w})\underline{w}_x = \frac{\Delta t}{2} (\underline{w}_{xx}/\lambda^2 - \underline{w}_{tt}) \tag{18}$$

Unless this viscous form is equivalent to (15), or for that matter to the viscous form of any other first order conservative scheme, it will yield inconsistent weak solutions. In other words, let

$$D = T(\underline{u}_{xx}/\lambda^2 - \underline{u}_{tt}) - (\underline{w}_{xx}/\lambda^2 - \underline{w}_{tt}) \tag{19}$$

then (13) and (16) will converge to the same weak solutions (to order Δt) only if $D \equiv 0$. This requirement will not in general be met. To enforce correct weak solutions, the correction terms D must be added to the RHS of the primitive formulation (14), which should read instead,

$$\underline{w}_t + A(\underline{w})\underline{w}_x = \frac{\Delta t}{2} D \tag{19}$$

5. EXAMPLES

The 1D isothermal Euler equations are given by

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + \rho c^2 \end{pmatrix}_x = 0 \tag{20}$$

Here ρ and u are density and velocity and $c^2 = 1$ is the constant sound speed. A right moving shoch with $u_R = 0$ satisfies the jump relations,

$$\begin{aligned} u_L^2 &= (\rho_L - \rho_R)^2 / \rho_L \rho_R \\ s &= \rho_L u_L / (\rho_L - \rho_R) \end{aligned} \tag{21}$$

Multiplying (20) by the transformation matrix

$$T = \begin{pmatrix} 1 & 0 \\ -u/\rho & 1/\rho \end{pmatrix}$$

gives a primitive formulation in terms of ρ and u

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ 1/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x \tag{22}$$

Although equation (23) may be rewritten in pseudo-conservation form

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ u^2/2 + ln(\rho) \end{pmatrix}_x \tag{23}$$

it does not represent any genuine physical conservation and will give non-physical weak solutions. Indeed, the jump relations for system (23) read (compare (21)),

$$u_L^2 = \frac{2(\rho_L - \rho_R)}{\rho_L + \rho_R} / \ln(\rho_L / \rho_R)$$
$$s = \rho_L u_L / (\rho_L - \rho_R) \tag{24}$$

A third formulation, in terms of $\ln(\rho)$ and u takes a symmetric form,

$$\begin{pmatrix} \ln(\rho) \\ u \end{pmatrix}_t + \begin{pmatrix} u & 1 \\ 1 & u \end{pmatrix} \begin{pmatrix} \ln(\rho) \\ u \end{pmatrix}_x = 0 \tag{25}$$

for which the transformation matrix is

$$T = \begin{pmatrix} 1/\rho & 0 \\ -u/\rho & 1/\rho \end{pmatrix}$$

Denote the above systems by system I,II and III. System I is, in this case, the consistent system. The correction terms for systems II and III are respectively

$$D_{II} = \frac{2}{\rho} \begin{pmatrix} 0 \\ \rho_x u_x / \lambda^2 - \rho_t u_t \end{pmatrix}$$
$$D_{III} = \begin{pmatrix} -1/\rho^2 (\rho_x \rho_x / \lambda^2 - \rho_t \rho_t) \\ \frac{2}{\rho} (\rho_x u_x / \lambda^2 - \rho_t u_t) \end{pmatrix} \tag{26}$$

That the first component in D_{II} is zero should come as no surprise, since this is an equation for the conserved quantity ρ and requires no correction of order Δt. For computational convenience, the time derivatives in (26) may be replaced by spatial derivatives using (22) or (25).

The complete 1D Euler equations in conservation form read

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ uE + up \end{pmatrix}_x = 0 \tag{27}$$

Here E is the specific total energy and p the pressure, obtainable from

$$p = (\gamma-1)[E - \frac{1}{2}\rho u^2]$$

using the ideal gas assumption. The primitive form using ρ, u and p reads

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_x = 0 \tag{28}$$

is obtained by the transformation

$$T = \begin{pmatrix} 1 & 0 & 0 \\ -u/\rho & 1/\rho & 0 \\ \frac{(\gamma-1)u^2}{2} & -(\gamma-1)u & (\gamma-1) \end{pmatrix}$$

and the correction terms are

$$D = \begin{pmatrix} 0 \\ \frac{2}{\rho}\,(\rho_x u_x/\lambda^2 - \rho_t u_t) \\ (\gamma-1)\rho\,[u_x u_x/\lambda^2 - u_t u_t] \end{pmatrix}$$

Consider the first order upwind approximation to (13)

$$\underline{u}_j^{n+1} = \underline{u}_j^n - \lambda\,(\,(\tilde{A}^c)^+_{j-1/2}(\underline{u}_j - \underline{u}_{j-1}) + (\tilde{A}^c)^-_{j+1/2}(\underline{u}_{j+1} - \underline{u}_j)) \tag{29}$$

Here, $A^c = \partial \underline{f}/\partial \underline{u}$ is the Jacobian matrix, $(A^c)^{\pm}$ denote its positive and negative parts and ($\tilde{\ }$) indicates locally averaged values. The superscript c denotes to conservative formulation. The viscous form of the first order upwind is

$$\underline{u}_t + \underline{f}(\underline{u})_x = \frac{\Delta t}{2}\,((|A^c|\underline{u}_x)_x/\lambda - \underline{u}_{tt})) \tag{30}$$

Let $\underline{w}$ be a set of primitive variables and let $T = \partial \underline{w}/\partial \underline{u}$ be the Jacobian of the transformation. The viscous form of the primitive formulation reads

$$\underline{W}_t + \underline{A}^{p}(\underline{W})\underline{W}_x = \frac{\Delta t}{2}\,\left(\left(|A^{p}|\underline{W}_x\right)_x / \lambda - \underline{W}_{tt}\right) \tag{31}$$

The superscript p denotes primitive formulation. Then

$$\begin{aligned} A^{c} &= T^{-1}A^{p}\,T \\ |A^{c}| &= T^{-1}|A^{p}|\,T \end{aligned} \tag{32}$$

The correction terms for the first order upwind are

$$D = \Big(\,T(|A^{c}|\underline{u}_x)_x - (|A^{p}|\underline{W}_x)_x\,\Big)/\lambda \;-\; (T\underline{u}_{tt} - \underline{W}_{tt})$$

or after rearrangement,

$$D = \Big(\,T\,(T^{-1})_x|A^{p}|\underline{W}_x\,\Big)/\lambda \;-\; T\,(T^{-1})_t\underline{W}_t \tag{33}$$

For the 1D Euler equations in the particular set $\underline{W} = (\rho,u,p)$, given in (28) the correction terms are

$$D = \begin{pmatrix} 0 \\[2ex] \dfrac{1}{2\rho}\left(\dfrac{\rho_x u_x c_1 + (1/c^2)\,u_x p_x c_2 + ((\rho/c)u_x^2 + (1/\rho c)\rho_x p_x)c_4}{\lambda} - 4\,\rho_t u_t\right) \\[2ex] \dfrac{(\gamma-1)}{2}\left(\dfrac{\rho u_x u_x c_3 + (1/c)u_x p_x c_4}{\lambda} - 2\,\rho u_t u_t\right) \end{pmatrix} \tag{34}$$

where

$$\begin{aligned} c_1 &= |u-c| + 2|u| + |u+c| \\ c_2 &= |u-c| - 2|u| + |u+c| \\ c_3 &= |u-c| + |u+c| \\ c_4 &= |u-c| - |u+c| \end{aligned}$$

6. NUMERICAL EXPERIMENTS

In all the following Figures, the dashed profiles were obtained by a conservative calculation, hence consistent solutions. The solid profiles were obtained by a primitive calculation. Figures (2) and (3) describe experiments with the 1D isothermal Euler equations, given in genuine conservation form in equation (20) and in two alternative primitive forms in equations (22) and (25). The conservative form was approximated by the LxF scheme (14) and the primitive forms by the 'LxF-Type' scheme (17). The correction terms are given by equation (26), where time derivatives are replaced by spatial derivatives, nodal values are replaced by central averages and x derivatives by centered differences. Initial data for this test were $(\rho_L, u_L) = (0.4, 1.0)$ and $(\rho_R, u_R) = (0.1, 0.0)$. The data were chosen to yield distinctively different jump conditions for the first two systems, given by equations (21) and (24) respectively. As is clear from the Figures, adding the correction terms to the primitive formulations reduces the errors to truncation level. It may also be noticed that the error in Figure (3) is slightly larger than that in Figure (2). This may be attributed to the fact that the variable $ln(\rho)$ is very sensitive to small changes in ρ in the density range over which the test was conducted and that consequently the respective formulation suffers larger truncation errors. In Figure (4), the 1D Euler, equations, given in conservation form by (27) and in primitive form by (28) are approximated by the first order upwind scheme (29), using Roe's averages [8] for the conservation form and simple arithmetic averages for the primitive form. The correction terms are given by equation (34), where again, time derivatives are replaced by spatial derivatives, local values are centrally averaged and x derivatives are replaced by centered differences. The solution was found *not* to be sensitive to the manner in which the correction terms were approximated. Figure (4) depicts Sod's shock-tube problem, with initial data $(\rho_L, u_L, p_L) = (1.0, 0.0, 1.0)$ and $(\rho_R, u_R, p_R) = (0.125, 0.0, 0.1)$. Again, the correction terms reduce the errors to truncation level. Inspection of the correction terms in (34) reveals that all the products that appear in them contain either u_x or p_x or both. Both these derivatives vanish near contact surfaces, indicating that the correction terms only act *away* from these regions. Applying the correction terms cannot thus affect the resolution of contact surfaces. This was exploited in the tests shown in Figures (5) and (6), where the correction terms (34) are used in conjunction with the second order upwind scheme and superbee flux limiter [9]. Figure (5) depicts Sod's shock-tube problem. Figure (6) depicts a more severe shock-tube test, with initial data $(\rho_L, u_L, p_L) = (1.0, 0.0, 1.0)$ and $(\rho_R, u_R, p_R) = (0.125, 0.0, 0.1)$, leading to a shock wave of pressure ratio 4:1. Indeed, the crisp representation of the contact surface is not damaged in any way by the correction terms while the errors due to shock formation are again removed. This novel feature is peculiar to choices of primitive formulations that include u and p, both of which are constant across contacts. It cannot, in general, be expected of other primitive forms.

7. CONCLUSIONS

It has been shown that primitive formulations of conservation laws do not possess uniquely defined weak solutions. Jump relations across shocks were shown to depend not only on the limiting left and right states but also on the viscous path connecting the two. A technique has been described to enforce consistent weak solutions on primitive formulations. The method is based on deriving high order correction terms, that render the viscous form of the conservative and primitive formulations equivalent. The resulting primitive algorithm is conservative to the order of the approximation. The explicit form of the correction terms is scheme-dependent. Expressions were obtained for the first order LxF and upwind schemes. This technique was implemented to the 1D Euler equations in problems containing fairly strong shocks. It has been demonstrated that applying the correction terms reduced conservation errors to truncation levels and that conservative and primitive flow calculations were of comparable quality. This method shows great promise with other primitive formulations.

8. ACKNOWLEDGEMENTS

I am happy to have had stimulating and fruitful discussions with E. Tadmor and A. Harten. I would like to thank P.L. Roe for his help in clarifying some issues concerning analytic viscous forms, from which this work has benefitted.

REFERENCES

(1) Colombeaux J.F. and Le Roux A.Y., (1987), *Numerical methods for Hyperbolic Systems in Non-Conservation Forms Using Products of Distributions*, Adv. Comp. Meth. for PDE's, VI, R. Vitchnevetsky and R.S.Stepleman (Eds.), Publ. IMACS, pp. 28-37.

(2) Colombeaux J.F., Le Roux A.Y. et al., (1989), *Microscopic Profiles of Shock Waves and Ambiguities in Multiplications of Distributions*, SIAM J NUM ANAL, 26, pp. 871-883.

(3) Karni S., (1990), *Two Phase Flow Calculation by a Consistent Primitive Algorithm*, to be submitted to Computers & Fluids.

(4) Lax P.D., (1954), *Weak Solutions of Non-Linear Hyperbolic Equations and their Numerical Computation*, Comm. Pure Appl. Maths., VII, pp. 159-193.

(5) Lax P.D., (1972), *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, Society for Industrial and Applied Mathematics.

(6) Moretti G., (1979), *The λ Scheme*, COMP & FLUIDS, 7, p. 191.

(7) Pego R.L., (1972), *Viscosities and Linearized Stability for Shock Profiles*, PhD Thesis in Applied Mathematics, University of California, Berkeley.

(8) Roe P.L., (1981), *Approximate Riemann Solvers, Parameter Vectors and Difference Schemes*, J COMP PHYS, 43, pp. 357-372.

(9) Roe P.L., (1982), *Fluctuations and Signals - A Framework for Numerical Evolution Problems*, Num Meth for Fluid Dynamics, K.W. morton and M.J. Baines (Eds.), Accademic Press, New-York, pp. 219-257.

(10) Zwas G. and Roseman J., (1973), *The Effect of Non-Linear Transformation on the Computation of Weak Solutions*, J COMP PHYS, 12, pp. 179-186.

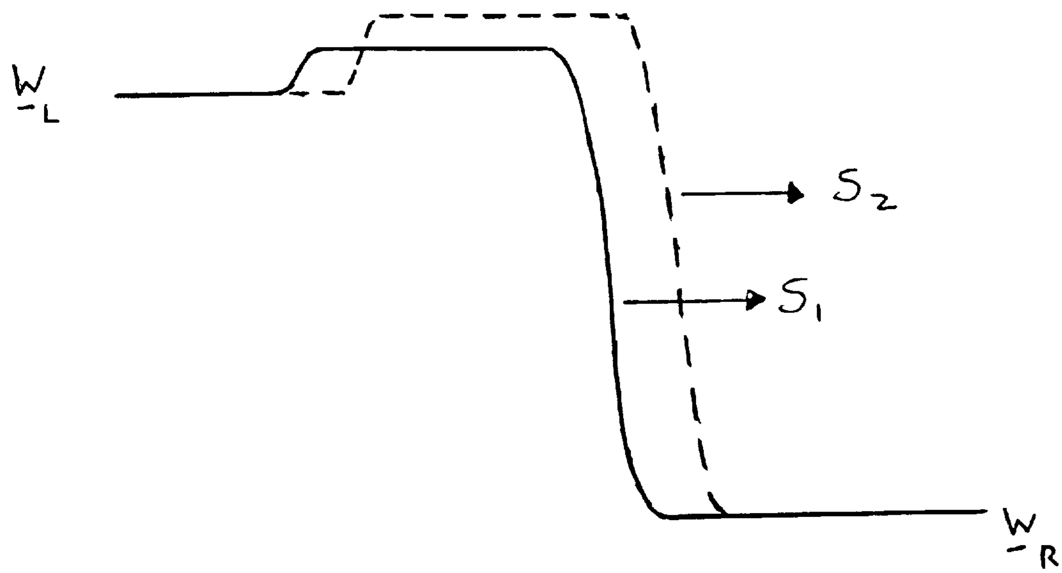

Figure (1) - Hyperbolic systems in primitive form:
Different shock relations for different viscous paths.

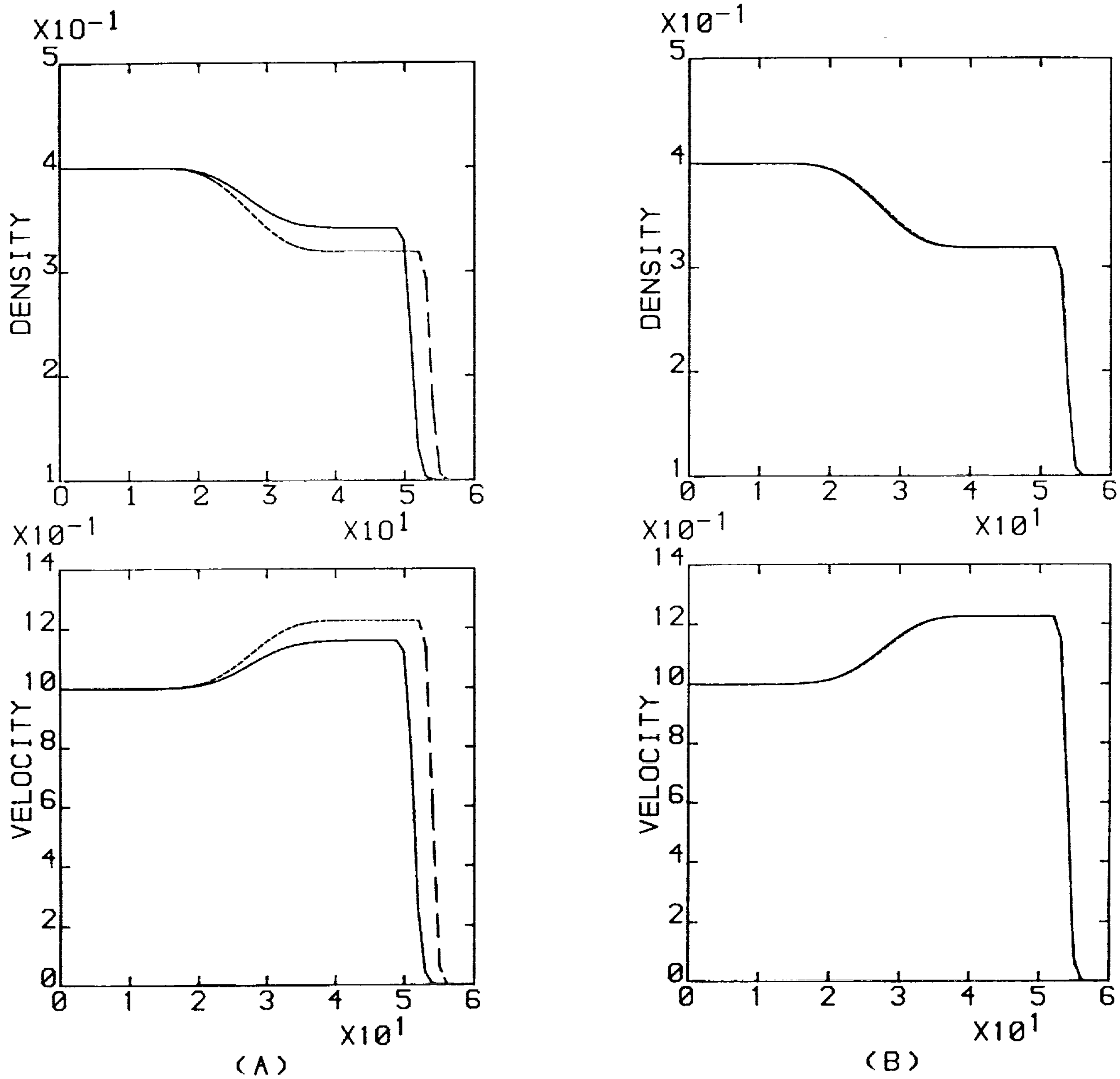

Figure (2) - 1D Isothermal Euler Equations:
Dashed line by conservation form (20)
Solid Line by primitive form (22)
(A) Without and (B) with correction terms.

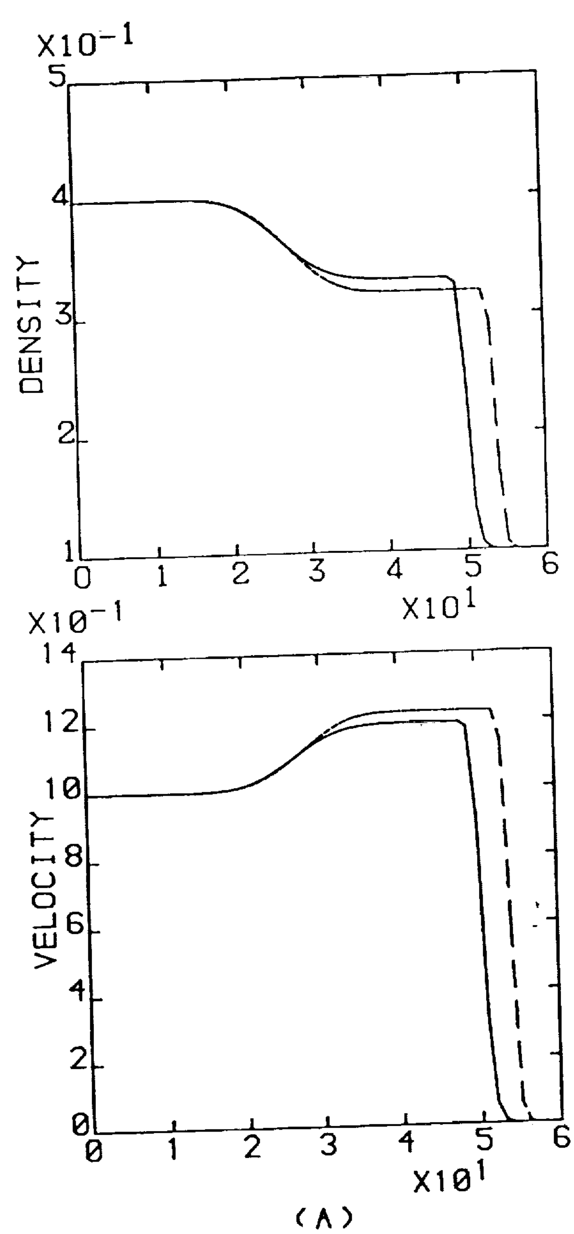

Figure (3) - 1D Isothermal Euler Equations:
Dashed line by conservation form (20)
Solid Line by primitive form (25)
(A) Without and (B) with correction terms.

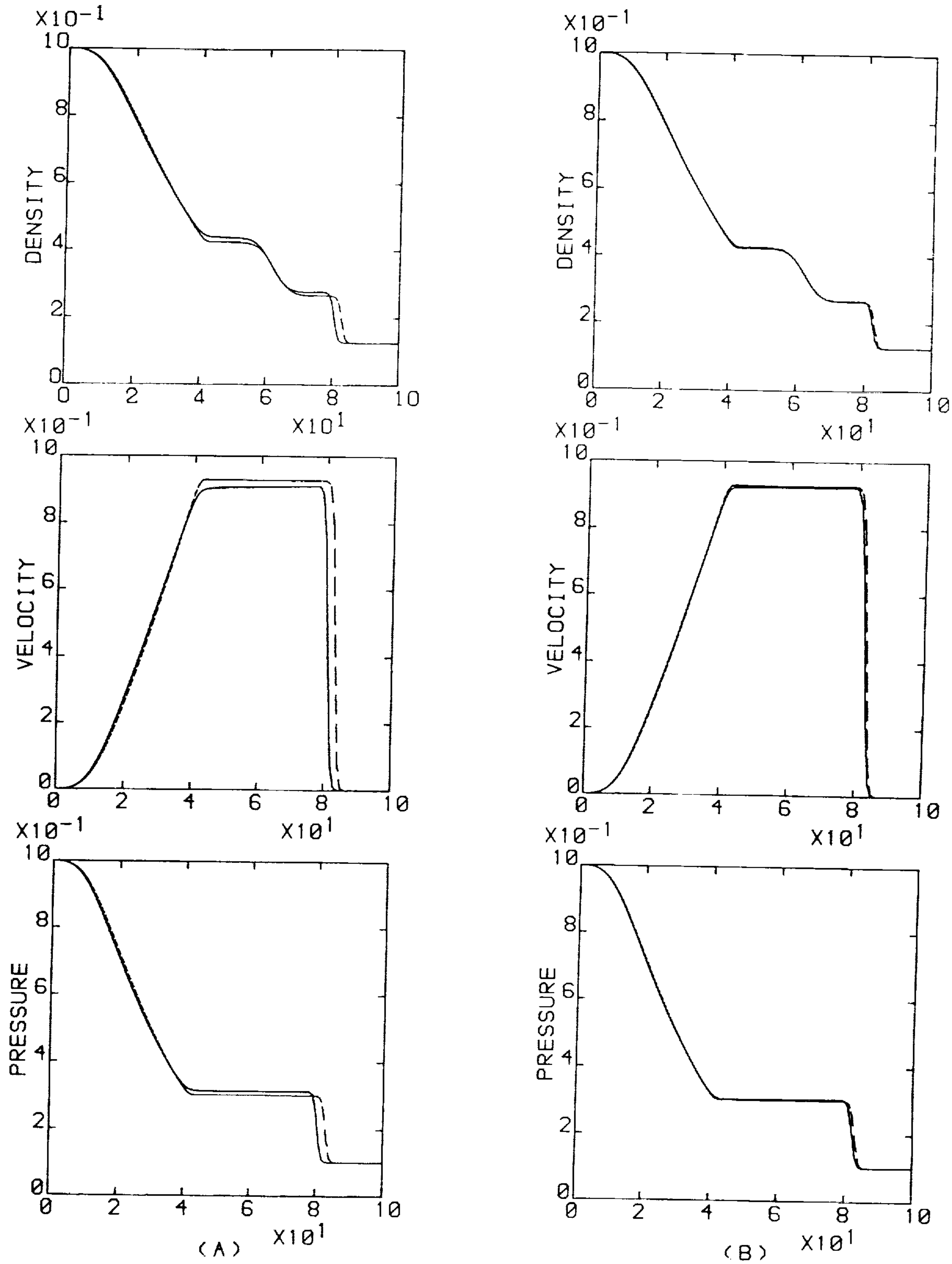

Figure (4) - 1D Euler Equations - Sod's shock tube problem by first order upwind scheme:

Dashed line by conservation form (27)

Solid Line by primitive form (28)

(A) Without and (B) with correction terms.

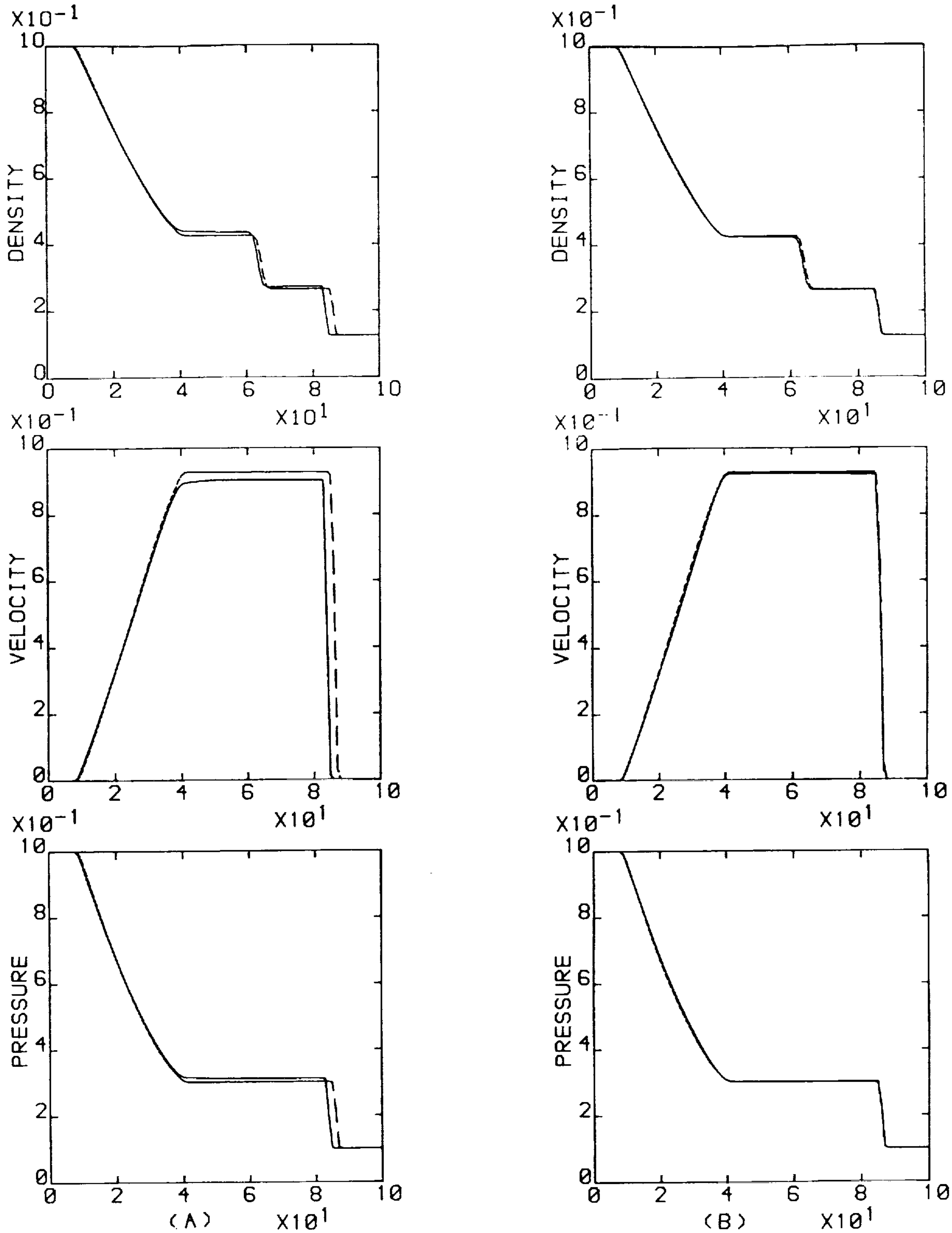

Figure (5) - 1D Euler Equations - Sod's shock tube problem by second order upwind scheme:

Dashed line by conservation form (27)

Solid Line by primitive form (28)

(A) Without and (B) with correction terms.

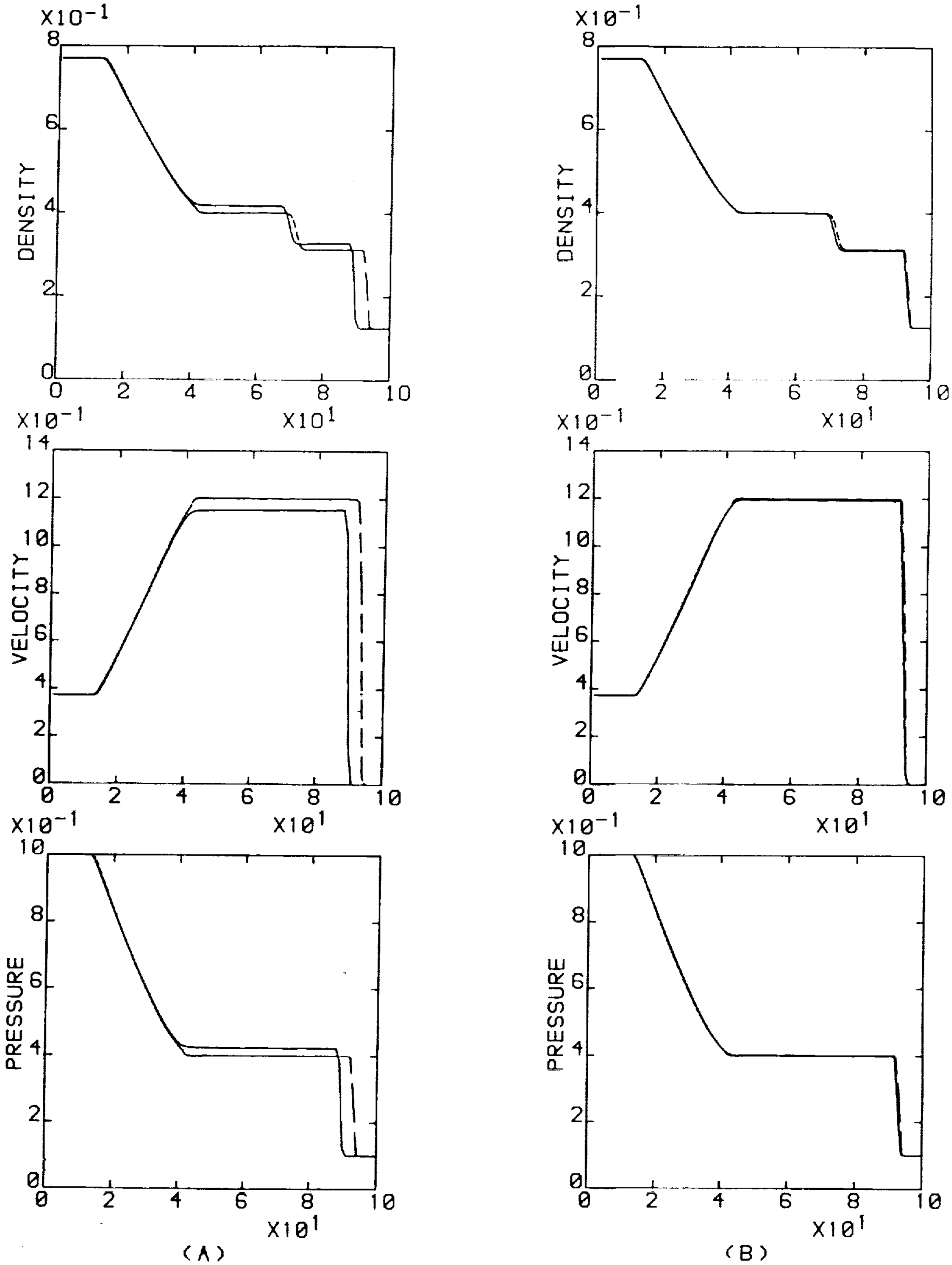

Figure (6) - 1D Euler Equations - Strong shock tube problem by second order upwind scheme:

Dashed line by conservation form (27)

Solid Line by primitive form (28)

(A) Without and (B) with correction terms.

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.
NASA CR-182029
ICASE Report No. 90-28

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle

VISCOUS SHOCK PROFILES AND PRIMITIVE FORMULATIONS

5. Report Date
March 1990

6. Performing Organization Code

7. Author(s)

S. Karni

8. Performing Organization Report No.
90-28

10. Work Unit No.
505-90-21-01

9. Performing Organization Name and Address
Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665-5225

11. Contract or Grant No.
NAS1-18605

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665-5225

13. Type of Report and Period Covered
Contractor Report

14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor:
Richard Barnwell

Final Report

Submitted to SIAM Journal of Numerical Analysis

16. Abstract

We consider weak solutions of hyperbolic systems in primitive (non-conservation) form for which a consistent conservation form exists. We show that primitive formulations, shock relations are not uniquely defined by the states to either side of the shock but also depend on the viscous path connecting the two. Scheme-dependent high order correction terms are derived that enforce consistent viscous shock profiles. The resulting primitive algorithm is conservative to the order of approximation. One dimensional Euler calculations of flows containing strong shocks clearly show that conservation errors in primitive flow calculations are of comparable quality.

17. Key Words (Suggested by Author(s))
Conservation Laws, Primitive Formulations, Weak solutions, Numerical Viscosity, Shock Profile

18. Distribution Statement
02 - Aerodynamics
64 - Numerical Analysis

Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	24	A03