A FRAMEWORK FOR MODELLING KINEMATIC MEASUREMENTS IN GRAVITY FIELD APPLICATIONS

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ABSTRACT

To assess the resolution of the local gravity field from kinematic measurements, a state model for motion in the gravity field of the earth is formulated. The resulting set of equations can accommodate gravity gradients, specific force, acceleration, velocity and position as input data and can take into account approximation errors as well as sensor errors.

1. PROBLEM STATEMENT

The last few years have seen major advances in kinematic methods of gravimetry. Shipborne gravimetry, already a reliable tool, will be further enhanced by using accurate position and velocity information from differential GPS. Airborne gravimetry in either the fixed wing aircraft or helicopter mode experienced a resurgence over the last few years and is at the point where it provides gravity information of acceptable accuracy for wavelengths down to 10 or 15 km. Airborne gravity gradiometry has entered the testing stage and holds great potential for short wavelength resolution. Compared to even fifteen years ago, there is now a variety of sensors on the market and it appears that a judicious combination will yield information on different parts of the gravity spectrum. To assess different sensor configurations, a model is needed which allows the combination of kinematic measurements from gravity gradiometers, dynamic gravity meters, inertial sensors, differential GPS, laser altimeters, precise pressure altimeters and similar devices. The model must allow for the interaction of gravitational and inertial measurements and must be able to take sensor biases and measurement noise into account. The formulation of such a model using state space techniques is the topic of this extended abstract. A detailed derivation with a comprehensive list of references will be published in the near future.

2. THE STATE SPACE MODEL OF KINEMATIC GEODESY

Newton's second law for motion in the gravitational field of the Earth, expressed in an inertial frame of reference (i), will be taken as the starting point

$$\ddot{\mathbf{r}}_i = \mathbf{f}_i + \bar{\mathbf{g}}_i \tag{1}$$

where $\mathbf{r}_i$ is the position vector from the origin of the inertial frame to the moving object and $\ddot{\mathbf{r}}_i$ is the second time derivative of this vector, $\mathbf{f}_i$ is the specific force vector, and $\bar{\mathbf{g}}_i$ is the vector of all gravitational accelerations acting on the moving object.

The set of nonlinear second-order differential equations (1) can be transformed into a set of first-order equations of the form

$$\begin{pmatrix} \dot{\mathbf{r}}_i \\ \dot{\mathbf{v}}_i \end{pmatrix} = \begin{pmatrix} \mathbf{v}_i \\ \mathbf{f}_i + \bar{\mathbf{g}}_i \end{pmatrix} . \tag{2}$$

In general, measurements will not be taken in an inertial frame of reference but in an arbitrary body frame (b). They can, be transformed into an inertial reference frame by

$$\mathbf{f}_i = \mathbf{R}_{ib}\, \mathbf{f}_b \tag{3}$$

where $\mathbf{R}_{ib}$ is a three-dimensional orthogonal matrix transforming $\mathbf{f}_b$ from the body frame (b) to the inertial frame (i). Note that the subscripts denote the direction of the rotation, not the element in the matrix. It is obvious from equation (3) that measurements $\mathbf{f}_b$ can only be used if $\mathbf{R}_{ib}$ is known

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or can be measured. Thus, a set of three first-order differential equations for the rotation rates $\mathbf{R}_{ib}$ has to be added to equation (2). They are of the form

$$\dot{\mathbf{R}}_{ib} = \mathbf{R}_{ib}\,\Omega_b^{ib} \tag{4}$$

where Ω_b^{ib} is the skew-symmetric matrix of angular velocities.

Similarly, gravitation is usually not given in an inertial frame but in the Conventional Terrestrial frame, which will be denoted by (e) in the following. We thus have the transformation

$$\bar{\mathbf{g}}_i = \mathbf{R}_{ie}\,\bar{\mathbf{g}}_e\ . \tag{5}$$

Since the rotation rate of the Earth can be considered constant for the applications discussed here, no additional equations for $\mathbf{R}_{ie}$ have to be added. Using equations (3) to (5) in (2) leads to the state equations

$$\dot{\mathbf{x}}_i = \begin{pmatrix} \dot{\mathbf{r}}_i \\ \dot{\mathbf{v}}_i \\ \dot{\mathbf{R}}_{ib} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_i \\ \mathbf{R}_{ib}\,\mathbf{f}_b + \mathbf{R}_{ie}\,\bar{\mathbf{g}}_e \\ \mathbf{R}_{ib}\,\Omega_b^{ib} \end{pmatrix} \tag{6}$$

describing rigid-body motion in three-dimensional Euclidean space by three rotational and three translational parameters.

The implicit assumption in equation (6) is that gravitation $\bar{\mathbf{g}}_e$ is known. In that case, the measurement of the specific force $\mathbf{f}_b$ and the rotation rates $\mathbf{R}_{ib}$ is sufficient to determine position, velocity, and attitude as functions of time. If gravitation is not known as a function of time, additional measurements are necessary. They can be of two types, inertial or gravitational. In the first case, $\dot{\mathbf{v}}_i$ is determined independently, in the second case $\bar{\mathbf{g}}_i$ is obtained. In both cases, gravitation and inertia can be separated in the second set of equations in (6). Thus, in principle, position, velocity, attitude and gravitation can be obtained by measuring $\mathbf{f}_b$, $\mathbf{R}_{ib}$ and either $\dot{\mathbf{v}}_i$ or $\bar{\mathbf{g}}_i$ independently. The second case which is that of gravity gradiometry will now be discussed.

To transform equation (6) into a system that also admits gravity gradiometer measurements, a set of equations describing the change of the gravitational vector with time has to be added. It is obtained by differentiating equation (5)

$$\begin{aligned} \dot{\bar{\mathbf{g}}}_i &= \dot{\mathbf{R}}_{ie}\,\bar{\mathbf{g}}_e + \mathbf{R}_{ie}\,\dot{\bar{\mathbf{g}}}_e = \mathbf{R}_{ie}\,(\Omega_e^{ie}\,\bar{\mathbf{g}}_e + \dot{\bar{\mathbf{g}}}_e) \\ &= \mathbf{R}_{ie}\,(\Omega_e^{ie}\,\bar{\mathbf{g}}_e + \mathbf{G}_e\,\mathbf{v}_e) = \Omega_i^{ie}\,\bar{\mathbf{g}}_i + \mathbf{G}_i\,(\mathbf{v}_i - \Omega_i^{ie}\,\mathbf{r}_i) \end{aligned} \tag{7}$$

where

$$\mathbf{G}_e = \frac{\partial \bar{\mathbf{g}}_e}{\partial \mathbf{r}_e} = \frac{\partial^2 V_e}{\partial \mathbf{r}_e\,\partial \mathbf{r}_e} \tag{8}$$

is the matrix of second-order gradients of the gravitational potential V_e. The state vector which includes gradient measurements of type (7) is therefore of the form

$$\dot{\mathbf{x}}_i = \begin{pmatrix} \dot{\mathbf{r}}_i \\ \dot{\mathbf{v}}_i \\ \dot{\mathbf{R}}_{ib} \\ \dot{\bar{\mathbf{g}}} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_i \\ \mathbf{R}_{ib}\,\mathbf{f}_b + \bar{\mathbf{g}}_i \\ \mathbf{R}_{ib}\,\Omega_b^{ib} \\ \Omega_i^{ie}\,\bar{\mathbf{g}}_i + \mathbf{G}_i\left(\mathbf{v}_i - \Omega_i^{ie}\,\mathbf{r}_i\right) \end{pmatrix} \tag{9}$$

3 CHANGE OF COORDINATE FRAME AND LINEARIZATION

In many cases, results are required in an earth-fixed coordinate frame and transformation of equations (9) into such frames are needed. They obviously involve the representation of the corresponding states in a rotating frame. The transformations are somewhat tedious and are given here without proof for the conventional terrestrial frame (e) and for the local-level frame (l).

In the first case, we get

$$\dot{\mathbf{x}}_e = \begin{pmatrix} \dot{\mathbf{r}}_e \\ \dot{\mathbf{v}}_e \\ \dot{\mathbf{R}}_{eb} \\ \dot{\mathbf{g}}_e \end{pmatrix} \begin{pmatrix} \mathbf{v}_e \\ \mathbf{R}_{eb}\,\mathbf{f}_b - 2\Omega_e^{ie}\,\mathbf{v}_e + \mathbf{g}_e \\ \mathbf{R}_{eb}\,\Omega_b^{eb} \\ \left(\mathbf{R}_{eb}\,\mathbf{G}_b\,\mathbf{R}_{be} - \Omega_e^{ie}\,\Omega_e^{ie}\right)\mathbf{v}_e \end{pmatrix} \tag{10}$$

where $\mathbf{g}_e$ is the gravity vector. The transformation into the local-level frame 1, using curvilinear $(\varphi, \lambda, \mathbf{h})$-coordinates, follows along similar lines, and results in the state vector

$$\dot{\mathbf{x}}_l = \begin{pmatrix} \dot{\mathbf{r}}_l \\ \dot{\mathbf{v}}_l \\ \dot{\mathbf{R}}_{lb} \\ \dot{\mathbf{g}}_l \end{pmatrix} = \begin{pmatrix} \mathbf{D}\mathbf{v}_l \\ \mathbf{R}_{lb}\,\mathbf{f}_b - \left(2\Omega_l^{ie} + \Omega_l^{el}\right)\mathbf{v}_l + \mathbf{g}_l \\ \mathbf{R}_{lb}\,\Omega_b^{lb} \\ \left(\mathbf{R}_{lb}\,\mathbf{G}_b\,\mathbf{R}_{bl} - \Omega_l^{ie}\,\Omega_l^{ie}\right)\mathbf{v}_l - \Omega_l^{el}\,\mathbf{g}_l \end{pmatrix} \tag{11}$$

where

$$\mathbf{D} = \begin{pmatrix} 1/R & 0 & 0 \\ 0 & 1/(R\cos\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The state equations derived up to this point are all nonlinear. The first step towards the solution of the differential equation system is usually linearization about a reference trajectory. The reference trajectory is obtained by introducing a gravity model into equation (9) and by integrating the gravity corrected measurements. Equation (9) can be written in the general form

$$\dot{\mathbf{x}} = \mathbf{f}\,\{\mathbf{r},\ \mathbf{v},\ \mathbf{w},\ \bar{\mathbf{g}}\} \tag{12}$$

where $\mathbf{w}$ denotes the angular velocities in the Ω^{ib} and the Ω^{ie} matrices. Subscripts have been omitted for convenience. It can be rewritten as

$$\dot{\mathbf{x}}^o + \mathbf{d}\mathbf{x}^o = \mathbf{f}\,\{\mathbf{r}^o + \mathbf{dr},\ \mathbf{v}^o + \mathbf{dv},\ \mathbf{w}^o + \mathbf{dw},\ \bar{\mathbf{g}}^o + \mathbf{dg}\} \tag{13}$$

where the superscript o denotes the reference trajectory and $\mathbf{d}$ the perturbation. By separating the reference trajectory

$$\dot{\mathbf{x}}^o = \mathbf{f}\,\{\mathbf{r}^o,\ \mathbf{v}^o,\ \mathbf{w}^o,\ \bar{\mathbf{g}}^o\} \tag{14}$$

from equation (12) and considering only first-order perturbations, the following set of equations is obtained

$$\begin{pmatrix} d\dot{r} \\ d\dot{v} \\ d\dot{w} \\ d\dot{g} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & -- & -- & -- \\ F_{31} & -- & -- & -- \\ F_{41} & -- & -- & F_{44} \end{pmatrix} \begin{pmatrix} dr \\ dv \\ dw \\ dg \end{pmatrix} \tag{15}$$

which is a set of linear homogeneous state equations of the form

$$\dot{x} = Fx \quad . \tag{16}$$

The matrices F_{11} to F_{33} are well known from the literature. F_{41} to F_{44} are obtained by developing the perturbation solution for $\dot{g}$. In the local-level frame, we get the reference solution

$$\dot{g}^o = \{G^o(r^o) - \Omega^{ie}(r^o)\,\Omega^{ie}(r^o)\}\, v^o + \Omega^{el}(r^o)\, g^o(r^o) \tag{17}$$

and the perturbation solution

$$d\dot{g} = \{G^o - \Omega^{ie}\Omega^{ie}\}\, dv + dG\, v^o - \Omega^{el}\, dg - d\Omega^{el} g^o \tag{18}$$

where

$$dG = (G - G^o) + \left\{\frac{\partial G^o}{\partial r} dr + \frac{\partial G^o}{\partial v} dv + \frac{\partial G^o}{\partial w} dw\right\} \tag{19}$$

Using equations (18) and (19) in (15) will give the desired solution.

4 SENSOR ERROR MODEL

The linearized model given by equation (16) is the kinematic description of rigid body motion in the gravity field of the Earth. Except for approximation errors, it is a rigorous model. If sensors are used to determine the trajectory and the gravity vector, the above model has to be augmented by a set of error states for each of the sensors used and by system noise. The resulting equations will be of the general form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + Gu \tag{20}$$

where x_1 contains the states in equation (15) and x_2 the error states, and where F_1 and F_2 are the corresponding dynamics matrices, and **Gu** describes the system noise. The F_2-matrix has a block-diagonal structure because the error states of the individual sensors are uncorrelated.

The error states are modeled in a stochastic manner by random biases, random walk models, or simple Gauss-Markov processes. Some judgement is required to decide whether a specific error source is modeled into x_2 or into **Gu.** If real-time estimation is performed with the model (20), a small state vector is often desirable.

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