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# ON THE CONTINUOUS DEPENDENCE WITH RESPECT TO SAMPLING OF THE LINEAR QUADRATIC REGULATOR PROBLEM FOR DISTRIBUTED PARAMETER SYSTEMS

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## ABSTRACT

The convergence of solutions to the discrete or sampled time linear quadratic regulator problem and associated Riccati equation for infinite dimensional systems to the solutions to the corresponding continuous time problem and equation, as the length of the sampling interval (the sampling rate) tends toward zero (infinity) is established. Both the finite and infinite time horizon problems are studied. In the finite time horizon case, strong continuity of the operators which define the control system and performance index together with a stability and consistency condition on the sampling scheme are required. For the infinite time horizon problem, in addition, the sampled systems must be stabilizable and detectable, uniformly with respect to the sampling rate. Classes of systems for which this condition can be verified are discussed. Results of numerical studies involving the control of a heat/diffusion equation, a hereditary of delay system, and a flexible beam are presented and discussed.

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**1. Introduction.** In this paper we consider the convergence of closed-loop solutions to discrete or sampled time linear quadratic (LQ) optimal control problems and the associated Riccati equations for infinite dimensional systems defined on Hilbert spaces to the solutions to the corresponding continuous time problems and Riccati equations, as the length of the sampling interval tends toward zero. With the advent and proliferation of micro-computers, and control tasks becoming ever more complex (for example, the stabilization of large flexible spacecraft), the roles played by discrete or sampled time control design techniques and distributed parameter systems have become increasingly more important. It has become necessary, therefore, to develop extensions of many of the familiar results for finite dimensional systems to an infinite dimensional setting. One area that has recently received a great deal of attention has been the LQ theory. Certain aspects of the linear-quadratic approach to control design for both continuous and sampled time infinite dimensional systems have been studied extensively. In particular, these aspects include, for example, the linear state feedback structure of the optimal control law, the optimal LQG estimator and compensator problems, boundary control, and finite dimensional approximation (for specific references, see below). But to the best of our knowledge, however, the inter-relation between the continuous and discrete time theories, which in the finite dimensional case is well understood, has not as of yet, been looked at in the context of infinite dimensional systems. Such a study would be useful, for example, because typically in engineering practice, the discrete and continuous time LQ theories are applied interchangeably without regard to as to whether or not the actual system is continuous or discrete in nature. In particular, due to hardware constraints, most systems occurring in engineering practice are in fact discrete. However, if the sampling is considered to be rapid enough, the system may be treated as continuous when an optimal control law, state estimator, or compensator is designed. Our work is largely motivated by the fact that the results we shall present here will serve to, in some sense, justify this approach.

We note that in finite dimensions, where strong and uniform norm convergence of linear operators are equivalent, the continuous dependence with respect to sampling of the solution to the linear quadratic control problem and associated Riccati equation is straight forward. Indeed, in [Le] the continuous time theory is established by first deriving the discrete time results, which are fundamentally algebraic in nature, and then taking the limit as the length of the sampling interval tends toward zero. However, in infinite dimensions, as is typically the case, the desired convergence is less obvious. This is especially true in the case of the infinite time horizon problem. It is this problem that we address here.

We consider both the finite and infinite time horizon problems. In the case of the finite time horizon problem, under the assumption of strong continuity of the operators which define the control system and performance index, together with a stability and consistency hypothesis on the sampling scheme, we are able to deduce the desired convergence. We must develop an appropriate framework to facilitate the comparison of discrete and continuous time operator families. For this purpose we rely

heavily upon Kato's [K] treatment of discrete semigroups. In the case of the infinite time horizon problem we must additionally assume stabilizability and detectability of the discrete time systems with some degree of uniformity in the sampling rate. The notion of stabilizability/detectability uniform with respect to sampling will be made precise in Section 3 below. We are able to establish that if the continuous time system is stabilizable and detectable via finite rank feedback, and if zero-order hold sampling is employed, then the resulting discrete time systems are uniformly stabilizable and detectable for sufficiently small sampling interval. We also have a result concerning the uniform stabilizability and detectability of parabolic systems. However, this result will not be discussed here, but rather in a forthcoming manuscript.

Our treatment is functional analytic in nature, and is similar in spirit to the many recent studies of convergence of solutions to LQ control and estimation problems and the associated Riccati equations under state (space) approximation (i.e. finite difference, modal, or finite element, for example). See, for example, [BK], [BW], [G],[GA], [GR], and [W]. For the discrete time LQ theory for infinite dimensional systems, we rely heavily on the well known results contained in [HH], [LCB], and [Z].

In addition to our theoretical results, we have included the results of some of our numerical convergence studies. We present and discuss our findings for the infinite time horizon LQ optimal control problems for a one dimensional heat or diffusion equation, a one dimensional hereditary or delay system, and a hybrid system of ordinary and partial differential equations describing the small amplitude transverse vibration of a cantilevered Voigt-Kelvin viscoelastic beam with tip mass.

An outline of the remainder of the paper is as follows. In section 2 we treat the finite time horizon problem. The infinite time horizon problem is considered in the third section. Our numerical results are presented and discussed in Section 4, while a brief fifth section contains a summary and some concluding remarks.

**2. LQR Problems with Finite Time Horizon.** In this section we consider the linear quadratic regulator (LQR) problem over a finite time interval. The basic notation and our general assumptions are introduced in the statements of both the continuous time and corresponding sampled time problems given below. The existence and uniqueness of the optimal control as well as its closed loop feedback structure can be obtained using a variety of approaches. Here we opt to consider the optimal control problem as the minimization of a strictly coercive quadratic form on the admissible control space. This approach yields an explicit representation for the solution of the usual Riccati equations (for both the continuous and sampled time problems) in terms of the underlying system and penalty operators which define the problems. Since the particular focus of our effort here is the consideration of sampled time problems as approximations to a continuous time problem, specialized notions and characterizations of convergence must be introduced. Once this is done, our fundamental result for the finite time horizon problem can be stated in terms of these specialized notions of convergence as follows. The convergence of the optimal control and the optimal feedback laws for the sampled systems to the optimal control and feedback law for the continuous time problem as the length of the sampling interval

tends to zero, follows directly from the convergence of the open-loop sampling of the underlying linear control system and quadratic performance index. We consider the open-loop sampling of the infinite dimensional LQR problem in an abstract setting so that our results can be applied to a wide range of sampling procedures.

Let  $H$  and  $U$  be Hilbert spaces with inner products  $\langle \cdot, \cdot \rangle_H$  and  $\langle \cdot, \cdot \rangle_U$  respectively. Let  $t_0, t_f \in \mathbb{R}$  be given with  $t_0 < t_f$ , and let  $T = \{T(t, s) : t_0 \leq s \leq t \leq t_f\}$  be an evolution system on  $H$ . For each  $t \in [t_0, t_f]$ , let  $B(t) \in L(U, H)$ ,  $Q(t) \in L(H)$ , and  $R(t) \in L(U)$ , and let  $G \in L(H)$ . We consider the continuous time LQR problem given by

(P) Determine a control input  $\bar{u} \in L_2(t_0, t_f; U)$  which minimizes the quadratic performance index

$$J(u; t_0, x(t_0), G) = \langle Gx(t_f), x(t_f) \rangle_H + \int_{t_0}^{t_f} \{ \langle Q(t)x(t), x(t) \rangle_H + \langle R(t)u(t), u(t) \rangle_U \} dt$$

where for each  $t \in [t_0, t_f]$  the state  $x(t) \in H$  is given by

$$(2.1) \quad x(t) = T(t, s)x(s) + \int_s^t T(t, \tau)B(\tau)u(\tau)d\tau, \quad t_0 \leq s \leq t \leq t_f.$$

We make the following standard assumptions on the operator families  $\{T, B, G, Q, R\}$  which determine problem (P).

(C1) The evolution system  $T$  is strongly continuous on  $H$  and therefore is uniformly exponentially bounded, with constants  $M > 0$  and  $\omega \in \mathbb{R}$ . That is

$$\|T(t, s)\|_{L(H)} \leq Me^{\omega(t-s)}, \quad t_0 \leq s \leq t \leq t_f.$$

(C2) The operator valued functions  $B, Q$ , and  $R$  are strongly continuous and therefore are uniformly bounded on  $[t_0, t_f]$ . That is, there exists a constant  $C > 0$  for which

$$\|B(t)\|_{L(U, H)} \leq C, \quad \|Q(t)\|_{L(H)} \leq C, \quad \|R(t)\|_{L(U)} \leq C,$$

$$t \in [t_0, t_f].$$

(C3) The operator  $G$  and the operators  $Q(t)$  and  $R(t)$  for each  $t \in [t_0, t_f]$  are self-adjoint and nonnegative definite. Moreover, there exists a constant  $r > 0$  for which  $R(t) \geq rI$ ,  $t \in [t_0, t_f]$ .

The strong continuity assumption in (C2) is not necessary for the well-posedness of the LQR problem. However, some assumptions on the continuity of the operators  $B, Q, R$  will be needed to obtain uniform convergence with respect to sampling.

The closed-loop linear state feedback form of the solution to problem (P) can be shown to exist and be explicitly constructed by considering the minimization of appropriately constructed strictly coercive quadratic forms on the Hilbert spaces  $\mathcal{U}_s = L_2(s, t_f; U)$ ,  $s \in [t_0, t_f]$  (see, for example, [G]). Since it will play a prominent role in

our discussions to follow, we briefly outline this approach here. For each  $s \in [t_0, t_f]$  define the operators  $\mathcal{B}_s \in L(H, \mathcal{U}_s)$  and  $\mathcal{R}_s \in L(\mathcal{U}_s)$  by

$$(2.2) \quad (\mathcal{B}_s \phi)(t) = B(t)^* \left\{ T(t_f, t)^* G T(t_f, s) + \int_t^{t_f} T(\eta, t)^* Q(\eta) T(\eta, s) d\eta \right\} \phi,$$

for  $\phi \in H$ ,  $t \in [s, t_f]$ , and

$$(2.3) \quad (\mathcal{R}_s u_s)(t) = R(t) u_s(t) + B(t)^* T(t_f, t)^* G \int_s^{t_f} T(t_f, \eta) B(\eta) u_s(\eta) d\eta \\ + B(t)^* \int_t^{t_f} \left\{ T(\eta, t)^* Q(\eta) \int_s^\eta T(\eta, \tau) B(\tau) u_s(\tau) d\tau \right\} d\eta,$$

for  $t \in [s, t_f]$  and  $u_s \in \mathcal{U}_s$ . It is not difficult to verify that the adjoint operator  $\mathcal{B}_s^* \in L(\mathcal{U}_s, H)$  of  $\mathcal{B}_s$  is given by

$$(2.4) \quad \mathcal{B}_s u_s = T(t_f, s)^* G \int_s^{t_f} T(t_f, t) B(t) u_s(t) dt \\ + \int_s^{t_f} T(\tau, s)^* Q(\tau) \left\{ \int_s^\tau T(\tau, \eta) B(\eta) u_s(\eta) d\eta \right\} d\tau,$$

and that for  $u_s \in \mathcal{U}_s$ , we have

$$J(u_s; s, x(s), G) = \langle G T(t_f, s) x(s), T(t_f, s) x(s) \rangle_H \\ + \int_s^{t_f} \langle Q(t) T(t, s) x(s), T(t, s) x(s) \rangle_H dt - \langle \mathcal{R}_s^{-1} \mathcal{B}_s x(s), \mathcal{B}_s x(s) \rangle_{\mathcal{U}_s} \\ + \langle \mathcal{R}_s(u_s + \mathcal{R}_s^{-1} \mathcal{B}_s x(s)), u_s + \mathcal{R}_s^{-1} \mathcal{B}_s x(s) \rangle_{\mathcal{U}_s}.$$

It follows that for  $x(s) \in H$  given,  $J(\cdot; s, x(s), G)$  is minimized by choosing  $u_s = \bar{u}_s = -\mathcal{R}_s^{-1} \mathcal{B}_s x(s) \in \mathcal{U}_s$ . We then obtain

$$\min_{\mathcal{U}_s} J(\cdot; s, x(s), G) = J(\bar{u}_s; s, x(s), G) \\ = \langle G T(t_f, s) x(s), T(t_f, s) x(s) \rangle_H + \int_s^{t_f} \langle Q(t) T(t, s) x(s), T(t, s) x(s) \rangle_H dt \\ - \langle \mathcal{R}_s^{-1} \mathcal{B}_s x(s), \mathcal{B}_s x(s) \rangle_{\mathcal{U}_s} \\ = \langle \Pi(s) x(s), x(s) \rangle_H$$

where the self-adjoint operator valued function  $\Pi : [t_0, t_f] \mapsto L(H)$  is defined by

$$(2.5) \quad \Pi(s) \phi = T(t_f, s)^* G T(t_f, s) \phi + \int_s^{t_f} T(t, s)^* Q(t) T(t, s) \phi dt \\ - \mathcal{B}_s^* \mathcal{R}_s^{-1} \mathcal{B}_s \phi, \quad \phi \in H.$$

Using the definitions given above, the following theorem concerning the existence and characterization of the closed-loop solution to problem (P) can be established.

**THEOREM 2.1.** *Suppose that assumptions (C1)–(C3) are satisfied. Then for any initial state  $x(t_0) \in H$  given, there exists a unique solution  $\bar{u}$  to problem (P). The optimal control  $\bar{u}$  is given in linear state feedback form by*

$$\bar{u}(t) = -R(t)^{-1} B(t)^* \Pi(t) \bar{x}(t), t \in [t_0, t_f]$$



where  $\bar{x}$  is the optimal trajectory. The operator valued function  $\Pi$  is given by (2.5) and it is the unique self-adjoint solution to the Riccati integral equation

$$(2.6) \quad \Pi(t) = T(t_f, t)^* G T(t_f, t) + \int_t^{t_f} T(\tau, t)^* \{ Q(\tau) - \Pi(\tau) B(\tau) R(\tau)^{-1} B(\tau)^* \Pi(\tau) \} T(\tau, t) d\tau,$$

$t \in [t_0, t_f]$ . We have

$$(2.7) \quad \min_{\bar{u}} J(\cdot; t_0, x(t_0), G) = J(\bar{u}; t_0, x(t_0), G) = \langle \Pi(t_0) x(t_0), x(t_0) \rangle_H.$$

We consider next the discrete or sampled time problem. Let  $k_0, k_f \in Z$  with  $k_f > k_0$  and let  $h \in R$  with  $h > 0$ . For  $k \in Z$  with  $k_0 \leq k \leq k_f - 1$  let  $A_h(k) \in L(H)$ , and let  $\{T_h(k, j) : k_0 \leq j \leq k \leq k_f\}$  be the discrete time evolution system on  $H$  given by

$$(2.8) \quad T_h(k, k) = I, \quad T_h(k, j) = A_h(k-1) \cdot A_h(k-2) \cdots A_h(j) = \prod_{i=j}^{k-1} A_h(i), \quad k_0 \leq j < k \leq k_f.$$

Let  $\{B_h(k)\}_{k=k_0}^{k_f-1}$ ,  $\{Q_h(k)\}_{k=k_0}^{k_f-1}$ , and  $\{R_h(k)\}_{k=k_0}^{k_f-1}$  be sequences in  $L(U, H)$ ,  $L(H)$  and  $L(U)$  respectively, and let  $G_h \in L(H)$ . The LQR problem is then given by

(P<sub>h</sub>) Determine a control input  $\bar{u}_h \in l_2(k_0, k_f - 1; U)$  which minimizes the quadratic performance index

$$J_h(u_h; k_0, x_h(k_0), G_h) = \langle G_h x_h(k_f), x_h(k_f) \rangle_H + h \sum_{k=k_0}^{k_f-1} \{ \langle Q_h(k) x_h(k), x_h(k) \rangle_H + \langle R_h(k) u_h(k), u_h(k) \rangle_U \}$$

where for each  $k \in Z$  with  $k_0 < k \leq k_f$ , the state  $x_h(k) \in H$  is given by

$$(2.9) \quad x_h(k) = T_h(k, j) x_h(j) + h \sum_{i=j}^{k-1} T_h(k, i+1) B_h(i) u_h(i),$$

for  $k_0 \leq j < k \leq k_f$ .

For the discrete time case, we make the following assumptions.

(D1) For each  $h > 0$  the operators  $A_h(k)$ ,  $B_h(k)$ ,  $Q_h(k)$ , and  $R_h(k)$  are bounded in  $k$  for  $k_0 \leq k \leq k_f - 1$ . Thus, there exists a constant  $C_h$  for which

$$\|A_h(k)\|_{L(H)} \leq C_h, \quad \|B_h(k)\|_{L(U, H)} \leq C_h, \\ \|Q_h(k)\|_{L(H)} \leq C_h, \quad \|R_h(k)\|_{L(U)} \leq C_h,$$

for  $k_0 \leq k \leq k_f - 1$ .

(D2) The operator  $G_h$  and the operators  $Q_h(k)$  and  $R_h(k)$  for  $k_0 \leq k \leq k_f - 1$  are self-adjoint and nonnegative. Moreover, there exists a constant  $r_h > 0$  for which  $R_h(k) \geq r_h I, k_0 \leq k \leq k_f - 1$ .

Note that assumption (D1) together with (2.8) yield that the discrete time evolution system  $\{T_h(k, j) : k_0 \leq j \leq k \leq k_f\}$  is uniformly exponentially bounded with

$$\|T_h(k, j)\|_{L(H)} \leq C_h^{k-j}, \quad k_0 \leq j \leq k \leq k_f.$$

Note also that the discrete time evolution equation (2.9) is equivalent to the discrete time dynamical system given by

$$(2.10) \quad x_h(k+1) = A_h(k)x_h(k) + hB_h(k)u_h(k), \quad k_0 \leq k \leq k_f - 1, x_h(k_0) \in H.$$

For each  $h > 0$  and  $j = k_0, k_0 + 1, \dots, k_f - 1$ , let  $\mathcal{U}_{h,j} = l_2(j, k_f - 1; U)$  endowed with the inner product

$$\langle u_{h,j}, v_{h,j} \rangle_{\mathcal{U}_{h,j}} = h \sum_{k=j}^{k_f-1} \langle u_{h,j}(k), v_{h,j}(k) \rangle_U.$$

Define the operators  $\mathcal{B}_{h,j} \in L(H, \mathcal{U}_{h,j})$  and  $\mathcal{R}_{h,j} \in L(\mathcal{U}_{h,j})$  by

$$(2.11) \quad (\mathcal{B}_{h,j}\phi)(k) = B_h(k)^* T_h(k_f, k+1)^* G_h T_h(k_f, j)\phi \\ + B_h(k)^* \left\{ h \sum_{i=k+1}^{k_f-1} T_h(i, k+1)^* Q_h(i) T_h(i, j) \right\} \phi,$$

for  $\phi \in H, k = j, j+1, \dots, k_f - 1$ , and

$$(2.12) \quad (\mathcal{R}_{h,j}u_{h,j})(k) = R_h(k)u_{h,j}(k) \\ + B_h(k)^* T_h(k_f, k+1)^* G_h h \sum_{i=j}^{k_f-1} T_h(k_f, i+1) B_h(i) u_{h,j}(i) \\ + B_h(k)^* h \sum_{i=k+1}^{k_f-1} T_h(i, k+1)^* Q_h(i) \left\{ h \sum_{l=j}^{i-1} T_h(i, l+1) B_h(l) u_{h,j}(l) \right\},$$

$u_{h,j} \in \mathcal{U}_{h,j}, k = j, j+1, \dots, k_f - 1$ , respectively, where in the above expressions and throughout the remainder of the paper we adopt the convention that  $\sum_{i=\mu}^{\nu} a_i = 0$  whenever  $\nu < \mu$ . It is not difficult to verify that the adjoint of  $\mathcal{B}_{h,j}$ , the operator  $\mathcal{B}_{h,j}^* \in L(\mathcal{U}_{h,j}, H)$  is given by

$$(2.13) \quad \mathcal{B}_{h,j}^* u_{h,j} = T_h(k_f, j)^* G_h h \sum_{k=j}^{k_f-1} T_h(k_f, k+1) B_h(k) u_{h,j}(k) \\ + h \sum_{k=j+1}^{k_f-1} T_h(k, j)^* Q_h(k) \left\{ h \sum_{i=j}^{k-1} T_h(k, i+1) B_h(i) u_{h,j}(i) \right\},$$

for  $u_{h,j} \in \mathcal{U}_{h,j}$ .

Proceeding as we did in the continuous time case, we find that for  $j = k_0, \dots, k_f - 1$ ,  $x_h(j) \in H$ , and  $u_{h,j} \in \mathcal{U}_{h,j}$

$$\begin{aligned} J_h(u_{h,j}; j, x_h(j), G_h) &= \langle G_h T_h(k_f, j) x_h(j), T_h(k_f, j) x_h(j) \rangle_H \\ &\quad + h \sum_{k=j}^{k_f-1} \langle Q_h(k) T_h(k, j) x_h(j), T_h(k, j) x_h(j) \rangle_H \\ &\quad - \langle \mathcal{R}_{h,j}^{-1} \mathcal{B}_{h,j} x_h(j), \mathcal{B}_{h,j} x_h(j) \rangle_{\mathcal{U}_{h,j}} \\ &\quad + \langle \mathcal{R}_{h,j}(u_{h,j} + \mathcal{R}_{h,j}^{-1} \mathcal{B}_{h,j} x_h(j)), u_{h,j} + \mathcal{R}_{h,j}^{-1} \mathcal{B}_{h,j} x_h(j) \rangle_{\mathcal{U}_{h,j}}, \end{aligned}$$

where the existence of the inverse of  $\mathcal{R}_{h,j}$  is guaranteed by assumption (D2). It is immediately clear that for  $j \in Z$  with  $j \in [k_0, k_f - 1]$  and  $x_h(j) \in H$  given,  $J_h(\cdot; j, x_h(j), G_h)$  is minimized when  $u_{h,j} = \bar{u}_{h,j} = -\mathcal{R}_{h,j}^{-1} \mathcal{B}_{h,j} x_h(j)$ . It follows that

$$\begin{aligned} \min_{\mathcal{U}_{h,j}} J_h(\cdot; j, x_h(j), G_h) &= J_h(\bar{u}_{h,j}; j, x_h(j), G_h) \\ &= \langle G_h T_h(k_f, j) x_h(j), T_h(k_f, j) x_h(j) \rangle_H \\ &\quad + h \sum_{k=j}^{k_f-1} \langle Q_h(k) T_h(k, j) x_h(j), T_h(k, j) x_h(j) \rangle_H \\ &\quad - \langle \mathcal{R}_{h,j}^{-1} \mathcal{B}_{h,j} x_h(j), \mathcal{B}_{h,j} x_h(j) \rangle_{\mathcal{U}_{h,j}} \\ &= \langle \Pi_h(j) x_h(j), x_h(j) \rangle_H, \end{aligned}$$

where the sequence of self-adjoint operators in  $L(H)$ ,  $\{\Pi_h(k)\}_{k=k_0}^{k_f-1}$ , are given by

$$\begin{aligned} (2.14) \quad \Pi_h(j) \phi &= T_h(k_f, j)^* G_h T_h(k_f, j) \phi + h \sum_{k=j}^{k_f-1} T_h(k, j)^* Q_h(k) T_h(k, j) \phi \\ &\quad - \mathcal{B}_{h,j}^* \mathcal{R}_{h,j}^{-1} \mathcal{B}_{h,j} \phi, \end{aligned}$$

for  $k = k_0, \dots, k_f - 1$  and  $\phi \in H$ . We note that it is completely consistent to define  $\Pi_h(k_f) = G_h$ .

Using the above definitions, it is possible to establish the following well known result (see, for example, [LCB], [Z], and [GR]) for the discrete time LQR problem  $(P_h)$ .

**THEOREM 2.2.** *Suppose that assumptions (D1) and (D2) are satisfied. Then for any given initial state  $x_h(k_0) \in H$  there exists a unique solution  $\bar{u}_h \in l_2(k_0, k_f - 1; U)$  to problem  $(P_h)$ . It is given in linear state feedback form by*

$$\bar{u}_h(k) = -\hat{R}_h(k)^{-1} B_h(k)^* \Pi_h(k+1) A_h(k) \bar{x}_h(k), \quad k = k_0, \dots, k_f - 1,$$

where  $\hat{R}_h(k) = R_h(k) + h B_h(k)^* \Pi_h(k+1) B_h(k)$ , for  $k = k_0, \dots, k_f - 1$ , and the optimal trajectory  $\bar{x}_h$  is given by (2.9) (equivalently (2.10)) with  $u_h = \bar{u}_h$ . The sequence of

operators in  $L(H)$ ,  $\{\Pi_h(k)\}_{k=k_0}^{k_f-1}$  are given by (2.14) with  $\Pi_h(k_f) = G_h$  and can be obtained recursively via the Riccati difference equation

$$(2.15) \quad \begin{aligned} \Pi_h(k) = & A_h(k)^* \Pi_h(k+1) A_h(k) \\ & - h A_h(k)^* \Pi_h(k+1) B_h(k) \hat{R}_h(k)^{-1} B_h(k)^* \Pi_h(k+1) A_h(k) \\ & + h Q_h(k), \end{aligned}$$

$k = k_f - 1, \dots, k_0$ ,  $\Pi_h(k_f) = G_h$ . We have

$$(2.16) \quad \begin{aligned} \min_{\bar{u}_h, k_0} J_h(\cdot; k_0, x_h(k_0), G_h) &= J_h(\bar{u}_h; k_0, x_h(k_0), G_h) \\ &= \langle \Pi_h(k_0) x_h(k_0), x_h(k_0) \rangle_H. \end{aligned}$$

For appropriate choices of the families of operators  $T_h, B_h, Q_h$ , and  $R_h$ , we are interested in studying the convergence of solutions to the problems  $(P_h)$  to the solution of problem  $(P)$  as the length of the sampling interval,  $h$ , tends toward zero. In particular, we want to investigate the convergence of the discrete families of Riccati operators  $\{\Pi_h(k) : k_0 \leq k \leq k_f\}$  to the continuous family of operators  $\{\Pi(t) : t_0 \leq t \leq t_f\}$ .

In order to reduce the necessary degree of technical detail, we make the simplifying assumption that  $t_0 = 0$ . There is of course no loss of generality in doing this since any system can be transformed to one on a time interval starting at the origin. Set  $k_0 = 0$  and for each  $h > 0$  let  $k_f = k_{f,h} = [t_f/h]$  where for  $a \in R$ ,  $[a]$  is used to denote the greatest integer less than or equal to  $a$ . Let  $t_{f,h} = h k_{f,h}$  and note that  $\lim_{h \rightarrow 0+} t_{f,h} = t_f$ .

In order to compare discrete and continuous families of operators, it is useful to identify certain  $l_2$  sequence spaces with subspaces of  $L_2$ . For  $X$  a Hilbert space and all  $h > 0$ , let  $L_{2,h}(0, t_{f,h}; X)$  be the subspace of  $L_2(0, t_{f,h}; X)$  defined by

$$L_{2,h}(0, t_{f,h}; X) = \{\phi \in L_2(0, t_{f,h}; X) : \phi \text{ is constant on each of the intervals } [0, h), [h, 2h), \dots, [(k_{f,h} - 1)h, t_{f,h}]\}.$$

Note that the subspace  $L_{2,h}(0, t_{f,h}; X)$  of  $L_2(0, t_{f,h}; X)$  is isometrically isomorphic to the space  $l_2(0, k_{f,h} - 1; X)$  endowed with the inner product

$$\langle \{\phi_j\}_{j=0}^{k_{f,h}-1}, \{\psi_j\}_{j=0}^{k_{f,h}-1} \rangle = h \sum_{j=0}^{k_{f,h}-1} \langle \phi_j, \psi_j \rangle_X.$$

Let  $\mathcal{U} = L_2(0, t_f; U)$  and let  $\mathcal{U}_h = L_{2,h}(0, t_{f,h}; U)$ . Let  $P_h \in L(\mathcal{U}, \mathcal{U}_h)$  be the orthogonal projection-like mapping of  $\mathcal{U}$  onto  $\mathcal{U}_h$  defined by

$$(P_h \phi)(t) = \sum_{j=0}^{k_{f,h}-1} (\phi_h)_j \chi_{I_j}(t), \quad 0 \leq t \leq t_{f,h},$$

for  $\phi \in \mathcal{U}$  where for  $j = 0, 1, \dots, k_{f,h} - 1$ ,  $\chi_{I_j}$  is the characteristic function for the interval  $I_j = [jh, (j+1)h)$  and

$$(\phi_h)_j = h^{-1} \int_{I_j} \phi(t) dt.$$

It is not difficult to show (see Appendix A) that

- (i) the net  $\{\|P_h\|_{L(\mathcal{U}, \mathcal{U}_h)}\}_{h>0}$  is uniformly bounded;
- (ii)  $\lim_{h \rightarrow 0^+} \|P_h \phi\|_{\mathcal{U}_h} = \|\phi\|_{\mathcal{U}}$ ,  $\phi \in \mathcal{U}$ , and
- (iii) for each  $\psi \in \mathcal{U}_h$  there exists a  $\phi \in \mathcal{U}$  such that  $\psi = P_h \phi$  and  $\|\phi\|_{\mathcal{U}} = \|\psi\|_{\mathcal{U}_h}$ .

Following Kato [K, §IX.4] we say that a net  $\{\phi_h\}_{h>0}$ ,  $\phi_h \in \mathcal{U}_h$  converges to  $\phi \in \mathcal{U}$  ( $\phi_h \rightarrow \phi$ , or  $\lim_{h \rightarrow 0^+} \phi_h = \phi$ ) if

$$\lim_{h \rightarrow 0^+} \|\phi_h - P_h \phi\|_{\mathcal{U}_h} = 0.$$

Also, if for  $h > 0$ ,  $\Phi_h \in L(\mathcal{U}_h)$ , then we say that  $\Phi_h$  converges strongly to  $\Phi \in L(\mathcal{U})$  if  $\Phi_h P_h \phi \rightarrow \Phi \phi$ ,  $\phi \in \mathcal{U}$ ; that is if

$$\lim_{h \rightarrow 0^+} \|\Phi_h P_h \phi - P_h \Phi \phi\|_{\mathcal{U}_h} = 0, \quad \phi \in \mathcal{U}.$$

With strong operator convergence defined in this way, it can be shown that  $\Phi_h P_h \phi \rightarrow \Phi \phi$ ,  $\phi \in \mathcal{U}$  implies that the net  $\{\|\Phi_h\|_{L(\mathcal{U}_h)}\}$  is uniformly bounded and that if  $\Phi_h P_h \phi \rightarrow \Phi \phi$ , and  $\Psi_h P_h \phi \rightarrow \Psi \phi$ ,  $\phi \in \mathcal{U}$ , then  $\Phi_h \Psi_h P_h \phi \rightarrow \Phi \Psi \phi$ ,  $\phi \in \mathcal{U}$ , etc. We note of course that an analogous definition of strong convergence can be made for bounded operators having only one or the other of its domain and co-domain being  $\mathcal{U}_h$ . That is, for example, if  $\Phi_h \in L(X, \mathcal{U}_h)$  and  $\Phi \in L(X, \mathcal{U})$  where  $X$  is a normed linear space, then we say that  $\Phi_h$  converges strongly to  $\Phi$  if  $\Phi_h x \rightarrow \Phi x$ ,  $x \in X$ , or

$$\lim_{h \rightarrow 0^+} \|\Phi_h x - P_h \Phi x\|_{\mathcal{U}_h} = 0.$$

Following the treatment of discrete semigroups in Kato [K], we make the following formal definition.

**DEFINITION 2.1.** The discrete time families of bounded linear operators  $\Phi_h = \{\Phi_h(k_n, k_{n-1}, \dots, k_1) : 0 \leq k_1 \leq k_2 \leq \dots \leq k_n \leq K_h\}$ ,  $h > 0$  from a Banach space  $X$  into a Banach space  $Y$  will be said to (strongly) approximate a continuous time family of operators  $\Phi = \{\Phi(t_n, t_{n-1}, \dots, t_1) : 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T\}$  with  $\Phi(t_n, \dots, t_1) \in L(X, Y)$  for  $t = (t_n, \dots, t_1) \in \Delta(n, T) = \{(t_n, t_{n-1}, \dots, t_1) \in \mathbb{R}^n : 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T\}$ , at  $\hat{t} = (\hat{t}_n, \dots, \hat{t}_1) \in \Delta(n, T)$ , if

- (i) There exists at least one net of multi-indices  $\{\hat{k}_h\}_{h>0}$ ,  $\hat{k} = (\hat{k}_{n,h}, \dots, \hat{k}_{1,h}) \in \mathbb{Z}^n$  with  $0 \leq \hat{k}_{1,h} \leq \dots \leq \hat{k}_{n,h} \leq K_h$  and  $\lim_{h \rightarrow 0^+} h \hat{k}_h = \hat{t}$ .
- (ii) For all nets  $\{\hat{k}_h\}_{h>0}$ , satisfying (i) above,

$$\lim_{h \rightarrow 0^+} \|\Phi_h(\hat{k}_h)x - \Phi(\hat{t})x\|_Y = 0, x \in X.$$

The families  $\Phi_h$ ,  $h > 0$  will be said to approximate  $\Phi$  on the set  $\Delta(n, T)$ , if  $K_h = \lceil T/h \rceil$  and if  $\Phi_h$  approximates  $\Phi$  at each  $\hat{t} \in \Delta(n, T)$ .

When the discrete time families  $\Phi_h$ ,  $h > 0$  approximate the continuous time family  $\Phi$  at time  $\hat{t}$  (on the set  $\Delta(n, T)$ ) we shall write  $\Phi_h \rightarrow \Phi$  at time  $\hat{t}$  (on the set  $\Delta(n, T)$ ).

**DEFINITION 2.2.** For  $h > 0$  and  $\Phi_h = \{\Phi_h(k_n, k_{n-1}, \dots, k_1) : 0 \leq k_1 \leq k_2 \leq \dots \leq k_n \leq K_h\}$  a discrete time family of bounded linear operators, we define an

associated continuous time family of operators,  $\tilde{\Phi}_h = \{\tilde{\Phi}_h(t_n, t_{n-1}, \dots, t_1) : 0 \leq t_1 < t_2 + h < \dots < t_n + h < (K_h + 1)h\}$  via  $\tilde{\Phi}_h(t_n, \dots, t_1) = \Phi_h([t_n/h], \dots, [t_1/h])$  for  $t = (t_n, \dots, t_1) \in \Delta_h(n, K_h) = \{(t_n, t_{n-1}, \dots, t_1) \in R^n : 0 \leq t_1 < t_2 + h < \dots < t_n + h < (K_h + 1)h\}$ .

Note that when  $K_h = [T/h]$ ,  $\Delta(n, T) \subset \Delta_h(n, K_h)$  for all  $h > 0$ .

The proof of the following theorem can be argued in much the same manner as were the proofs of Lemmas IX.3.4 and IX.3.5 in Kato [K].

**THEOREM 2.3.** *Suppose that the continuous time family of bounded linear operators  $\Phi$  is strongly continuous on  $\Delta(n, T)$  and that  $\Phi_h$ ,  $h > 0$  are discrete time families for which  $\Phi_h \rightarrow \Phi$  on the set  $\Delta(n, T)$ . Suppose further that for each  $h > 0$ ,  $\tilde{\Phi}_h$  is the continuous time family on  $\Delta_h(n, K_h)$  corresponding to the discrete time family  $\Phi_h$  constructed according to Definition 2.2 above. Then*

- (i) *The families  $\Phi_h$ ,  $h > 0$  are uniformly bounded in  $h$  in  $L(X, Y)$ ; that is there exists a constant  $M > 0$  independent of  $h$  for which*

$$\|\Phi_h(k_n, k_{n-1}, \dots, k_1)\|_{L(X, Y)} \leq M, \quad 0 \leq k_1 \leq k_2 \leq \dots \leq k_n \leq K_h, h > 0,$$

- (ii)  *$\tilde{\Phi}_h \rightarrow \Phi$  uniformly in  $t$  for  $t \in \Delta(n, T)$ ; that is*

$$\lim_{h \rightarrow 0^+} \|\tilde{\Phi}_h(t)x - \Phi(t)x\|_Y = 0, x \in X,$$

*uniformly in  $t$  for  $t = (t_n, \dots, t_1) \in \Delta(n, T)$ .*

*Conversely, if  $K_h = [T/h]$  and  $\tilde{\Phi}_h \rightarrow \Phi$  uniformly in  $t$  for  $t \in \Delta(n, T)$ , then  $\Phi_h \rightarrow \Phi$  on the set  $\Delta(n, T)$ .*

Let the continuous time families  $T = \{T(t, s) : 0 \leq s \leq t \leq T\} \subset L(H)$ ,  $B = \{B(t) : 0 \leq t \leq t_f\} \subset L(U, H)$ ,  $Q = \{Q(t) : 0 \leq t \leq t_f\} \subset L(H)$  and  $R = \{R(t) : 0 \leq t \leq t_f\} \subset L(U)$  be as given in the statement of the continuous time LQR problem (P) (i.e., in particular assume that the conditions (C1)–(C3) hold). For  $h > 0$ , let  $k_{f,h} = [t_f/h]$  and let  $A_h = \{A_h(k) : 0 \leq k \leq k_{f,h} - 1\} \subset L(H)$ ,  $B_h = \{B_h(k) : 0 \leq k \leq k_{f,h} - 1\} \subset L(U, H)$ ,  $Q_h = \{Q_h(k) : 0 \leq k \leq k_{f,h} - 1\} \subset L(H)$ , and  $R_h = \{R_h(k) : 0 \leq k \leq k_{f,h} - 1\} \subset L(U)$  be discrete time families of bounded linear operators which satisfy conditions (D1) and (D2) and which satisfy the following conditions.

- (A1)  $B_h \rightarrow B$ ,  $Q_h \rightarrow Q$ ,  $R_h \rightarrow R$ , and  $B_h^* \rightarrow B^*$  on the set  $\Delta(1, t_f)$  where  $B^* = \{B(t)^* : 0 \leq t \leq t_f\}$  and  $B_h^* = \{B_h(k)^* : 0 \leq k \leq k_{f,h}\}$ .
- (A2) (a) (Stability) The discrete time families of operators  $T_h = \{T_h(k, j) : 0 \leq j \leq k \leq k_{f,h}\} \subset L(H)$  given by

$$T_h(k, j) = \begin{cases} \prod_{i=j}^{k-1} A_h(i), & j < k, \\ I, & j = k \end{cases}$$

are uniformly bounded in  $L(H)$  for  $h > 0$ .

(b) (Consistency)

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \|\tilde{T}_h(t+h, t)\phi - T(t+h, t)\phi\| = 0, \quad \phi \in H,$$

and

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \|\tilde{T}_h(t+h, t)^*\phi - T(t+h, t)^*\phi\| = 0, \quad \phi \in H,$$

uniformly in  $t$  for  $t \in [0, t_f]$ .

(A3) The scalars  $r_h$  given in the statement of condition (D2) are bounded away from zero uniformly in  $h$ . That is  $r_h \geq r > 0$ ,  $h > 0$ .

LEMMA 2.1. Condition (A2) implies that  $T_h \rightarrow T$  and  $T_h^* \rightarrow T^*$  on the set  $\Delta(2, t_f)$ .

*Proof.* We consider the convergence  $T_h \rightarrow T$  only; the adjoint convergence is completely analogous. Following the proof of the well known Lax-Equivalence Theorem [RM], the result is an immediate consequence of condition (A2), the strong continuity of the continuous time family  $T$ , and the identity

$$T_h(k, j)\phi - T(kh, jh)\phi = \begin{cases} \sum_{i=j}^{k-1} T_h(k, i+1)\{A_h(i) - T((i+1)h, ih)\}T(ih, jh)\phi, & k > j, \\ 0, & k = j, \end{cases}$$

$$0 \leq j \leq k \leq k_{f,h}, \phi \in H. \quad \square$$

We shall also assume that  $G \in L(H)$  is as in condition (C3) and that for each  $h > 0$  the operator  $G_h \in L(H)$  satisfies condition (D2). We require that the additional approximation condition

(A4)  $\lim_{h \rightarrow 0^+} G_h\phi = G\phi$ ,  $\phi \in H$ ,  
be satisfied as well.

For  $h > 0$  and  $s \in [0, t_f]$  define  $\tilde{B}_{h,s} \in L(H, \mathcal{U}_h)$  by

$$(2.17) \quad (\tilde{B}_{h,s}\phi)(t) = \begin{cases} \text{For } t \in [0, t_{f,h} - h] : \\ \chi_{[s/h]h, t_{f,h}]}(t) \tilde{B}_h(t)^* \{ \tilde{T}_h(t_{f,h}, t+h)^* G_h \tilde{T}_h(t_{f,h}, s) \\ + \int_{[(t+h)/h]h}^{t_{f,h}} \tilde{T}_h(\eta, t+h)^* \tilde{Q}_h(\eta) \tilde{T}_h(\eta, s) d\eta \} \phi, \\ \text{For } t \in [t_{f,h} - h, t_{f,h}] : \\ (\tilde{B}_{h,s}\phi)(t_{f,h} - h), \end{cases}$$

when  $s \in [0, t_{f,h})$ , and by  $(\tilde{B}_{h,s}) = 0$  when  $s \in [t_{f,h}, t_f]$ , for  $\phi \in H$ . Note that for  $j = 0, 1, 2, \dots, k_{f,h} - 1$ ,  $k = j, j+1, \dots, k_{f,h} - 1$ , and  $\phi \in H$

$$(2.18) \quad (\tilde{B}_{h,s}\phi)(t) = (\mathcal{B}_{h,j}\phi)(k),$$

for  $s \in [jh, (j+1)h)$  and  $t \in [kh, (k+1)h)$ , where for  $j = 0, 1, 2, \dots, k_{f,h} - 1$ ,  $\mathcal{B}_{h,j} \in L(U, \mathcal{U}_{h,j})$  is given by (2.11).

Similarly, for  $s \in [0, t_f]$  define  $\tilde{B}_{h,s}^* \in L(\mathcal{U}_h, H)$  by

$$(2.19) \quad \tilde{B}_{h,s}^* u_h = \tilde{T}_h(t_{f,h}, s)^* G_h \int_{[s/h]h}^{t_{f,h}} \tilde{T}_h(t_{f,h}, t+h) \tilde{B}_h(t) u_h(t) dt \\ + \int_{[(s+h)/h]h}^{t_{f,h}} \tilde{T}_h(\tau, s)^* \tilde{Q}_h(\tau) \left\{ \int_{[s/h]h}^{[\tau/h]h} \tilde{T}_h(\tau, \eta+h) \tilde{B}_h(\eta) u_h(\eta) d\eta \right\} d\tau,$$

when  $s \in [0, t_{f,h})$ , and by  $\tilde{B}_{h,s}^* u_h = 0$  when  $s \in [t_{f,h}, t_f]$  for  $u_h \in \mathcal{U}_h$ . Note that for  $j = 0, 1, 2, \dots, k_{f,h} - 1$  and  $u_{h,j} \in \mathcal{U}_{h,j}$  we have

$$(2.20) \quad \tilde{B}_{h,s}^* u_h = B_{h,j}^* u_{h,j}$$

for  $s \in [jh, (j+1)h)$  when  $u_h \in \mathcal{U}_h$  is given by

$$(2.21) \quad u_h(t) = \begin{cases} 0, & 0 \leq t < jh \\ u_{h,j}(k), & kh \leq t < (k+1)h, \end{cases}$$

$k = j, j+1, \dots, k_{f,h} - 1$ , and  $B_{h,j}^* \in L(\mathcal{U}_{h,j}, H)$  is given by (2.13). Note also that  $\tilde{B}_{h,s}^* = (\tilde{B}_{h,s})^*$ . That is that  $\tilde{B}_{h,s}^* \in L(\mathcal{U}_h, H)$  given by (2.19) is the Hilbert space adjoint of the operator  $\tilde{B}_{h,s} \in L(H, \mathcal{U}_h)$  given by (2.17) for all  $s \in [0, t_f]$ .

For  $s \in [0, t_f]$  define  $\tilde{\mathcal{R}}_{h,s} \in L(\mathcal{U}_h)$  by

$$(2.22) \quad (\tilde{\mathcal{R}}_{h,s} u_h)(t) = \begin{cases} \text{For } t \in [0, t_{f,h} - h] : \\ \tilde{R}_h(t) u_h(t) + \chi_{[s/h]h, t_{f,h}}(t) \tilde{B}_h(t)^* \\ \left\{ \tilde{T}_h(t_{f,h}, t+h)^* G_h \int_{[s/h]h}^{t_{f,h}} \tilde{T}_h(t_{f,h}, \eta+h) \tilde{B}_h(\eta) u_h(\eta) d\eta \right. \\ \left. + \int_{[(t+h)/h]h}^{t_{f,h}} \tilde{T}_h(\eta, t+h)^* \tilde{Q}_h(\eta) \right. \\ \left. \cdot \left[ \int_{[s/h]h}^{[\eta/h]h} \tilde{T}_h(\eta, \tau+h) \tilde{B}_h(\tau) u_h(\tau) d\tau \right] d\eta \right\}, \\ \text{For } t \in [t_{f,h} - h, t_{f,h}] : \\ (\tilde{\mathcal{R}}_{h,s} u_h)(t_{f,h} - h), \end{cases}$$

when  $s \in [0, t_{f,h})$  and by  $(\tilde{\mathcal{R}}_{h,s} u_h)(t) = \tilde{R}_h(t) u_h(t)$ ,  $0 \leq t < t_{f,h}$ , when  $s \in [t_{f,h}, t_f]$ , for  $u_h \in \mathcal{U}_h$ . We again have that for  $j = 0, 1, 2, \dots, k_{f,h} - 1$ ,  $k = j, j+1, \dots, k_{f,h} - 1$ , and  $u_{h,j} \in \mathcal{U}_{h,j}$ ,

$$(2.23) \quad (\tilde{\mathcal{R}}_{h,s} u_h)(t) = (\mathcal{R}_{h,j} u_{h,j})(k)$$

for  $s \in [jh, (j+1)h)$  and  $t \in [kh, (k+1)h)$ , where  $u_h \in \mathcal{U}_h$  is given by (2.21) and  $\mathcal{R}_{h,j} \in L(\mathcal{U}_{h,j})$  is given by (2.12). We note also that  $\tilde{\mathcal{R}}_{h,s}$  is self-adjoint and positive definite on  $\mathcal{U}_h$  and that if we let  $\tilde{\mathcal{U}}_{h,j}$  denote the subspace of  $\mathcal{U}_h$  obtained from  $\mathcal{U}_{h,j}$  via the natural embedding (i.e. via (2.21)), then,  $\tilde{\mathcal{R}}_{h,s}$  is a bijection from  $\tilde{\mathcal{U}}_{h,j}$  onto  $\tilde{\mathcal{U}}_{h,j}$ . It follows therefore from (2.17), (2.19), (2.22) that for  $j = 0, 1, 2, \dots, k_{f,h} - 1$

$$(2.24) \quad B_{h,j}^* \mathcal{R}_{h,j}^{-1} B_{h,j} \phi = \tilde{B}_{h,s}^* \tilde{\mathcal{R}}_{h,s}^{-1} \tilde{B}_{h,s} \phi$$



for each  $\phi \in H$  and all  $s \in [jh, (j+1)h)$ , and that

$$(2.25) \quad \tilde{B}_{h,s}^* \tilde{\mathcal{R}}_{h,s}^{-1} \tilde{B}_{h,s} \phi = 0$$

for all  $s \in [t_{f,h}, t_f]$ .

Setting  $\tilde{\Pi}_h(s) = \Pi_h(k)$ ,  $kh \leq s < (k+1)h$ , for  $s \in [0, t_f]$ , from (2.11), (2.12), (2.13), (2.14), and (2.24) we find that

$$(2.26) \quad \begin{aligned} \tilde{\Pi}_h(s)\phi &= \tilde{T}_h(t_{f,h}, s)^* G_h \tilde{T}_h(t_{f,h}, s)\phi \\ &\quad + \int_{[s/h]h}^{t_{f,h}} \tilde{T}_h(t, s)^* \tilde{Q}_h(t) \tilde{T}_h(t, s)\phi dt - \tilde{B}_{h,s}^* \tilde{\mathcal{R}}_{h,s}^{-1} \tilde{B}_{h,s} \phi \end{aligned}$$

for each  $\phi \in H$ . Note that (2.25) implies that  $\tilde{\Pi}_h(t) = G_h$  for  $t \in [t_{f,h}, t_f]$ .

For  $B_s \in L(H, \mathcal{U}_s)$ ,  $\mathcal{R}_s \in L(\mathcal{U}_s)$ , and  $B_s^* \in L(\mathcal{U}_s, H)$  given by (2.2), (2.3), and (2.4), respectively, define  $\tilde{B}_s \in L(H, \mathcal{U})$ ,  $\tilde{\mathcal{R}}_s \in L(\mathcal{U})$ , and  $\tilde{B}_s^* \in L(\mathcal{U}, H)$  by

$$(2.27) \quad (\tilde{B}_s \phi)(t) = \begin{cases} 0, & 0 \leq t < s, \\ (B_s \phi)(t), & s \leq t \leq t_f, \end{cases}$$

$$(2.28) \quad (\tilde{\mathcal{R}}_s u)(t) = \begin{cases} R(t)u(t), & 0 \leq t < s, \\ (\mathcal{R}_s u)(t), & s \leq t \leq t_f, \end{cases}$$

and

$$(2.29) \quad \tilde{B}_s^* u = B_s^* u$$

for  $\phi \in H$  and  $u \in \mathcal{U}$ . It is not difficult to show that  $\tilde{B}_s^* = (\tilde{B}_s)^*$  (i.e. that  $\tilde{B}_s^* \in L(\mathcal{U}, H)$  is the Hilbert space adjoint of  $\tilde{B}_s \in L(H, \mathcal{U})$ ), that  $\tilde{\mathcal{R}}_s$  is self-adjoint positive definite on  $\mathcal{U}$ , and that if we let  $\tilde{\mathcal{U}}_s$  denote the subspace of  $\mathcal{U}$  obtained via the natural embedding of  $\mathcal{U}_s$  into  $\mathcal{U}$ , then  $\tilde{\mathcal{R}}_s$  is a bijection from  $\tilde{\mathcal{U}}_s$  onto  $\tilde{\mathcal{U}}_s$ . Consequently, it follows that

$$(2.30) \quad B_s^* \mathcal{R}_s^{-1} B_s \phi = \tilde{B}_s^* \tilde{\mathcal{R}}_s^{-1} \tilde{B}_s \phi$$

for all  $\phi \in H$  and  $s \in [0, t_f]$ . From (2.5) we obtain that

$$(2.31) \quad \begin{aligned} \Pi(s)\phi &= T(t_f, s)^* G T(t_f, s)\phi + \int_s^{t_f} T(t, s)^* Q(t) T(t, s)\phi dt \\ &\quad - \tilde{B}_s^* \tilde{\mathcal{R}}_s^{-1} \tilde{B}_s \phi, \end{aligned}$$

for all  $\phi \in H$  and  $s \in [0, t_f]$ .

Our fundamental convergence or approximation result for the finite time horizon problem is given in the following theorem and its corollary.

**THEOREM 2.4.** *Suppose that the families of operators  $\{T, B, Q, R\}$  satisfy conditions (C1)–(C3) and that for all  $h > 0$ , the families of operators  $\{T_h, B_h, Q_h, R_h\}$  satisfy conditions (D1) and (D2). Suppose further that the approximation assumptions*

(A1)–(A4) are satisfied. Then the discrete time family of operators  $\Pi_h = \{\Pi_h(k) : 0 \leq k \leq k_{f,h}\}$  given by (2.14) or (2.15) strongly approximates the continuous time family of operators  $\Pi = \{\Pi(t) : 0 \leq t \leq t_f\}$  given by (2.5) or (2.6) on the set  $\Delta(1, t_f)$ . That is,  $\Pi_h \rightarrow \Pi$  on the set  $\Delta(1, t_f)$ .

*Proof.* The desired result will follow from Theorem 2.3 if we can argue that  $\lim_{h \rightarrow 0+} \tilde{\Pi}_h(t)\phi = \Pi(t)\phi$ , uniformly in  $t$ , for  $t \in (0, t_f]$ , for each  $\phi \in H$ , where  $\tilde{\Pi}_h$  and  $\Pi$  are given by (2.26) and (2.31), respectively.

From assumption (A3), we have that the operators  $\tilde{\mathcal{R}}_{h,s}^{-1}$  are bounded uniformly in  $h > 0$  and  $s \in [0, t_f]$ . Somewhat technical, but rather elementary arguments can be used to show that  $\tilde{\mathcal{B}}_{h,s}\phi \rightarrow \tilde{\mathcal{B}}_s\phi$ , for all  $\phi \in H$ ,  $\tilde{\mathcal{R}}_{h,s}P_h u \rightarrow \tilde{\mathcal{R}}_s u$  for  $u \in \mathcal{U}$ , and  $\tilde{\mathcal{B}}_{h,s}^*P_h u \rightarrow \tilde{\mathcal{B}}_s^* u$ , for  $u \in \mathcal{U}$ , uniformly in  $s$  for  $s \in [0, t_f]$ , where  $\tilde{\mathcal{B}}_{h,s}$ ,  $\tilde{\mathcal{B}}_s$ ,  $\tilde{\mathcal{R}}_{h,s}$ ,  $\tilde{\mathcal{R}}_s$ ,  $\tilde{\mathcal{B}}_{h,s}^*$ , and  $\tilde{\mathcal{B}}_s^*$  are given by (2.17), (2.27), (2.22), (2.28), (2.19), and (2.29) respectively (see Appendix B). This together with the identity

$$\tilde{\mathcal{R}}_{h,s}^{-1}P_h - P_h\tilde{\mathcal{R}}_s^{-1} = \tilde{\mathcal{R}}_{h,s}^{-1}\{\tilde{\mathcal{R}}_{h,s}P_h - P_h\tilde{\mathcal{R}}_s\}\tilde{\mathcal{R}}_s^{-1}$$

yield that  $\lim_{h \rightarrow 0+} \tilde{\mathcal{B}}_{h,s}^*\tilde{\mathcal{R}}_{h,s}^{-1}\tilde{\mathcal{B}}_{h,s}\phi = \tilde{\mathcal{B}}_s^*\tilde{\mathcal{R}}_s^{-1}\tilde{\mathcal{B}}_s\phi$ , for  $\phi \in H$ , uniformly in  $s$  for  $s \in [0, t_f]$ . The desired convergence can then be obtained from assumptions (A1), (A2), (A4) and equations (2.26), (2.31).  $\square$

Let  $F = \{F(t) : 0 \leq t \leq t_f\}$  and  $S = \{S(t, s) : 0 \leq s \leq t \leq t_f\}$  be respectively the continuous time families of optimal closed-loop feedback gain operators and optimal closed-loop state transition operators for the continuous time LQR problem (P). That is, for  $t \in [0, t_f]$

$$F(t) = R(t)^{-1}B(t)^*\Pi(t) \in L(H, \mathcal{U}),$$

and for  $0 \leq s \leq t \leq t_f$

$$\begin{aligned} (2.32) \quad S(t, s)\phi &= T(t, s)\phi - \int_s^t T(t, \eta)B(\eta)F(\eta)S(\eta, s)\phi d\eta \\ &= T(t, s)\phi - \int_s^t T(t, \eta)B(\eta)(\mathcal{R}_s^{-1}\mathcal{B}_s^*\phi)(\eta)d\eta, \end{aligned}$$

for  $\phi \in H$  (see [G]). Similarly, for the discrete time problem, let the discrete time families,  $F_h = \{F_h(k) : 0 \leq k \leq k_{f,h} - 1\} \subset L(H, \mathcal{U})$  and  $S_h = \{S_h(k, j) : 0 \leq j \leq k \leq k_{f,h}\} \subset L(H)$  be given by

$$F_h(k) = \hat{R}_h(k)^{-1}B_h(k)^*\Pi_h(k+1)A_h(k),$$

where

$$\hat{R}_h(k) = R_h(k) + hB_h(k)^*\Pi_h(k+1)B_h(k),$$

$k = 0, 1, \dots, k_{f,h} - 1$ , and

$$\begin{aligned} (2.33) \quad S_h(k, j)\phi &= T_h(k, j)\phi - h \sum_{i=j}^{k-1} T_h(k, i+1)B_h(i)F_h(i)S_h(i, j)\phi \\ &= T_h(k, j)\phi - h \sum_{i=j}^{k-1} T_h(k, i+1)B_h(i)(\mathcal{R}_{h,j}^{-1}\mathcal{B}_{h,j}^*\phi)(i), \end{aligned}$$

$0 \leq j \leq k \leq k_{f,h}$ , for  $\phi \in H$ .

**COROLLARY 2.1.** *Suppose that the hypotheses of Theorem 2.4 above are satisfied and let  $\{\bar{u}, \bar{x}\}$  and  $\{\bar{u}_h, \bar{x}_h\}$  be the optimal control/trajectory pairs for the LQR problems (P) and (P<sub>h</sub>), respectively, corresponding to the initial data  $x(0) = x_h(0) = x_0 \in H$ . Then*

- (i)  $F_h \rightarrow F$ ;
- (ii)  $S_h \rightarrow S$ ;
- (iii)  $\lim_{h \rightarrow 0^+} \|\bar{u}_h(k_h) - \bar{u}(t)\|_U = 0$ , and  $\lim_{h \rightarrow 0^+} \|\bar{x}_h(k_h) - \bar{x}(t)\|_H = 0$ , for  $t \in [0, t_f]$  and for all nets  $\{k_h\}_{h>0}$  for which  $\lim_{h \rightarrow 0^+} h k_h = t$ .
- (iv)  $\lim_{h \rightarrow 0^+} \bar{J}_h = \bar{J}$ .

*Proof.* Statement (i) and (iv) (recall (2.7) and (2.16)) are immediate consequences of Theorem 2.4. Statement (iii) follows from statement (i) and (ii) since  $\bar{u}(t) = -F(t)\bar{x}(t)$ ,  $\bar{x}(t) = S(t, 0)x_0$ ,  $t \in [0, t_f]$ , and  $\bar{u}_h(k) = -F_h(k)\bar{x}_h(k)$ ,  $0 \leq k \leq k_{f,h} - 1$ ,  $\bar{x}_h(k) = S_h(k, 0)x_0$ ,  $0 \leq k \leq k_{f,h}$ . Thus we need only to verify statement (ii).

We rewrite (2.32) as

$$S(t, s)\phi = T(t, s)\phi - \int_s^t T(t, \eta)B(\eta)(\tilde{\mathcal{R}}_s^{-1}\tilde{B}_s^*\phi)(\eta)d\eta,$$

and from (2.33) we obtain

$$\tilde{S}_h(t, s)\phi = \tilde{T}_h(t, s)\phi - \int_{[s/h]h}^{[t/h]h} \tilde{T}_h(t, \eta + h)\tilde{B}_h(\eta)(\tilde{\mathcal{R}}_{h,s}^{-1}\tilde{B}_{h,s}^*\phi)(\eta)d\eta.$$

The result now follows as in the proof of Theorem 2.4 □

**REMARK** In actual practice, given the continuous time LQR problem (P), the net of discrete time problems  $\{(P_h)\}$  is typically obtained by considering zero-order hold (i.e. piecewise constant) control inputs and output sampling. In this case we would obtain  $A_h(k) = T((k+1)h, kh)$ ,  $B_h(k) = h^{-1} \int_{kh}^{(k+1)h} T((k+1)h, s)B(s)ds$ ,  $Q_h(k) = h^{-1} \int_{kh}^{(k+1)h} Q(s)ds$ ,  $R_h(k) = h^{-1} \int_{kh}^{(k+1)h} R(s)ds$ , and  $G_h = G$ . When conditions (C1)–(C3) on the continuous time families  $T, B, Q$ , and  $R$  are satisfied, it is immediately clear that the discrete time families  $T_h, B_h, Q_h$  and  $R_h$ , and the operator  $G_h$  satisfy conditions (D1) and (D2) and the approximation conditions (A1)–(A4). More generally, other discretizations are also admissible. For example, in the time invariant case, the semigroup  $\{T(t) : t \geq 0\}$  could be discretely approximated using A-stable Padé approximants to the exponential (see [HK]). In particular, if  $T(t) = \exp(tA)$ ,  $t \geq 0$ , then one might set  $T_h(k) = (I - hA)^{-k}$  (implicit Euler) or  $T_h(k) = (I - hA/2)^{-k}(I + hA/2)^k$  (Crank-Nicolson). The stability and consistency of these discretizations (i.e. assumption (A2)) can be verified using the theory and techniques developed in [HK]. Finally, along these same lines, the convergence of simultaneous but independent state and time discretization in the context of the LQR theory should also be looked at. We note that appropriately “coupled” state and time discretization can be handled using the theory and framework which has been developed above.

**3. The Infinite Time Horizon Problem.** The linear quadratic regulator problem over an infinite time interval can be viewed as an extension of the finite time interval problem. The state equations (2.1) and (2.9) governing the dynamics of the continuous time and discrete time control systems, respectively, remain the same. The continuous and discrete time operator families  $\{T, B, Q, R\}$ , and  $\{T_h, B_h, Q_h, R_h\}$  are assumed to be defined on the infinite time intervals  $[t_0, +\infty) \subset R$  and  $[k_0, +\infty) \subset Z$ , respectively. The cost functionals are taken to be

$$(3.1) \quad J_\infty(u; t_0, x(t_0)) = \int_{t_0}^{\infty} \{ \langle Q(t)x(t), x(t) \rangle_H + \langle R(t)u(t), u(t) \rangle_U \} dt \\ = \lim_{t_f \rightarrow \infty} J(u; t_0, x(t_0), 0)$$

and

$$(3.2) \quad J_{h,\infty}(u_h; k_0, x_h(k_0)) = h \sum_{k=k_0}^{\infty} \{ \langle Q_h(k)x_h(k), x_h(k) \rangle_H + \langle R_h(k)u_h(k), u_h(k) \rangle_U \} \\ = \lim_{k_f \rightarrow \infty} J_h(u_h; k_0, x_h(k_0), 0)$$

Under appropriate stabilizability and detectability assumptions on the continuous time and the discrete time control systems, the existence and the uniqueness of the optimal controls  $\bar{u}, \bar{u}_h$  minimizing (3.1) and (3.2), respectively, can be obtained. Moreover, these optimal controls can be written in a closed-loop state feedback form (see Theorem 3.1 below). We are again interested in investigating the convergence of the optimal controls and the optimal feedback laws for the sampled systems as the length of the sampling interval tends toward zero.

Our fundamental result can be outlined as follows. Assume that the conditions (A1)–(A4) for the convergence of the open-loop control problems are satisfied on every finite time interval  $[t_0, t_f]$ . Suppose further that the stabilizability and the detectability of the continuous time system are uniformly preserved by the sampled time systems (see Definition 3.3-(iii) and 3.4-(iii)). Then the optimal controls  $\bar{u}_h$  and the optimal state feedback laws  $F_h$  for the sampled time systems converge to the optimal control  $\bar{u}$  and optimal feedback law  $F$  for the continuous time system, respectively, as the length,  $h$ , of the sampling interval tends toward zero. We note that the problem of uniform preservation of stabilizability and detectability under sampling is in general, a difficult one. Here we shall treat this question only in a limited sense. We shall have to assume finite rank feedback stabilizability and finite rank detectability (see Condition (F)) for the continuous time system, although we have some conjectures about other reasonably broad classes of systems for which these conditions can be verified. We address this question in greater detail below.

As in the finite time horizon problems, the functionals  $J_\infty$  and  $J_{h,\infty}$  can be viewed as quadratic forms on  $L_2(t_0, \infty; U)$  and  $l_2(k_0, \infty; U)$ , respectively. However, since  $J_\infty$  and  $J_{h,\infty}$  are, in general, not bounded (for example, the uncontrolled system may not be asymptotically stable, hence the cost for the control input  $u = 0$  may be infinity), one must deal with some rather tedious technical details. Therefore, the

infinite horizon LQR problem is commonly viewed as the asymptotic limit of the finite time horizon problems with the final time,  $t_f$ , tending to infinity. The existence and the uniqueness of the optimal controls and feedback laws can then be obtained by considering the limiting behavior of the optimal controls and the optimal feedback laws for the finite time horizon problems. From the definitions of the functionals  $J_\infty$  and  $J_{h,\infty}$ , it is natural to view the infinite time horizon problem as the limit of the finite horizon problems with the final state penalty operators  $G$  and  $G_h$  taken to be zero. However, we note that if the optimal trajectories  $\bar{x}(t)$  and  $\bar{x}_h(k)$  are known to be asymptotically stable (i.e.,  $\bar{x}(t) \rightarrow 0$ ,  $\bar{x}_h(k) \rightarrow 0$ , as  $t$  and  $k$  tend to infinity), then the infinite horizon LQR problem can also be considered as the limit of finite horizon problems with  $G_h, G \geq 0$ . Once again, for simplicity, we shall assume henceforth, without loss of generality, that  $t_0 = k_0 = 0$ .

**DEFINITION 3.1.** (Cost functional stabilizability)

- (i) The continuous time system associated with the operator pair  $\{T, B\}$  is said to be cost functional stabilizable with respect to the performance index  $J_\infty$  given by (3.1), if for each  $\phi \in H$ , there exists a constant  $M(\phi)$  such that for any  $s \geq 0$ , there exists a control input  $u_s \in L_2(s, \infty; U)$  with  $J_\infty(u_s; s, \phi) \leq M(\phi)$ .
- (ii) The sampled time system associated with the operator pair  $\{T_h, B_h\}$  is said to be cost functional stabilizable with respect to the discrete performance index  $J_{h,\infty}$  given by (3.2), if for each  $\phi \in H$ , there exists a constant  $M_h(\phi)$  such that for any  $j \geq 0$ , there exists a control input sequence  $u_{h,j} \in l_2(j, \infty; U)$  with  $J_{h,\infty}(u_{h,j}; j, \phi) \leq M_h(\phi)$ .
- (iii) The sampled systems are said to be uniformly cost functional stabilizable for all  $0 < h \leq h_0$ , if, for each  $\phi \in H$ , the constants  $M_h(\phi)$  defined in (ii) are independent of the length of the sampling interval  $h$ , for all  $h \leq h_0$  for some  $h_0 > 0$ .

For any given final time  $t_f$  and final index  $k_{f,h}$ , let  $\Pi_{t_f}(\cdot; G)$  and  $\Pi_{k_{f,h}}(\cdot; G_h)$  denote the Riccati operators given by (2.5) and (2.14) corresponding to the final state penalty operators  $G$  and  $G_h$ , respectively. In the case  $G = G_h = 0$ , using (2.7) and (2.16), it is easy to verify that (see for example, [DI]) for each given  $t \geq 0$  and  $k \geq 0$ , the functions  $t_f \mapsto \Pi_{t_f}(t; 0)$  and  $k_{f,h} \mapsto \Pi_{h,k_{f,h}}(k; 0)$  are nondecreasing, self-adjoint, nonnegative operator valued functions. The assumed cost functional stabilizability of the continuous and discrete time control systems then provides an upper bound for  $\Pi_{t_f}$  and  $\Pi_{h,k_{f,h}}$ . Indeed, we have

$$\langle \Pi_{t_f}(t; 0)\phi, \phi \rangle_H \leq M(\phi), \quad \phi \in H,$$

and

$$\langle \Pi_{h,k_{f,h}}(k; 0)\phi, \phi \rangle_H \leq M_h(\phi), \quad \phi \in H,$$

for all  $t_f$  and  $k_{f,h}$ . Thus, the strong limits of  $\Pi_{t_f}(t; 0)$  and  $\Pi_{h,k_{f,h}}(k; 0)$  exist for each  $t \geq 0$  and  $k \geq 0$  as  $t_f$  and  $k_{f,h}$  tend to infinity. Let us denote these strong limiting,

operator valued functions by  $\Pi_\infty(\cdot; 0)$  and  $\Pi_{h,\infty}(\cdot; 0)$ , respectively. The existence and uniqueness of the solutions to the continuous and discrete time optimal control problems is given in the following well known theorem; see, for example, [BW], [G], [GR], [LCB], [HH], and [Z].

**THEOREM 3.1.** *Assume that the continuous time system and the sampled time systems for all  $h$  sufficiently small are cost functional stabilizable. Then for any  $s \geq 0$  and  $j \geq 0$ , and initial states  $x(s) = \phi$  and  $x_{j,h} = \phi$ , there exist unique optimal controls  $\bar{u}$ , and  $\bar{u}_h$  which minimize the cost functionals  $J_\infty(\cdot; s, x(s); 0)$  over  $L_2(s, \infty; U)$  and  $J_{h,\infty}(\cdot; j, x_h(j); 0)$  over  $l_2(j, \infty; U)$ , respectively. The optimal controls can be written in linear state feedback form as*

$$\bar{u}(t) = -R(t)^{-1}B(t)^*\Pi_\infty(t; 0)\bar{x}(t) = -F(t)\bar{x}(t),$$

and

$$\bar{u}_h(k) = -\hat{R}_h(k)^{-1}B_h(k)^*\Pi_{h,\infty}(k+1; 0)A_h(k)\bar{x}_h(k) = -F_h(k)\bar{x}_h(k),$$

where  $\bar{x}$  and  $\bar{x}_h$  are the corresponding optimal trajectories and  $\hat{R}_h(k) = R_h(k) + hB_h(k)^*\Pi_{h,\infty}(k+1; 0)B_h(k)$ . The operator-valued function  $\Pi_\infty(\cdot; 0)$  is bounded on the interval  $[0, \infty)$  and satisfies the Riccati integral equation

$$(3.3) \quad \begin{aligned} \Pi_\infty(s; 0)\phi &= T(t, s)^*\Pi_\infty(t; 0)T(t, s)\phi \\ &+ \int_s^t T(\tau, s)^*[Q(\tau) - \Pi_\infty(\tau; 0)(BR^{-1}B^*)(\tau)\Pi_\infty(\tau; 0)]T(\tau, s)\phi d\tau, \end{aligned}$$

for all  $\phi \in H$  and  $(t, s) \in \Delta(2, \infty)$ . Similarly, the operator-valued sequence  $\Pi_{h,\infty}(\cdot; 0)$  is bounded for  $0 \leq k < \infty$  and satisfies the Riccati difference equation

$$(3.4) \quad \begin{aligned} \Pi_{h,\infty}(k; 0) &= A_h(k)^*\Pi_{h,\infty}(k+1; 0)A_h(k) + hQ_h(k) \\ &- hA_h(k)^*\Pi_{h,\infty}(k+1; 0)B_h(k)\hat{R}_h(k)^{-1}B_h(k)^*\Pi_{h,\infty}(k+1; 0)A_h(k). \end{aligned}$$

If the sampled time systems are uniformly cost functional stabilizable for  $0 < h \leq h_0$ , then the operator-valued sequences  $\Pi_{h,\infty}(\cdot; 0)$  are uniformly bounded for all sampling period  $h$  with  $0 < h \leq h_0$ .

We assume that the general conditions (A1)–(A4) for the convergence of the open loop problems hold on any given finite time interval. From the feedback form of the optimal controls given in Theorem 3.1, it is not difficult to see that on a given finite time interval  $[0, t_f]$ , the uniform convergence of the optimal controls  $\bar{u}_h$ , the optimal trajectories  $\bar{x}_h$ , and the optimal feedback gains  $F_h$  for the sampled time control problems would follow directly from the uniform convergence of  $\Pi_{h,\infty}(\cdot; 0)$ . Our investigation is therefore, focused on the convergence of  $\Pi_{h,\infty}(\cdot; 0)$  to  $\Pi_\infty(\cdot; 0)$  as  $h$  tends toward zero. Using the notation introduced in the previous section, we note that for each  $t \geq 0$ , an obvious sufficient condition for the convergence of  $\Pi_{h,\infty}(t; 0)$  to  $\Pi_\infty(t; 0)$  is the convergence of  $\Pi_{t_f}(t; G)$  to  $\Pi_\infty(t; 0)$  and the uniform convergence

in  $h$  of  $\tilde{\Pi}_{h,k_{f,h}}(t; G_h)$  to  $\tilde{\Pi}_{h,\infty}(t; 0)$  (with  $k_{f,h} = [t_f/h]$ ) as  $t_f$  tends to infinity for some  $G \geq 0$  and corresponding  $G_h \geq 0$ . Indeed, from the triangle inequality, for  $\phi \in H$ , we have

$$\begin{aligned} \|\tilde{\Pi}_{h,\infty}(t; 0)\phi - \Pi_\infty(t; 0)\phi\|_H &\leq \|\tilde{\Pi}_{h,\infty}(t; 0)\phi - \tilde{\Pi}_{h,k_{f,h}}(t; G_h)\phi\|_H \\ &+ \|\tilde{\Pi}_{h,k_{f,h}}(t; G_h)\phi - \Pi_{t_f}(t; G)\phi\|_H + \|\Pi_{t_f}(t; G)\phi - \Pi_\infty(t; 0)\phi\|_H. \end{aligned}$$

Then for an arbitrary  $\epsilon > 0$ , a sufficiently large  $t_f$  can be chosen such that the first and the last terms on the right hand side of the above inequality are smaller than  $\epsilon/3$  for all  $h$ . By applying the theory of the previous section on the interval  $[0, t_f]$ , there exists  $h_0 > 0$  small enough such that for all  $0 < h \leq h_0$ , the second term on the right hand side of the above inequality is bounded by  $\epsilon/3$ . Thus, the desired convergence immediately follows.

As we have pointed out, if the trajectories of the systems are asymptotically stable, then as  $t_f$  tends to infinity, the cost functionals  $J_\infty$  and  $J_{h,\infty}$  are also limits of the cost functionals  $J, J_h$  for the finite time interval problems on  $[0, t_f]$  with final state penalties  $G$  and  $G_h$  different from zero. In particular, if the optimal trajectory of the infinite horizon problem is asymptotically stable, the convergence rates of  $\bar{J}_h(\bar{u}_h; k, \phi, G_h) = \langle \Pi_{h,k_{f,h}}(k; G_h)\phi, \phi \rangle_H$  with  $G_h \geq M_h(\phi)$  and  $\bar{J}(\bar{u}; t, \phi, G) = \langle \Pi_{t_f}(t; G)\phi, \phi \rangle_H$  with  $G \geq M(\phi)$  can be estimated by the decay rate of the optimal trajectory  $\bar{x}$  for the infinite horizon problem. Toward this end, let  $S = \{S(t, s) : 0 \leq s \leq t < \infty\}$  be the continuous time evolution system given by

$$(3.5) \quad S(t, s)\phi = T(t, s)\phi - \int_s^t T(t, \tau)B(\tau)F(\tau)S(\tau, s)\phi ds, \quad \text{for } \phi \in H.$$

The evolution system  $S$  is also referred to as the perturbation of  $T$  by  $-BF$ . It is not difficult to verify that  $S(t, 0)\phi$  corresponds to the optimal trajectory for the continuous time infinite horizon problem with initial state  $\phi \in H$ . Similarly, let the discrete time evolution system  $S_h = \{S_h(i, j) : 0 \leq j \leq i < \infty\}$  be defined as

$$(3.6) \quad S_h(i, j) = \begin{cases} I & i = j, \\ \prod_{k=j}^{i-1} \{A_h(k) - hB_h(k)\hat{R}_h(k)^{-1}B_h(k)^*\Pi_{h,\infty}(k+1; 0)A_h(k)\}, & i > j. \end{cases}$$

Thus,  $S_h(k, 0)\phi$  is the optimal trajectory for the discrete time infinite horizon problem with initial state  $\phi \in H$ .

**DEFINITION 3.2.** (Exponential stability of the optimal feedback systems)

- (i) The optimal continuous time feedback system (3.5) is said to be exponentially stable, if there exist constants  $M$  and  $\alpha > 0$  such that for all  $0 \leq s \leq t < \infty$ ,  $\|S(t, s)\|_{L(H)} \leq M \exp\{-\alpha(t-s)\}$ .
- (ii) The discrete time optimal feedback system (3.6) is said to be exponentially stable, if there exist constants  $M_h$  and  $\alpha_h > 0$  such that, for all  $0 \leq j \leq i < \infty$ ,  $\|S_h(i, j)\|_{L(H)} \leq M_h \exp\{-\alpha_h(i-j)h\}$ .

- (iii) The sampled time optimal feedback systems are said to be uniformly exponentially stable for all  $0 < h \leq h_0$ , if the constants  $M_h$  and  $\alpha_h > 0$  in (ii) above are independent of  $h$  for  $0 < h \leq h_0$ .

The following result is an important property of the solutions of the Riccati equations on the infinite time interval when the optimal feedback systems are exponentially stable. The proof can be found in [BW], [DI], [G] for the continuous time problem, and in [GR] (Theorem 2.9) for the discrete time problem.

LEMMA 3.1. *Assume that the continuous time control system and the sampled time control system with sampling period  $h$  are cost functional stabilizable. If the corresponding optimal feedback systems are exponentially stable, then  $\Pi_\infty(\cdot; 0)$ , and  $\Pi_{h,\infty}(\cdot; 0)$  are the unique bounded solutions of the corresponding Riccati equations (3.3) and (3.4) on the infinite time interval. Furthermore, if  $G$  and  $G_h$  are chosen such that  $G \geq \Pi_\infty(t; 0)$  and  $G_h \geq \Pi_{h,\infty}(k; 0)$  for all  $t$  and  $k$ , then the solutions of the Riccati equations on the finite time interval,  $\Pi_{t_f}(t; G)$  and  $\Pi_{h,k_{f,h}}(k; G_h)$ , satisfy*

$$\langle \Pi_{t_f}(t; G)\phi - \Pi_\infty(t; 0)\phi, \phi \rangle_H \leq \langle GS(t_f, t)\phi, S(t_f, t)\phi \rangle_H,$$

and

$$\langle \Pi_{h,k_{f,h}}(k; G_h)\phi - \Pi_{h,\infty}(k; G_h)\phi, \phi \rangle_H \leq \langle G_h S_h(k_{f,h}, k)\phi, S_h(k_{f,h}, k)\phi \rangle_H,$$

respectively, for all  $t \leq t_f$ ,  $k \leq k_{f,h}$ , and  $\phi \in H$ .

LEMMA 3.2. *Assume that the sampled systems are uniformly cost functional stabilizable with the optimal feedback systems uniformly exponentially stable for  $0 < h \leq h_0$ . Then, the operators  $G$  and  $G_h$  can be chosen as described in Lemma 3.1 with  $G_h \leq C \cdot I$  for some constant  $C$  independent of  $h$ . As  $t_f$  tends to infinity,  $\Pi_{t_f}(\cdot; G)$  converges to  $\Pi_\infty(\cdot; 0)$  uniformly on any bounded subinterval  $[a, b]$  of  $[0, \infty)$  and the convergence of  $\tilde{\Pi}_{h,k_{f,h}}(\cdot; G_h)$  with  $k_{f,h} = [t_f/h]$  to  $\tilde{\Pi}_{h,\infty}(\cdot; 0)$  is uniform in  $h$  for all  $0 < h \leq h_0$  on any bounded subinterval  $[a, b]$  of  $[0, \infty)$  in the uniform operator norm.*

*Proof.* We prove only the discrete time assertion. The continuous time case is completely analogous, if not simpler. The assumption of uniform cost functional stabilizability implies that the operators  $G_h$  can be chosen as stated in the theorem. Then let  $M$  and  $\alpha$  be the constants in Definition 3.2-(iii). For a given  $\epsilon > 0$  and  $t \in [a, b]$ , we can take  $t_f$  large enough such that  $CM^2 \exp\{-2\alpha(t_f - t - h_0)\} \leq \epsilon$ . Let  $k_h = [t/h]$ , then  $(k_{f,h} - k_h)h \geq t_f - t - h_0$  for all  $0 < h \leq h_0$ . Since  $\Pi_{h,k_{f,h}}(k_h; G_h) \geq \Pi_{h,\infty}(k_h; 0)$ , we have

$$\begin{aligned} \|\tilde{\Pi}_{h,k_{f,h}}(t; G_h) - \tilde{\Pi}_{h,\infty}(t; 0)\|_{L(H)} &= \|\Pi_{h,k_{f,h}}(k_h; G_h) - \Pi_{h,\infty}(k_h; 0)\|_{L(H)} \\ &= \sup_{\|\phi\|_H \leq 1} \langle (\Pi_{h,k_{f,h}}(k_h; G_h) - \Pi_{h,\infty}(k_h; 0))\phi, \phi \rangle_H \\ &\leq \sup_{\|\phi\|_H \leq 1} \langle G_h S_h(k_{f,h}, k_h)\phi, S_h(k_{f,h}, k_h)\phi \rangle_H \\ &\leq CM^2 e^{-2\alpha(k_{f,h} - k_h)h} \leq \epsilon. \end{aligned}$$



□

As a consequence of the above lemma, we obtain our first convergence result.

**THEOREM 3.2.** *Let Conditions (A1)–(A4) for the operator families  $\{T_h, B_h, Q_h, R_h\}$  hold on any finite subinterval of  $[0, \infty)$ . Assume further that the continuous time system and the sampled time systems with  $0 < h \leq h_0$  are uniformly cost functional stabilizable, and that the optimal closed-loop evolution systems are uniformly exponentially stable. Then, the Riccati operators  $\Pi_{h,\infty}(t; 0)$  converge strongly to  $\Pi_\infty(t; 0)$  and the convergence is uniform on any bounded subinterval of  $[0, \infty)$ .*

*Proof.* Let  $\phi \in H$  and let  $[a, b]$  be a bounded subinterval of  $[0, \infty)$ . We choose an operator  $G$  such that  $G \geq \Pi_\infty(t; 0)$  and  $G \geq \Pi_{h,\infty}(k; 0)$  for all  $t \in [0, \infty) \subset R$ ,  $k \in [0, \infty) \subset Z$  and  $0 < h \leq h_0$ . By Lemma 3.2,  $t_f$  can be taken large enough such that for all  $k_{f,h} = [t_f/h]$ , we have

$$\|\Pi_{t_f}(t; G)\phi - \Pi_\infty(t; 0)\phi\|_H \leq \frac{\epsilon}{3} \quad \text{and} \quad \|\tilde{\Pi}_{h,k_{f,h}}(t; G)\phi - \tilde{\Pi}_{h,\infty}(t; 0)\phi\|_H \leq \frac{\epsilon}{3},$$

for all  $t \in [a, b]$  and all  $0 < h \leq h_0$ . By Theorem 2.4 of Section 2, we can find  $h$  small enough such that

$$\|\tilde{\Pi}_{h,k_{f,h}}(t; G)\phi - \Pi_{t_f}(t; G)\phi\|_H \leq \frac{\epsilon}{3},$$

for all  $t \in [a, b]$ . Therefore, we have

$$\begin{aligned} \|\tilde{\Pi}_{h,\infty}(t; 0)\phi - \Pi_\infty(t; 0)\phi\|_H &\leq \|\tilde{\Pi}_{h,\infty}(t; 0)\phi - \tilde{\Pi}_{h,k_{f,h}}(t; G)\phi\|_H \\ &\quad + \|\tilde{\Pi}_{h,k_{f,h}}(t; G)\phi - \Pi_{t_f}(t; G)\phi\|_H + \|\Pi_{t_f}(t; G)\phi - \Pi_\infty(t; 0)\phi\|_H \leq \epsilon, \end{aligned}$$

for all  $t \in [a, b]$ . □

We note that although the exponential stability of the optimal feedback systems is only a sufficient condition for the uniqueness of the solutions to the Riccati equations, it also implies the stability of the solutions to the Riccati equations under small perturbations (see [BW], [DI]) which is important in the context of approximation. Consequently the remainder of our discussions here are concerned with conditions which guarantee the exponential stability and uniform exponential stability of the optimal feedback systems.

A useful characterization of exponentially stable evolution systems is given in a result due to Datko in the continuous time case (see [D]) and Zabczyk in the discrete time case (see [Z]). We state it here in both its continuous and discrete time forms as a lemma.

**LEMMA 3.3.**

(i) *Let  $T$  be a strongly continuous evolution system. If there exists constants  $C_1, C_2$ , and  $\omega > 0$  such that*

$$\|T(t, s)\|_{L(H)} \leq C_1 e^{\omega(t-s)}, \quad \text{and} \quad \int_s^\infty \|T(t, s)\phi\|_H^2 dt \leq C_2 \|\phi\|_H^2,$$

for all  $\phi \in H$  and  $0 \leq s \leq t < \infty$ , then, we can find constants  $M$  and  $\alpha > 0$ , depending only on  $C_1, C_2$ , and  $\omega$ , such that  $\|T(t, s)\|_{L(H)} \leq M \exp\{-\alpha(t-s)\}$ , for all  $0 \leq s \leq t < \infty$ .

(ii) Let  $T_h$  be the discrete time evolution system defined by

$$T_h(i, j) = \begin{cases} I, & i = j, \\ \prod_{k=j}^{i-1} A_h(k), & i > j. \end{cases}$$

If there exist constants  $C_{1,h}, \omega_h$  and  $C_{2,h}$  such that

$$\|T_h(i, j)\|_{L(H)} \leq C_{1,h} e^{\omega_h(i-j)h}, \quad \text{and} \quad h \sum_{i=k}^{\infty} \|T_h(i, k)\phi\|_H^2 \leq C_{2,h} \|\phi\|_H^2,$$

for all  $0 \leq k < \infty$  and  $\phi \in H$ , then, we can find constants  $M_h$  and  $\alpha_h > 0$ , depending only on  $C_{1,h}, \omega_h$  and  $C_{2,h}$ , such that for all  $0 \leq j \leq i < \infty$

$$\|T_h(i, j)\|_{L(H)} \leq M_h e^{-\alpha_h(i-j)h}.$$

The converse of this lemma is obviously true. By the uniform cost functional stabilizability, the solutions of the Riccati equations (3.3) and (3.4),  $\Pi_\infty$  and  $\Pi_{h,\infty}$  are uniformly bounded, and therefore, the evolution systems  $S$  and  $S_h$  given by (3.5) and (3.6), respectively, are uniformly exponentially bounded if  $T$  and  $T_h$  are uniformly exponentially bounded. Moreover, we have

$$\int_s^\infty \|Q(t)^{1/2} S(t, s)\phi\|_H^2 dt \leq M \|\phi\|_H^2$$

and

$$h \sum_{k=j}^{\infty} \|Q_h(k)^{1/2} S_h(k, j)\phi\|_H^2 \leq M \|\phi\|_H^2,$$

for some constant  $M$  and for all  $\phi \in H$ . If the operators  $Q(t)$  and  $Q_h(k)$  are uniformly strictly coercive (i.e., there exists a constant  $q > 0$ , such that  $Q(t) \geq qI$  and  $Q_h(k) \geq qI$ , for  $t \geq 0$  and  $k \geq 0$ ), we can immediately conclude that  $S$  and  $S_h$  are uniformly exponentially stable.

A more general case in which the boundedness of the cost functional implies the stability of the feedback system, is the case of detectable systems.

**DEFINITION 3.3.** (Detectability)

- (i) A continuous time control system is said to be detectable with respect to the cost functional (3.1), if there exists a bounded operator-valued function  $V(\cdot) : [0, \infty) \mapsto L(H)$  such that the evolution system  $T_V$ , corresponding to the perturbation of  $T$  by  $VQ^{1/2}(\cdot)$ , is exponentially stable.
- (ii) A sampled time control system is said to be detectable with respect to the cost functional (3.2), if there exists a bounded sequence of operators  $\{V_h(k)\}_{k=0}^\infty \subset L(H)$  such that the discrete time evolution system  $T_{V,h}$  given by

$$T_{V,h}(i, j) = \begin{cases} I, & i = j, \\ \prod_{k=j}^{i-1} (A_h(k) + hV_h(k)Q_h(k)^{1/2}), & i > j, \end{cases}$$

is exponentially stable.

- (iii) The sampled time systems are said to be uniformly detectable for  $0 < h \leq h_0$ , if there exist constants  $C_1, C_2$ , and  $\alpha > 0$ , independent of  $h$  such that the operator-valued sequences  $\{V_h(k)\}_{k=0}^\infty$  in (ii) satisfy  $\|V_h(k)\|_{L(H)} \leq C_1$  and

$$\|T_{V,h}(i,j)\|_{L(H)} \leq C_2 e^{-\alpha(i-j)h}, \quad 0 \leq j \leq i < \infty,$$

for all sampling rates  $0 < h \leq h_0$ .

Under appropriate conditions, the detectability of the control systems implies the stability of the optimal feedback systems. Indeed, toward this end, we require the following boundedness assumption.

- (B) The continuous time evolution system  $T$  and the discrete time evolution system  $T_h$  are uniformly exponentially bounded on  $\Delta(2, \infty)$ . That is, there exist constants  $M$  and  $\omega$  such that

$$\|T(t,s)\|_{L(H)} \leq M e^{\omega(t-s)}, \quad \|T_h(i,j)\|_{L(H)} \leq M e^{\omega(i-j)h},$$

for  $0 \leq s \leq t < \infty$  and  $0 \leq j \leq i < \infty$ . The operator families  $B, Q$ , and  $R$  and the piecewise constant operator families  $\tilde{B}_h, \tilde{Q}_h$ , and  $\tilde{R}_h$  are uniformly bounded in norm by a given constant  $C$  on the entire interval  $[0, \infty)$  for all sampling rates  $h > 0$ . Furthermore, there exists a constant  $r > 0$  such that  $R(t) \geq rI$  and  $\tilde{R}_h(t) \geq rI$  for all  $t \geq 0$ , and  $h > 0$ .

**THEOREM 3.3.** *Consider a detectable continuous time control system and a detectable sampled time system which are both cost functional stabilizable. Assume that the evolution systems  $T, T_h$  are exponentially bounded, and the operator families  $\{B, Q, R\}$  and  $\{B_h, Q_h, R_h\}$  are bounded in norm on the infinite time interval. Then, the optimal feedback systems for both systems are exponentially stable. Furthermore, suppose that constants  $C, \omega, r > 0$ , and  $\alpha > 0$  can be found such that the following conditions are satisfied.*

- (i) *The operator families  $\{B, Q, R, \Pi_\infty(\cdot; 0), V\}$  and  $\{B_h, Q_h, R_h, \Pi_{h,\infty}(\cdot; 0), V_h\}$  are bounded in norm by  $C$ ;*
- (ii) *For all  $t \geq 0, k \geq 0$ ,  $R(t) \geq rI$  and  $R_h \geq rI$ ;*
- (iii) *The evolution systems  $T, T_h, T_V$ , and  $T_{V,h}$  satisfy*

$$\|T(t,s)\|_{L(H)} \leq C e^{\omega(t-s)}, \quad \|T_V(t,s)\|_{L(H)} \leq C e^{-\alpha(t-s)},$$

and

$$\|T_h(i,j)\|_{L(H)} \leq C e^{\omega(i-j)h}, \quad \|T_V(i,j)\|_{L(H)} \leq C e^{-\alpha(i-j)h}.$$

*Then there exists constants  $M$  and  $\beta > 0$  depending only on the constants  $C, r, \alpha$  and  $\omega$  such that*

$$\|S(t,s)\|_{L(H)} \leq M e^{-\beta(t-s)}, \quad \|S_h(i,j)\|_{L(H)} \leq M e^{-\beta(i-j)h}.$$

Moreover, under Assumption (B), if the sampled systems are uniformly detectable and uniformly cost functional stabilizable for  $0 < h \leq h_0$ , then, the optimal closed-loop systems are uniformly exponentially stable for  $0 < h \leq h_0$ .

*Proof.* In the case of continuous time system, a proof is given by Da Prato and Ichikawa in [DI]. The dependence of the exponential bound for the optimal closed-loop system on the constants indicated above is proved in [W]. The arguments for the discrete time case are very similar to those used in the continuous time case. Indeed, let  $S_h$  correspond to the perturbation of  $T_{V,h}$  by  $\Delta_h = \{\Delta_h(k) = -B_h(k)F_h(k) + V_h(k)Q_h(k)^{1/2}\}$  in the sense that

$$\tilde{S}_h(t, s)\phi = \tilde{T}_{V,h}(t, s)\phi + \int_{[s/h]}^{[t/h]} \tilde{T}_{V,h}(t, \tau) \tilde{\Delta}_h(\tau) \tilde{S}_h(\tau, s)\phi d\tau.$$

Let us define

$$f_h(k, i) = -R_h(k)^{1/2}F_h(k)\phi, \quad \text{and} \quad g_h(k, i) = Q_h(k)^{1/2}S_h(k, i)\phi,$$

for  $k \geq i \geq 0$ . Then cost functional stabilizability implies that

$$\|f_h(\cdot, i)\|_{l_2(i, \infty; U)} \leq C\|\phi\|_H, \quad \|g_h(\cdot, i)\|_{l_2(i, \infty; H)} \leq C\|\phi\|_H.$$

The evolution system  $T_{V,h}$  is bounded;  $\|T_{V,h}(i, j)\|_{L(H)} \leq C \exp\{-\alpha(i-j)h\}$ . Thus we obtain

$$\begin{aligned} \|\tilde{S}_h(t, s)\phi\|_H &\leq C e^{-\alpha(t-s)} + \int_{[s/h]}^{[t/h]} C e^{-\alpha(t-\tau)} (\|\tilde{B}_h(\tau) \tilde{R}_h(\tau)^{-1/2}\|_{L(U, H)} \|\tilde{f}_h(\tau, s)\|_U \\ &\quad + \|\tilde{V}_h(\tau)\|_{L(H)} \|\tilde{g}_h(\tau, s)\|_H) d\tau, \end{aligned}$$

and by Young's inequality (see, [A, Theorem 4.30, p.90]), we have

$$\int_s^\infty \|\tilde{S}_h(t, s)\phi\|_H^2 dt \leq K \|\phi\|_H^2, \phi \in H,$$

for some constant  $K$ . Applying Lemma 3.3, we obtain the exponential stability of  $S_h$ . The dependence of the exponential bound for  $S_h$  on the indicated constants of course follows from the dependence of the constant  $K$  on the indicated constants as prescribed in the lemma. In this way it is easy to see how under Assumption (B), uniform detectability and cost function stabilizability will imply the uniform exponential stability of the closed-loop systems.  $\square$

Another closely related control theoretic concept is the stabilizability.

**DEFINITION 3.4.** (Stabilizability)

- (i) A continuous time system is said to be stabilizable, if there exists a bounded operator-valued function  $K(\cdot) : [t_0, \infty) \mapsto L(H, U)$  such that the evolution system  $T_K$  corresponding to the perturbation of  $T$  by  $BK$  is exponentially stable.

- (ii) A sampled system is said to be stabilizable, if there exists a bounded sequence of operators  $\{K_h(k)\}_{k=0}^{\infty} \subset L(H, U)$  such that the discrete evolution operator  $T_{K,h}$  given by

$$T_{K,h}(i, j) = \begin{cases} I, & i = j, \\ \prod_{k=j}^{i-1} (A_h(k) + hB_h(k)K_h(k)), & i > j, \end{cases}$$

is exponentially stable.

- (iii) The sampled time systems for are said to be uniformly stabilizable for  $0 < h \leq h_0$  if there exist constants  $C_1, C_2, \alpha > 0$  independent of the sampling period  $h$ , such that  $K_h$  and  $T_{K,h}$  satisfy

$$\|K_h(k)\|_{L(H,U)} \leq C_1, \quad \|T_{K,h}(i, j)\|_{L(H)} \leq C_2 e^{-\alpha(i-j)h},$$

for all  $0 \leq k < \infty, 0 \leq j \leq i < \infty$ .

Using Theorem 3.3 it is easy to verify that cost functional stabilizability and detectability imply stabilizability (take  $K = F, K_h = F_h$ , for example). Conversely, stabilizability clearly implies cost functional stabilizability. Therefore, under the uniform detectability assumption, cost functional stabilizability and stabilizability are equivalent. In general, uniform stabilizability and uniform detectability are required for the convergence of  $\Pi_{h,\infty}$  to  $\Pi_{\infty}$  as  $h$  tends toward zero.

**THEOREM 3.4.** *Let Assumption (B) hold. Suppose further that Conditions (A1)–(A4) hold on any bounded subinterval of  $[0, \infty)$ . If the continuous time system and the sampled time systems are uniformly stabilizable and uniformly detectable, then, the unique solution  $\Pi_{h,\infty}$  of the infinite horizon Riccati difference equation (3.4) converges to the solution  $\Pi_{\infty}$  of the infinite horizon Riccati integral equation (3.3) as  $h$  tends toward zero. The convergence is uniform in time on any bounded subinterval of  $[0, \infty)$ .*

*Proof.* By Theorem 3.3, uniform stabilizability and uniform detectability imply exponential stability of the optimal feedback systems (i.e., Definition 3.2), uniformly over all sampled systems with  $0 < h \leq h_0$ . Therefore, by Theorem 3.2, we obtain the desired convergence.  $\square$

Most control systems of interest in engineering practice are stabilizable and detectable. In fact, in modeling many control systems of practical interest, a realistic description of the physical system frequently necessitates stabilizability and detectability of the system model (see, for example, [BKS], [BKSW]). Investigation of stabilizability and detectability of particular classes of evolution systems has generated several interesting mathematical problems (see, for example, [C], [L]). However, in the context of approximation, we usually assume that the original control system is stabilizable and detectable. An important issue here is whether or not a given time discretization algorithm is capable of preserving, uniformly, these properties, and therefore provide discrete time convergent approximations for the optimal feedback operators. In the remainder of this section, we attempt to address this issue for

some particular discretization algorithms and indicate some other possible approaches to this problem.

Assume that the control system defined in Equation (2.1) is stabilizable and detectable with respect to the cost functional (3.1). Thus, there exist bounded operator-valued functions  $K(\cdot) : [t_0, \infty) \mapsto L(H; U)$  and  $V(\cdot) : [t_0, \infty) \mapsto L(H)$  such that the evolution systems  $T_K, T_V$ , corresponding to the perturbations of  $T$  by  $BK$  and  $VQ^{1/2}$ , respectively, are exponentially stable. That is, there exist constants  $M, \alpha > 0$  such that  $\|T_K(t, s)\|_{L(H)} \leq M \exp\{-\alpha(t - s)\}$  and  $\|T_V(t, s)\|_{L(H)} \leq M \exp\{-\alpha(t - s)\}$ , for all  $0 \leq s \leq t < \infty$ . By definition, the evolution operators  $T_K$ , and  $T_V$  satisfy

$$(3.7) \quad T_K(t, s)\phi = T(t, s)\phi + \int_s^t T(t, \eta)B(\eta)K(\eta)T_K(\eta, s)\phi d\eta,$$

$$(3.8) \quad T_V(t, s)\phi = T(t, s)\phi + \int_s^t T(t, \eta)V(\eta)Q^{1/2}(\eta)T_V(\eta, s)\phi d\eta,$$

for all  $\phi \in H$  and for all  $0 \leq s \leq t < \infty$ . Consider the zero-order hold discretization described in the Section 2. For each  $k \geq 0$ , the operators  $A_h(k), B_h(k)$  are defined by

$$(3.9) \quad A_h(k) = T((k+1)h, kh),$$

$$(3.10) \quad B_h(k) = \frac{1}{h} \int_{kh}^{(k+1)h} T((k+1)h, \eta)B(\eta)d\eta,$$

with the discrete evolution systems  $T_{K,h}, T_{V,h}$  then given by

$$(3.11) \quad T_{K,h}(i, j) = \begin{cases} I, & i = j, \\ \prod_{k=j}^{i-1} \{A_h(k) + hB_h(k)K(kh)\}, & i > j, \end{cases}$$

$$(3.12) \quad T_{V,h}(i, j) = \begin{cases} I, & i = j, \\ \prod_{k=j}^{i-1} \{A_h(k) + hV(kh)Q_h(k)^{1/2}\}, & i > j. \end{cases}$$

If the discrete time evolution systems  $T_{K,h}, T_{V,h}$  are uniformly exponentially stable for all  $0 < h \leq h_0$  for some  $h_0 > 0$ , then, these sampled time systems are uniformly stabilizable and uniformly detectable. Using (3.7) and (3.8), the evolution systems  $T_K$ , and  $T_V$  satisfy

$$T_K(ih, jh) = \prod_{k=j}^{i-1} \left[ T((k+1)h, kh) + \int_{kh}^{(k+1)h} T((k+1)h, \eta)B(\eta)K(\eta)T_K(\eta, kh)d\eta \right],$$

and

$$T_V(ih, jh) = \prod_{k=j}^{i-1} \left[ T((k+1)h, kh) + \int_{kh}^{(k+1)h} T((k+1)h, \eta)V(\eta)Q(\eta)^{1/2}T_V(\eta, kh)d\eta \right],$$

for  $0 \leq j \leq i < \infty$ . Therefore,  $T_{K,h}$ , and  $T_{V,h}$  can be considered as perturbations of  $T_K$  and  $T_V$ , respectively. In fact, we have

$$(3.13) \quad T_{K,h}(i, j) = \prod_{k=j}^{i-1} \{T_K((k+1)h, kh) + h\Phi_h(k)\},$$

$$(3.14) \quad T_{V,h}(i, j) = \prod_{k=j}^{i-1} \{T_V((k+1)h, kh) + h\Psi_h(k)\},$$

for  $0 \leq j < i < \infty$ , where

$$\Phi_h(k) = \frac{1}{h} \left( \int_{kh}^{(k+1)h} T((k+1)h, \eta) B(\eta) [K(kh) - K(\eta) T_V(\eta, kh)] d\eta \right),$$

and

$$\Psi_h(k) = \left( V(kh) Q_h(k)^{1/2} - \frac{1}{h} \int_{kh}^{(k+1)h} T((k+1)h, \eta) V(\eta) Q(\eta)^{1/2} T_V(\eta, kh) d\eta \right),$$

for  $k \geq 0$ . Let  $0 < \omega \leq \alpha$  and define  $\hat{T}_{K,h}(i, j) = \exp\{\omega(i-j)h\} T_{K,h}(i, j)$ ,  $0 \leq j \leq i < \infty$  and  $\hat{T}_K(t, s) = \exp\{\omega(t-s)\} T_K(t, s)$ ,  $0 \leq s \leq t < \infty$ . We define  $\hat{T}_V$  and  $\hat{T}_{V,h}$  analogously. It is not difficult to verify that

$$\|\hat{T}_K(t, s)\|_{L(H)} \leq M, \quad \|\hat{T}_V(t, s)\|_{L(H)} \leq M.$$

Multiplying both sides of (3.13) and (3.14) by  $\exp\{\omega(i-j)h\}$ , and rewriting these equations in a variation of constants form, we obtain

$$\begin{aligned} \hat{T}_{K,h}(i, j) &= \hat{T}_K(ih, jh) + h \sum_{k=j}^{i-1} \hat{T}_K(ih, (k+1)h) e^{\omega h} \Phi_h(k) \hat{T}_{K,h}(k, j), \\ \hat{T}_{V,h}(i, j) &= \hat{T}_V(ih, jh) + h \sum_{k=j}^{i-1} \hat{T}_V(ih, (k+1)h) e^{\omega h} \Psi_h(k) \hat{T}_{V,h}(k, j). \end{aligned}$$

If there exists a constant  $h_0 > 0$  such that for all  $h \leq h_0$ ,  $\exp\{\omega h\} \|\Phi_h(k)\|_{L(H)} \leq \omega/2M$  and  $\exp\{\omega h\} \|\Psi_h(k)\|_{L(H)} \leq \omega/2M$ , then,

$$\begin{aligned} \|\hat{T}_{K,h}(i, j)\|_{L(H)} &\leq M + h \sum_{k=j}^{i-1} M \cdot \frac{\omega}{2M} \|\hat{T}_{K,h}(k, j)\|_{L(H)}, \\ \|\hat{T}_{V,h}(i, j)\|_{L(H)} &\leq M + h \sum_{k=j}^{i-1} M \cdot \frac{\omega}{2M} \|\hat{T}_{V,h}(k, j)\|_{L(H)}. \end{aligned}$$

The discrete Gronwall inequality then yields

$$\begin{aligned} \|\hat{T}_{K,h}(i, j)\|_{L(H)} &\leq M e^{\omega(i-j)h/2}, \\ \|\hat{T}_{V,h}(i, j)\|_{L(H)} &\leq M e^{\omega(i-j)h/2}. \end{aligned}$$

Therefore,  $T_{K,h}$  and  $T_{V,h}$  are uniformly exponentially stable for all  $0 < h \leq h_0$ .

It is not difficult to see that for each  $k \geq 0$ ,  $\Phi_h(k)$  and  $\Psi_h(k)$  converge strongly to zero as  $h$  tends toward zero. We can obtain convergence in norm if the rank of the operator valued functions  $\Phi_h(k)$  and  $\Psi_h(k)$  is finite.

**DEFINITION 3.5.** (Finite rank operator-valued function) Let  $X$  and  $Y$  be Hilbert spaces with inner products  $\langle \cdot, \cdot \rangle_X$  and  $\langle \cdot, \cdot \rangle_Y$ , respectively. An operator-valued function  $W(\cdot) : [0, \infty) \mapsto L(X, Y)$  is said to be continuous and to have finite rank, if

there exist continuous vector-valued functions  $f_k(\cdot) : [0, \infty) \mapsto X$  and  $g_k(\cdot) : [0, \infty) \mapsto Y$ ,  $k = 1, \dots, n$  with  $n < \infty$ , such that for all  $x \in X$ ,

$$W(t)x = \sum_{k=1}^n \langle f_k(t), x \rangle_Y g_k(t).$$

We define the following condition.

(F) (Finite Rank Stabilizability and Detectability Condition) There exist finite rank continuous operator-valued functions  $K(\cdot), V(\cdot)$  such that the perturbed evolution systems  $T_K$  and  $T_V$  are exponentially stable.

LEMMA 3.4. *Suppose that Conditions (A1)–(A4) hold. If the finite rank condition (F) is satisfied, then on any finite subinterval of  $[0, \infty)$ , the operator-valued functions  $\tilde{\Phi}_h$ , and  $\tilde{\Psi}_h$  constructed from  $\Phi_h$  and  $\Psi_h$  in the usual manner, converge uniformly to zero in the uniform operator norm as  $h$  tends toward zero.*

*Proof.* We consider  $\tilde{\Psi}_h$  only, the argument for  $\tilde{\Phi}_h$  is analogous. Using the finite rank condition, we write

$$V(t)\phi = \sum_{k=1}^n \langle f_k(t), \phi \rangle_H g_k(t),$$

with  $f_k$  and  $g_k$  continuous for  $k = 1, \dots, n$ . It follows that

$$V(ih)Q_h(i)^{1/2}\phi = \sum_{k=1}^n \langle Q_h(i)^{1/2}f_k(ih), \phi \rangle_H g_k(ih),$$

for  $i \geq 0$ , and

$$T(t, \eta)V(\eta)Q(\eta)^{1/2}T_V(\eta, s)\phi = \sum_{k=1}^n \langle T_V(\eta, s)^*Q(\eta)^{1/2}f_k(\eta), \phi \rangle_H T(t, \eta)g_k(\eta),$$

for  $0 \leq s \leq \eta \leq t < \infty$ . Therefore, we have

$$\begin{aligned} \tilde{\Psi}_h(t)\phi &= V([t/h]h)\tilde{Q}_h(t)^{1/2}\phi \\ &\quad - \frac{1}{h} \int_{[t/h]h}^{([t/h]+1)h} T([t/h] + 1)h, \eta) V(\eta)Q(\eta)^{1/2}T_V(\eta, [t/h]h)\phi d\eta \\ &= \frac{1}{h} \sum_{k=1}^n \int_{[t/h]h}^{([t/h]+1)h} \left\{ \langle \tilde{Q}_h(t)^{1/2}f_k([t/h]h), \phi \rangle_H g_k([t/h]h) \right. \\ &\quad \left. - \langle T_V(\eta, [t/h]h)^*Q(\eta)^{1/2}f_k(\eta), \phi \rangle_H T([t/h] + 1)h, \eta)g_k(\eta) \right\} d\eta. \end{aligned}$$

By adding and subtracting the term  $\langle T_V(\eta, [t/h]h)^*Q(\eta)^{1/2}f_k(\eta), \phi \rangle_H g_k([t/h]h)$  under each of the above integral signs, and using the Schwartz inequality, we obtain



the following estimate

$$\begin{aligned}
\|\tilde{\Psi}_h(t)\phi\|_H &\leq \frac{1}{h} \sum_{k=1}^n \int_{[t/h]h}^{([t/h]+1)h} \{w_h(t, \eta) \|g([t/h]h)\|_H \\
&\quad + v_h(t, \eta) \|T_V(\eta, [t/h]h)^* Q(\eta)^{1/2} f_k(\eta)\|_H\} \|\phi\|_H d\eta \\
&= \frac{1}{h} \sum_{k=1}^n \int_{[t/h]h}^{([t/h]+1)h} u_h(t, \eta) \|\phi\|_H d\eta
\end{aligned}$$

where

$$\begin{aligned}
w_h(t, \eta) &= \|\tilde{Q}_h(t)^{1/2} f_k([t/h]h) - T_V(\eta, [t/h]h)^* Q(\eta)^{1/2} f_k(\eta)\|_H, \\
v_h(t, \eta) &= \|g([t/h]h) - T([t/h] + 1, \eta)g(\eta)\|_H, \quad \text{and} \\
u_h(t, \eta) &= w_h(t, \eta) \|g([t/h]h)\|_H + v_h(t, \eta) \|T_V(\eta, [t/h]h)^* Q(\eta)^{1/2} f_k(\eta)\|_H.
\end{aligned}$$

Since the functions  $f_k$  and  $g_k$  are continuous on any bounded subinterval  $[a, b]$  of  $[0, \infty)$ , and for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\|f_k(t) - f_k(s)\|_H \leq \epsilon$  and  $\|g_k(t) - g_k(s)\|_H \leq \epsilon$  for all  $t, s \in [a, b]$  with  $|t - s| \leq \delta$  and  $k = 1, \dots, n$ . Then, the boundedness of the operator families  $T, T_V, V, Q_h, Q$  and the uniform strong convergence of  $\tilde{Q}_h$  to  $Q$  implies that for any bounded subinterval  $[a, b]$  of  $[0, \infty)$ , and for any given constant  $\epsilon > 0$ , we can find  $h_0 > 0$  such that for all  $0 < h \leq h_0$  and  $t \in [a, b]$ , the functions  $u_h(t, \eta) \leq \epsilon$  for  $\eta \in [t, t + h]$  and  $t \in [a, b]$ . Consequently  $\|\tilde{\Psi}_h(t)\|_{L(H)} \leq \epsilon$  for all  $t \in [a, b]$ .  $\square$

We can extend the uniform convergence on finite time intervals to uniform convergence on the infinite time interval by assuming certain periodicity (in particular time invariance) of the evolution system  $T$  and the operator-valued functions  $B, Q, \tilde{Q}_h, K$  and  $V$ . In fact, the periodicity assumption implies that  $\tilde{\Phi}_h, \tilde{\Psi}_h$  are also periodic functions of time.

**THEOREM 3.5.** *Assume that the evolution system  $T$  and the operator-valued functions  $B, Q, R$  are strongly continuous and periodic with the same period,  $\theta$ . Suppose further that the periodicity of  $Q$  is preserved by  $\tilde{Q}_h$  for the sampled time systems. If the finite rank condition (F) holds for some  $\theta$ -periodic functions  $K$  and  $V$ , then the discretization defined in (3.9) and (3.10) generates uniformly stabilizable and uniformly detectable sampled control systems for sampling periods  $h$  with  $0 < h \leq h_0$  for some constant  $h_0 > 0$ .*

The periodicity assumption is trivially satisfied in a large number of practical examples, in particular, it is satisfied for all time invariant systems. However, the finite rank assumption says, in essence, that only a finite number of modes of the state vector are unstable in the absence of control. Indeed, in the case of evolution systems corresponding to a hyperbolic partial differential equation, there exists examples in which if the finite rank condition is not satisfied, all sampled systems are not stabilizable even though the continuous time control system is stabilizable. For parabolic, compact, evolution systems, the spectral properties of the evolution system provide

valuable additional structure. In this case, an approach which does not require the finite rank condition, similar to the one used in [R], can be applied. The results using this type of argument will be reported on elsewhere. However, even in the case of parabolic, compact, evolution systems, since the unstable spectrum consists of only a finite number of isolated points, it would be interesting to know whether these systems can be stabilized via finite rank feedback. If the answer is affirmative, then the arguments presented here may not be as restrictive as they seem. For other discretization schemes, the uniform stabilizability and uniform detectability of the generated sampled systems remains, in most cases, an open question.

**4. Examples and Numerical Results.** In this section we present and briefly discuss some of our numerical findings which serve to illustrate our convergence results in the context of a variety of distributed parameter control systems. In particular, we consider the infinite horizon optimal control or regulation of a heat or diffusion equation, a delay or hereditary system, and a flexible structure in the form of a cantilevered Voigt-Kelvin viscoelastic beam with tip mass.

In all of the examples to follow, we consider time invariant systems only, and obtain the discrete or sampled time operators from the corresponding continuous time operators via  $T_h = T(h)$ ,  $B_h = h^{-1} \int_0^h T(t)Bdt$ ,  $Q_h = Q$ , and  $R_h = R$ , for  $h > 0$  (i.e., via zero-order hold sampling). In order to solve the resulting infinite dimensional continuous and discrete time LQR problems, we introduced some form of state discretization (i.e. either modal or spline based Ritz-Galerkin techniques) which were known to yield convergence in the closed-loop problem. By choosing the state discretization sufficiently fine, we could assume that we obtained a reasonably accurate finite dimensional approximation to the solution of the infinite dimensional LQR problems.

The resulting finite dimensional continuous and discrete time LQR problems (more precisely, the matrix algebraic Riccati equations) were solved using either eigenvector (in the continuous time case, also known as Potter's method, see [KS]) or Schur vector (for the discrete time problems, see [PLS]) decomposition of the Hamiltonian matrix. All computations for the first two examples were carried out on an IBM PC AT. The flexible structure problem was solved on an IBM3090, although it too could have been solved on a personal computer.

In each of the examples below, the control systems are time invariant and the control space  $U$  is finite dimensional. In fact,  $U = R$ . Thus, the optimal feedback gains,  $F$  and  $F_h$ , are elements in  $L(H, R)$ . That is, they are bounded linear functionals on  $H$ . Consequently, they admit representors, respectively  $f$  and  $f_h$ , in  $H$  with  $F\varphi = \langle f, \varphi \rangle_H$  and  $F_h\varphi = \langle f_h, \varphi \rangle_H$ , for  $\varphi \in H$ . The elements  $f$  and  $f_h$  in  $H$  are referred to as the optimal continuous or discrete time functional feedback control gains. The finite dimensionality of the control space  $U$  also implies the uniform stabilizability of the sampled systems when the continuous time systems are stabilizable (recall Theorem 3.5). Our convergence result implies that  $\lim_{h \rightarrow 0^+} F_h\varphi = F\varphi$  for  $\varphi \in H$ . Note that when  $U$  is finite dimensional, this is equivalent to  $\lim_{h \rightarrow 0^+} F_h = F$  in the uniform norm topology on  $L(H, U)$  and  $\lim_{h \rightarrow 0^+} f_h = f$  in  $H$ . It is this latter

convergence of the functional gains which we shall exhibit in our plots below.

**Example 4.1.** We consider the scalar or one dimensional heat or diffusion control system

$$\frac{\partial}{\partial t}x(t, \eta) = a \frac{\partial^2}{\partial \eta^2}x(t, \eta) + b\chi_{[\epsilon_1, \epsilon_2]}(\eta)u(t), \quad 0 < \eta < 1, t > 0,$$

with the Dirichlet boundary conditions

$$x(t, 0) = x(t, 1) = 0, \quad t > 0,$$

at  $\eta = 0$  and  $\eta = 1$  where  $a > 0, b \in R, 0 \leq \epsilon_1 < \epsilon_2 \leq 1$ , and  $\chi$ , denotes the characteristic function on the set  $S$ . We take the performance index to be

$$J(u) = \int_0^\infty \left\{ \int_0^1 q x(t, \eta)^2 d\eta + r u(t)^2 \right\} dt$$

with  $q \geq 0$  and  $r > 0$ .

In this case we have  $H = L_2(0, 1), U = R, A : \text{Dom}(A) \subset H \mapsto H$  given by

$$A\varphi = aD^2\varphi \text{ for } \varphi \in \text{Dom}(A) = H^2(0, 1) \cap H_0^1(0, 1),$$

$B \in L(R, H)$  given by  $(Bv)(\eta) = b\chi_{[\epsilon_1, \epsilon_2]}(\eta)v, 0 < \eta < 1, v \in R, Q \in L(H)$  given by  $Q = qI$ , and  $R \in L(U)$  given by  $R = rI$ , where  $I$  denotes the identity map on  $R$ . We note that  $\{T(t) : t \geq 0\}$ , the semigroup of bounded linear operators on  $H$  with infinitesimal generator  $A$ , is parabolic and uniformly exponentially stable. Thus the continuous time pairs,  $\{A, B\}$  and  $\{Q, A\}$  are trivially stabilizable and detectable and the discrete time pairs,  $\{T_h, B_h\}$  and  $\{Q_h, T_h\}$  are uniformly stabilizable and detectable as well.

Setting  $a = 0.1, b = 1.0, q = 1.0, r = 1.0, \epsilon_1 = 0.21$ , and  $\epsilon_2 = 0.275$ , we obtained the plot of the functional gains  $f$  and  $f_h$  in  $L_2(0, 1)$ , for various values of  $h > 0$ , given in Figures 4.1 and 4.2. Those in Figure 4.1 were obtained via a modal (i.e.  $\sin(k\pi x), k = 1, 2, \dots, N$ ) state discretization with  $N = 20$  modal elements. For the gains in Figure 4.2, we used linear B-spline elements (i.e. “hat” functions) defined with respect to a uniform partition of  $[0, 1]$  into  $N = 20$  subintervals of equal length. Convergence of these state approximations and the corresponding closed-loop solutions to the control problem is well known (see, for example, [G], [GR]).

**Example 4.2.** In this example, we consider the scalar, single input hereditary control system

$$(4.1) \quad \dot{x}(t) = a_0x(t) + a_1x(t-1) + bu(t)$$

where  $a_0, a_1, b \in R$ . We take the performance index to be

$$J(u) = \int_0^\infty \{qx^2(t) + ru^2(t)\} dt$$

with  $q \geq 0$  and  $r > 0$ .

FIG. 4.1. Functional gains for heat equation with modal approximation

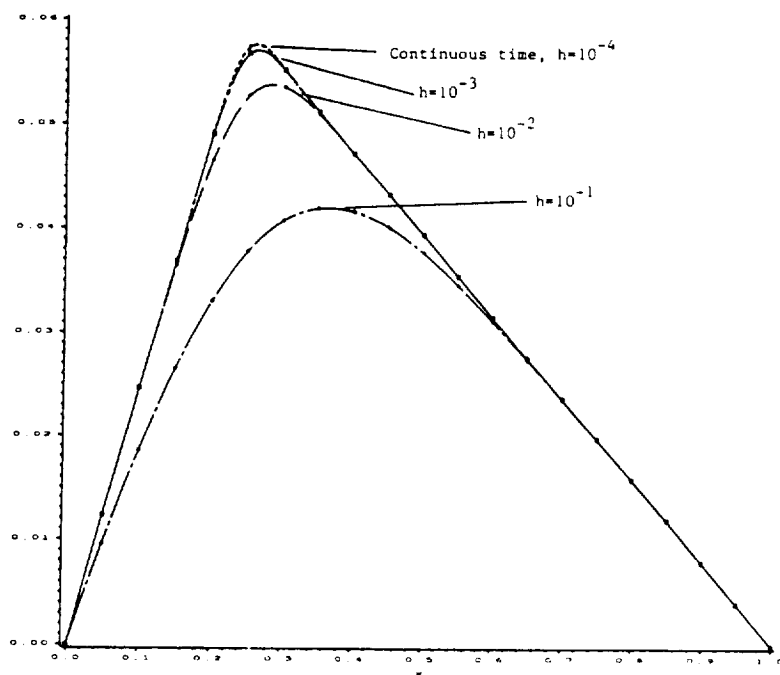


FIG. 4.2. Functional gains for heat equation with spline approximation

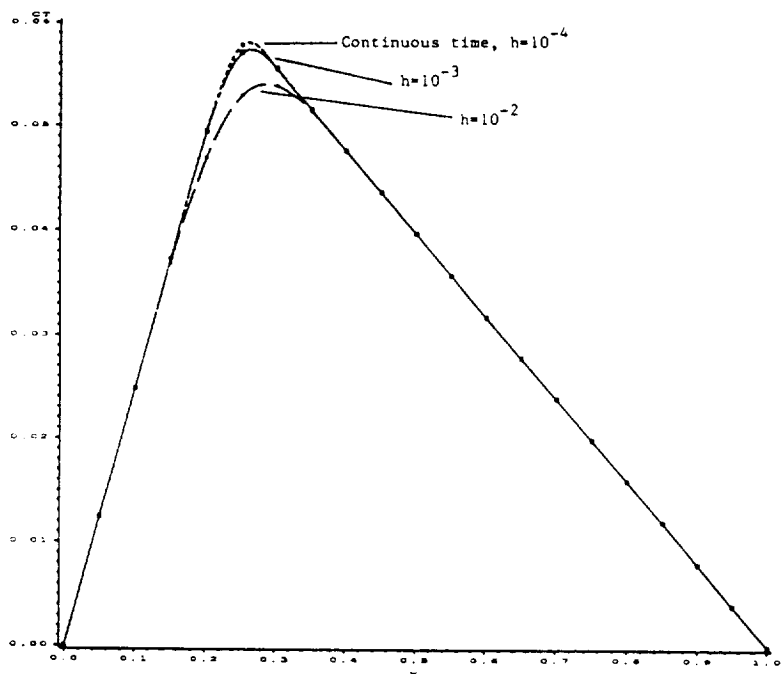


TABLE 1  
Head gains for hereditary system.

Sampling period $h$	Head gain $f^0$
$10^{-1}$	3.76185
$10^{-2}$	4.35007
$10^{-3}$	4.41577
$10^{-4}$	4.42241
$10^{-5}$	4.42308
$10^{-6}$	4.42314
Continuous time	4.42315

The abstract Hilbert space formulation for linear hereditary control systems is well known (see, for example, [BB]). We let  $H = R \times L_2(-1, 0)$ ,  $U = R$  and set  $A : \text{Dom}(A) \subset H \mapsto H$  to be  $A(\eta, \varphi) = (a_0\eta + a_1\varphi(-1), D\varphi)$  for  $(\eta, \varphi) \in \text{Dom}(A) = \{(\xi, \psi) \in H : \psi \in H^1(-1, 0), \xi = \psi(0)\}$ . The operator  $A$  is the infinitesimal generator of the  $C_0$ -semigroup of bounded linear operators on  $H$ ,  $\{T(t) : t \geq 0\}$ , given by  $T(t)(\eta, \varphi) = (x(t), x_t)$  where  $x$  is the solution to (4.1) with  $u \equiv 0$  and corresponding to the initial data  $x(0) = \eta$ ,  $x(\theta) = \varphi(\theta)$ ,  $-1 \leq \theta \leq 0$ , and  $x_t \in L_2(-1, 0)$  is the past history of  $x$  from  $t$  back to  $t - 1$ . That is  $x_t(\theta) = x(t + \theta)$ ,  $-1 \leq \theta \leq 0$ . We let  $B \in L(R, H)$ ,  $Q \in L(H)$ , and  $R \in L(U)$  be given by  $Bv = (bv, 0)$ ,  $Q(\eta, \varphi) = (q\eta, 0)$ , and  $Rv = rv$ , respectively.

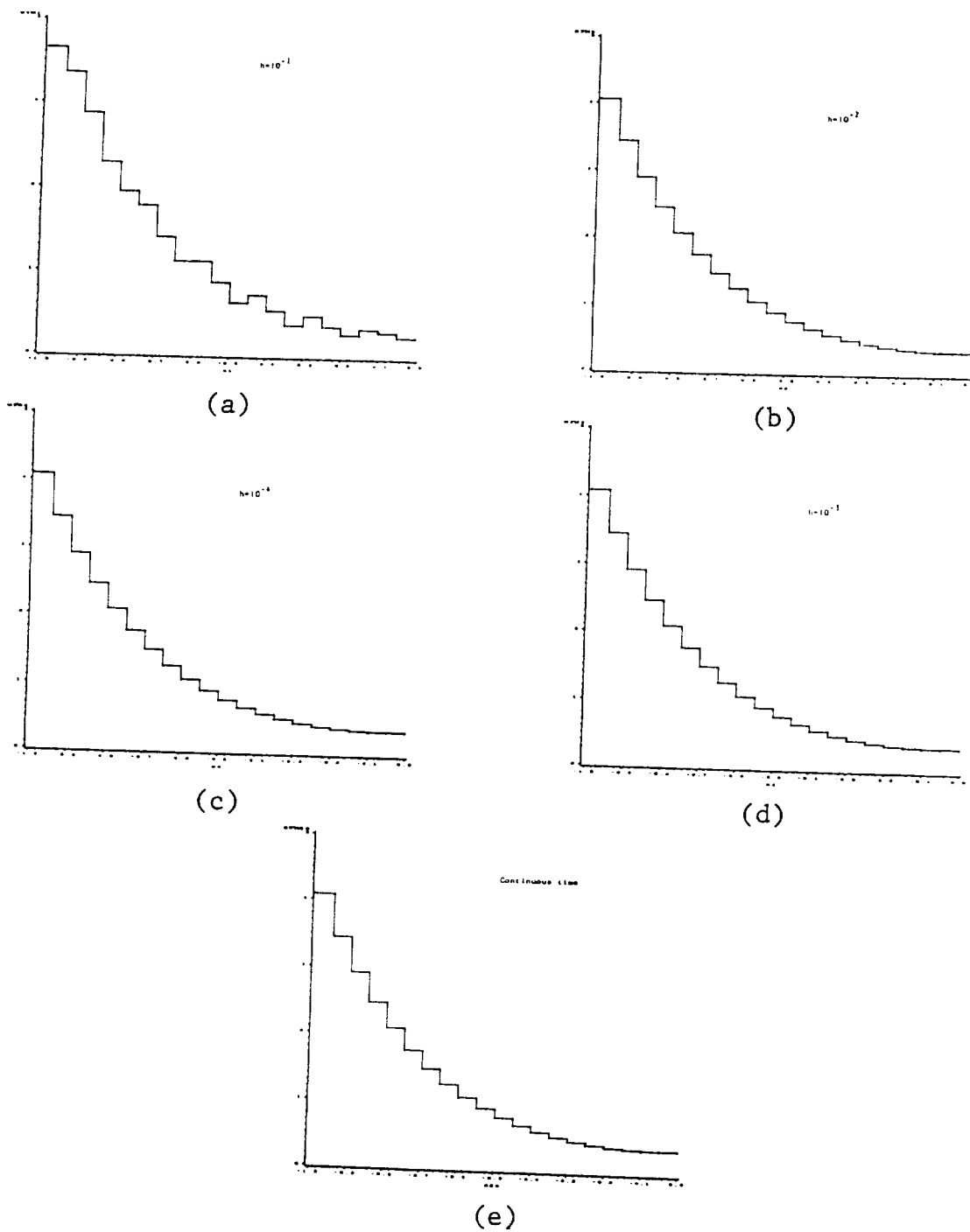
To solve both the continuous and discrete time LQR problems we employed a piecewise constant/linear spline hybrid finite element scheme developed by Ito and Kappel in [IK]. Setting  $a_0 = a_1 = b = q = r = 1$ , and with a state discretization level in the Ito-Kappel scheme taken to be  $N = 20$ , we obtained the  $R \times L_2(-1, 0)$  functional gains,  $f = (f^0, f^1)$  and  $f_h = (f_h^0, f_h^1)$  for various values of  $h > 0$ , tabulated and plotted in Table 1 and Figure 4.3 below. We note that for this choice of the parameters  $a_0$  and  $a_1$ , the open loop system has an eigenvalue with positive real part. Consequently the system (4.1) is open-loop unstable. It is not difficult to argue that the pairs  $\{A, B\}$  and  $\{Q, A\}$  are respectively stabilizable and detectable. Also, since the operators  $B$  and  $Q$  are of finite rank, there exists  $h_0 > 0$  such that for all sampling periods  $h \leq h_0$ , the sampled control systems are uniformly stabilizable and detectable in  $h$ .

**Example 4.3.** We consider the control of the small amplitude transverse vibration of a cantilevered Voigt-Kelvin viscoelastic beam with tip-mass. The relevant dynamics are described by the hybrid system of ordinary and partial differential equations

$$\begin{aligned} \rho \frac{\partial^2}{\partial t^2} x(t, \eta) + cI \frac{\partial^5}{\partial \eta^4 \partial t} x(t, \eta) + EI \frac{\partial^4}{\partial \eta^4} x(t, \eta) &= 0, \eta \in (0, 1), \\ m \frac{\partial^2}{\partial t^2} x(t, 1) - cI \frac{\partial^4}{\partial \eta^3 \partial t} x(t, 1) - EI \frac{\partial^3}{\partial \eta^3} x(t, 1) &= bu(t), \end{aligned}$$

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FIG. 4.3. Functional gains for delay equation. (a)  $h=0.1$ , (b)  $h=0.01$ , (c)  $h=0.001$ , (d)  $h=0.0001$ , (e) Continuous time.



for  $t > 0$ , the essential (or stable) boundary conditions at  $\eta = 0$

$$x(t, 0) = 0, \quad \frac{\partial}{\partial \eta} x(t, 0) = 0, \quad t > 0,$$

and the natural (or unstable) boundary condition at  $\eta = 1$ ,

$$cI \frac{\partial^3}{\partial \eta^2 \partial t} x(t, 1) - EI \frac{\partial^2}{\partial \eta^2} x(t, 1) = 0, \quad t > 0.$$

In the above equations  $\rho > 0$  is the linear mass density of the beam,  $I > 0$  is the beam's cross sectional moment of inertia,  $c > 0$  is the viscosity coefficient,  $E > 0$  is Young's modulus,  $m > 0$  is the mass of the tip mass, and  $b \in \mathbb{R}$  is a constant.

We take an energy based performance index:

$$\begin{aligned} J(u) = & \int_0^\infty \left\{ \frac{1}{2} EI \int_0^1 \left( \frac{\partial^2}{\partial \eta^2} x(t, \eta) \right)^2 d\eta + \frac{1}{2} m \left( \frac{\partial}{\partial t} x(t, 1) \right)^2 \right. \\ & \left. + \int_0^1 \frac{1}{2} \rho \left( \frac{\partial}{\partial t} x(t, \eta) \right)^2 d\eta + r u(t)^2 \right\} dt. \end{aligned}$$

Once again the abstract Hilbert space formulation of this problem is standard. We let  $H = H_L^2(0, 1) \times \mathbb{R} \times L_2(0, 1)$  where  $H_L^2(0, 1) = \{\varphi \in H^2(0, 1) : \varphi(0) = D\varphi(0) = 0\}$ , and endow  $H$  with the energy inner product

$$\begin{aligned} \langle (\varphi_1, \eta_1, \psi_1), (\varphi_2, \eta_2, \psi_2) \rangle_H = & EI \int_0^1 D^2 \varphi_1 D^2 \varphi_2 \\ & + m \eta_1 \eta_2 + \rho \int_0^1 \psi_1 \psi_2. \end{aligned}$$

The operator  $A : \text{Dom}(A) \subset H \mapsto H$  is given by  $A(\varphi, \eta, \psi) = (\psi, cID^3\psi(1) + EID^3\varphi(1), -cID^4\psi - EID^4\varphi)$  for  $(\varphi, \eta, \psi) \in \text{Dom}(A) = \{(\varphi, \eta, \psi) \in H : \psi \in H_L^2(0, 1), \eta = \psi(1), cID^2\psi + EID^2\varphi \in H^2(0, 1), cID^2\psi(1) + EID^2\varphi(1) = 0\}$ . We take  $U = \mathbb{R}$  and define  $B \in L(\mathbb{R}, H)$  by  $Bv = (0, bv, 0)$ . We let  $Q \in L(H)$  and  $R \in L(U)$  be given by  $Q = (1/2)I_H$  and  $R = rI_U$ , where  $I_H$ , and  $I_U$  denote, respectively, the identity operators on  $H$  and  $U$ .

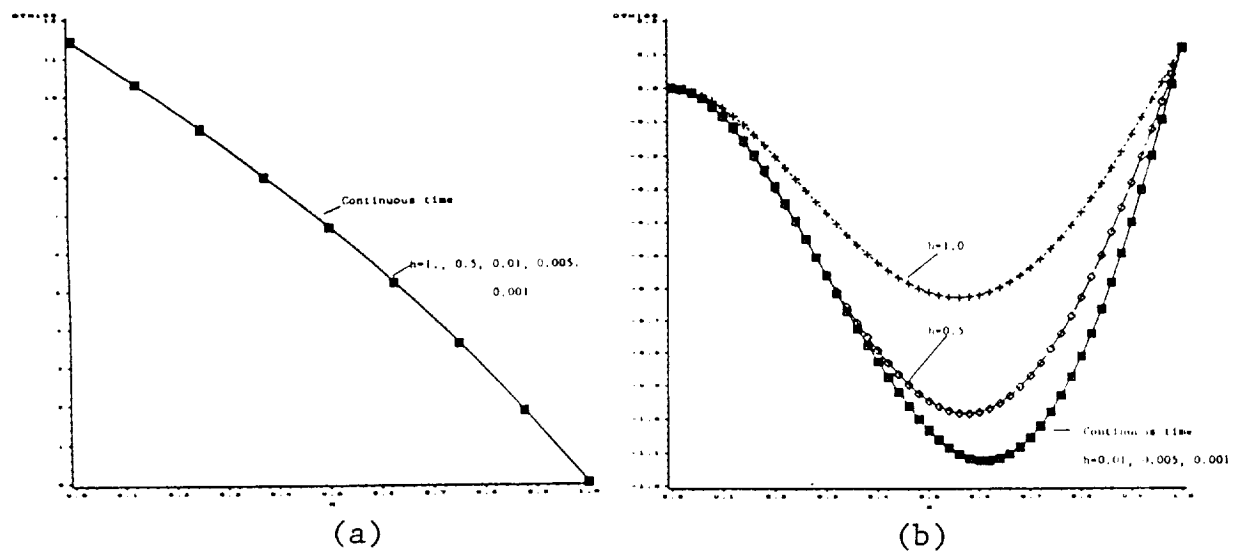
It can be shown (see [GA]) that  $A$  is the infinitesimal generator of a uniformly exponentially stable analytic semigroup. Thus once again stabilizability and detectability for the continuous time problems trivially follows as does the uniform stabilizability and detectability for the discrete time problems.

We employed a standard cubic spline based Ritz-Galerkin finite element scheme to approximate or finite dimensionalize the continuous and discrete time LQR problems (see [GA], [GR]). Setting  $\rho = 0.1$ ,  $EI = 1.3333 \times 10^{-4}$ ,  $cI = 1.3333 \times 10^{-7}$ ,  $m = 1$ ,  $b = 1$ ,  $q = 1$ , and  $r = 1$  and with  $N = 9$  cubic spline elements, we obtained the functional gains  $f = (f^0, f^1, f^2)$ ,  $f_h = (f_h^0, f_h^1, f_h^2) \in H$  exhibited in Table 2 and Figure 4.4 below.

TABLE 2  
Tip gains for beam equation.

Sampling period $h$	Tip Gain, $f^1$
1.000	0.12181
0.500	0.12003
0.010	0.11798
0.005	0.11796
0.001	0.11794
Continuous time	0.11793

FIG. 4.4. Functional gains for beam equation. (a) Displacement, we plot  $D^2 f^0$  to exhibit the  $H^2$  convergence; (b) Velocity.





**5. Summary and Concluding Remarks.** We have investigated and established the convergence of solutions to discrete or sampled time linear quadratic regulator problems and the associated Riccati equations for infinite dimensional systems to the solutions to the corresponding continuous time problem and associated Riccati equation, as the length of the sampling interval tends toward zero. We have considered both the finite and infinite time horizon problems and carried out numerical studies involving a variety of distributed parameter control systems in order to observe how well our theoretical results predict what actually takes place in practice. In the context of the finite time horizon problem, the assumption of strong continuity on the operators which define the control system and performance index, together with a stability and consistency hypothesis on the sampling scheme, are sufficient to establish the strong convergence of the Riccati operators, feedback gains, optimal control laws, and optimal trajectories, with some degree of uniformity in time over the compact interval of interest. For the infinite time horizon problem, we require the additional assumption of stabilizability and detectability, uniformly with respect to the length of the sampling interval. We have shown that this condition can be verified when zero-order hold sampling is employed and the continuous time system is stabilizable and detectable by finite rank feedback. We also have a result for parabolic systems, but this will be reported on elsewhere.

Several interesting open questions related to the results we have presented here remain open. For example the inter-relation between stabilizability/detectability for the continuous and sampled time systems in a more general setting and under more general sampling schemes (A-Stable Pade, for example) requires further study. Also, convergence under simultaneous and independent state (space) discretization (i.e. finite difference or finite element approximation) and temporal sampling should be investigated. It would not be difficult to extend the results we presented here to handle certain "coupled" state and time discretizations. Finally, a study similar to the present one could be carried out for the LQG estimator and compensator problems. We have not as of yet looked at these problems, but suspect that similar results to those given above could be obtained.

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## Appendix A.

- (i) *Continuity of  $P_h$* : Let  $\phi \in \mathcal{U}$ . Then since  $P_h\phi$  is constant on each of the

intervals  $I_j = [jh, (j+1)h)$ , we have

$$\begin{aligned}
\|P_h\phi\|_{\mathcal{U}_h}^2 &= \int_0^{t_{f,h}} \|(P_h\phi)(t)\|^2 dt \\
&= \sum_{j=0}^{k_{f,h}-1} h \left\| h^{-1} \int_{I_j} \phi(s) ds \right\|^2 \\
&\leq \sum_{j=0}^{k_{f,h}-1} h^{-1} \int_{I_j} \|\phi(s)\|^2 ds \int_{I_j} ds \\
&= \int_0^{t_{f,h}} \|\phi(s)\|^2 ds \leq \|\phi\|_{\mathcal{U}}^2.
\end{aligned}$$

(ii) *Convergence of  $P_h$* : First, let us consider a continuous function  $\phi \in \mathcal{U}$ . Then, for all  $t \in [0, t_f)$ , there exists  $h_0(t) > 0$  such that for all  $h \leq h_0(t)$ , we have  $t \in [0, t_{f,h}]$ . That is, for all  $t \in [0, t_f]$ ,

$$(P_h\phi)(t) = h^{-1} \int_I \phi(s) ds$$

for  $h$  small enough, where  $I$  is an interval of length  $h$  containing  $t$ . The Bochner integral is equal to the Riemann integral for continuous functions, and therefore, for all  $t \in [0, t_f)$ , it follows that

$$\lim_{h \rightarrow 0^+} (P_h\phi)(t) = \phi(t).$$

Now consider an extrapolation operator,  $E_h : \mathcal{U}_h \mapsto \mathcal{U}$ , defined for all  $u \in \mathcal{U}_h$  as:  $E_h u(t) = u(t)$ , for  $t \in [0, t_{f,h}]$ ,  $E_h u(t) \equiv 0$  for  $t \in (t_{f,h}, t_f]$ . It is evident that  $(E_h P_h \phi)(t)$  converges to  $\phi(t)$  as  $h$  tends to 0 for all  $t \in [0, t_{f,h}]$ . Since

$$\int_0^{t_f} \|(E_h P_h \phi)(s)\|^2 ds = \|P_h \phi\|_{\mathcal{U}_h}^2,$$

from (i), we conclude, via the dominated convergence theorem, that

$$\lim_{h \rightarrow 0^+} \|P_h \phi\|_{\mathcal{U}_h} = \|\phi\|_{\mathcal{U}}.$$

Then from the uniform boundedness of  $P_h$  and the density of the continuous functions in  $\mathcal{U}$ , we obtain

$$\lim_{h \rightarrow 0^+} \|P_h \phi\|_{\mathcal{U}_h} = \|\phi\|_{\mathcal{U}}$$

holds for all  $\phi \in \mathcal{U}$ .

(iii) For all  $u \in \mathcal{U}_h$  we have  $E_h u \in \mathcal{U}$ ,  $P_h E_h u = u$ , and  $\|P_h E_h u\|_{\mathcal{U}_h} = \|E_h u\|_{\mathcal{U}}$ .

## Appendix B.

We shall show that

$$\lim_{h \rightarrow 0^+} \|\tilde{\mathcal{B}}_{h,s} \phi - P_h \tilde{\mathcal{B}}_s \phi\|_{\mathcal{U}_h} = 0,$$

uniformly in  $s$  for  $s \in [0, t_f]$ , and  $\phi \in H$ . Analogous arguments can be used to show that

$$\lim_{h \rightarrow 0^+} \|\tilde{\mathcal{R}}_{h,s} P_h u - P_h \tilde{\mathcal{R}}_s u\|_{U_s} = 0$$

and that

$$\lim_{h \rightarrow 0^+} \|\tilde{\mathcal{B}}_{h,s}^* P_h u - \tilde{\mathcal{B}}_s^* u\|_H = 0$$

uniformly in  $s$  for  $s \in [0, t_f]$ , and each  $u \in \mathcal{U}$ .

From the definitions of  $\mathcal{B}_s$  and  $\tilde{\mathcal{B}}_s$  given in (2.2) and (2.11) respectively, it is easy to observe the following for any  $\phi \in H$ .

- (i)  $\|(\tilde{\mathcal{B}}_s \phi)(t)\|_U$  is continuous with respect to  $s$  for all  $t \geq s$ . Therefore, from the uniform boundedness of  $\|(\tilde{\mathcal{B}}_s \phi)(t)\|_U$  for all  $0 \leq s \leq t \leq t_f$ , we conclude that  $\tilde{\mathcal{B}}_s \phi$  is a continuous function of  $s$  in the  $\mathcal{U}$  norm.
- (ii) By condition (A1)–(A4), it is easy to see that

$$\Phi(t, s) = \|\tilde{\mathcal{B}}_{s,h} \phi(t) - \tilde{\mathcal{B}}_s \phi(t)\|_U$$

converges to zero, uniformly in  $\Delta(2, t_f)$ . By the uniform boundedness of  $\|\tilde{\mathcal{B}}_{h,s} \phi(t)\|_U$  for all  $h$ , we conclude that

$$\|\tilde{\mathcal{B}}_{s,h} \phi - \tilde{\mathcal{B}}_s \phi\|_{\mathcal{U}_h}^2 = \int_{[s/h]h}^s \|\tilde{\mathcal{B}}_{s,h} \phi(t)\|_U^2 dt + \int_{[s/h]h}^{t_f, h} \|\tilde{\mathcal{B}}_{s,h} \phi(t) - \tilde{\mathcal{B}}_s \phi(t)\|_U^2 dt$$

converges to zero, uniformly in  $s$ .

Since we have

$$\|\tilde{\mathcal{B}}_{h,s} \phi - P_h \tilde{\mathcal{B}}_s \phi\|_{\mathcal{U}_h} \leq \|\tilde{\mathcal{B}}_{h,s} \phi - \tilde{\mathcal{B}}_s \phi\|_{\mathcal{U}_h} + \|(P_h - I) \tilde{\mathcal{B}}_s \phi\|_{\mathcal{U}_h},$$

the strong continuity of  $\tilde{\mathcal{B}}_s$ , and the strong convergence of  $P_h$  to the identity on  $\mathcal{U}$ , yield the desired uniform convergence result.



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