N90-23084

Approximation in LQG Control of a Thermoelastic Rod

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Abstract

Control and estimator gains are computed for linear-quadratic-Gaussian (LQG) optimal control of the axial vibrations of a thermoelastic rod. The computations are based on a modal approximation of the partial differential equations representing the rod, and convergence of the approximations to the control and estimator gains is the main issue.

1 Introduction

The axial vibrations of a uniform rod are represented by a one-dimensional wave equation with constant coefficients, and thermoelastic damping in the rod is represented by a one-dimensional heat equation coupled to the wave equation. The solutions to the wave and heat equations are, respectively, the axial displacement and temperature fields in the rod. [See 1, 2].

The length of the rod in this paper is normalized to 1. For active control, a single force is distributed parallel to the rod, uniformly over the portion $s_0 \le s \le s_1$ of the rod. The equations of motion of the plant are then

$$\rho w_{tt} = (\lambda + 2\mu) w_{ss} - \alpha(3\lambda + 2\mu)\theta_s + bu + b\eta_1, \qquad t > 0, \quad 0 < s < 1, \tag{1.1}$$

$$\rho c \theta_t = k\theta_{ss} - \theta_0 \alpha (3\lambda + 2\mu) w_{ts}, \qquad t > 0, \quad 0 < s < 1, \tag{1.2}$$

where

$$b(s) = \begin{cases} 1, & s_0 \le s \le s_1, \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

*This author was supported by AFOSR Grant 870373.
†This author was supported by AFOSR Grant 870356.

In these equations, $w(t) = w(t,s)$ is the axial displacement, $\theta(t) = \theta(t,s)$ is the temperature distribution and $u(t)$ is the control force. We assume that the actuator force has the form $u + \eta_1$ where $u(t)$ is the known control function and η_1 is zero-mean Guassian white noise with intensity q_1. The constants $\rho, \alpha, \lambda, \mu, \theta_0, c$ and k are physical constants with values to be given later.

We assume that we have a sensor that measures the displacement at the left end of the rod segment over which the actuator force is distributed. This measurement is then

$$y(t) = w(t, s_0) + \eta_0 \tag{1.4}$$

where η_0 is zero-mean Guassian white noise with intensity 1.

In this paper, we use the boundary conditions

$$\begin{aligned} w(t,0) = w(t,1) = 0, \\ \theta_s(t,0) = \theta_s(t,1) = 0, \end{aligned} \tag{1.5}$$

which mean that the rod is clamped and insulated at both ends. Because of the insulated, or Neumann, bondary conditions for the heat equation, zero is an eigenvalue of the open-loop system and the corresponding eigenvector represents a constant temperature distribution. The span of this eigenvector is an uncontrollable and unobservable subspace, and the orthogonal complement of this subspace contains only states for which the average temperature along the rod is zero. Whatever the initial conditions and the control function, the average temperature in the rod therefore is a constant function of time. We will denote this average temperature by $\bar{\theta}$.

2 The Control Problem

We set $\tilde{\theta} = \theta - \bar{\theta}$ and define the state vector

$$x(t) = (w(t), w_t(t), \tilde{\theta}(t)). \tag{2.1}$$

We take the state space to be the Hilbert space

$$E = H_0^1(0,1) \times L_2(0,1) \times L_2(0,1) \tag{2.2}$$

where $H_0^1(0,1)$ is the first-order Sobolev space containing functions that vanish at both ends of the interval. The system in (1.1) – (1.4) then has the form

$$\dot{x} = Ax + Bu + B\eta_1, \tag{2.3}$$

$$y = Cx + \eta_0, \tag{2.4}$$

where A generates a strongly continuous semigroup of contraction operators on E (see [3]). We note that $|x(t)|_E^2$ is the sum of twice the mechanical energy in the rod and the integral over the rod of $\tilde{\theta}^2$ at time t.

The LQG optimal control problem in this paper is to find u to minimize

$$J = \lim_{\bar{t}\to\infty} \mathcal{E}\{\frac{1}{\bar{t}} \int_0^{\bar{t}} [\langle Qx(t), x(t)\rangle_E^2 + u^2(t)]dt\} \tag{2.5}$$

where

$$Qx = (w, w_t, 0). \tag{2.6}$$

The operator Q is chosen so that mechanical energy is penalized but temperature variations from $\bar{\theta}$ are not penalized. As usual, this problem separates into a deterministic linear-quadratic regulator problem and a state estimation, or filtering, problem. Each of these problems has a unique solution because the open-loop system can be shown to be uniformly exponentially stable.

The optimal control law has the form

$$u(t) = -F\hat{x}(t) \tag{2.7}$$

where the state estimate $\hat{x}$ satisfies

$$\dot{\hat{x}} = A\hat{x} + Bu + \hat{F}(y - C\hat{x}). \tag{2.8}$$

The gain operators F and $\hat{F}$ are given by

$$F = B^*P \tag{2.9}$$

and

$$\hat{F} = \hat{P}C^* \tag{2.10}$$

where P and $\hat{P}$ are the unique nonnegative self-adjoint bounded linear operators on E satisfying the Riccati operator equations [4,5]

$$A^*P + PA - PBB^*P + Q = 0 \tag{2.11}$$

and

$$A\hat{P} + \hat{P}A^* - \hat{P}C^*C\hat{P} + q_1BB^* = 0. \tag{2.12}$$

The gain operators can be represented in terms of elements of E; i.e.,

$$Fx = \langle (f_1, f_2, f_3), x \rangle_E, \qquad (f_1, f_2, f_3) \in E, \tag{2.13}$$

$$\hat{F} = (\hat{f}_1, \hat{f}_2, \hat{f}_3) \in E. \tag{2.14}$$

3 Approximation

We approximate the infinite dimensional system in (2.3) and (2.4) with a sequence of finite dimensional control systems of the form

$$\dot{x}_n = A_n x_n + B_n u + B_n \eta_1, \tag{3.1}$$

$$y_n = C_n x_n + \eta_0. \tag{3.2}$$

In [10], we compared two Galerkin approximations: a finite element scheme in which linear splines were the basis vectors and a modal scheme in which the open-loop eigenvectors of the distributed system were the basis vectors. The modal scheme gave faster convergence for approximations to control gains like those in (2.9) and (2.13). We use the modal scheme here.

It is easy to see that the eigenspaces of the open-loop thermoelastic rod with the boundary conditions in (1.5) are three-dimensional subspaces each spanned by a two-dimensional eigenspace of the undamped wave equation and a one-dimensional eigenspace of the heat equation. The eigenvectors are sine waves for the wave equation and cosine waves for the heat equation. Since the modal approximation amounts to projection onto a sequence of complete orthogonal subspaces, it is easy to show that the approximations to the open-loop semigroup and adjoint semigroup converge strongly, as commonly needed in numerical solution of infinite dimensional Riccati equations for distributed systems [4-7].

For each n, we approximate the solutions to the infinite dimensional Riccati equations (2.11) and (2.12) by solving a pair of finite dimensional Riccati equations involving A_n, B_n and C_n. With the solutions to these matrix Riccati equations, we approximate the control and estimator gains as in [5]. In particular, we compute approximations to the functional gains (f_1, f_2, f_3) and $(\hat{f}_1, \hat{f}_2, \hat{f}_3)$ in (2.13) and (2.14).

4 Numerical Results

We chose the constants in (1.1) and (1.2) for an aluminum rod of length $100in$ (see [8, 9]). We normalized the length to 1, so that the constants take the numerical values

$$\begin{aligned} \rho &= 9.82 \times 10^{-2} \quad \lambda = 2.064 \times 10^{-1} \quad \mu = 1.11 \times 10^{-1} \\ \alpha &= 1.29 \times 10^{-3} \quad c = 5.40 \times 10^{-1} \\ k &= 7.02 \times 10^{-7} \quad \theta_0 = 68. \end{aligned}$$

The actuator force is spread uniformly between $s_0 = .385$ and $s_1 = .486$, and the intensity of η_1 is $q_1 = 1$.

Figures 1-3 show the approximations to the control functional gains (f_1, f_2, f_3) for a range of approximation orders n, and Figures 4-6 show the approximations to the estimator functional gains $(\hat{f}_1, \hat{f}_2, \hat{f}_3)$. (The number of modal subspaces used is n.) Because thermoelastic damping is very light in aluminum, as in most metals, many modes of the rod must be controlled actively. Therefore, many modal subspaces must be used in the approximations before the functional gains converge.

5 References

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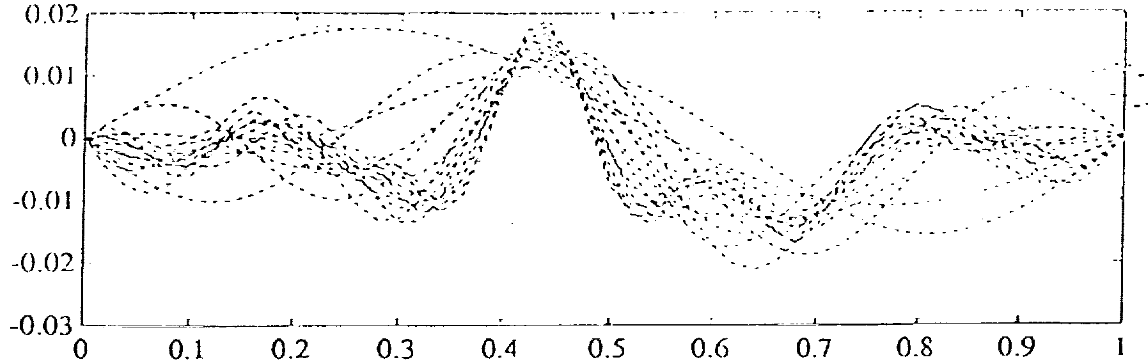

Figure 1(a): f1, the control gain, n = 2 - 17

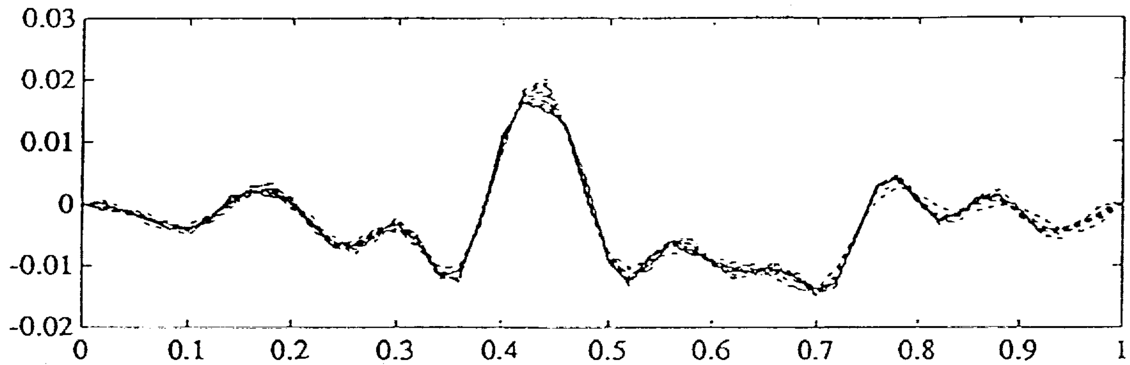

Figure 1(b): f1, the control gain, n = 18 - 33

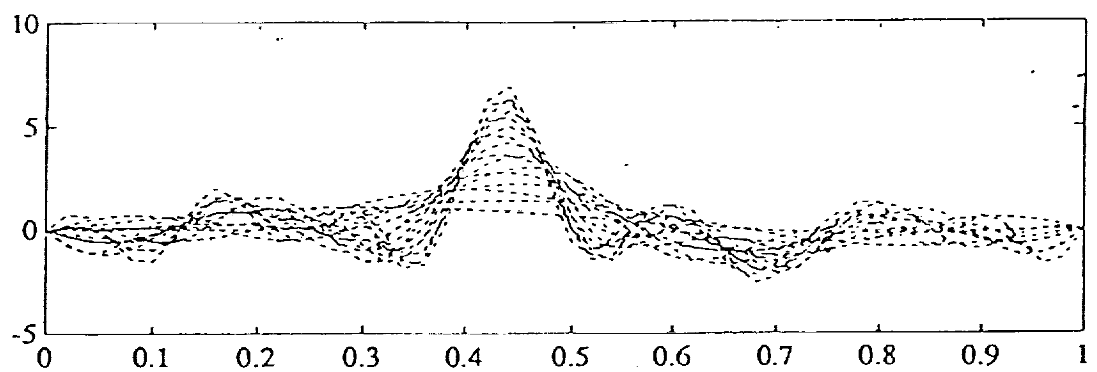

Figure 2(a): f2, the control gain, n = 2 - 17

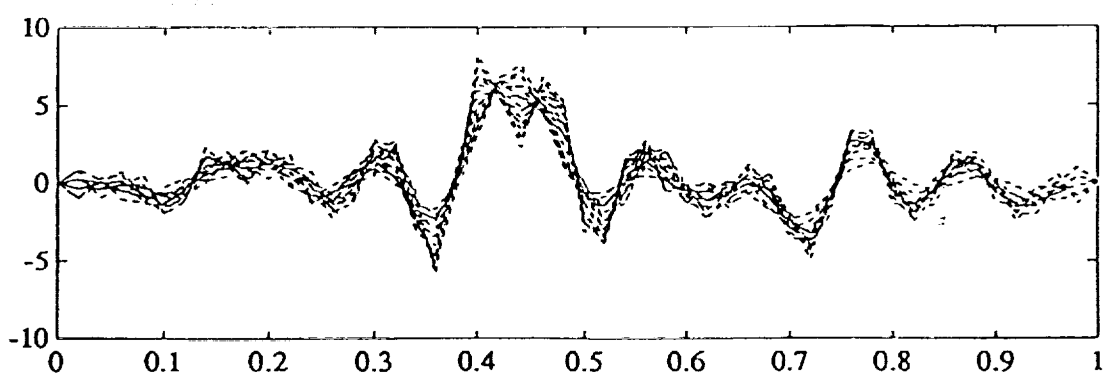

Figure 2(b): f2, the control gain, n = 18 - 33

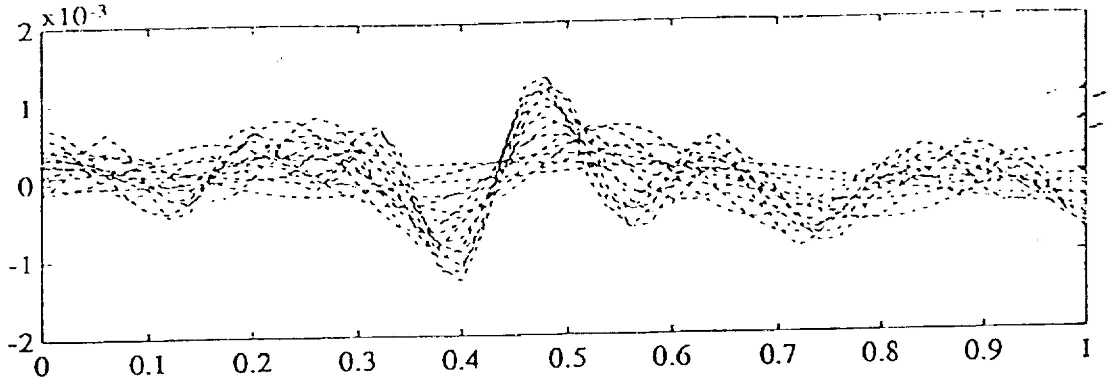

Figure 3(a): f3, the control gain, n = 2 - 17

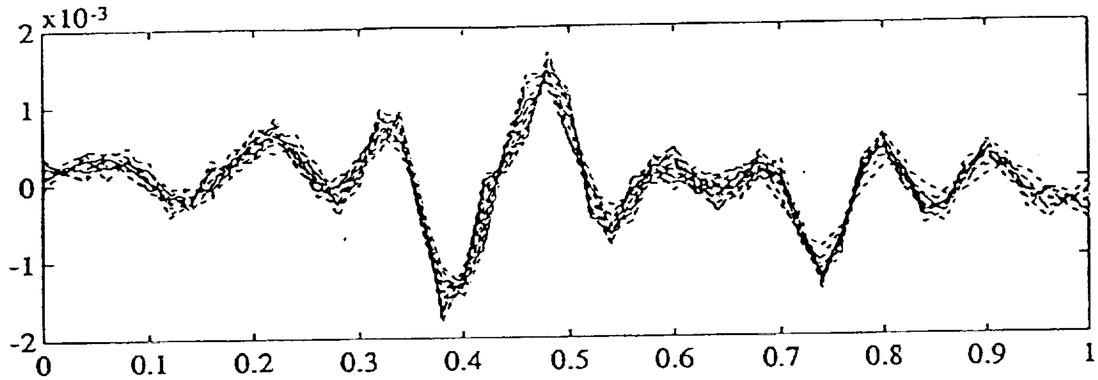

Figure 3(b): f3, the control gain, n = 18 - 33

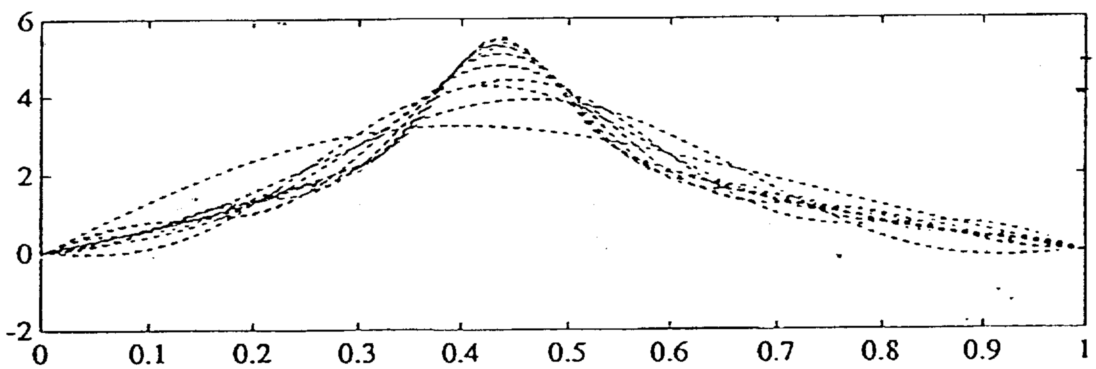

Figure 4(a): f1, the estimator gain, n = 2 - 17

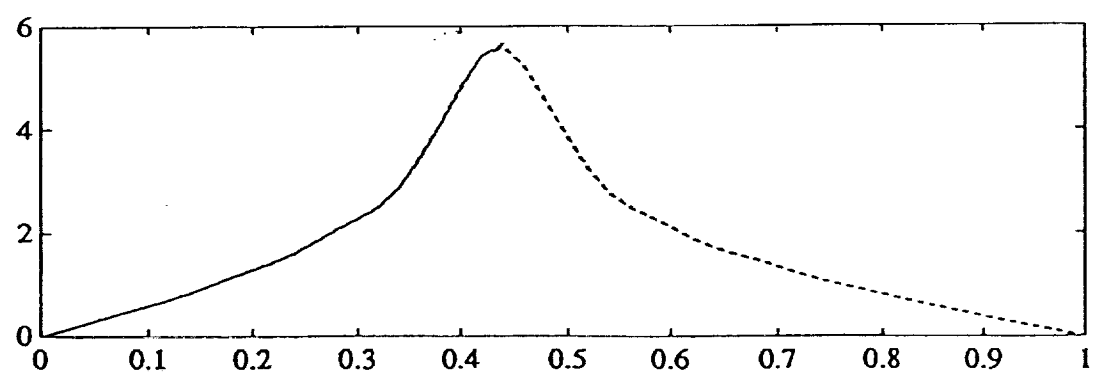

Figure 4(b): f1, the estimator gain, n = 18 - 33

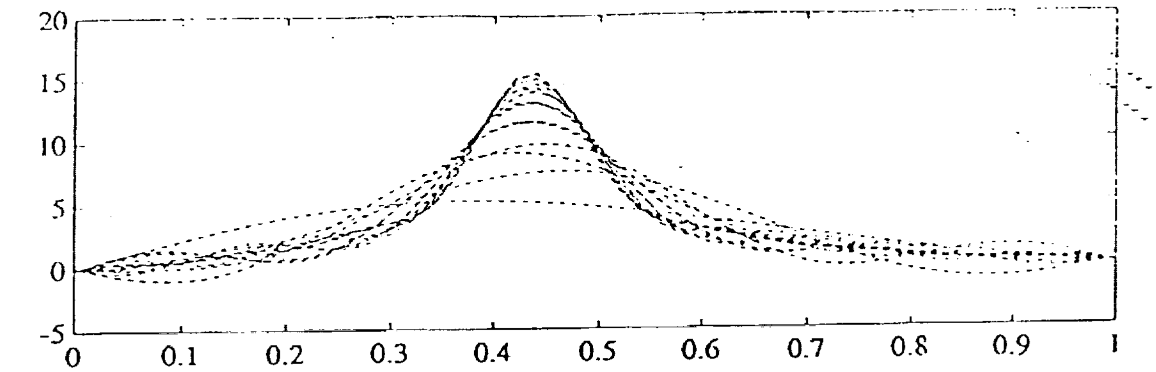

Figure 5(a): f2, the estimator gain, n = 2 - 17

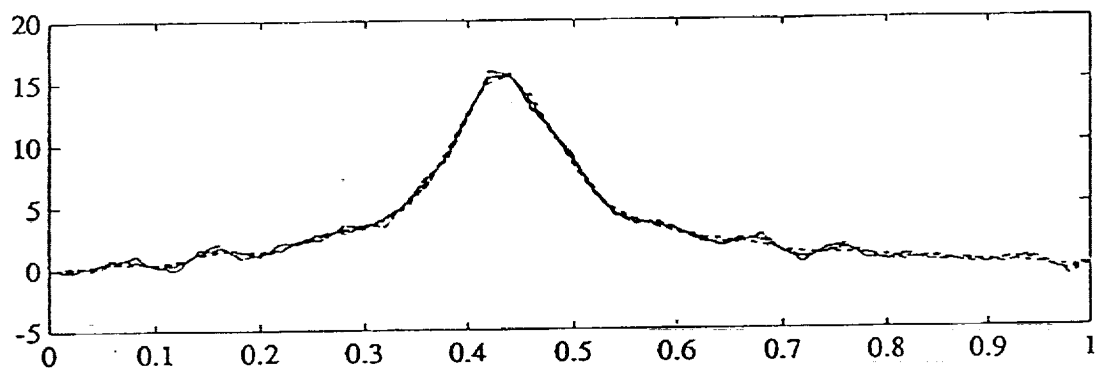

Figure 5(b): f2, the estimator gain, n = 18 - 33

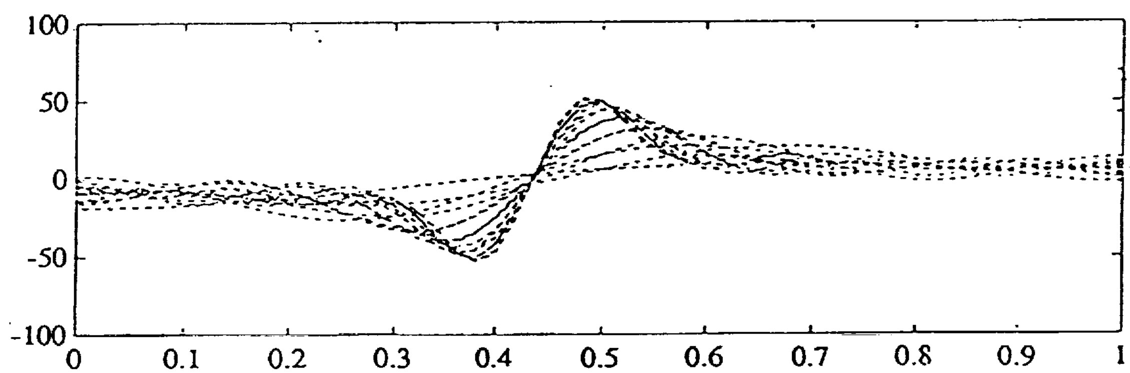

Figure 6(a): f3, the estimator gain, n = 2 - 17

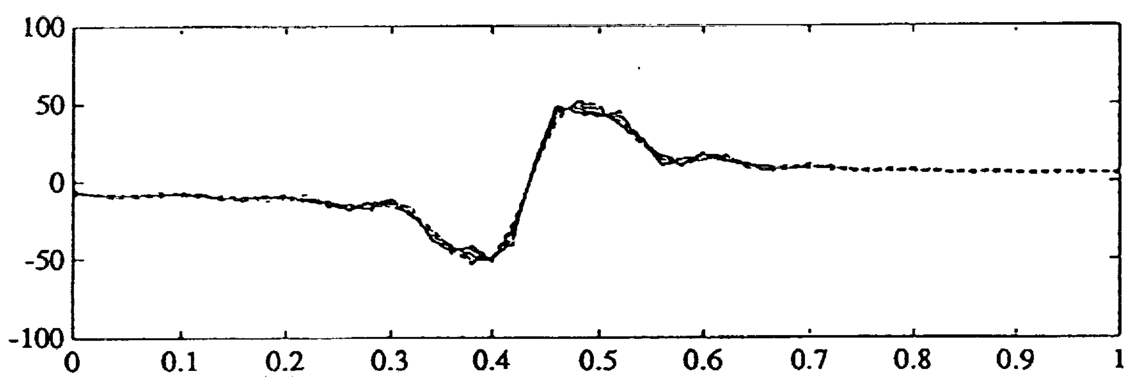

Figure 6(b): f3, the estimator gain, n = 18 - 33