N90-23038

MINIMUM ATTAINABLE RMS ATTITUDE ERROR USING CO-LOCATED RATE SENSORS

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Abstract

In this paper we announce a closed form analytical expression for the minimum attainable attitude error (as well as the error rate) in a flexible beam by feedback control using co-located rate sensors. For simplicity, we consider a beam clamped at one end with an offset mass (antenna) at the other end where the controls and sensors are located. Both control moment generators and force actuators are provided. The results apply to any beam-like lattice-type truss, and provide the kind of performance criteria needed under CSI — Controls-Structures-Integrated optimization.

† Research Supported in part under NAS1-18585 Task Assignment 49.

1. Introduction

One of the challenges in the Design Challenge For Flexible Flight Structure Control System Design formulated in the inaugural paper on SCOLE [1] was to hold the antenna pointing error within ±0.02 degrees after slewing by appropriate feedback control. In this paper we derive a closed form expression for the minimal achievable mean square pointing error using co-located rate sensors. A slightly simplified form of the SCOLE article (which eliminates rigid-body modes) is used: a cantilevered beam with an offset mass where the controls — both c.m.g.'s and force actuators — and the rate sensors are located. Our results are in terms of continuum model parameters — the uniform Bernoulli version is used. The beam dynamics are given in Section 2. The main results are in Section 3. We note that a technique for deriving equivalent Bernoulli beam parameters for various types of trusses is described by Noor and Anderson in [4]. Recently Noor and Russell [5] presented equivalent anisotropic Timoshenko beam models for beam-like lattice trusses with an arbitrary degree of modal coupling, which appear to yield excellent agreement with modal frequencies derived from finite element models. Our theory is able to handle these Timoshenko models, and moreover we can also use it for rigid-body modes, although they are not included here. Thus our results can be used for any beam-like lattice truss structure.

2. The Model

We consider a uniform Bernoulli beam clamped at one end with an offset mass (antenna) at the other end which also houses the sensors and actuators. See Figure 1. We allow for both force actuators and moment actuators. The sensors are rate gyros. Because of the clamping at one end, no rigid-body modes are involved and hence no attitude sensors are needed.

We allow bending in two mutually perpendicular planes containing the beam axis, as well as torsion in the plane perpendicular to the beam axis, all uncoupled. The continuum model (uniform Beroulli beam) dynamics can then be described by the following partial differential equations (similar to those in [2, 3]). Let the beam extend along the z-axis, $0 < s < L$, and let $u_\phi(s, t)$, $u_\theta(s, t)$, denote the bending displacements and $u_\psi(s, t)$ the torsion angle about the beam axis. Let in the usual notation (cf. [1]), EI_ϕ, EI_θ denote the flexural stiffness and GI_ψ the torsional rigidity. Let ρ denote the mass per unit area and A the cross-sectional area. Then we have:

$$\rho A \frac{\partial^2 u_\phi(s, t)}{\partial t^2} + EI_\phi \frac{\partial^4 u_\phi(s, t)}{\partial s^4} = 0, \qquad 0 < s < L;\ 0 < t$$

$$\rho A \frac{\partial^2 u_\theta(s, t)}{\partial t^2} + EI_\theta \frac{\partial^4 u_\theta(s, t)}{\partial s^4} = 0, \qquad 0 < s < L;\ 0 < t$$

$$\rho I_\psi \frac{\partial^2 u_\psi(s, t)}{\partial t^2} - GI_\psi u_\psi''(s, t) = 0, \qquad 0 < s < L;\ 0 < t$$

with the clamped boundary conditions at $s = 0$:

$$u_\phi(0, t) = u_\theta(0, t) = u_\psi(0, t) = 0$$

$$u_\phi'(0, t) = u_\theta'(0, t) = 0\,.$$

The antenna center of gravity is located at

$$(r_x, r_y, L)\,.$$

The distance from the beam tip to antenna center of gravity is denoted by

$$|r| = \sqrt{r_x^2 + r_y^2}\,.$$

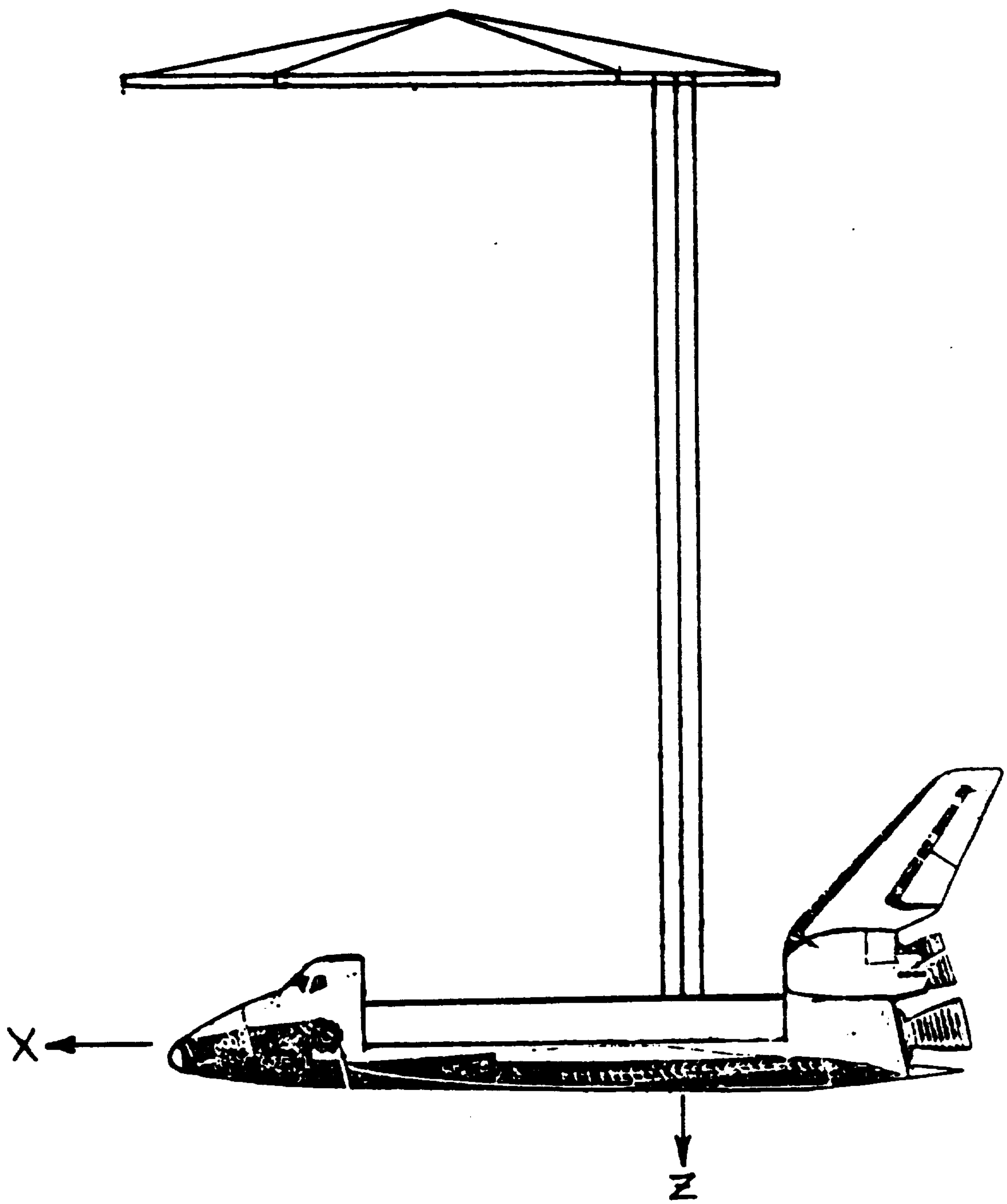

Figure 1: Shuttle/Antenna Configuration

The force balance equation at $s = L$ yields

$$m \begin{vmatrix} 1 & 0 & r_x \\ 0 & 1 & r_y \end{vmatrix} \begin{vmatrix} \ddot{u}_\phi(L,t) \\ \ddot{u}_\theta(L,t) \\ \ddot{u}_\psi(L,t) \end{vmatrix} + \begin{vmatrix} f_1(t) \\ f_2(t) \end{vmatrix} = \begin{vmatrix} EI_\phi u_\phi'''(L,t) \\ EI_\theta u_\theta'''(L,t) \end{vmatrix}$$

where m is the antenna mass and $f_1(\cdot)$, $f_2(\cdot)$ are the applied control forces. The torque balance equations yield

$$0 = \begin{vmatrix} EI_\phi u_\phi''(L,t) \\ EI_\theta u_\theta''(L,t) \\ GI_\psi u_\psi'(L,t) \end{vmatrix} + \hat{I}_a \dot{\omega} + M(t) + r \otimes \begin{vmatrix} f_1(t) \\ f_2(t) \\ 0 \end{vmatrix}$$

$$+ r \otimes \begin{vmatrix} \ddot{u}_\phi(L,t) + r_x \ddot{u}_\psi(L,t) \\ \ddot{u}_\theta(L,t) + r_y \ddot{u}_\psi(L,t) \end{vmatrix}$$

where the superdots indicate time derivatives and the primes the derivatives with respect to the spatial variable s; $\otimes$ denotes the vector cross-product and ω the angular rate vector

$$\omega = \begin{vmatrix} \dot{u}_\phi'(L,t) \\ \dot{u}_\theta'(L,t) \\ \dot{u}_\psi(L,t) \end{vmatrix},$$

$\hat{I}_a$ denotes the moment of inertia of the antenna about the beam tip ($s = L$) and finally, $M(t)$ denotes the applied control moment.

It is convenient to denote by $b(t)$ the boundary vector:

$$b(t) = \begin{vmatrix} u_\phi(L,t) \\ u_\theta(L,t) \\ u_\phi'(L,t) \\ u_\theta'(L,t) \\ u_\psi(L,t) \end{vmatrix}.$$

The boundary rate vector would thus be $\dot{b}(t)$. Hence our sensor model is:

$$v(t) = \dot{b}(t) + N_o(t)$$

where we assume that $N_o(t)$ is white Gaussian noise with spectral density matrix $d_o I$, where I is the identity (5×5) matrix. Similarly we assume that the control actuators are also characterized by additive white Gaussian noise. Denoting the applied control vector by $u(t)$:

$$u(t) = \begin{vmatrix} u_1(L,t) \\ u_2(L,t) \\ u_3(L,t) \\ u_4(L,t) \\ u_5(L,t) \end{vmatrix} .$$

we have

$$\begin{vmatrix} f_1(t) \\ f_2(t) \\ M(t) \end{vmatrix} = u(t) + N_s(t)$$

where $N_s(t)$ is white Gaussian with spectral density d_s. We shall also use M_b to denote the actuator mass/inertia matrix

$$M_b = \begin{vmatrix} m & 0 & 0 & 0 & mr_x \\ 0 & m & 0 & 0 & mr_y \\ 0 & 0 & & & \\ 0 & 0 & & \hat{I}_a & \\ mr_x & mr_y & & & \end{vmatrix}$$

where

$$\hat{I}_a = I_a + \begin{vmatrix} r_y^2 & -r_x r_y & 0 \\ -r_x r_y & r_x^2 & 0 \\ 0 & 0 & r_x^2 + r_y^2 \end{vmatrix}$$

where I_a is the antenna moment of inertia about its center of gravity. For any control input $u(\cdot)$ (which must perforce be a "feedback" control, based on the sensor data $v(\cdot)$) the mean square pointing error is then expressed by:

$$\lim_{T\to\infty} \frac{1}{T}\left\{ \int_0^T u_\phi(L,t)^2\,dt + \int_0^T u_\theta(L,t)^2\,dt + |r|^2 \int_0^T u_\psi(L,t)^2\,dt \right\}$$

and the mean square pointing *rate* is given by

$$\lim_{T\to\infty} \frac{1}{T}\left\{ \int_0^T \dot{u}_\phi(L, t)^2\, dt + \int_0^T \dot{u}_\theta(L, t)^2\, dt + |r|^2 \int_0^T \dot{u}_\psi(L, t)^2\, dt \right\}.$$

From the results in [6] it follows that the minimal attainable mean square pointing error is given by

$$aM_b^{-1}a^*$$

where

$$a = \text{row vector } (1, 1, 0, 0, |r|)$$

$$a^* = \text{transpose of } a$$

3. Main Results

We need some notation first. The mean square attitude response, whatever the feedback control used is defined by

$$\lim_{T\to\infty} \frac{1}{T}\left\{ \int_0^T u_\phi(t, \ell)^2\, dt + \int_0^T u_\theta(t, \ell)^2\, dt + |r|^2 \int_0^T u_\psi(t, \ell)^2\, dt \right\}. \tag{3.1}$$

This is recognized as the mean square displacement of the center of gravity of the antenna which is then also proportional to the mean square "pointing" error — see [1] for the relationships.

Next let u denote any (vector) of control inputs — a constant "step" input:

$$u = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{vmatrix}. \qquad (3.1)$$

Solve the equations

$$\left.\begin{aligned} EI_\phi u_\phi''''(s) &= 0 \\ EI_\theta u_\theta''''(s) &= 0 \\ GI_\psi u_\psi''(s) &= 0 \end{aligned}\right\} \quad 0 < s < L, \tag{3.2}$$

subject to the end conditions

$$\left.\begin{aligned} EI_\phi u_\phi'''(L) &= u_1 \\ EI_\theta u_\theta'''(L) &= u_2 \\ EI_\phi u_\phi''(L) + u_3 &= 0 \\ EI_\theta u_\theta''(L) + u_4 &= 0 \\ GI_\psi u_\psi'(L) + u_5 &= 0 \end{aligned}\right\} . \tag{3.3}$$

Note that the solution can be recognized as the steady-state response of the system to the step-input u, assuming that there is some damping. We only need to calculate the response to three specialized inputs:

Calculate the response to $u_\phi(L)$ to the special case, Case 1, where:

$$u_1 = 1$$

$$u_i = 0 , \qquad 2 \le i \le 5 .$$

Calculate the response $u_\theta(L)$ to the special case, Case 2, where:

$$u_1 = 0$$

$$u_2 = 1$$

$$u_3 = u_4 = u_5 = 0 .$$

Calculate the response $u_\psi(L)$ to the special case, Case 3, where

$$u_1 = u_2 = u_3 = u_4 = 0$$

$$u_6 = 1 .$$

Then the minimal achievable mean-square response whatever the choice of the feedback and whatever the mean-square control effort, is given by

$$\sqrt{d_s d_o}\ (u_\phi(L)^2 + u_\theta(L)^2 + r^2 u_\psi(L)^2) . \tag{3.4}$$

This is our main result. Unfortunately the derivation is beyond the scope of this report and will be published elsewhere. To proceed further with (3.4) we calculate the solution of (3.2), (3.3) explicitly. Thus for any u, the solution is of the form

$$u_\phi(s) = a_3 s^3 + a_2 s^2 , \qquad 0 < s < L$$

$$u_\theta(s) = b_3 s^3 + b_2 s^2 , \qquad 0 < s < L$$

$$u_\psi(s) = c_1 s , \qquad 0 < s < L$$

where

$$a_3 = -\frac{u_1}{6EI_\phi}$$

$$b_3 = -\frac{u_2}{6EI_\theta}$$

$$a_2 = \frac{1}{2}\left[\frac{u_3}{EI_\phi} + \frac{u_1 \ell}{EI_\phi}\right]$$

$$b_2 = \frac{1}{2}\left[\frac{u_4}{EI_\theta} + \frac{u_1 L}{EI_\theta}\right]$$

$$c_1 = \frac{u_5}{GI_\psi} .$$

Thus for Case 1 we have

$$u_\phi(L)^2 = \frac{L^3}{3EI_\phi}$$

and for Case 2 we have

$$u_\theta(L)^2 = \frac{L^3}{3EI_\theta}$$

and for Case 3:

$$u_\psi(L)^2 = \frac{L}{GI_\psi} .$$

Hence the mean-square attitude error

$$= \sqrt{d_s d_o}\left(\frac{L^3}{3EI_\phi} + \frac{L^3}{3EI_\theta} + \frac{|r|^2 L}{GI_\psi}\right) . \qquad (3.5)$$

Note the appearance of the noise parameters in (3.5) in product form.

The technique for calculating the minimal mean square atttiude error in more complex models than that illustrated is the same: calculate the mean square step response (assuming some damping) to unit step inputs.

In conclusion we suggest this result (3.5) can be the basis for combined structures-controls optimization — CSI, since the required structural parameters can be calculated for a lattice truss from the material gage and physical dimensions as in [4, 5]. We omit the details of these calculations.

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