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Exact Closed-Form Expressions for the Performance of the Split-Symbol Moments Estimator of Signal-to-Noise Ratio

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Previously, the performance of the split-symbol moments estimator (SSME) of signal-to-noise ratio (SNR) has been evaluated by means of approximate expressions for the estimator mean and variance. These are asymptotic formulas in the sense that they become accurate as the number of estimator samples gets large. In the present article, exact closed-form expressions are obtained for the same quantities. These expressions confirm the accuracy of the previously derived asymptotic results, and, unlike the asymptotic formulas, they are useful even when the number of samples is small. It is also shown that the conventional split-symbol estimator can be trivially scaled to form a signal-to-noise ratio estimator which is precisely unbiased (as long as the estimate is based on more than two split-symbols).

I. Introduction

The so-called split-symbol moments estimator (SSME) of the received symbol signal-to-noise ratio (SNR) was first suggested in [1]. In [2], approximate expressions were derived for the mean and variance of the SSME. These approximate expressions were later used as the basis for a more detailed performance analysis in [3], which included the effects of intersymbol interference as a result of filtering the received symbol stream.

The approximations in [2] result from expanding the exact equations for the mean and variance in a series involving the central moments of the SNR estimator and then ignoring all central moments higher than second order. This type of approximation relies on the fact that successively higher central moments of the estimator go to zero faster than lower-order central moments as the number of samples gets large.

A natural question is: Under what conditions is it valid to throw away the higher-order terms? The results quoted for the mean and variance of the SSME (Eqs. 28 and 31 of [2]) include terms which go to zero as the number of sampled split-symbols, n, gets large. These terms are proportional to $1/n$. How large does n have to be before the $1/n$ terms are guaranteed to dominate the $1/n^2$ terms? Does the answer to this question depend on the magnitude of the true SNR? For example, the expression for the signal-to-noise ratio of the estimator (Eq. 32 of [2])

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exhibits peculiar behavior when the true SNR is very small. In this case, the equation says that the signal-to-noise ratio of the estimator actually decreases with n until n gets to be about as large as 1/SNR.

In this article, exact closed-form expressions are obtained for the mean and variance of of the SSME when the symbol stream is unfiltered. These exact results confirm the accuracy of the previously known asymptotic expressions. Unlike the asymptotic formulas, the exact expressions are useful even when the number of sampled split-symbols, n, is small. Furthermore, the basic techniques developed for this derivation might also be useful in analyzing the more complicated case of filtered data.

II. Derivation of the Result

Using the notation of [3], the split-symbol moments estimator of SNR is denoted by SNR^* and expressed as

$$SNR^* = \frac{m_p}{2\left(\frac{1}{4}m_{ss} - m_p\right)} \tag{1}$$

where m_p and m_{ss} are average values over n symbols of the product and sum squared, respectively, of the outputs of half-symbol accumulators,

$$m_p = \frac{1}{n}\sum_{j=1}^{n} Y_{\alpha j} Y_{\beta j} \tag{2a}$$

$$m_{ss} = \frac{1}{n}\sum_{j=1}^{n} (Y_{\alpha j} + Y_{\beta j})^2 \tag{2b}$$

$$Y_{\alpha j} = \sum_{i=0}^{\frac{N_s}{2}-1} y_{ij} \tag{3a}$$

$$Y_{\beta j} = \sum_{i=\frac{N_s}{2}}^{N_s-1} y_{ij} \tag{3b}$$

$$\begin{aligned} y_{ij} &= \sqrt{S} d_j + n_{ij} \\ &= \frac{m}{N_s} d_j + n_{ij} \qquad (3c) \\ & i = 0, \ldots, N_s - 1 \quad j = 1, \ldots, n \end{aligned}$$

Here, N_s is the number of samples per symbol and is assumed to be an even integer, as in [3], and $m = N_s\sqrt{S}$. The datum y_{ij} is assumed to be an unfiltered sample of the jth received symbol $d_j (d_j = \pm 1)$. The noise samples n_{ij} are assumed to be independent with zero mean and equal variance σ^2/N_s. The parameters m and σ are the conditional mean and standard deviation (given d_j), respectively, of the whole-symbol accumulator outputs $(Y_{\alpha j} + Y_{\beta j})$. The symbol signal-to-noise ratio SNR is given in terms of m and σ by

$$SNR = \frac{m^2}{2\sigma^2} \tag{4}$$

Conditioned on the sequence of symbol values $\mathbf{d} = (d_1, \ldots, d_n)$, the y_{ij}'s are independent Gaussian random variables, as are the vectors of half-symbol accumulator outputs $\mathbf{Y}_\alpha = (Y_{\alpha 1}, \ldots, Y_{\alpha n})$ and $\mathbf{Y}_\beta = (Y_{\beta 1}, \ldots, Y_{\beta n})$. The statistics of $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\beta$ are given by

$$E\{Y_{\alpha j}|\mathbf{d}\} = E\{Y_{\beta j}|\mathbf{d}\} = \frac{m}{2} d_j \tag{5a}$$

$$E\{Y_{\alpha j}^2|\mathbf{d}\} = E\{Y_{\beta j}^2|\mathbf{d}\} = \frac{m^2}{4} + \frac{\sigma^2}{2} \tag{5b}$$

$$E\{Y_{\alpha j} Y_{\beta j}|\mathbf{d}\} = \frac{m^2}{4} \tag{5c}$$

$$E\{Y_{\alpha j} Y_{\alpha j'}|\mathbf{d}\} = E\{Y_{\beta j} Y_{\beta j'}|\mathbf{d}\} = \frac{m^2}{4} d_j d_{j'} \quad j' \neq j \tag{5d}$$

$$E\{Y_{\alpha j} Y_{\beta j'}|\mathbf{d}\} = \frac{m^2}{4} d_j d_{j'} \quad j' \neq j \tag{5e}$$

We begin the derivation of our result by rewriting Eq. (1) for SNR^* in the form

$$2\ SNR^* + 1 = \frac{\sum_{j=1}^{n} u_j^2}{\sum_{j=1}^{n} v_j^2} = \frac{U}{V} \tag{6}$$

where

$$u_j = Y_{\alpha j} + Y_{\beta j} \tag{7a}$$

$$v_j = Y_{\alpha j} - Y_{\beta j} \tag{7b}$$

$$U = \sum_{j=1}^{n} u_j^2 \tag{8a}$$

$$V = \sum_{j=1}^{n} v_j^2 \tag{8b}$$

The usefulness of expressing SNR^* in the form of Eq. (6) results from the fact that U and V are conditionally independent random variables. Given the data vector $\mathbf{d}$, the random vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ are jointly Gaussian with

$$E\{u_j|\mathbf{d}\} = m d_j \qquad E\{u_j^2|\mathbf{d}\} = m^2 + \sigma^2 \tag{9a}$$

$$E\{v_j|\mathbf{d}\} = 0 \qquad E\{v_j^2|\mathbf{d}\} = \sigma^2 \tag{9b}$$

$$E\{u_j v_j|\mathbf{d}\} = 0 = E\{u_j|\mathbf{d}\} E\{v_j|\mathbf{d}\} \tag{9c}$$

$$E\{u_j v_{j'}|\mathbf{d}\} = E\{v_j v_{j'}|\mathbf{d}\} = 0 \qquad j' \neq j \tag{9d}$$

$$E\{u_j u_{j'}|\mathbf{d}\} = m^2 d_j d_{j'} = E\{u_j|\mathbf{d}\} E\{u_{j'}|\mathbf{d}\} \qquad j' \neq j \tag{9e}$$

Equations (9a–e) imply that $\mathbf{u}$ and $\mathbf{v}$ are uncorrelated given $\mathbf{d}$. Because $\mathbf{u}$ and $\mathbf{v}$ are jointly Gaussian and uncorrelated, conditioned on $\mathbf{d}$, they are therefore conditionally independent given $\mathbf{d}$. Likewise the random variables U and V are conditionally independent given $\mathbf{d}$, because they are functions of $\mathbf{u}$ and $\mathbf{v}$, respectively [4]. Thus, all of the moments of $2\ SNR^* + 1$ can be factored as follows:

$$E\left\{(2\ SNR^* + 1)^k\right\} = E_{\mathbf{d}}\left[E\{U^k|\mathbf{d}\} E\{V^{-k}|\mathbf{d}\}\right] \tag{10}$$

To evaluate the mean and standard deviation of SNR^* exactly, it is only necessary to compute the conditional moments $E\{U|\mathbf{d}\}$, $E\{U^2|\mathbf{d}\}$, $E\{V^{-1}|\mathbf{d}\}$, and $E\{V^{-2}|\mathbf{d}\}$, and then calculate the expectation over $\mathbf{d}$. All four of these conditional moments can be evaluated in closed form. The U-moments are simply the (positive) moments of a noncentral chi-squared random variable.

$$E\{U|\mathbf{d}\} = nE\{u_j^2|\mathbf{d}\} = n(m^2 + \sigma^2) \tag{11}$$

$$\begin{aligned} E\{U^2|\mathbf{d}\} &= nE\{u_j^4|\mathbf{d}\} + n(n-1)\left[E\{u_j^2|\mathbf{d}\}\right]^2 \\ &= n\left[(m^4 + 6m^2\sigma^2 + 3\sigma^4) + n(n-1)(m^2+\sigma^2)^2\right] \\ &= n^2(m^2+\sigma^2)^2 + n(4m^2\sigma^2 + 2\sigma^4) \end{aligned} \tag{12}$$

In Eq. (12), the first equality follows from the independence of the individual u_j's, and the second equality follows from applying the general formula for the fourth moment of a nonzero-mean Gaussian random variable [5].

The V-moments are the negative moments of a central chi-squared random variable with n degrees of freedom. In the Appendix, it is shown that

$$E\{V^{-1}|\mathbf{d}\} = \frac{1}{(n-2)\sigma^2} \qquad (\text{for } n > 2) \tag{13}$$

$$E\{V^{-2}|\mathbf{d}\} = \frac{1}{(n-2)(n-4)\sigma^4} \qquad (\text{for } n > 4) \tag{14}$$

These V-moments are infinite if n is smaller than the indicated limits.

Note that all of the expressions in Eqs. (11), (12), (13), and (14) are independent of $\mathbf{d}$. Plugging these results into Eq. (10) leads to the following expressions for the moments of the estimator SNR^* (assuming $n > 2$ or $n > 4$ as appropriate):

$$E\{2\ SNR^* + 1\} = \frac{n(m^2+\sigma^2)}{(n-2)\sigma^2} = (2\ SNR + 1)\frac{n}{n-2} \tag{15}$$

$$\begin{aligned} E\left\{(2\ SNR^* + 1)^2\right\} &= \left[(m^2+\sigma^2)^2 + \frac{1}{n}(4m^2\sigma^2 + 2\sigma^4)\right] \\ &\quad \times \frac{n^2}{(n-2)(n-4)\sigma^4} \\ &= \left[(2\ SNR + 1)^2 + \frac{1}{n}(8\ SNR + 2)\right] \\ &\quad \times \frac{n^2}{(n-2)(n-4)} \end{aligned} \tag{16}$$

The mean and variance of SNR^* are calculated from Eqs. (15) and (16) as

$$\begin{aligned} E\{SNR^*\} &= \frac{1}{2}\left[E\{2\ SNR^* + 1\} - 1\right] \\ &= SNR + \frac{1}{n-2}(2\ SNR + 1) \qquad (\text{for } n > 2) \end{aligned} \tag{17a}$$

$$\begin{aligned} \text{var}\{SNR^*\} &= \frac{1}{4}\text{var}\{2\ SNR^* + 1\} \\ &= \frac{1}{4}\left[E\left\{(2\ SNR^* + 1)^2\right\} - \left(E\{2\ SNR^* + 1\}\right)^2\right] \\ &= \left[\frac{n}{n-2}\right]^2 \frac{1}{n-4}\left[2\ SNR^2 + (4\ SNR + 1)\frac{n-1}{n}\right] \\ &(\text{for } n > 4) \end{aligned} \tag{17b}$$

In comparison, in [2] the following approximate asymptotic results were obtained:

$$E\{SNR^*\} \approx SNR + \frac{1}{n}(2\ SNR + 1) \tag{18a}$$

$$\text{var}\{SNR^*\} \approx \frac{1}{n}\left[2\ SNR^2 + 4\ SNR + 1\right] \tag{18b}$$

By comparing Eq. (17) with Eq. (18) it is seen that the asymptotic expressions of [2] are accurate if $\frac{n-1}{n}$, $\frac{n-2}{n}$, and $\frac{n-4}{n}$ can all be approximated by 1.

In addition to the mean and variance, higher-order moments of the SSME may also be evaluated by applying known formulas for the moments of noncentral chi-squared random variables (U-moments) and for the moments of the reciprocal of a central chi-squared random variable (V-moments), and then solving Eq. (10) algebraically for the corresponding moments of SNR^*.

Finally, it is seen from Eq. (15) that it is easy to convert SNR^* into an unbiased estimator SNR^{**} (for $n > 2$) by making the definition

$$2\ SNR^{**} + 1 = (2\ SNR^* + 1)\frac{n-2}{n} \tag{19a}$$

i.e.,

$$SNR^{**} = SNR^* - \frac{1}{n}(2\ SNR^* + 1) \tag{19b}$$

Then, assuming $n > 2$ for Eq. (20a) and $n > 4$ for Eq. (20b),

$$E\{SNR^{**}\} = SNR \qquad (\text{for } n > 2) \tag{20a}$$

$$var\{SNR^{**}\} = \frac{1}{n-4}\left[2\ SNR^2 + (4\ SNR + 1)\frac{n-1}{n}\right]$$

$$(\text{for } n > 4) \tag{20b}$$

This should be the preferred form for the split-symbol moments estimator because, in addition to being unbiased, SNR^{**} also has a slightly smaller variance than SNR^*.

III. Conclusion

The technique introduced in this article enables one to compute exact closed-form expressions for the performance of the split-symbol moments estimator, for the case of unfiltered, undegraded data. The essential trick required for this computation was to algebraically manipulate the estimator formula into the form of a quotient of conditionally independent random variables. The calculations in this article confirm the accuracy of previously derived asymptotic expressions for unfiltered data. In addition, the exact performance expressions, unlike the asymptotic formulas, are useful even when the number of sampled split-symbols, n, is small. The exact formulas show that as long as $n > 2$ the conventional split-symbol estimator can be trivially scaled to form a signal-to-noise ratio estimator which is precisely unbiased. Furthermore, the same techniques developed here may be applied to obtain simplified expressions (though not closed form) for the more complicated case when the data is filtered.

Appendix

The Moments of the Reciprocal of a Chi-Squared Random Variable

Conditioned on $\mathbf{d}$, the random variable V is a central chi-squared random variable with n degrees of freedom. According to [6], its conditional probability function is therefore given by

$$p(V|\mathrm{d}) = \frac{V^{(n/2)-1}e^{-V/2\sigma^2}}{(2\sigma^2)^{n/2}\,\Gamma(n/2)} \qquad V \geq 0 \qquad \text{(A-1)}$$

where $\Gamma(\cdot)$ is the gamma function. Thus, the conditional (negative) moments of V can be calculated (for $k > 0$) as

$$E\left(V^{-k}|\mathrm{d}\right) = \int_0^\infty \frac{V^{(n/2)-k-1}e^{-V/2\sigma^2}}{(2\sigma^2)^{n/2}\,\Gamma(n/2)}\,dV$$

$$= \left(2\sigma^2\right)^{-k}\frac{\Gamma[(n/2)-k]}{\Gamma(n/2)}$$

$$\times \int_0^\infty \frac{V^{(n/2)-k-1}e^{-V/2\sigma^2}}{(2\sigma^2)^{(n/2)-k}\,\Gamma[(n/2)-k]}\,dV$$

$$= \left(2\sigma^2\right)^{-k}\frac{\Gamma[(n/2)-k]}{\Gamma(n/2)} \qquad \text{if } n > 2k$$

$$= \frac{1}{(n-2)(n-4)\cdots(n-2k)\sigma^{2k}} \qquad \text{(A-2)}$$

The next-to-last equality in Eq. (A-2) follows from noting that, if $n > 2k$, the rescaled integral in the preceding line is simply the integral of the probability density function of a chi-squared random variable with $n - 2k$ degrees of freedom. The last equality in Eq. (A-2) follows from the gamma function recurrence formula, $\Gamma(z + 1) = z\Gamma(z)$. Finally, it is easy to see that the integral in Eq. (A-2), and hence the corresponding moment of V, is infinite if $n \leq 2k$, because in this case $V^{(n/2)-k-1}$ becomes infinite near $V = 0$ at least as fast as V^{-1}.

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