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A Parallel Structure Transient Response Algorithm Using Independent Substructure Response Computation

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Abstract

An algorithm for parallel computation of transient response for structures is presented in which responses of substructures are computed independently for dozens of time steps at a time, and these substructure responses are then corrected to obtain the response of the overall coupled structure. The correction of the uncoupled substructure responses only requires the responses computed for interfaces at occasional points in time, and is done independently for different substructures in a very efficient procedure. A numerical example is presented to demonstrate the method and show the accuracy of the method.

Introduction

A significant amount of effort has been directed recently toward the development of methods for subdividing the computational effort associated with the solution of large transient response problems. The general approach of subdividing the computation associated with a given problem on the basis of a subdivision of the problem domain into subdomains has come to be known as domain decomposition in the last few years.[1,2] For transient response problems in structural dynamics, some efforts in this direction have been motivated by the need to solve problems for systems consisting of two or more well-defined subsystems, such as the Shuttle orbiter and its payloads, using modal data that have already been obtained for each of the subsystems rather than computing new modal data for the combined system.[3–5] Other work has been done in the context of the element-by-element approach to finite element analysis.[6,7] More recently, Ortiz *et al.* have proposed methods specifically intended for concurrent computation of transient response based on a subdivision of the problem domain into subdomains.[8,9] In their approach, an implicit integration scheme is used to obtain response for each subdomain for a given time step, and the results of these computations are averaged at interfaces to yield an approximation of the response of the overall system. Hajjar and Abel have investigated the accuracy of these methods for certain structural dynamics transient response problems, and have concluded that their accuracy is inadequate for these problems when practical time step sizes are used.[10]

In all of the transient response methods mentioned above, computation of response on the substructure level can only be done independently for one time step at a time. In contrast to this, an algorithm was presented recently by these authors which allows independent computation of substructure response for an arbitrary number of time steps at a time.[11] After independent substructure responses have been computed, they are corrected based on the interface motion computed for substructures at each time step, to obtain the response of the combined structure. Allowing the response to be computed independently for a number of time steps at a time reduces the interdependence between processors assigned to different substructures significantly, which can be important when the

amount of computation required for different substructures is unequal. Also, if there are more substructures than processors, the cost of swapping different substructures in and out of processors will be reduced if it can be done less frequently.

In the present paper, an extension of the algorithm presented in Ref. 11 is presented in which independent substructure response computation can proceed for much longer periods of time. Independent substructure responses are corrected on the basis of computed interface motion sampled at occasional points in time. The correction procedure for obtaining the response of the structure from the computed substructure responses is extremely efficient once the transient response computation is under way, although there is some computational overhead required to set up the correction capability. A numerical example is presented which illustrates the method and shows the accuracy that is obtained.

A Method Using Substructure-Level Response Computation

The algorithm presented in this paper is for computing the transient response of structures whose motion is governed by the equation

$$M\ddot{\boldsymbol{u}} + C\dot{\boldsymbol{u}} + K\boldsymbol{u} = \boldsymbol{F}(t) \tag{1}$$

where M, C, and K are taken to be constant mass, damping, and stiffness matrices, $\ddot{\boldsymbol{u}}$, $\dot{\boldsymbol{u}}$, and $\boldsymbol{u}$ are acceleration, velocity, and displacement vectors, and $\boldsymbol{F}(t)$ is a vector of forces exciting the system. As mentioned in the Introduction, the transient response of a given structure is computed in this algorithm by solving transient response problems for the substructures defined by decomposing the structure. To introduce the notation that will be used in this paper, a mass matrix for a structure composed of two substructures is shown below, after a possible reordering of rows and columns:

$$M = \begin{bmatrix} M_{LL}^{(1)} & M_{LS}^{(1)} & 0 \\ M_{SL}^{(1)} & M_{SS}^{(1)} + M_{SS}^{(2)} & M_{SL}^{(2)} \\ 0 & M_{LS}^{(2)} & M_{LL}^{(2)} \end{bmatrix}. \tag{2}$$

The superscripts in parentheses tell which substructure a given matrix partition is associated with, and the subscripts S and L refer to matrix partitions associated with *shared*, or interface, and *local*, or internal degrees of freedom. For some of the development in this paper, a structure composed of only two substructures is considered in an effort to simplify the presentation. However, the methods presented will be applicable for an arbitrary number of substructures.

Because responses will be obtained for each of the substructures a structure is composed of, some convention must be adopted for representing the structure response in terms of the substructure responses, particularly at the interfaces. In this paper, the approach taken is similar to the standard approach for the assembly of element matrices in the finite element method. The response of the structure in interface degrees of freedom is represented as the sum of the interface responses for the substructures sharing the interface, e.g.,

$$\boldsymbol{u} = \begin{Bmatrix} \boldsymbol{u}_L^{(1)} \\ \boldsymbol{u}_S^{(1)} + \boldsymbol{u}_S^{(2)} \\ \boldsymbol{u}_L^{(2)} \end{Bmatrix}, \tag{3}$$

so that each substructure's interface response is only one component of the total interface response of the structure. Of course, if this convention is adopted, substructure transient response problems must be defined and solved in such a way that the response of the structure obtained by assembling together the substructure responses is accurate.

Substructure response problems can be defined for independent computation by extracting equations from the structure equations of motion, and they will be of the form

$$\begin{bmatrix} \hat{M}_{SS}^{(k)} & M_{SL}^{(k)} \\ M_{LS}^{(k)} & M_{LL}^{(k)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_S^{(k)} \\ \ddot{u}_L^{(k)} \end{Bmatrix} + \begin{bmatrix} \hat{C}_{SS}^{(k)} & C_{SL}^{(k)} \\ C_{LS}^{(k)} & C_{LL}^{(k)} \end{bmatrix} \begin{Bmatrix} \dot{u}_S^{(k)} \\ \dot{u}_L^{(k)} \end{Bmatrix} + \begin{bmatrix} \hat{K}_{SS}^{(k)} & K_{SL}^{(k)} \\ K_{LS}^{(k)} & K_{LL}^{(k)} \end{bmatrix} \begin{Bmatrix} u_S^{(k)} \\ u_L^{(k)} \end{Bmatrix} = \begin{Bmatrix} \hat{F}_S^{(k)} \\ F_L^{(k)} \end{Bmatrix}, \tag{4}$$

where "hat" symbols identify matrix or vector partitions for which a policy for assigning the corresponding partitions in the structure equations of motion to the different substructures must be determined. Again, reordering of rows and columns may be necessary to collect all "shared" degrees of freedom together for a given substructure. Simply computing substructure responses that satisfy these equations and assembling them together will not result in an accurate representation of the response of the overall structure, because the interaction between substructures is neglected in such an approach. It must be noted that in the response of the structure, each substructure has two sources of excitation. One is the external applied force, which appears on the right hand side of the equation above, and the other is due to interaction with adjacent substructures at the interfaces. This suggests a two-step approach for computing the responses of substructures in the response of the coupled structure. The first step consists of obtaining independent substructure responses that satisfy the substructure equations of motion above. These responses neglect any interaction between substructures. Then the second step consists of correcting these substructure responses to obtain responses of substructures in the motion of the coupled structure. It will be shown that this second step can be accomplished with a surprisingly small amount of effort, and with very little information from the independent substructure responses.

If independent responses satisfying the substructure equations of motion are computed, and assembled together and inserted into the structure equations of motion, a residual $r(t)$ will be obtained. For a two-substructure structure the residual will be given by

$$\begin{aligned} r(t) &\equiv M\ddot{u} + C\dot{u} + Ku - F \\ &= \begin{bmatrix} M_{LL}^{(1)} & M_{LS}^{(1)} & 0 \\ M_{SL}^{(1)} & M_{SS}^{(1)} + M_{SS}^{(2)} & M_{SL}^{(2)} \\ 0 & M_{LS}^{(2)} & M_{LL}^{(2)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_L^{(1)} \\ \ddot{u}_S^{(1)} + \ddot{u}_S^{(2)} \\ \ddot{u}_L^{(2)} \end{Bmatrix} \\ &+ \begin{bmatrix} C_{LL}^{(1)} & C_{LS}^{(1)} & 0 \\ C_{SL}^{(1)} & C_{SS}^{(1)} + C_{SS}^{(2)} & C_{SL}^{(2)} \\ 0 & C_{LS}^{(2)} & C_{LL}^{(2)} \end{bmatrix} \begin{Bmatrix} \dot{u}_L^{(1)} \\ \dot{u}_S^{(1)} + \dot{u}_S^{(2)} \\ \dot{u}_L^{(2)} \end{Bmatrix} \\ &+ \begin{bmatrix} K_{LL}^{(1)} & K_{LS}^{(1)} & 0 \\ K_{SL}^{(1)} & K_{SS}^{(1)} + K_{SS}^{(2)} & K_{SL}^{(2)} \\ 0 & K_{LS}^{(2)} & K_{LL}^{(2)} \end{bmatrix} \begin{Bmatrix} u_L^{(1)} \\ u_S^{(1)} + u_S^{(2)} \\ u_L^{(2)} \end{Bmatrix} - \begin{Bmatrix} F_L^{(1)} \\ F_S \\ F_L^{(2)} \end{Bmatrix}. \end{aligned} \tag{5}$$

By making use of Eq. (4), the residual can be obtained as

$$\begin{Bmatrix} r_L^{(1)}(t) \\ r_S(t) \\ r_L^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} M_{LS}^{(1)}\ddot{u}_S^{(2)} + C_{LS}^{(1)}\dot{u}_S^{(2)} + K_{LS}^{(1)}u_S^{(2)} \\ r_S(t) \\ M_{LS}^{(2)}\ddot{u}_S^{(1)} + C_{LS}^{(2)}\dot{u}_S^{(1)} + K_{LS}^{(2)}u_S^{(1)} \end{Bmatrix}, \tag{6}$$

where

$$r_S(t) = (M_{SS}^{(1)} + M_{SS}^{(2)})(\ddot{u}_S^{(1)} + \ddot{u}_S^{(2)}) - \hat{M}_{SS}^{(1)}\ddot{u}_S^{(1)} - \hat{M}_{SS}^{(2)}\ddot{u}_S^{(2)}$$

$$+ (C_{SS}^{(1)} + C_{SS}^{(2)})(\dot{u}_S^{(1)} + \dot{u}_S^{(2)}) - \hat{C}_{SS}^{(1)}\dot{u}_S^{(1)} - \hat{C}_{SS}^{(2)}\dot{u}_S^{(2)}$$
$$+ (K_{SS}^{(1)} + K_{SS}^{(2)})(u_S^{(1)} + u_S^{(2)}) - \hat{K}_{SS}^{(1)}u_S^{(1)} - \hat{K}_{SS}^{(2)}u_S^{(2)}$$
$$- F_S + \hat{F}_S^{(1)} + \hat{F}_S^{(2)}. \tag{7}$$

Note that the residual associated with one substructure is given entirely in terms of the interface motion computed for adjacent substructures. Note also that $r_S(t)$ is defined in terms of the "hat" partitions of Eq. (4), and can be obtained as a null vector, if these "hat" partitions are chosen to satisfy the following:

$$\begin{aligned} \hat{M}_{SS}^{(1)} &= \hat{M}_{SS}^{(2)} = M_{SS}^{(1)} + M_{SS}^{(2)}, \\ \hat{C}_{SS}^{(1)} &= \hat{C}_{SS}^{(2)} = C_{SS}^{(1)} + C_{SS}^{(2)}, \\ \hat{K}_{SS}^{(1)} &= \hat{K}_{SS}^{(2)} = K_{SS}^{(1)} + K_{SS}^{(2)}, \\ \hat{F}_S^{(1)} &+ \hat{F}_S^{(2)} = F_S. \end{aligned} \tag{8}$$

With this as motivation, the "hat" partitions are taken to be defined this way in this paper. A physical interpretation of this choice is that for each of the independent substructure response problems, the structure is modeled as if it were clamped one node beyond the interfaces, and the excitation acting on the structure at the interfaces is divided between the substructures that share the interfaces.

The residual in the equations associated with a given substructure can be seen to be a result of including the interface motion of adjacent substructures in the given substructure's equations of motion. This interface motion for adjacent substructures was neglected in the solution of the independent substructure response problems. In order to obtain the true response of the structure, the substructure responses must be corrected to account for adjacent substructures' interface motion, so that when the substructure responses are assembled into the structure equations of motion, the residual is zero.

For the correction to the first substructure's response, note that if the interface motion for the second substructure were given, the residual in the structure equations of motion associated with the first substructure would be defined. The first substructure's response would have to be corrected by adding a response of the first substructure to the negative of the residual resulting from the interface motion of the second substructure. The second substructure's response would have to be corrected in a similar manner, if the interface motion for the first substructure were given. However, the interface motion for both substructures is not known *a priori*, because all of the interface motion will be changed as a result of the corrections to the substructure responses. The responses of both substructures will have to be corrected simultaneously, so that the response of each substructure to the negative residual due to the other's *corrected* interface motion will be added to the independently computed substructure response. The following paragraphs present a method for accomplishing this.

Because the residual is defined in terms of interface motion, it is convenient to introduce a vector $v^{(k)}(t)$ containing the interface accelerations, velocities, and displacements for the kth substructure as

$$v^{(k)}(t) = \begin{Bmatrix} \ddot{u}_S^{(k)}(t) \\ \dot{u}_S^{(k)}(t) \\ u_S^{(k)}(t) \end{Bmatrix}. \tag{9}$$

With this definition, the correction of the first substructure's response to account for the second substructure's interface motion will be the response to an excitation of the form

$$f^{(1)}(t) = \begin{bmatrix} -M_{LS}^{(1)} & -C_{LS}^{(1)} & -K_{LS}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} v^{(2)}(t), \tag{10}$$

where the degrees of freedom are ordered as in the structure equations of motion. If the second substructure's interface motion $v^{(2)}(t)$ is given only at the beginning and the end of a time interval consisting of p time steps of length Δt, the interface displacement $u_S^{(2)}(t)$ can be approximated over the time interval $0 \leq t \leq p\Delta t$ by interpolation. Hence, $u_S^{(2)}(t)$ is assumed to take the form

$$\begin{aligned} u_S^{(2)}(t) &= [\psi_1(t)I \quad \psi_2(t)I \quad \psi_3(t)I]v^{(2)}(0) \\ &+ [\psi_4(t)I \quad \psi_5(t)I \quad \psi_6(t)I]v^{(2)}(p\Delta t) \end{aligned} \tag{11}$$

where I represents a unit matrix and $\psi_i(t)$, $i = 1, \ldots, 6$ are interpolation functions that must satisfy the following end conditions:

$$\begin{array}{ll} \ddot{\psi}_1(0) = 1, & \dot{\psi}_1(0) = \psi_1(0) = \ddot{\psi}_1(p\Delta t) = \dot{\psi}_1(p\Delta t) = \psi_1(p\Delta t) = 0, \\ \dot{\psi}_2(0) = 1, & \ddot{\psi}_2(0) = \psi_2(0) = \ddot{\psi}_2(p\Delta t) = \dot{\psi}_2(p\Delta t) = \psi_2(p\Delta t) = 0, \\ \psi_3(0) = 1, & \ddot{\psi}_3(0) = \dot{\psi}_3(0) = \ddot{\psi}_3(p\Delta t) = \dot{\psi}_3(p\Delta t) = \psi_3(p\Delta t) = 0, \\ \ddot{\psi}_4(p\Delta t) = 1, & \ddot{\psi}_4(0) = \dot{\psi}_4(0) = \psi_4(0) = \dot{\psi}_4(p\Delta t) = \psi_4(p\Delta t) = 0, \\ \dot{\psi}_5(p\Delta t) = 1, & \ddot{\psi}_5(0) = \dot{\psi}_5(0) = \psi_5(0) = \ddot{\psi}_5(p\Delta t) = \psi_5(p\Delta t) = 0, \\ \psi_6(p\Delta t) = 1, & \ddot{\psi}_6(0) = \dot{\psi}_6(0) = \psi_6(0) = \ddot{\psi}_6(p\Delta t) = \dot{\psi}_6(p\Delta t) = 0. \end{array} \tag{12}$$

Quintic polynomials were used for the results obtained in this paper. Expressions for $\dot{u}_S^{(2)}(t)$ and $\ddot{u}_S^{(2)}(t)$ for defining the excitation for correcting the first substructure's response are easily obtained by differentiating the interpolation functions.

With $u_S^{(2)}(t)$ defined in terms of $v^{(2)}(0)$ and $v^{(2)}(p\Delta t)$, the corrected interface motion for the first substructure at the end of the time interval will be the sum of the response to the independent response problem and the response based on $v^{(2)}(t)$, $0 \leq t \leq p\Delta t$. Hence, it will have the form

$$v^{(1)}(p\Delta t) = v_{ind}^{(1)}(p\Delta t) + S_{12}v^{(2)}(0) + T_{12}v^{(2)}(p\Delta t), \tag{13}$$

where each column of the matrices S_{12} and T_{12} contains the first substructure's interface response at $t = p\Delta t$ to a negative residual specified by a column of the first or second matrix, respectively, on the right-hand side of Eq. (11). Using a similar approach, the corrected interface response of the second substructure at the time $t = p\Delta t$ can be expressed in terms of the first substructure's interface motion as

$$v^{(2)}(p\Delta t) = v_{ind}^{(2)}(p\Delta t) + S_{21}v^{(1)}(0) + T_{21}v^{(1)}(p\Delta t). \tag{14}$$

As mentioned above, corrected interface motion for an adjacent substructure is not known before the reconciliation is accomplished. All that is known in the two equations above is the interface motion of both substructures at $t = 0$, from initial conditions, and the interface motion obtained from the solution of the independent substructure transient response problems. However, given the set of linear equations in Eqs. (13) and (14), it is straightforward to solve for the unknowns, with the result that

$$\begin{Bmatrix} v^{(1)}(p\Delta t) \\ v^{(2)}(p\Delta t) \end{Bmatrix} = \left[I - \begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix} \right]^{-1} \begin{Bmatrix} v_{ind}^{(1)}(p\Delta t) + S_{12}v^{(2)}(0) \\ v_{ind}^{(2)}(p\Delta t) + S_{21}v^{(1)}(0) \end{Bmatrix}. \tag{15}$$

More compactly, the reconciled interface motion is given by

$$\begin{Bmatrix} v^{(1)}(p\Delta t) \\ v^{(2)}(p\Delta t) \end{Bmatrix} = [I - T]^{-1} [S \quad I] \begin{Bmatrix} v^{(1)}(0) \\ v^{(2)}(0) \\ v_{ind}^{(1)}(p\Delta t) \\ v_{ind}^{(2)}(p\Delta t) \end{Bmatrix}, \tag{16}$$

where the matrices S and T are readily identified. The corrected motion for the first substructure's local degrees of freedom at $t = p\Delta t$ is given by

$$\begin{Bmatrix} u_L^{(1)}(p\Delta t) \\ \dot{u}_L^{(1)}(p\Delta t) \end{Bmatrix} = \begin{Bmatrix} u_{Lind}^{(1)}(p\Delta t) \\ \dot{u}_{Lind}^{(1)}(p\Delta t) \end{Bmatrix} + [\, S_L^{(1)} \quad T_L^{(1)} \,] \begin{Bmatrix} v^{(2)}(0) \\ v^{(2)}(p\Delta t) \end{Bmatrix}, \tag{17}$$

where columns of the matrices $S_L^{(1)}$ and $T_L^{(1)}$ contain responses in local degrees of freedom to interpolation functions for representing interface motion. These two matrices are naturally obtained at the same time that the matrices S_{12} and T_{12} are obtained, from the solution of the same substructure response problems. The corrected motion in local degrees of freedom for the second substructure is obtained in the same manner. Once the motion in both local and shared degrees of freedom has been corrected for $t = p\Delta t$, the initial conditions have been obtained for ongoing computation of response for the next p time steps.

The developments presented here are easily applied to structures composed of more than two substructures. For example, if there are three substructures, the matrices S and T in Eq. (16) take the form

$$S = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & T_{23} \\ T_{31} & T_{32} & 0 \end{bmatrix}, \tag{18}$$

and modification of the rest of the procedure presented for two substructures is straightforward.

Infrequent Reconciliation of Substructure Responses

In the method of the preceding section, responses are computed independently for different substructures for p time steps at a time, and then the independent substructure responses are corrected to obtain substructure responses in the response of the overall coupled structure. In this section, a procedure for carrying out the reconciliation of independent substructure responses after a number of p-step time intervals is developed. This procedure will allow substructure responses to be computed independently for long periods of time without correcting for interaction between substructures.

The interface motion for the second substructure over the time interval $p\Delta t \le t \le 2p\Delta t$ can be approximated in terms of the interpolation functions introduced in the preceding section and the interface motion at the beginning and end of the time interval as

$$u_S^{(2)}(t) = [\psi_1(t^*)I \quad \psi_2(t^*)I \quad \psi_3(t^*)I]v^{(2)}(p\Delta t) + [\psi_4(t^*)I \quad \psi_5(t^*)I \quad \psi_6(t^*)I]v^{(2)}(2p\Delta t), \tag{19}$$

where $t^* = t - p\Delta t$. Recalling that substructure responses have two components including the response to external excitation, which is represented in the independent substructure responses, and the response due to interaction with adjacent substructures, which is represented in the correction to the independent substructure responses, the interface response of the first substructure at the time $t = 2p\Delta t$ will have the form

$$v^{(1)}(2p\Delta t) = v_{ind}^{(1)}(2p\Delta t) + S_{12}(2p\Delta t)v^{(2)}(0) + T_{12}(2p\Delta t)v^{(2)}(p\Delta t) + T_{12}(p\Delta t)v^{(2)}(2p\Delta t). \tag{20}$$

Here, the columns of $S_{12}(2p\Delta t)$ contain responses of the first substructure at $t = 2p\Delta t$ based on the second substructure's interface motion, which is given in terms of the interpolation functions ψ_1, ψ_2, and ψ_3 for $0 \le t \le p\Delta t$, and is extended as zero for $p\Delta t \le t \le 2p\Delta t$. Similarly, the columns of $T_{12}(2p\Delta t)$ contain responses of the first substructure at $t = 2p\Delta t$ based on interface motion of the second substructure which is given in terms of the interpolation functions ψ_4, ψ_5, and ψ_6 for $0 \le t \le p\Delta t$, and is extended in terms of ψ_1, ψ_2, and ψ_3 for $p\Delta t \le t \le 2p\Delta t$. The matrix $T_{12}(p\Delta t)$ is simply the matrix T_{12} of the preceding section.

The interface motion for both substructures at $t = 2p\Delta t$ can be written as

$$\begin{Bmatrix} v^{(1)}(2p\Delta t) \\ v^{(2)}(2p\Delta t) \end{Bmatrix} = \left(\begin{Bmatrix} v^{(1)}_{ind}(2p\Delta t) \\ v^{(2)}_{ind}(2p\Delta t) \end{Bmatrix} + S(2p\Delta t) \begin{Bmatrix} v^{(1)}(0) \\ v^{(2)}(0) \end{Bmatrix} + T(2p\Delta t) \begin{Bmatrix} v^{(1)}(p\Delta t) \\ v^{(2)}(p\Delta t) \end{Bmatrix} \right) + T(p\Delta t) \begin{Bmatrix} v^{(1)}(2p\Delta t) \\ v^{(2)}(2p\Delta t) \end{Bmatrix} \tag{21}$$

with S and T matrices defined in terms of 0, S_{12}, S_{21}, 0, etc., as in the last section. Solving for $v^{(1)}(2p\Delta t)$ and $v^{(2)}(2p\Delta t)$ gives the result

$$v(2p\Delta t) = [I - T(p\Delta t)]^{-1} [\, S(2p\Delta t) \quad T(2p\Delta t) \quad I \,] \begin{Bmatrix} v(0) \\ v(p\Delta t) \\ v_{ind}(2p\Delta t) \end{Bmatrix} \tag{22}$$

where

$$v(ip\Delta t) \equiv \begin{Bmatrix} v^{(1)}(ip\Delta t) \\ v^{(2)}(ip\Delta t) \end{Bmatrix}. \tag{23}$$

Recalling that

$$v(p\Delta t) = [I - T(p\Delta t)]^{-1} [\, S(p\Delta t) \quad I \,] \begin{Bmatrix} v(0) \\ v_{ind}(p\Delta t) \end{Bmatrix}, \tag{24}$$

and letting $A \equiv [I - T(p\Delta t)]^{-1}$, $S_i \equiv S(ip\Delta t)$, and $T_i \equiv T(ip\Delta t)$, $v(2p\Delta t)$ can be obtained in terms of initial conditions and independent substructure responses as

$$v(2p\Delta t) = A[\, (S_2 + T_2 A S_1) \quad T_2 A \quad I \,] \begin{Bmatrix} v(0) \\ v_{ind}(p\Delta t) \\ v_{ind}(2p\Delta t) \end{Bmatrix}. \tag{25}$$

The corrected interface motion at $t = 3p\Delta t$ can be found using the same approach. When the interface motion for the different substructures is assumed in terms of interpolation functions as in Eq. (19), linear equations involving $v(3p\Delta t)$ can be written as in Eq. (21). These equations can be solved for $v(3p\Delta t)$, yielding the result

$$v(3p\Delta t) = A[\, S_3 \quad T_3 \quad T_2 \quad I \,] \begin{Bmatrix} v(0) \\ v(p\Delta t) \\ v(2p\Delta t) \\ v_{ind}(3p\Delta t) \end{Bmatrix}. \tag{26}$$

Interpolation functions are simply extended as zero into the time interval $2p\Delta t \leq t \leq 3p\Delta t$ in the generation of responses for matrices S_3 and T_3. Inserting the expressions for $v(p\Delta t)$ and $v(2p\Delta t)$ from Eqs. (24) and (25) gives $v(3p\Delta t)$ in terms of initial conditions and independent substructure responses as

$$v(3p\Delta t) = A[\, (S_3 + T_2 A S_2 + (T_3 A + (T_2 A)^2) S_1) \quad (T_3 A + (T_2 A)^2) \quad (T_2 A) \quad I \,] \begin{Bmatrix} v(0) \\ v_{ind}(p\Delta t) \\ v_{ind}(2p\Delta t) \\ v_{ind}(3p\Delta t) \end{Bmatrix}. \tag{27}$$

This result can be generalized for finding the corrected interface motion at a time $t = mp\Delta t$, with the result that

$$v(mp\Delta t) = A \left[\left(\sum_{i=0}^{m-1} B_i S_{m-i} \right) \quad B_{m-1} \quad B_{m-2} \quad \cdots \quad B_0 \right] \begin{Bmatrix} v(0) \\ v_{ind}(p\Delta t) \\ \vdots \\ v_{ind}(mp\Delta t) \end{Bmatrix}, \tag{28}$$

where $B_0 \equiv I$, and the other B_i matrices are defined by the recursive formula

$$B_i \equiv \sum_{l=0}^{i-1} (T_{i-l+1}\bar{A})B_l, \tag{29}$$

so that $B_1 = T_2A$, $B_2 = T_3A + (T_2A)^2$, $B_3 = T_4A + T_3AT_2A + T_2AT_3A + (T_2A)^3$, etc. Defining a matrix C_m as

$$C_m \equiv A\left[\left(\sum_{i=0}^{m-1} B_i S_{m-i}\right) \quad B_{m-1} \quad B_{m-2} \quad \cdots \quad B_0\right], \tag{30}$$

the corrected interface motion can be obtained separately for each substructure by partitioning C_m into upper and lower halves $\bar{C}_m^{(1)}$ and $\bar{C}_m^{(2)}$, and multiplying each by the vector on the right hand side of Eq. (28). For parallel computation, if different processors are assigned to different substructures, the processor for the kth substructure only needs to have access to $\bar{C}_m^{(k)}$ and the interface motion computed independently for all substructures for every pth time step.

After interface motion has been corrected for $t = mp\Delta t$, the motion for local degrees of freedom for each substructure can be corrected. As an example, the corrected local motion for the first substructure will be given by

$$\begin{Bmatrix} \boldsymbol{u}_L^{(1)}(mp\Delta t) \\ \dot{\boldsymbol{u}}_L^{(1)}(mp\Delta t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{u}_{Lind}^{(1)}(mp\Delta t) \\ \dot{\boldsymbol{u}}_{Lind}^{(1)}(mp\Delta t) \end{Bmatrix} + [\, S_{Lm}^{(1)} \quad T_{Lm}^{(1)} \quad \cdots \quad T_{L1}^{(1)} \,] \begin{Bmatrix} \boldsymbol{v}^{(2)}(0) \\ \boldsymbol{v}^{(2)}(p\Delta t) \\ \vdots \\ \boldsymbol{v}^{(2)}(mp\Delta t) \end{Bmatrix}, \tag{31}$$

where the matrices $S_{Li}^{(k)}$ and $T_{Li}^{(k)}$ contain responses in local degrees of freedom to interface motion given in terms of interpolation functions, and are analogous to the S_i and T_i matrices used above in terms of subscript numbering. The vector of the second substructure's corrected interface motion at every pth time step is given in terms of the independently computed interface responses as

$$\begin{Bmatrix} \boldsymbol{v}^{(2)}(0) \\ \boldsymbol{v}^{(2)}(p\Delta t) \\ \vdots \\ \boldsymbol{v}^{(2)}(mp\Delta t) \end{Bmatrix} = \begin{bmatrix} 0 & \mathrm{I} & 0 & 0 & 0 & \cdots & 0 \\ [& \bar{C}_1^{(2)} &] & 0 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ [& & \bar{C}_m^{(2)} & & & &] \end{bmatrix} \begin{Bmatrix} \boldsymbol{v}(0) \\ \boldsymbol{v}_{ind}(p\Delta t) \\ \vdots \\ \boldsymbol{v}_{ind}(mp\Delta t) \end{Bmatrix}. \tag{32}$$

Therefore, the product of the matrix on the right hand side of Eq. (31) and the matrix on the right hand side of Eq. (32) is the matrix by which the vector of independently computed interface responses must be multiplied to obtain the correction for the motion in local degrees of freedom for the first substructure. The same approach is taken to find the correction for the motion in local degrees of freedom for the second substructure.

To summarize, the developments presented in this section permit the independent computation of response for different substructures for a total time interval of length $mp\Delta t$. The interface motion for all of the substructures at the end of this time interval can be corrected using Eq. (28), and then the motion for local degrees of freedom for each of the substructures can be corrected as shown above. Once these corrections are made, initial conditions are obtained so that independent computation of substructure responses can proceed again for another $mp\Delta t$. The amount of computation required for the corrections is very small compared to the amount of computation required for obtaining the independent substructure responses. The computational "overhead" that is required for this method consists of obtaining substructure responses to interface motion specified in terms of interpolation

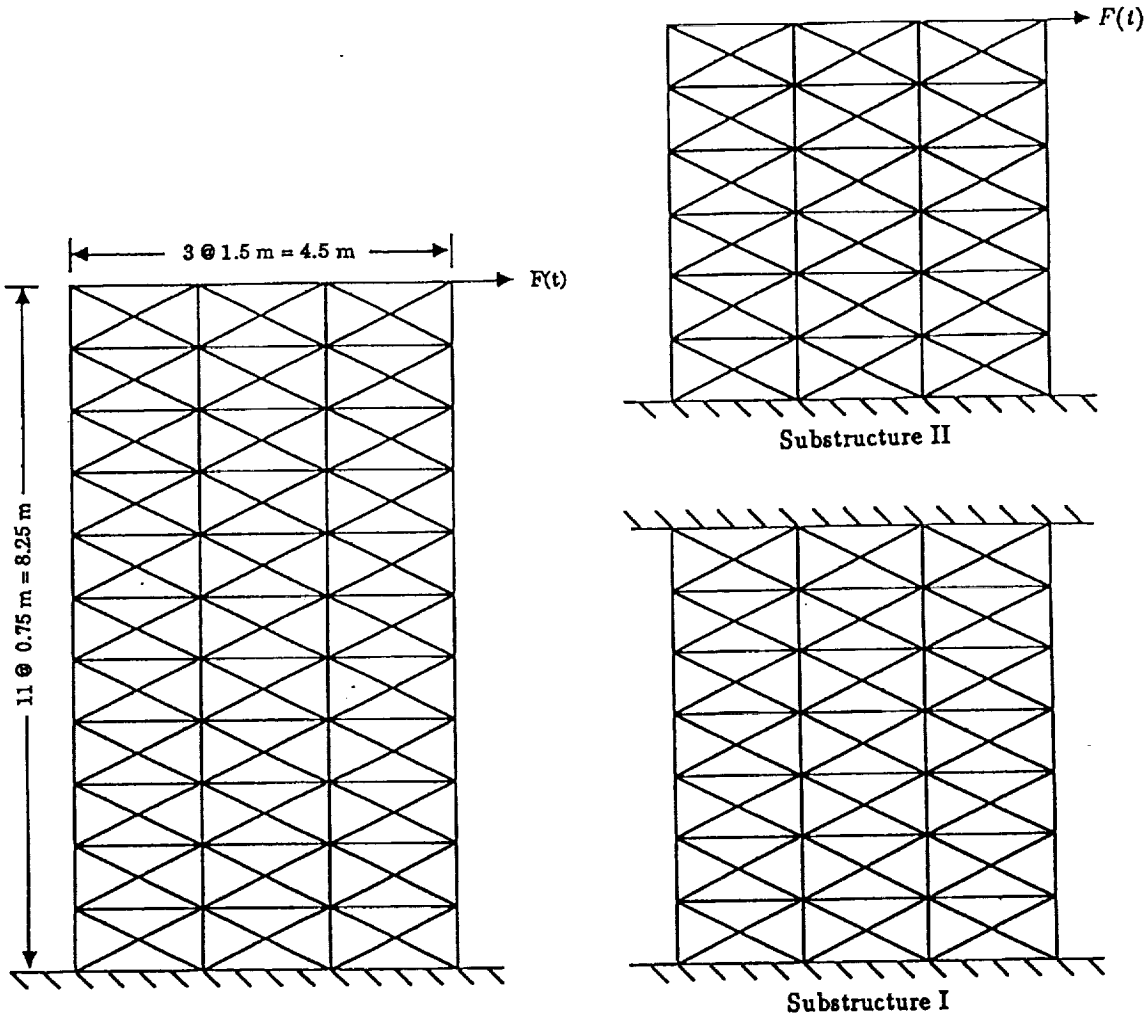

Figure 1: Plane truss used in the numerical example, and its division into substructures.

functions, and carrying out the matrix operations outlined above to obtain the matrices required for making corrections. This overhead is justified if the transient response of the structure must be computed for a long time. The amount of computation required both for the "overhead" operations and for the corrections is determined by the dimensions of the matrices involved, which is determined in turn by how many shared and local degrees of freedom are associated with each of the substructures.

Numerical Example

The algorithm presented in this paper is demonstrated on an example structure which is shown in Fig. 1. The structure is a plane truss composed of 143 aluminum members, each of which has an elastic modulus of $E = 70 \times 10^9$ N/m^2, a cross-sectional area of $A = 4 \times 10^{-4}$ m^2, and a density of $\rho = 2710$ kg/m^3. The dimensions are as shown. A force is applied to the top right corner of the truss starting at $t = 0$, and it is given by

$$F(t) = 5(1 - \cos \Omega t) \text{ (Newtons)}, \tag{33}$$

where $\Omega = 590.3$ radians per second, which is between the second and third natural frequencies of the structure. The truss has eighty-eight degrees of freedom, and is assumed to have proportional

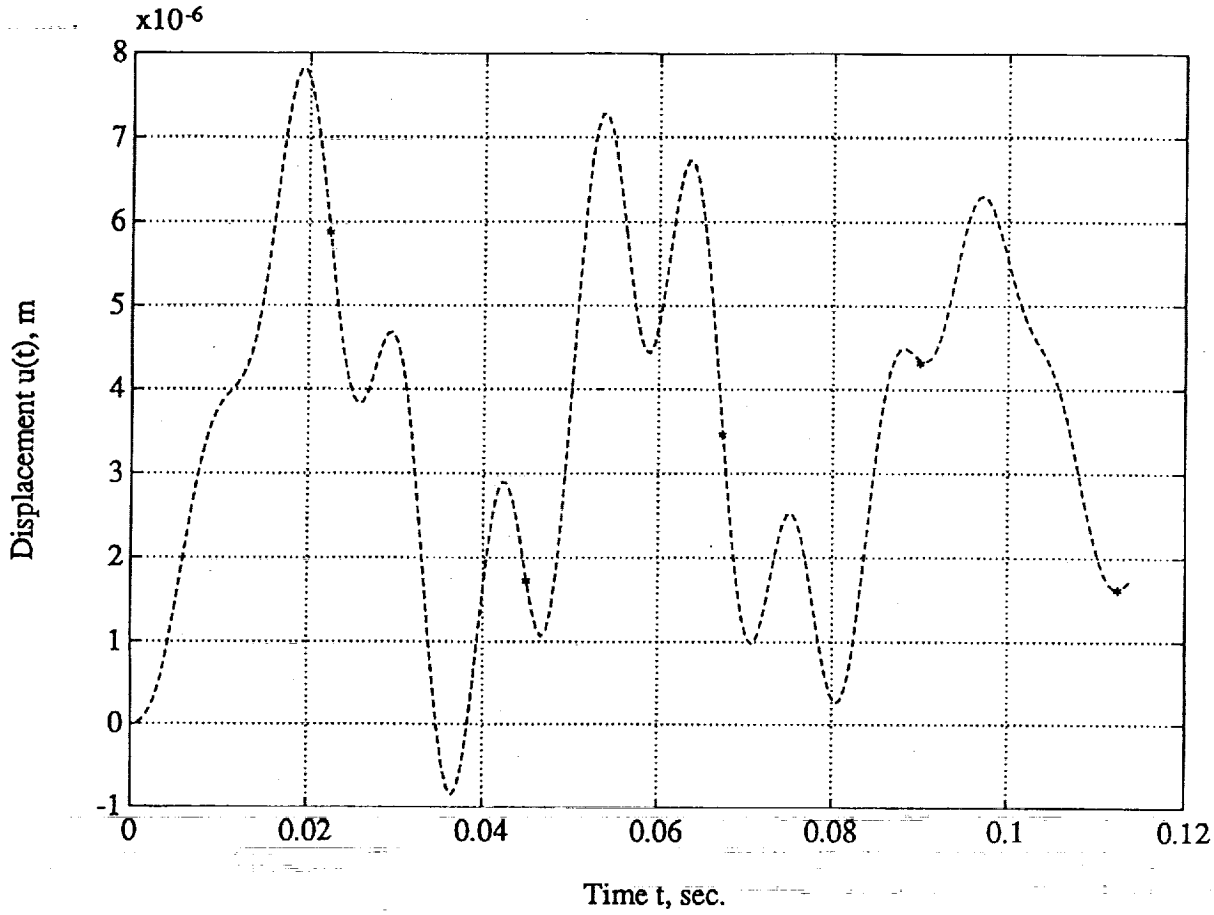

Figure 2: Plots of exact response (dashed line) and computed response (asterisks).

damping of the form $C = \alpha M + \beta K$, where α and β are chosen to give modal damping factors between one and five percent. For application of the algorithm presented in this paper, the structure was partitioned at the top of the sixth bay into two substructures, which are also shown in Fig. 1. Note that each substructure is modeled in the algorithm as being effectively clamped one truss bay beyond the interface, as shown in the figure.

In Fig. 2, the horizontal displacement of the structure at the point where the excitation is applied is plotted. The dashed line is a plot of the exact response, obtained from a mode-by-mode exact solution, and the asterisks represent values that were obtained using the algorithm of this paper. The responses of the two substructures were obtained using an algorithm that finds the exact response to a piecewise linear approximation of the excitation.[12] A time step of $\Delta t = 3.74 \times 10^{-4}$ seconds was used, which is equal to about one twenty-eighth of the period of the excitation, and is also approximately equal to the period of the highest mode of the structure. For larger time steps, the error becomes visible on a plot scaled as in Fig. 2, when the piecewise linear algorithm is used on the structure as a whole. In this example, substructure responses were computed independently for sixty time steps at a time, and then corrections to the independent substructure responses were made based on the interface motion computed for every tenth time step. Therefore, the quintic interpolation polynomials for interface motion were defined over time intervals of length $p\Delta t$ with p equal to ten, and there were six of these time intervals in each time period over which independent substructure

responses were computed.

Because the response of the structure was only corrected for every sixtieth time step, the asterisks on the plot in Fig. 2 are sixty time steps apart. It should be noted, however, that the response for any degree of freedom at any time can be obtained in a straightforward manner with a small amount of additional computation. From the plot of Fig. 2, it is evident that the accuracy obtained in this example is quite adequate for most purposes, even though the corrections to independent substructure responses were made based on a very limited amount of information. The only approximations made in obtaining these results were in the piecewise linear approximation of the excitation and the piecewise quintic approximations of the interface motion.

Summary

In this paper, an algorithm is presented for computing the transient response of structures by computing the transient responses of substructures. The algorithm is well suited for parallel implementation, where a different processor would be assigned to each substructure. The fact that computation can proceed independently for different substructures for dozens of time steps at a time reduces the interdependence between processors, which can be of considerable importance when different substructures require different amounts of computational effort per time step. The correction of independently computed substructure responses to obtain the response of the structure acting as a whole requires only the interface motion computed for substructures at occasional points in time. This correction of substructure responses can be done independently for different substructures once the interface motion for all of the substructures has been computed, and this correction requires very little effort. Because of this, the total amount of computation required using this approach will be only slightly greater than the amount required to solve the transient response problem for the structure as a whole for many problems. A surprisingly high level of accuracy is obtained using this algorithm, in view of how little information is required for making corrections to the independent substructure responses.

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