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INNOVATIVE DESIGN OF COMPOSITE STRUCTURES: AXISYMMETRIC DEFORMATIONS OF UNSYMMETRICALLY LAMINATED CYLINDERS LOADED IN AXIAL COMPRESSION

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ABSTRACT

The study focuses on the axisymmetric deformation response of unsymmetrically laminate cylinders loaded in axial compression by known loads. A geometrically nonlinear analysis is used. Though buckling is not studied, the deformations can be considered to be the prebuckling response. Attention is directed at three 16 layer laminates: a $(90_8/0_8)_T$; a $(0_8/90_8)_T$; and a $(0/90)_{4s}$. The symmetric laminate is used as a basis for comparison, while the two unsymmetric laminates were chosen because they have equal but opposite bending-stretching effects. Particular attention is given to the influence of the thermally-induced preloading deformations that accompany the cool-down of any unsymmetric laminate from the consolidation temperature. Simple support and clamped boundary conditions are considered. It is concluded that: (1) The radial deformations of an unsymmetric laminate are significantly larger than the radial deformations of a symmetric laminate. For both symmetric and unsymmetric laminates the large deformations are confined to a boundary layer near the ends of the cylinder; (2) For this nonlinear problem the length of the boundary layer is a function of the applied load; (3) The sign of the radial deformations near the supported end of the cylinder depends strongly on the sense (sign) of the laminate asymmetry; (4) For unsymmetric laminates, ignoring the thermally-induced preloading deformations that accompany cool-down results in load-induced deformations that are under predicted; and (5) The support conditions strongly influence the response but the influence of the sense of asymmetry and the influence of the thermally-induced preloading deformations are independent of the support conditions.

INTRODUCTION

Composite cylindrical structures are known for their efficiency. In addition, they are well suited for fabrication by automated fiber-placement techniques such as filament winding. Axial loadings are common, as are pressure, and to a lesser extent, torsional and bending loads. Cylinders subjected to axial end loads are susceptible to buckling or collapse if the load reaches significant levels, or if the cylinder contains imperfections in the form of out-of-roundness, variations in thickness, material flaws, or combinations of these. In addition to their structural efficiency, composite cylindrical structures are somewhat more tolerant of unsymmetric lamination sequences than flat laminates. Despite the tendency of an unsymmetric laminate to deform significantly when cooled from its processing temperature, cylindrical forms must remain cylindrical due to the axisymmetric nature of the basic geometry. In many applications involving cylinders with hundreds of layers in the cylinder walls, the stacking arrangement is often unsymmetrical simply because it is easier to continue a winding sequence through the wall rather than reverse the sequence half way through the wall. For the many-layered case, the degree of asymmetry is small and is not generally accounted for in design and analysis procedures.

There is the general feeling that having any degree of asymmetry is not good. However, there is an increasing interest in unsymmetric laminates for structural applications. Unsymmetric laminates may indeed be the minimum weight design, the addition of more layers to make the laminate symmetric simply adding weight to the structure with no increase in performance. Also, manufacturing issues, just mentioned, may dictate that an unsymmetric laminate is less expensive to fabricate. And finally, there is a considerable degree of elastic coupling in unsymmetric laminates that is not present in symmetric laminates. For structural tailoring, these couplings may prove beneficial. This reports details the finding of a very basic study focused on determining some of the characteristics of unsymmetrically laminated cylinders, in particular, unsymmetrically laminated cylinders subjected to axial compressive loads. Considerable work has been done by others in the area of axially-loaded composite cylinders. However, in numerical examples in the past work that have involved unsymmetric laminates, no attention has been given to the fact that when an unsymmetrically laminated cylinder is actually constructed, the temperature change from the processing temperature to, say, room temperature results in a cylinder that has a deformed shape before it is even loaded [1-3]. That is, there is a preloading deformed shape that, when loaded, deforms further. When considering prebuckling deformations in a buckling analysis, for example, or the response in a collapse analysis, does inclusion of these thermally-induced preloading deformations have an effect on the predicted result? The purpose of the work here is to begin to accurately model this thermally-induced deformation in unsymmetric laminates and determine if the thermally-induced preloading deformations, in addition to the presence of the bending-stretching of unsymmetric laminates, 1) has an influence on response, and; 2) can be used to advantage. As an example of point 2, it might be possible that bending-stretching coupling could be used to increase the buckling load, or alter the sensitivity of the cylinder to post-buckling collapse.

The particular work reported on here focuses on a derivation of the equations governing the prebuckling deformations, including thermal effects, and the equations governing buckling. For the present, only perfect cylinders are considered, i.e., perfectly round, uniform material properties, etc. Buckling and collapse are not addressed. The next sections define the geometry, coordinate system, and nomenclature used to study cylinder response, and then proceed to derive the equilibrium and buckling equations. Though the latter are not used here, they are derived for future use. The derivation is based on variational principles and total potential energy. The first variation of the total potential energy is used to establish the equilibrium conditions, while the second variation, through the Trefftz stability criterion, is used to develop the buckling equations. Following the derivation the axisymmetric prebuckling responses of several simple unsymmetrically laminated cylinders to an axial compressive load are studied. The influence of the thermally-induced preloading deformations, and the influence of changing the sign of the bending-stretching coupling effects are studied. This prebuckling study is limited in that only three different cylinders are studied. These are: a $(0_8/90_8)_T$ laminate; a $(90_8/0_8)_T$ laminate; and a $(0/90)_{4S}$ laminate. The first two laminates exhibit extremes in asymmetry, while the third laminate is a simple symmetric laminate with the same thickness and the same number of layers with fibers in both the axial and circumferential directions as the two unsymmetric laminates.

GEOMETRY AND COORDINATE SYSTEM

The cylinder is assumed to be oriented in a global rectangular coordinate system with the X axis coincident with the centerline of the cylinder, as illustrated in fig. 1. The global Z axis is up, and the global Y axis is to the right. The origin of the coordinate system can be at the midspan of the cylinder, or at one end, depending on which location is convenient for the particular analysis. Here it shall be at midspan. The cylindrical coordinates used in the analysis consist of the axial coordinate, x, which is coincident with the X axis, θ, which is measured positive from the +Z axis toward the +Y axis, and r , which is measured outward from the X axis. The cylinder has mean radius R, measured to midwall, and thickness H. The coordinate z is measured outward from the mean radius. The displacement in the axial direction is denoted u(x, θ,r), that in the circumferential direction (positive in the direction of $+\theta$) as v(x, θ,r), and that in the radial direction (positive outward) as w(x, θ,r) . The temperature change considered is assumed to be spatially uniform and is denoted as ΔT, ΔT being positive for temperature increases. Though interest here will be with applied axial end loads, the equations will be derived for the case of applied end torsional and applied radial loads.

DERIVATION OF EQUILIBRIUM AND STABILITY EQUATIONS

The total potential energy of a cylinder deformed by known applied loads, and subjected to preloading deformations, is written as

$$\pi = \frac{1}{2}\int_r\int_\theta\int_x\left(\left(\sigma_x - \sigma_x^P\right)\varepsilon_x + \left(\sigma_\theta - \sigma_\theta^P\right)\varepsilon_\theta + \left(\tau_{x\theta} - \tau_{x\theta}^P\right)\gamma_{x\theta}\right)rd\theta drdx + \pi_{load} \tag{1}$$

where π_{load} is the potential of the applied load. The stress components superscripted with a "P" denote preloading effects. They could be due to imperfections in the cylinder geometry, for example, or due to thermally-induced deformations, as another example. They will be defined shortly. Using Donnell's assumptions for the kinematics of deformation,

$$
\begin{aligned}
u(x, \theta, r) &= u^{o}(x, \theta) + z\beta_x^{o}(x, \theta) \\
v(x, \theta, r) &= v^{o}(x, \theta) + z\beta_\theta^{o}(x, \theta) \\
w(x, \theta, r) &= w^{o}(x, \theta) .
\end{aligned}
\tag{2}
$$

In the above, as mentioned earlier,

$$z = r - R$$

and the superscript zero, as usual, signifies the displacements of the cylinder's reference surface (i.e., the surface at mean radius R), and the β's are the reference surface rotations given by

$$
\begin{aligned}
\beta_x^{o} &= -\frac{\partial w^{o}}{\partial x} \\
\beta_\theta^{o} &= -\frac{\partial w^{o}}{R\partial \theta} .
\end{aligned}
\tag{3}
$$

It should be noted that the displacements u^{o}, v^{o}, and w^{o} are measured relative to the perfectly round cylinder, not the initial shape of the cylinder due to any preloading effects. The pertinent strain-displacement relations in polar coordinates are

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_\theta &= \frac{\partial v}{r\partial \theta} + \frac{w}{r} + \frac{1}{2}\left(\frac{\partial w}{r\partial \theta}\right)^2 \\
\gamma_{x\theta} &= \frac{\partial v}{\partial x} + \frac{\partial u}{r\partial \theta} + \frac{\partial w}{r\partial \theta}\;\frac{\partial w}{\partial x} .
\end{aligned}
\tag{4}
$$

Substituting eqs. 2 and 3 into eq. 4 and considering only thin shells so that the approximation $r = R$ can be made with sufficient accuracy, the strains become

$$\begin{aligned}\varepsilon_x &= \varepsilon_x^o + z\kappa_x^o \\ \varepsilon_\theta &= \varepsilon_\theta^o + z\kappa_\theta^o \\ \gamma_{x\theta} &= \gamma_{x\theta}^o + z\kappa_{x\theta}^o ,\end{aligned} \tag{5}$$

with

$$\begin{aligned}\varepsilon_x^o &= \frac{\partial u^o}{\partial x} + \frac{1}{2}\beta_x^{o^2} \\ \varepsilon_\theta^o &= \frac{\partial v^o}{R\partial\theta} + \frac{w^o}{R} + \frac{1}{2}\beta_\theta^{o^2} \\ \gamma_{x\theta}^o &= \frac{\partial v^o}{\partial x} + \frac{\partial u^o}{R\partial\theta} + \beta_x^o\beta_\theta^o \\ \kappa_x^o &= \frac{\partial\beta_x^o}{\partial x} ; \quad \kappa_\theta^o = \frac{\partial\beta_\theta^o}{R\partial\theta} ; \quad \kappa_{x\theta}^o = \frac{\partial\beta_\theta^o}{\partial x} + \frac{\partial\beta_x^o}{R\partial\theta} .\end{aligned} \tag{6}$$

The stresses are given by the relations

$$\begin{aligned}\sigma_x &= \overline{Q}_{11}(\varepsilon_x - \varepsilon_x^P) + \overline{Q}_{12}(\varepsilon_\theta - \varepsilon_\theta^P) + \overline{Q}_{16}(\gamma_{x\theta} - \gamma_{x\theta}^P) \\ \sigma_\theta &= \overline{Q}_{12}(\varepsilon_x - \varepsilon_x^P) + \overline{Q}_{22}(\varepsilon_\theta - \varepsilon_\theta^P) + \overline{Q}_{26}(\gamma_{x\theta} - \gamma_{x\theta}^P) \\ \tau_{x\theta} &= \overline{Q}_{16}(\varepsilon_x - \varepsilon_x^P) + \overline{Q}_{26}(\varepsilon_\theta - \varepsilon_\theta^P) + \overline{Q}_{66}(\gamma_{x\theta} - \gamma_{x\theta}^P) ,\end{aligned} \tag{7}$$

where ε_x^P, ε_θ^P, and $\gamma_{x\theta}^P$ are the strains due to preloading effects. Equation 7 can be rewritten as

$$\begin{aligned}\sigma_x &= \overline{Q}_{11}\varepsilon_x + \overline{Q}_{12}\varepsilon_\theta + \overline{Q}_{16}\gamma_{x\theta} - \sigma_x^P \\ \sigma_\theta &= \overline{Q}_{12}\varepsilon_x + \overline{Q}_{22}\varepsilon_\theta + \overline{Q}_{26}\gamma_{x\theta} - \sigma_\theta^P \\ \tau_{x\theta} &= \overline{Q}_{16}\varepsilon_x + \overline{Q}_{26}\varepsilon_\theta + \overline{Q}_{66}\gamma_{x\theta} - \tau_{x\theta}^P ,\end{aligned} \tag{8}$$

where

$$\begin{aligned}\sigma_x^P &= \left(\overline{Q}_{11}\varepsilon_x^P + \overline{Q}_{12}\varepsilon_\theta^P + \overline{Q}_{16}\gamma_{x\theta}^P\right) \\ \sigma_\theta^P &= \left(\overline{Q}_{12}\varepsilon_x^P + \overline{Q}_{22}\varepsilon_\theta^P + \overline{Q}_{26}\gamma_{x\theta}^P\right) \\ \tau_{x\theta}^P &= \left(\overline{Q}_{16}\varepsilon_x^P + \overline{Q}_{26}\varepsilon_\theta^P + \overline{Q}_{66}\gamma_{x\theta}^P\right) .\end{aligned} \tag{9}$$

If the preloading effects are due to thermally-induced deformations, for example, then

$$\begin{aligned}\varepsilon_x^P &= \alpha_x \Delta T \\ \varepsilon_\theta^P &= \alpha_\theta \Delta T \\ \gamma_{x\theta}^P &= \alpha_{x\theta} \Delta T\end{aligned} \tag{10}$$

where α_x, α_θ, and $\alpha_{x\theta}$ are the coefficients of thermal expansion in the cylindrical coordinate system. If this is the case, then the stress-strain relation of eq. 7 can be written as

$$\begin{aligned}\sigma_x &= \overline{Q}_{11}(\varepsilon_x - \alpha_x \Delta T) + \overline{Q}_{12}(\varepsilon_\theta - \alpha_\theta \Delta T) + \overline{Q}_{16}(\gamma_{x\theta} - \alpha_{x\theta} \Delta T) \\ \sigma_\theta &= \overline{Q}_{12}(\varepsilon_x - \alpha_x \Delta T) + \overline{Q}_{22}(\varepsilon_\theta - \alpha_\theta \Delta T) + \overline{Q}_{26}(\gamma_{x\theta} - \alpha_{x\theta} \Delta T) \\ \tau_{x\theta} &= \overline{Q}_{16}(\varepsilon_x - \alpha_x \Delta T) + \overline{Q}_{26}(\varepsilon_\theta - \alpha_\theta \Delta T) + \overline{Q}_{66}(\gamma_{x\theta} - \alpha_{x\theta} \Delta T),\end{aligned} \tag{11}$$

or

$$\begin{aligned}\sigma_x &= \overline{Q}_{11}\varepsilon_x + \overline{Q}_{12}\varepsilon_\theta + \overline{Q}_{16}\gamma_{x\theta} - \sigma_x^T \\ \sigma_\theta &= \overline{Q}_{12}\varepsilon_x + \overline{Q}_{22}\varepsilon_\theta + \overline{Q}_{26}\gamma_{x\theta} - \sigma_\theta^T \\ \tau_{x\theta} &= \overline{Q}_{16}\varepsilon_x + \overline{Q}_{26}\varepsilon_\theta + \overline{Q}_{66}\gamma_{x\theta} - \tau_{x\theta}^T,\end{aligned} \tag{12}$$

where

$$\begin{aligned}\sigma_x^T &= (\overline{Q}_{11}\alpha_x + \overline{Q}_{12}\alpha_\theta + \overline{Q}_{16}\alpha_{x\theta})\Delta T \\ \sigma_\theta^T &= (\overline{Q}_{12}\alpha_x + \overline{Q}_{22}\alpha_\theta + \overline{Q}_{26}\alpha_{x\theta})\Delta T \\ \tau_{x\theta}^T &= (\overline{Q}_{16}\alpha_x + \overline{Q}_{26}\alpha_\theta + \overline{Q}_{66}\alpha_{x\theta})\Delta T.\end{aligned} \tag{13}$$

In this situation the superscript "T" denotes the fact that the preloading effects are thermally-induced. The stresses σ_x^T, σ_θ^T, $\tau_{x\theta}^T$ would then have the physical interpretation of being the stresses at a point if the composite is <u>fully constrained</u> from any deformation.

For the case of known axial loads applied at either end of the cylinder

$$\pi_{load} = \int_\theta N_x^-(\theta) u^\circ\left(-\frac{L}{2}, \theta\right) R d\theta - \int_\theta N_x^+(\theta) u^\circ\left(\frac{L}{2}, \theta\right) R d\theta. \tag{14}$$

The quantity $N_x^-(\theta)$ is the load at $x = -\frac{L}{2}$ and $N_x^+(\theta)$ is the load at $x = +\frac{L}{2}$. The integrals are taken around ends of the cylinder and the dimension of the N_x's is force per unit circumferential length. For the case of a known torsional load applied at either end of the cylinder,

$$\pi_{load} = \int_\theta N_{x\theta}^-(\theta)v^o\left(-\frac{L}{2}, \theta\right)Rd\theta - \int_\theta N_{x\theta}^+(\theta)v^o\left(\frac{L}{2}, \theta\right)Rd\theta, \tag{15}$$

and for a known outward radial load

$$\pi_{load} = -\int_\theta\int_x q(x, \theta)w^o(x, \theta)Rd\theta dx. \tag{16}$$

In general, the end loads can be functions of θ and the radial load can be a function of x and θ The $N_{x\theta}$'s are tangential load per unit circumferential length, and $q(x, \theta)$ is a load per unit area. Other loadings on the ends of the cylinder can be included, e.g., an applied moment, but they will not be considered here.

Substituting the expressions for the strains, eq. 5, into the energy expressions for the cylinder, eq. 1, and including the three loading terms being considered, results in

$$\begin{aligned}\pi(u^o, v^o, w^o) = \frac{1}{2}\int_z\int_\theta\int_x \Big(& (\sigma_x - \sigma_x^P)(\varepsilon_x^o + z\kappa_x^o) + (\sigma_\theta - \sigma_\theta^P)(\varepsilon_\theta^o + z\kappa_\theta^o) \\ & + (\tau_{x\theta} - \tau_{x\theta}^P)(\gamma_{x\theta}^o + z\kappa_{x\theta}^o)\Big)Rd\theta dz dx \\ & + \int_\theta N_x^-(\theta)u^o\left(-\frac{L}{2}, \theta\right)Rd\theta - \int_\theta N_x^+(\theta)u^o\left(\frac{L}{2}, \theta\right)Rd\theta \\ & + \int_\theta N_{x\theta}^-(\theta)v^o\left(-\frac{L}{2}, \theta\right)Rd\theta - \int_\theta N_{x\theta}^+(\theta)v^o\left(\frac{L}{2}, \theta\right)Rd\theta \\ & - \int_\theta\int_x q(x, \theta)w^o(x, \theta)Rd\theta dx .\end{aligned} \tag{17}$$

In the above use has been made of the facts that

$$r = z + R \simeq R$$
$$dr = dz \tag{18}$$

and the integration on r has been replaced with integration on z . Integrating on z leads to

$$\begin{aligned}
\pi(u^o, v^o, w^o) = \frac{1}{2}\int_\theta\int_x &\Big((N_x - N_x^P)\varepsilon_x^o + (N_\theta - N_\theta^P)\varepsilon_\theta^o + (N_{x\theta} - N_{x\theta}^P)\gamma_{x\theta}^o + (M_x - M_x^P)\kappa_x^o \\
&+ (M_\theta - M_\theta^P)\kappa_\theta^o + (M_{x\theta} - M_{x\theta}^P)\kappa_{x\theta}^o\Big)Rd\theta dx \\
&+ \int_\theta N_x^-(\theta)u^o\Big(-\frac{L}{2}, \theta\Big)Rd\theta - \int_\theta N_x^+(\theta)u^o\Big(\frac{L}{2}, \theta\Big)Rd\theta \\
&+ \int_\theta N_{x\theta}^-(\theta)v^o\Big(-\frac{L}{2}, \theta\Big)Rd\theta - \int_\theta N_{x\theta}^+(\theta)v^o\Big(\frac{L}{2}, \theta\Big)Rd\theta \\
&- \int_\theta\int_x q(x, \theta)w^o(x, \theta)Rd\theta dx .
\end{aligned} \tag{19}$$

The stress resultants in eq. 19 are defined as

$$\begin{aligned}
N_x &\equiv \int_{-H/2}^{H/2} \sigma_x dz = A_{11}\varepsilon_x^o + A_{12}\varepsilon_\theta^o + A_{16}\gamma_{x\theta}^o + B_{11}\kappa_x^o + B_{12}\kappa_\theta^o + B_{16}\kappa_{x\theta}^o - N_x^P \\
N_\theta &\equiv \int_{-H/2}^{H/2} \sigma_\theta dz = A_{12}\varepsilon_x^o + A_{22}\varepsilon_\theta^o + A_{26}\gamma_{x\theta}^o + B_{12}\kappa_x^o + B_{22}\kappa_\theta^o + B_{26}\kappa_{x\theta}^o - N_\theta^P \\
N_{x\theta} &\equiv \int_{-H/2}^{H/2} \tau_{x\theta} dz = A_{16}\varepsilon_x^o + A_{26}\varepsilon_\theta^o + A_{66}\gamma_{x\theta}^o + B_{16}\kappa_x^o + B_{26}\kappa_\theta^o + B_{66}\kappa_{x\theta}^o - N_{x\theta}^P \\
M_x &\equiv \int_{-H/2}^{H/2} z\sigma_x dz = B_{11}\varepsilon_x^o + B_{12}\varepsilon_\theta^o + B_{16}\gamma_{x\theta}^o + D_{11}\kappa_x^o + D_{12}\kappa_\theta^o + D_{16}\kappa_{x\theta}^o - M_x^P \\
M_\theta &\equiv \int_{-H/2}^{H/2} z\sigma_\theta dz = B_{12}\varepsilon_x^o + B_{22}\varepsilon_\theta^o + B_{26}\gamma_{x\theta}^o + D_{12}\kappa_x^o + D_{22}\kappa_\theta^o + D_{26}\kappa_{x\theta}^o - M_\theta^P \\
M_{x\theta} &\equiv \int_{-H/2}^{H/2} z\tau_{x\theta} dz = B_{16}\varepsilon_x^o + B_{26}\varepsilon_\theta^o + B_{66}\gamma_{x\theta}^o + D_{16}\kappa_x^o + D_{26}\kappa_\theta^o + D_{66}\kappa_{x\theta}^o - M_{x\theta}^P ,
\end{aligned} \tag{20}$$

H being the cylinder wall thickness. In the above

$$
\begin{aligned}
N_x^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_x^P dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{11}\varepsilon_x^P + \overline{Q}_{12}\varepsilon_\theta^P + \overline{Q}_{16}\gamma_{x\theta}^P\right) dz \\
N_\theta^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_\theta^P dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{12}\varepsilon_x^P + \overline{Q}_{22}\varepsilon_\theta^P + \overline{Q}_{26}\gamma_{x\theta}^P\right) dz \\
N_{x\theta}^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \tau_{x\theta}^P dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{16}\varepsilon_x^P + \overline{Q}_{26}\varepsilon_\theta^P + \overline{Q}_{66}\gamma_{x\theta}^P\right) dz \\
M_x^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_x^P z dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{11}\varepsilon_x^P + \overline{Q}_{12}\varepsilon_\theta^P + \overline{Q}_{16}\gamma_{x\theta}^P\right) z dz \\
M_\theta^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_\theta^P z dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{12}\varepsilon_x^P + \overline{Q}_{22}\varepsilon_\theta^P + \overline{Q}_{26}\gamma_{x\theta}^P\right) z dz \\
M_{x\theta}^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \tau_{x\theta}^P z dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{16}\varepsilon_x^P + \overline{Q}_{26}\varepsilon_\theta^P + \overline{Q}_{66}\gamma_{x\theta}^P\right) z dz\,.
\end{aligned}
\tag{21}
$$

These expressions are the so-called equivalent preloading stress resultants. Again, if the preloading effects are thermally induced,

$$
\begin{aligned}
N_x^P = N_x^T &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_x^T dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{11}\alpha_x + \overline{Q}_{12}\alpha_\theta + \overline{Q}_{16}\alpha_{x\theta}\right)\Delta T dz \\
N_\theta^P = N_\theta^T &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_\theta^T dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{12}\alpha_x + \overline{Q}_{22}\alpha_\theta + \overline{Q}_{26}\alpha_{x\theta}\right)\Delta T dz \\
N_{x\theta}^P = N_{x\theta}^T &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \tau_{x\theta}^T dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{16}\alpha_x + \overline{Q}_{26}\alpha_\theta + \overline{Q}_{66}\alpha_{x\theta}\right)\Delta T dz \\
M_x^P = M_x^T &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_x z dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{11}\alpha_x + \overline{Q}_{16}\alpha_\theta + \overline{Q}_{12}\alpha_{x\theta}\right)\Delta T z dz \\
M_\theta^P = M_\theta^P &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \sigma_\theta z dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{12}\alpha_x + \overline{Q}_{22}\alpha_\theta + \overline{Q}_{26}\alpha_{x\theta}\right)\Delta T z dz \\
M_{x\theta}^P = M_{x\theta}^T &\equiv \int_{-\frac{H}{2}}^{\frac{H}{2}} \tau_{x\theta} z dz = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(\overline{Q}_{16}\alpha_x + \overline{Q}_{26}\alpha_\theta + \overline{Q}_{66}\alpha_{x\theta}\right)\Delta T z dz\,.
\end{aligned}
\tag{22}
$$

In this case these expressions are the so-called equivalent thermal stress resultants.

The notation

$$\pi = \pi(u^o, v^o, w^o) \tag{23}$$

is being used to emphasize the fact that the total potential energy is a function of the displacements (here the cylinder reference surface displacements). The governing conditions will be derived by examining variations in the total potential energy. These variations in the total potential energy will be due to variations in these displacements. To this end, consider the increment, or variation, of the total potential energy due to increments, or variations, in the displacements. Specifically,

$$u^o + \varepsilon u_1^o; \quad v^o + \varepsilon v_1^o; \quad w^o + \varepsilon w_1^o \ , \tag{24}$$

where ε is a small parameter and the quantities u_1^o, v_1^o, and w_1^o satisfy all the kinematic requirements of the problem. Using the notation

$$\pi + \Delta\pi = \pi(u^o + \varepsilon u_1^o, v^o + \varepsilon v_1^o, w^o + \varepsilon w_1^o) \ , \tag{25}$$

the incremented total potential can be expanded using eq. 19 as follows:

$$
\begin{aligned}
\pi + \Delta\pi = \frac{1}{2}\int_\theta\int_x &\{(N_x + \Delta N_x - N_x^P)(\varepsilon_x^o + \Delta\varepsilon_x^o) + (N_\theta + \Delta N_\theta - N_\theta^P)(\varepsilon_\theta^o + \Delta\varepsilon_\theta^o) \\
&+ (N_{x\theta} + \Delta N_{x\theta} - N_{x\theta}^P)(\gamma_{x\theta}^o + \Delta\gamma_{x\theta}^o) + (M_x + \Delta M_x - M_x^P)(\kappa_x^o + \Delta\kappa_x^o) \\
&+ (M_\theta + \Delta M_\theta - M_\theta^P)(\kappa_\theta^o + \Delta\kappa_\theta^o) + (M_{x\theta} + \Delta M_{x\theta} - M_{x\theta}^P)(\kappa_{x\theta}^o + \Delta\kappa_{x\theta}^o)\} R d\theta dx \\
&+ \int_\theta N_x^-(\theta)\left\{u^o\left(-\frac{L}{2},\theta\right) + \varepsilon u_1^o\left(-\frac{L}{2},\theta\right)\right\} R d\theta \\
&- \int_\theta N_x^+(\theta)\left\{u^o\left(\frac{L}{2},\theta\right) + \varepsilon u_1^o\left(\frac{L}{2},\theta\right)\right\} R d\theta \\
&+ \int_\theta N_{x\theta}^-(\theta)\left\{v^o\left(-\frac{L}{2},\theta\right) + \varepsilon v_1^o\left(-\frac{L}{2},\theta\right)\right\} R d\theta \\
&- \int_\theta N_{x\theta}^+(\theta)\left\{v^o\left(\frac{L}{2},\theta\right) + \varepsilon v_1^o\left(\frac{L}{2},\theta\right)\right\} R d\theta \\
&- \int_x\int_\theta q(x,\theta)\{w^o(x,\theta) + \varepsilon w_1^o(x,\theta)\} R d\theta dx .
\end{aligned} \tag{26}
$$

It should be noted that the equivalent preloading stress resultants do not have increments because they depend only on material properties and initial displacements (or perhaps the temperature), not the displacements due to the applied forces. Subtracting eq. 19 from eq. 26 leads to an expression for the increment in the total potential energy, namely

$$
\begin{aligned}
\Delta\pi = \frac{1}{2}\int_x\int_\theta &\{(N_x - N_x^P)\Delta\varepsilon_x^o + \Delta N_x\varepsilon_x^o + \Delta N_x\Delta\varepsilon_x^o \\
&+ (N_\theta - N_\theta^P)\Delta\varepsilon_\theta^o + \Delta N_\theta\varepsilon_\theta^o + \Delta N_\theta\Delta\varepsilon_\theta^o + (N_{x\theta} - N_{x\theta}^P)\Delta\gamma_{x\theta}^o + \Delta N_{x\theta}\gamma_{x\theta}^o + \Delta N_{x\theta}\Delta\gamma_{x\theta}^o \\
&+ (M_x - M_x^P)\Delta\kappa_x^o + \Delta M_x\kappa_x^o + \Delta M_x\Delta\kappa_x^o + (M_\theta - M_\theta^P)\Delta\kappa_\theta^o + \Delta M_\theta\kappa_\theta^o + \Delta M_\theta\Delta\kappa_\theta^o \\
&+ (M_{x\theta} - M_{x\theta}^P)\Delta\kappa_{x\theta}^o + \Delta M_{x\theta}\kappa_{x\theta}^o + \Delta M_{x\theta}\Delta\kappa_{x\theta}^o\} R d\theta dx \\
&+ \varepsilon\left\{\int_\theta N_x^-(\theta) u_1^o\left(-\frac{L}{2},\theta\right) R d\theta - \int_\theta N_x^+(\theta) u_1^o\left(\frac{L}{2},\theta\right) R d\theta\right\} \\
&+ \varepsilon\left\{\int_\theta N_{x\theta}^-(\theta) v_1^o\left(-\frac{L}{2},\theta\right) R d\theta - \int_\theta N_{x\theta}^+(\theta) v_1^o\left(\frac{L}{2},\theta\right) R d\theta\right\} \\
&- \varepsilon\left\{\int_x\int_\theta q(x,\theta) w_1^o(x,\theta) R d\theta dx\right\} .
\end{aligned} \tag{27}
$$

The various increments in eq. 27 are given by substituting the increments in the displacements, eq. 24, into the basic definitions of strain and curvature, eq. 6. That substitution leads to, for ε_x^o,

$$\varepsilon_x^o + \Delta\varepsilon_x^o = \frac{\partial(u^o + \varepsilon u_1^o)}{\partial x} + \frac{1}{2}(\beta_x^o + \varepsilon\beta_{x_1}^o)^2 . \tag{28}$$

Dropping the superscript for convenience,

$$\varepsilon_x + \Delta\varepsilon_x = \frac{\partial u}{\partial x} + \varepsilon\frac{\partial u_1}{\partial x} + \frac{1}{2}\beta_x^2 + \varepsilon\beta_x\beta_{x_1} + \frac{1}{2}\varepsilon^2\beta_{x_1}^2 . \tag{29}$$

Then, using eq. 6,

$$\Delta\varepsilon_x = \varepsilon\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + \frac{1}{2}\varepsilon^2\beta_{x_1}^2 . \tag{30}$$

This can be written as

$$\Delta\varepsilon_x = \varepsilon\varepsilon_{x1} + \varepsilon^2\varepsilon_{x_2} , \tag{31}$$

where ε_{x_1} and ε_{x_2} are defined to be

$$\varepsilon_{x_1} = \frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1} \tag{32}$$

$$\varepsilon_{x_2} = \frac{1}{2}\beta_{x_1}^2 . \tag{33}$$

In a similar fashion,

$$\varepsilon_\theta + \Delta\varepsilon_\theta = \frac{\partial(v + \varepsilon v_1)}{R d\theta} + \frac{w + \varepsilon w_1}{R} + \frac{1}{2}(\beta_\theta + \varepsilon\beta_{\theta_1})^2 \tag{34}$$

$$= \frac{\partial v}{R\partial\theta} + \varepsilon\frac{\partial v_1}{R\partial\theta} + \frac{w}{R} + \varepsilon\frac{w_1}{R} + \frac{1}{2}\beta_\theta^2 + \varepsilon\beta_\theta\beta_{\theta_1} + \frac{1}{2}\varepsilon^2\beta_{\theta_1}^2, \tag{35}$$

or

$$\Delta\varepsilon_\theta = \varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}. \tag{36}$$

where

$$\varepsilon_{\theta_1} = \left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1} \right) \tag{37}$$

$$\varepsilon_{\theta_2} = \frac{1}{2}\beta_{\theta_1}^2. \tag{38}$$

Also,

$$\gamma_{x\theta} + \Delta\gamma_{x\theta} = \frac{\partial(v + \varepsilon v_1)}{\partial x} + \frac{\partial(u + \varepsilon u_1)}{R\partial\theta} + (\beta_x + \varepsilon\beta_{x_1})(\beta_\theta + \varepsilon\beta_{\theta_1}) \tag{39}$$

$$\begin{aligned}\gamma_{x\theta} + \Delta\gamma_{x\theta} = {} & \frac{\partial v}{\partial x} + \varepsilon\frac{\partial v_1}{\partial x} + \frac{\partial u}{R\partial\theta} + \varepsilon\frac{\partial u_1}{R\partial\theta} + \beta_x\beta_\theta \\ & + \varepsilon(\beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\theta) + \varepsilon^2\beta_{x_1}\beta_{\theta_1}.\end{aligned} \tag{40}$$

This results in

$$\Delta\gamma_{x\theta} = \varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2} \tag{41}$$

where

$$\gamma_{x\theta_1} = \left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1} \right) \tag{42}$$

$$\gamma_{x\theta_2} = \beta_{x_1}\beta_{\theta_1}. \tag{43}$$

Finally,

$$\Delta\kappa_x = \varepsilon\kappa_{x_1}; \quad \Delta\kappa_\theta = \varepsilon\kappa_{\theta_1}; \quad \Delta\kappa_{x\theta} = \varepsilon\kappa_{x\theta_1} \tag{44}$$

where

$$\kappa_{x_1} = \frac{\partial \beta_{x_1}}{\partial x} \quad ; \tag{45}$$

$$\kappa_{\theta_1} = \frac{\partial \beta_{\theta_1}}{R\partial \theta} \quad ; \tag{46}$$

$$\kappa_{x\theta_1} = \left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial \theta} \right) . \tag{47}$$

In the above use has been made of the definitions

$$\beta_{x_1} = -\frac{\partial w_1}{\partial x} \; ; \quad \beta_{\theta_1} = -\frac{\partial w_1}{R\partial \theta} \; . \tag{48}$$

In terms of the strain increments, the increments in the stress resultants are:

$$\begin{aligned}
\Delta N_x &= A_{11}\Delta\varepsilon_x + A_{12}\Delta\varepsilon_\theta + A_{16}\Delta\gamma_{x\theta} + B_{11}\Delta\kappa_x + B_{12}\Delta\kappa_\theta + B_{16}\Delta\kappa_{x\theta} \\
\Delta N_\theta &= A_{12}\Delta\varepsilon_x + A_{22}\Delta\varepsilon_\theta + A_{26}\Delta\gamma_{x\theta} + B_{12}\Delta\kappa_x + B_{22}\Delta\kappa_\theta + B_{26}\Delta\kappa_{x\theta} \\
\Delta N_{x\theta} &= A_{16}\Delta\varepsilon_x + A_{26}\Delta\varepsilon_\theta + A_{66}\Delta\gamma_{x\theta} + B_{16}\Delta\kappa_x + B_{26}\Delta\kappa_\theta + B_{66}\Delta\kappa_{x\theta} \\
\Delta M_x &= B_{11}\Delta\varepsilon_x + B_{12}\Delta\varepsilon_\theta + B_{16}\Delta\gamma_{x\theta} + D_{11}\Delta\kappa_x + D_{12}\Delta\kappa_\theta + D_{16}\Delta\kappa_{x\theta} \\
\Delta M_\theta &= B_{12}\Delta\varepsilon_x + B_{22}\Delta\varepsilon_\theta + B_{26}\Delta\gamma_{x\theta} + D_{12}\Delta\kappa_x + D_{22}\Delta\kappa_\theta + D_{26}\Delta\kappa_{x\theta} \\
\Delta M_{x\theta} &= B_{16}\Delta\varepsilon_x + B_{26}\Delta\varepsilon_\theta + B_{66}\Delta\gamma_{x\theta} + D_{16}\Delta\kappa_x + D_{26}\Delta\kappa_\theta + D_{66}\Delta\kappa_{x\theta}
\end{aligned} \tag{49}$$

It is convenient to expand the increments in the stress resultants, and redefine those increments in terms of powers of ε. From eq. 49, incorporating the definitions for the strain increments, eqs. 28-47, the stress resultant increments are

$$\begin{aligned}
\Delta N_x = {} & A_{11}\left(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}\right) + A_{12}\left(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}\right) + A_{16}\left(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}\right) \\
& + B_{11}\varepsilon\kappa_{x_1} + B_{12}\varepsilon\kappa_{\theta_1} + B_{16}\varepsilon\gamma_{x\theta_1} \, .
\end{aligned} \tag{50a}$$

These terms can be redefined to give

$$\Delta N_x = \varepsilon N_{x_1} + \varepsilon^2 N_{x_2} \, , \tag{50b}$$

where

$$N_{x_1} = A_{11}\varepsilon_{x_1} + A_{12}\varepsilon_{\theta_1} + A_{16}\gamma_{x\theta_1} + B_{11}\kappa_{x_1} + B_{12}\kappa_{\theta_1} + B_{16}\kappa_{x\theta_1} \tag{50c}$$

and

$$N_{x_2} = A_{11}\varepsilon_{x_2} + A_{12}\varepsilon_{\theta_2} + A_{16}\gamma_{x\theta_2} \,. \tag{50d}$$

Expanding for future reference,

$$\begin{aligned} N_{x_1} = {} & A_{11}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + A_{12}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right) \\ & + A_{16}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) \\ & + B_{11}\frac{\partial\beta_{x_1}}{\partial x} + B_{12}\frac{\partial\beta_{\theta_1}}{R\partial\theta} + B_{16}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right) \end{aligned} \tag{50e}$$

and

$$N_{x_2} = \left(\frac{1}{2}A_{11}\beta_{x_1}^2 + \frac{1}{2}A_{12}\beta_{\theta_1}^2 + A_{16}\beta_{x_1}\beta_{\theta_1}\right) . \tag{50f}$$

Using this procedure for the remaining stress resultants,

$$\begin{aligned} \Delta N_\theta = {} & A_{12}\left(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}\right) + A_{22}\left(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}\right) + A_{26}\left(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}\right) \\ & + B_{12}\varepsilon\kappa_{x_1} + B_{22}\varepsilon\kappa_{\theta_1} + B_{26}\varepsilon\kappa_{x\theta_1} \,. \end{aligned} \tag{51a}$$

Redefining,

$$\Delta N_\theta = \varepsilon N_{\theta_1} + \varepsilon^2 N_{\theta_2} \,, \tag{51b}$$

with

$$N_{\theta_1} = A_{12}\varepsilon_{x_1} + A_{22}\varepsilon_{\theta_1} + A_{26}\gamma_{x\theta_1} + B_{12}\kappa_{x_1} + B_{22}\kappa_{\theta_1} + B_{26}\kappa_{x\theta_1} \tag{51c}$$

and

$$N_{\theta_2} = A_{12}\varepsilon_{x_2} + A_{22}\varepsilon_{\theta_2} + A_{26}\gamma_{x\theta_2} \, . \tag{51d}$$

Expanding for future reference,

$$\begin{aligned} N_{\theta_1} = & A_{12}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1} \right) + A_{22}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1} \right) \\ & + A_{26}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1} \right) \\ & + B_{12}\frac{\partial\beta_{x_1}}{\partial x} + B_{22}\frac{\partial\beta_{\theta_1}}{R\partial\theta} + B_{26}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta} \right) \end{aligned} \tag{51e}$$

and

$$N_{\theta_2} = \left(\frac{1}{2} A_{12}\beta_{x_1}^2 + \frac{1}{2} A_{22}\beta_{\theta_1}^2 + A_{26}\beta_{x_1}\beta_{\theta_1} \right) . \tag{51f}$$

Likewise,

$$\begin{aligned} \Delta N_{x\theta} = & A_{16}\left(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}\right) + A_{26}\left(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}\right) + A_{66}\left(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}\right) \\ & + B_{16}\varepsilon\kappa_{x_1} + B_{26}\varepsilon\kappa_{\theta_1} + B_{66}\varepsilon\kappa_{x\theta_1} \, , \end{aligned} \tag{52a}$$

or

$$\Delta N_{x\theta} = \varepsilon N_{x\theta_1} + \varepsilon^2 N_{x\theta_2}, \tag{52b}$$

where

$$N_{x\theta_1} = A_{16}\varepsilon_{x_1} + A_{26}\varepsilon_{\theta_1} + A_{66}\gamma_{x\theta_1} + B_{16}\kappa_{x_1} + B_{26}\kappa_{\theta_1} + B_{66}\kappa_{x\theta_1} \tag{52c}$$

and

$$N_{x\theta_2} = A_{16}\varepsilon_{x_2} + A_{26}\varepsilon_{\theta_2} + A_{66}\gamma_{x\theta_2} \, . \tag{52d}$$

Expanding,

$$N_{x\theta_1} = A_{16}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1} \right) + A_{26}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1} \right)$$
$$+ A_{66}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1} \right) \tag{52e}$$
$$+ B_{16}\frac{\partial\beta_{x_1}}{\partial x} + B_{26}\frac{\partial\beta_{\theta_1}}{R\partial\theta} + B_{66}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta} \right)$$

and

$$N_{x\theta_2} = \left(\frac{1}{2} A_{16}\beta_{x_1}^2 + \frac{1}{2} A_{26}\beta_{\theta_1}^2 + A_{66}\beta_{x_1}\beta_{\theta_1} \right) . \tag{52d}$$

The increments in the moments can be similiarly defined, namely,

$$\Delta M_x = B_{11}\left(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}\right) + B_{12}\left(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}\right) + B_{16}\left(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}\right)$$
$$+ D_{11}\varepsilon\kappa_{x_1} + D_{12}\varepsilon\kappa_{\theta_1} + D_{16}\varepsilon\kappa_{x\theta_1} , \tag{53a}$$

or

$$\Delta M_x = \varepsilon M_{x_1} + \varepsilon^2 M_{x_2} , \tag{53b}$$

with

$$M_{x_1} = B_{11}\varepsilon_{x_1} + B_{12}\varepsilon_{\theta_1} + B_{16}\gamma_{x\theta_1} + D_{11}\kappa_{x_1} + D_{12}\kappa_{\theta_1} + D_{16}\kappa_{x\theta_1} \tag{53c}$$

and

$$M_{x_2} = B_{11}\varepsilon_{x_2} + B_{12}\varepsilon_{\theta_2} + B_{16}\gamma_{x\theta_2} , \tag{53d}$$

where

$$M_{x_1} = B_{11}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + B_{12}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)$$
$$+ B_{16}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) \tag{53e}$$
$$+ D_{11}\frac{\partial\beta_{x_1}}{\partial x} + D_{12}\frac{\partial\beta_{\theta_1}}{R\partial\theta} + D_{16}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)$$

and

$$M_{x_2} = \left(\frac{1}{2}B_{11}\beta_{x_1}^2 + \frac{1}{2}B_{12}\beta_{\theta_1}^2 + B_{16}\beta_{x_1}\beta_{\theta_1}\right). \tag{53f}$$

In a similar manner,

$$\Delta M_\theta = B_{12}\left(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}\right) + B_{22}\left(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}\right) + B_{26}\left(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}\right)$$
$$+ D_{12}\varepsilon\kappa_{x_1} + D_{22}\varepsilon\kappa_{\theta_1} + D_{26}\varepsilon\kappa_{x\theta_1}, \tag{54a}$$

or

$$\Delta M_\theta = \varepsilon M_{\theta_1} + \varepsilon^2 M_{\theta_2}, \tag{54b}$$

with

$$M_{\theta_1} = B_{12}\varepsilon_{x_1} + B_{22}\varepsilon_{\theta_1} + B_{26}\gamma_{x\theta_1} + D_{12}\kappa_{x_1} + D_{22}\kappa_{\theta_1} + D_{26}\kappa_{x\theta_1} \tag{54c}$$

and

$$M_{\theta_2} = B_{12}\varepsilon_{x_2} + B_{22}\varepsilon_{\theta_2} + B_{26}\gamma_{x\theta_2}, \tag{54d}$$

where

$$
\begin{aligned}
M_{\theta_1} = {} & B_{12}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + B_{22}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right) \\
& + B_{26}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) \\
& + D_{12}\frac{\partial\beta_{x_1}}{\partial x} + D_{22}\frac{\partial\beta_{\theta_1}}{R\partial\theta} + D_{26}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)
\end{aligned}
\qquad (54e)
$$

and

$$
M_{\theta_2} = \left(\frac{1}{2}B_{12}\beta_{x_1}^2 + \frac{1}{2}B_{22}\beta_{\theta_1}^2 + B_{26}\beta_{x_1}\beta_{\theta_1}\right). \qquad (54f)
$$

Finally

$$
\begin{aligned}
\Delta M_{x\theta} = {} & B_{16}\left(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}\right) + B_{26}\left(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2}\right) + B_{66}\left(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}\right) \\
& + D_{16}\varepsilon\kappa_{x_1} + D_{26}\varepsilon\kappa_{\theta_1} + D_{66}\varepsilon\kappa_{x\theta_1}
\end{aligned}
\qquad (55a)
$$

where

$$
\Delta M_{x_\theta} = \varepsilon M_{x\theta_1} + \varepsilon^2 M_{x\theta_2} \qquad (55b)
$$

with

$$
M_{x\theta_1} = B_{16}\varepsilon_{x_1} + B_{26}\varepsilon_{\theta_1} + B_{66}\gamma_{x\theta_1} + D_{16}\kappa_{x_1} + D_{26}\kappa_{\theta_1} + D_{66}\kappa_{x\theta_1} \qquad (55c)
$$

and

$$
M_{x\theta_2} = B_{16}\varepsilon_{x_2} + B_{26}\varepsilon_{\theta_2} + B_{66}\gamma_{x\theta_2}. \qquad (55d)
$$

Expanding for future reference

$$M_{x\theta_1} = B_{16}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + B_{26}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)$$
$$+ B_{66}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) \tag{55e}$$
$$+ D_{16}\frac{\partial\beta_{x_1}}{\partial x} + D_{26}\frac{\partial\beta_{\theta_1}}{R\partial\theta} + D_{66}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)$$

and

$$M_{x\theta_2} = \left(\frac{1}{2}B_{16}\beta_{x_1}^2 + \frac{1}{2}B_{26}\beta_{\theta_1}^2 + B_{66}\beta_{x_1}\beta_{\theta_1}\right). \tag{55f}$$

With the various increments defined and expanded, the definitions can be substituted into eq. 27. This results in

$$\Delta\pi = \frac{1}{2}\int_x\int_\theta\{(N_x - N_x^P)(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}) + (\varepsilon N_{x_1} + \varepsilon^2 N_{x_2})\varepsilon_x$$
$$+ (\varepsilon N_{x_1} + \varepsilon^2 N_{x_2})(\varepsilon\varepsilon_{x_1} + \varepsilon^2\varepsilon_{x_2}) + (N_\theta - N_\theta^P)(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2})$$
$$+ (\varepsilon N_{\theta_1} + \varepsilon^2 N_{\theta_2})\varepsilon_\theta + (\varepsilon N_{\theta_1} + \varepsilon^2 N_{\theta_2})(\varepsilon\varepsilon_{\theta_1} + \varepsilon^2\varepsilon_{\theta_2})$$
$$+ (N_{x\theta} - N_{x\theta}^P)(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}) + (\varepsilon N_{x\theta_1} + \varepsilon^2 N_{x\theta_2})\gamma_{x\theta}$$
$$+ (\varepsilon N_{x\theta_1} + \varepsilon^2 N_{x\theta_2})(\varepsilon\gamma_{x\theta_1} + \varepsilon^2\gamma_{x\theta_2}) + (M_x - M_x^P)\varepsilon\kappa_{x_1}$$
$$+ (\varepsilon M_{x_1} + \varepsilon^2 M_{x_2})\kappa_x + (\varepsilon M_{x_1} + \varepsilon^2 M_{x_2})\varepsilon\kappa_{x_1}$$
$$+ (M_\theta - M_\theta^P)\varepsilon\kappa_{\theta_1} + (\varepsilon M_{\theta_1} + \varepsilon^2 M_{\theta_2})\kappa_\theta + (\varepsilon M_{\theta_1} + \varepsilon^2 M_{\theta_2})\varepsilon\kappa_{\theta_1} \tag{56}$$
$$+ (M_{x\theta} - M_{x\theta}^P)\varepsilon\kappa_{x\theta_1} + (\varepsilon M_{x\theta_1} + \varepsilon^2 M_{x\theta_2})\kappa_{x\theta} + (\varepsilon M_{x\theta_1} + \varepsilon^2 M_{x\theta_2})\varepsilon\kappa_{x\theta_1}\}R\,d\theta\,dx$$
$$+ \varepsilon\left\{\int_\theta N_x^-(\theta)u_1^o\left(-\frac{L}{2},\theta\right)R\,d\theta - \int_\theta N_x^+(\theta)u_1^o\left(\frac{L}{2},\theta\right)R\,d\theta\right\}$$
$$+ \varepsilon\left\{\int_\theta N_{x\theta}^-(\theta)v_1^o\left(-\frac{L}{2},\theta\right)R\,d\theta - \int_\theta N_{x\theta}^+(\theta)v_1^o\left(\frac{L}{2},\theta\right)R\,d\theta\right\}$$
$$- \varepsilon\left\{\int_x\int_\theta q(x,\theta)w_1^o(x,\theta)R\,d\theta\,dx\right\}.$$

Expanding and regrouping in powers of ε leads to

$$
\begin{aligned}
\Delta\pi = \frac{1}{2}\int_x\int_\theta\Big[&\varepsilon\{(N_x - N_x^P)\varepsilon_{x_1} + N_{x_1}\varepsilon_x + (N_\theta - N_\theta^P)\varepsilon_{\theta_1} + N_{\theta_1}\varepsilon_\theta \\
&+ (N_{x\theta} - N_{x\theta}^P)\gamma_{x\theta_1} + N_{x\theta_1}\gamma_{x\theta} + (M_x - M_x^P)\kappa_{x_1} + M_{x_1}\kappa_x \\
&+ (M_\theta - M_\theta^P)\kappa_{\theta_1} + M_{\theta_1}\kappa_\theta + (M_{x\theta} - M_{x\theta}^P)\kappa_{x\theta_1} + M_{x\theta_1}\kappa_{x\theta}\} \\
&+ \varepsilon^2\{(N_x - N_x^P)\varepsilon_{x_2} + N_{x_2}\varepsilon_x + N_{x_1}\varepsilon_{x_1} + (N_\theta - N_\theta^P)\varepsilon_{\theta_2} + N_{\theta_2}\varepsilon_\theta + N_{\theta_1}\varepsilon_{\theta_1} \\
&+ (N_{x\theta} - N_{x\theta}^P)\gamma_{x\theta_2} + N_{x\theta_2}\gamma_{x\theta} + N_{x\theta_1}\gamma_{x\theta_1} + M_{x_2}\kappa_x + M_{x_1}\kappa_{x_1} \\
&+ M_{\theta_2}\kappa_\theta + M_{\theta_1}\kappa_{\theta_1} + M_{x\theta_2}\kappa_{x\theta} + M_{x\theta_1}\kappa_{x\theta_1}\} \\
&+ \varepsilon^3\{N_{x_1}\varepsilon_{x_2} + N_{x_2}\varepsilon_{x_1} + N_{\theta_1}\varepsilon_{\theta_2} + N_{\theta_2}\varepsilon_{\theta_1} + N_{x\theta_1}\gamma_{x\theta_2} + N_{x\theta_2}\gamma_{x\theta_1} \\
&+ M_{x_2}\kappa_{x_1} + M_{\theta_2}\kappa_{\theta_1} + M_{x\theta_2}\kappa_{x\theta_1}\} + \varepsilon^4\{N_{x_2}\varepsilon_{x_2} + N_{\theta_2}\varepsilon_{\theta_2} \\
&+ N_{x\theta_2}\gamma_{x\theta_2}\}\Big]Rd\theta dx \\
&+ \varepsilon\left\{\int_\theta N_x^-(\theta)u_1^o\left(-\frac{L}{2},\theta\right)Rd\theta - \int_\theta N_x^+(\theta)u_1^o\left(\frac{L}{2},\theta\right)Rd\theta\right\} \\
&+ \varepsilon\left\{\int_\theta N_{x\theta}^-(\theta)v_1^o\left(-\frac{L}{2},\theta\right)Rd\theta - \int_\theta N_{x\theta}^+(\theta)v_1^o\left(\frac{L}{2},\theta\right)Rd\theta\right\} \\
&- \varepsilon\left\{\int_x\int_\theta q(x,\theta)w_1^o(x,\theta)Rd\theta dx\right\}.
\end{aligned}
\tag{57}
$$

The increment in the total potential energy can be written as

$$\Delta\pi = \varepsilon\pi_1 + \varepsilon^2\pi_2 + \varepsilon^3\pi_3 + \varepsilon^4\pi_4 . \tag{58}$$

The quantities π_1, π_2, π_3 , and π_4 are defined to be the first, second, third, and fourth variation, respectively. The equilibrium conditions for the cylinder are obtained from the condition

$$\pi_1(u_1, v_1, w_1) = 0 , \tag{59}$$

where the notation indicates π_1 is to be made stationary with respect to the displacements u_1, v_1, w_1. These displacements are the variations in the equilibrium displacements. The second variation is used to examine stability of the equilibrium displacements. According to the

Trefftz stability criterion, transition from a stable equilibrium configuration to an unstable one is characterized by

$$\delta\pi_2(u_1, v_1, w_1) = 0 . \tag{60}$$

This states the second variation of the total potential energy should be stationary with respect to variations in u_1, v_1, w_1. The above two conditions will now be examined.

First Variation

The first variation can be isolated from $\Delta\pi$ and is

$$\begin{aligned}
\pi_1 = \frac{1}{2}\int_x\int_\theta \Big\{ & (N_x - N_x^P)\varepsilon_{x_1} + N_{x_1}\varepsilon_x + \left(N_\theta - N_\theta^P\right)\varepsilon_{\theta_1} + N_{\theta_1}\varepsilon_\theta \\
& + \left(N_{x\theta} - N_{x\theta}^P\right)\gamma_{x\theta_1} + N_{x\theta_1}\gamma_{x\theta} + \left(M_x - M_x^P\right)\kappa_{x_1} + M_{x_1}\kappa_x \\
& + \left(M_\theta - M_\theta^P\right)\kappa_{\theta_1} + M_{\theta_1}\kappa_\theta + \left(M_{x\theta} - M_{x\theta}^P\right)\kappa_{x\theta_1} + M_{x\theta_1}\kappa_{x\theta}\Big\} R\,d\theta\,dx \\
& + \left\{\int_\theta N_x^-(\theta)u_1^o\left(-\frac{L}{2}, \theta\right)R\,d\theta - \int_\theta N_x^+(\theta)u_1^o\left(\frac{L}{2}, \theta\right)R\,d\theta\right\} \\
& + \left\{\int_\theta N_{x\theta}^-(\theta)v_1^o\left(-\frac{L}{2}, \theta\right)R\,d\theta - \int_\theta N_{x\theta}^+(\theta)v_1^o\left(\frac{L}{2}, \theta\right)R\,d\theta\right\} \\
& - \left\{\int_x\int_\theta q(x, \theta)w_1^o(x, \theta)R\,d\theta\,dx\right\} .
\end{aligned} \tag{61}$$

A more useful form of the first variation can be obtained by substituting for N_{x_1}, N_{θ_1}, $N_{x\theta_1}$, M_{x_1}, M_{θ_1}, and $M_{x\theta}$, from eqs. 50e, 51e, 52e, 53e, 54e, and 55e. If this is done and the various terms in this expanded form of π_1 regrouped, the result is

$$\pi_1 = \int_x \int_\theta \{N_x\varepsilon_{x_1} + N_\theta\varepsilon_{\theta_1} + N_{x\theta}\gamma_{x\theta_1} + M_x\kappa_{x_1}$$
$$+ M_\theta\kappa_{\theta_1} - M_{x\theta}\kappa_{x\theta_1}\}Rd\theta dx$$
$$+ \left\{\int_\theta N_x^-(\theta)u_1^\circ\left(-\frac{L}{2},\theta\right)Rd\theta - \int_\theta N_x^+(\theta)u_1^\circ\left(\frac{L}{2},\theta\right)Rd\theta\right\} \tag{62}$$
$$+ \left\{\int_\theta N_{x\theta}^-(\theta)v_1^\circ\left(-\frac{L}{2},\theta\right)Rd\theta - \int_\theta N_{x\theta}^+(\theta)v_1^\circ\left(\frac{L}{2},\theta\right)Rd\theta\right\}$$
$$- \left\{\int_x \int_\theta q(x,\theta)w_1^\circ(x,\theta)Rd\theta dx\right\}.$$

Note the quantities N_x^p ,... $M_{x\theta}^p$ have disappeared, as has the factor of 1/2.

If the strain and curvature increments ε_{x_1} ,... $\kappa_{x\theta_1}$ are written in terms of the displacement increments, using eqs. 32, 37, 42, and 45-47, the first variation takes the form

$$\pi_1 = \left\{\int_x \int_\theta \left\{N_x\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + N_\theta\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)\right.\right.$$
$$+ N_{x\theta}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) + M_x\frac{\partial\beta_{x_1}}{\partial x} + M_\theta\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$
$$+ M_{x\theta}\left.\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)\right\}Rd\theta dx \tag{63}$$
$$+ \int_\theta\left\{N_x^-u_1\left(-\frac{L}{2},\theta\right) - N_x^+u_1\left(\frac{L}{2},\theta\right)\right\}Rd\theta$$
$$\left. + \int_\theta\left\{N_{x\theta}^-v_1\left(-\frac{L}{2},\theta\right) - N_{x\theta}^+v_1\left(\frac{L}{2},\theta\right)\right\}Rd\theta - \int_x\int_\theta q(x)w_1(x,\theta)Rd\theta dx\right\}$$

This is one of the fundamental forms of the first variation for determining the response of a cylinder. This form can be used directly in approximate schemes such as the Rayleigh-Ritz method. However, here we are interested in the governing equilibrium equations and associated boundary conditions. The steps to derive these follow:

To determine the equilibrium equations, and the associated boundary conditions, differentiation of u_1,v_1 , and w_1 with respect to the spatial variables x and θ must eliminated. This is done using integration by parts on the various terms in eq. 63. This shall be done in the following term by term:

first term

$$\int_x\int_\theta N_x\frac{\partial u_1}{\partial x}Rd\theta dx=\int_\theta(N_xu_1)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}}Rd\theta \\ -\int_x\int_\theta\frac{\partial N_x}{\partial x}u_1Rd\theta dx\,. \tag{64}$$

second term

$$\int_x\int_\theta N_x\beta_x\beta_{x_1}Rd\theta dx=\int_x\int_\theta N_x\frac{\partial w}{\partial x}\frac{\partial w_1}{\partial x}Rd\theta dx \\ =\int_\theta\left(N_x\frac{\partial w}{\partial x}w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}}Rd\theta \\ -\int_x\int_\theta\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}\right)w_1Rd\theta dx\,. \tag{65}$$

In the above use has been made of eqs. 3 and 48.

third term

$$\int_x\int_\theta N_\theta\frac{\partial v_1}{R\partial\theta}Rd\theta dx=\int_x(N_\theta v_1)\Big|_{\theta=-\pi}^{\theta=+\pi}dx \\ -\int_x\int_\theta\frac{\partial N_\theta}{R\partial\theta}v_1Rd\theta dx\,. \tag{66}$$

Because the cylinder is complete, the response is a continuous function at the spatial variable θ. Thus

$$N_\theta(x, +\pi) \equiv N_\theta(x, -\pi)$$
$$v_1(x, +\pi) \equiv v_1(x, -\pi)\,. \tag{67}$$

As a result the first integral on x in eq. 66 sums to zero and we are left with the third term as

$$\int_x \int_\theta N_\theta \frac{\partial v_1}{R\partial\theta} R d\theta dx = -\int_x \int_\theta \frac{\partial N_\theta}{R\partial\theta} v_1 R d\theta dx\,. \tag{68}$$

fifth term

$$\begin{aligned}\int_x \int_\theta N_\theta \beta_\theta \beta_{\theta_1} R d\theta dx &= \int_x \int_\theta N_\theta \frac{\partial w}{R\partial\theta} \frac{\partial w_1}{R\partial\theta} R d\theta dx \\ &= -\int_x \int_\theta \frac{\partial}{R\partial\theta}\left(N_\theta \frac{\partial w}{R\partial\theta}\right) w_1 R d\theta dx\,,\end{aligned} \tag{69}$$

where use has been made of eqs. 3 and 48, and the fact that the first terms normally on the right when using integration by parts is zero because the cylinder is complete.

sixth term

$$\begin{aligned}\int_x \int_\theta N_{x\theta} \frac{\partial v_1}{\partial x} R d\theta dx &= \int_\theta (N_{x\theta} v_1)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ &\quad - \int_x \int_\theta \frac{\partial N_{x\theta}}{\partial x} v_1 R d\theta dx\,.\end{aligned} \tag{70}$$

seventh term

$$\int_x \int_\theta N_{x\theta} \frac{\partial u_1}{R\partial\theta} Rd\theta dx = -\int_x \int_\theta \frac{\partial N_{x\theta}}{R\partial\theta} u_1 Rd\theta dx \tag{71}$$

eighth term

$$\begin{aligned}\int_x \int_\theta N_{x\theta}\beta_x\beta_{\theta_1} Rd\theta dx &= \int_x \int_\theta N_{x\theta} \frac{\partial w}{\partial x}\frac{\partial w_1}{R\partial\theta} Rd\theta dx \\ &= -\int_x \int_\theta \frac{\partial}{R\partial\theta}\left(N_{x\theta}\frac{\partial w}{\partial x}\right) w_1 Rd\theta dx\end{aligned} \tag{72}$$

ninth term

$$\begin{aligned}\int_x \int_\theta N_{x\theta}\beta_\theta\beta_{x_1} Rd\theta dx &= \int_x \int_\theta N_{x\theta} \frac{\partial w}{R\partial\theta}\frac{\partial w_1}{\partial x} Rd\theta dx \\ &= \int_\theta \left(N_{x\theta}\frac{\partial w}{R\partial\theta} w_1\right)\Bigg|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ &\quad -\int_x \int_\theta \frac{\partial}{\partial x}\left(N_{x\theta}\frac{\partial w}{R\partial\theta}\right) w_1 Rd\theta dx\end{aligned} \tag{73}$$

tenth term

$$\begin{aligned}\int_x \int_\theta M_x \frac{\partial \beta_{x_1}}{\partial x} Rd\theta dx &= \int_\theta (M_x\beta_{x_1})\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ &\quad -\int_x \int_\theta \frac{\partial M_x}{\partial x}\beta_{x_1} Rd\theta dx\end{aligned} \tag{74}$$

These terms can be expanded further. Specifically, using eq. 48, the second term on the right side of eq. 74 becomes

$$-\int_x\int_\theta \frac{\partial M_x}{\partial x}\beta_{x_1} R d\theta dx = \int_x\int_\theta \frac{\partial M_x}{\partial x}\frac{\partial w_1}{\partial x} R d\theta dx \,. \tag{75}$$

Using integration by parts once more yields

$$\begin{aligned}-\int_x\int_\theta \frac{\partial M_x}{\partial x}\beta_{x_1} R d\theta dx = &\int_\theta \left(\frac{\partial M_x}{\partial x} w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ &-\int_x\int_\theta \frac{\partial^2 M_x}{\partial x^2} w_1 R d\theta dx\end{aligned} \tag{76}$$

The tenth term can thus be written as

$$\begin{aligned}\int_x\int_\theta M_x \frac{\partial \beta_{x_1}}{\partial x} R d\theta dx = &-\int_\theta \left(M_x \frac{\partial w_1}{\partial x}\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ &+\int_\theta \left(\frac{\partial M_x}{\partial x} w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ &-\int_x\int_\theta \frac{\partial^2 M_x}{\partial x^2} w_1 R d\theta dx \,.\end{aligned} \tag{77}$$

<u>eleventh term</u>

$$\int_x\int_\theta M_\theta \frac{\partial \beta_{\theta_1}}{R\partial\theta} R d\theta dx = -\int_x\int_\theta \frac{\partial M_\theta}{R\partial\theta}\beta_{\theta_1} R d\theta dx \,, \tag{78}$$

where continuity of the cylinder has been used. The expression on the right can be further integrated by using the definition of β_{θ_1} from eq. 48, i.e.,

$$-\int_x\int_\theta \frac{\partial M_\theta}{R\partial\theta}\beta_{\theta_1} R d\theta dx = \int_x\int_\theta \frac{\partial M_\theta}{R\partial\theta}\frac{\partial w_1}{R\partial\theta} R d\theta dx$$
$$= -\int_x\int_\theta \frac{\partial^2 M_\theta}{R^2\partial\theta^2} w_1 R d\theta dx \tag{79}$$

Thus the eleventh term becomes

$$\int_x\int_\theta M_\theta \frac{\partial\beta_{\theta_1}}{R\partial\theta} R d\theta dx = -\int_x\int_\theta \frac{\partial^2 M_\theta}{R^2\partial\theta^2} w_1 R d\theta dx \tag{80}$$

<u>twelfth term</u>

$$\int_x\int_\theta M_{x\theta} \frac{\partial\beta_{\theta_1}}{\partial x} R d\theta dx = \int_\theta (M_{x\theta}\beta_{\theta_1})\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta$$
$$-\int_x\int_\theta \frac{\partial M_{x\theta}}{\partial x}\beta_{\theta_1} R d\theta dx\,. \tag{81}$$

The second term on the right can be further integrated as

$$-\int_x\int_\theta \frac{\partial M_{x\theta}}{\partial x}\beta_{\theta_1} R d\theta dx = \int_x\int_\theta \frac{\partial M_{x\theta}}{\partial x}\frac{\partial w_1}{R\partial\theta} R d\theta dx$$
$$= -\int_x\int_\theta \frac{\partial^2 M_{x\theta}}{R\partial\theta dx} w_1 R d\theta dx \tag{82}$$

where again completeness of the cylinder has been used. Hence the twelfth term becomes

$$\int_x\int_\theta M_{x\theta} \frac{\partial\beta_{\theta_1}}{\partial x} R d\theta dx = \int_\theta (M_{x\theta}\beta_{\theta_1})\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta$$
$$-\int_x\int_\theta \frac{\partial^2 M_{x\theta}}{R\partial\theta\partial x} w_1 R d\theta dx\,. \tag{83}$$

The first term on the right in this resulting equation is special. It can be integrated by parts one more time. Considering the integral at $x = +L/2$, the term becomes

$$\int_\theta M_{x\theta}\left(+\frac{L}{2},\theta\right)\beta_{\theta_1}\left(+\frac{L}{2},\theta\right)Rd\theta = -\int_\theta M_{x\theta}\left(+\frac{L}{2},\theta\right)\frac{\partial w_1\left(+\frac{L}{2},\theta\right)}{R\partial\theta}Rd\theta \tag{84}$$

Integrating by parts

$$\begin{aligned}\int_\theta M_{x\theta}\left(+\frac{L}{2},\theta\right)\beta_{\theta_1}\left(+\frac{L}{2},\theta\right)Rd\theta = &- M_{x\theta}\left(+\frac{L}{2},+\pi\right)\frac{\partial w_1\left(+\frac{L}{2},+\pi\right)}{R\partial\theta} \\ &+ M_{x\theta}\left(+\frac{L}{2},-\pi\right)\frac{\partial w_1\left(+\frac{L}{2},-\pi\right)}{R\partial\theta} \\ &+ \int_\theta \frac{\partial M_{x\theta}\left(+\frac{L}{2},\theta\right)}{R\partial\theta} w_1\left(+\frac{L}{2},\theta\right)Rd\theta\end{aligned} \tag{85}$$

Because of completeness of the cylinder, the first two terms on the right sum to zero. The same procedure can be used at $x = -L/2$. The result is

$$\int_\theta \left(M_{x\theta}\beta_{\theta_1}\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta = \int_\theta \left(\frac{\partial M_{x\theta}}{R\partial\theta}w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta\,. \tag{86}$$

Thus the final form for the twelfth term becomes

$$\begin{aligned}\int_x\int_\theta M_{x\theta}\frac{\partial\beta_{\theta_1}}{\partial x}Rd\theta dx = &\int_\theta \left(\frac{\partial M_{x\theta}}{R\partial\theta}w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ &- \int_x\int_\theta \frac{\partial^2 M_{x\theta}}{R\partial\theta\partial x}w_1 Rd\theta dx\,.\end{aligned} \tag{87}$$

thirteenth term

$$\int_x \int_\theta M_{x\theta} \frac{\partial \beta_{x_1}}{R\partial\theta} Rd\theta dx = -\int_x \int_\theta \frac{\partial M_{x\theta}}{R\partial\theta} \beta_{x_1} Rd\theta dx\,, \tag{88}$$

where completeness of the cylinder has been used once more. The term on the right can be further integrated by the fact that

$$\begin{aligned} -\int_x \int_\theta \frac{\partial M_{x\theta}}{R\partial\theta} \beta_{x_1} Rd\theta dx &= \int_x \int_\theta \frac{\partial M_{x\theta}}{R\partial\theta} \frac{\partial w_1}{\partial x} Rd\theta dx \\ &= \int_\theta \left(\frac{\partial M_{x\theta}}{R\partial\theta} w_1 \right) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ &- \int_x \int_\theta \frac{\partial^2 M_{x\theta}}{R\partial\theta\partial x} w_1 Rd\theta dx \end{aligned} \tag{89}$$

The thirteenth term thus becomes

$$\begin{aligned} \int_x \int_\theta M_{x\theta} \frac{\partial \beta_{x_1}}{R\partial\theta} Rd\theta dx &= \int_\theta \left(\frac{\partial M_{x\theta}}{R\partial\theta} w_1 \right) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ &- \int_x \int_\theta \frac{\partial^2 M_{x\theta}}{Rd\theta dx} w_1 Rd\theta \partial x \end{aligned} \tag{90}$$

Using eqs. 64, 65, 68, 69, 70, 71, 72, 73, 77, 86, 87, and 90 in the expression for the first variation, eq. 63, and combining boundary terms (i.e., integrals on θ), results in

$$
\begin{aligned}
\pi_1 = &\int_x \int_\theta \Bigg[\left\{ -\frac{\partial N_x}{\partial x} - \frac{\partial N_{x\theta}}{R\partial\theta} \right\} u_1 + \left\{ -\frac{\partial N_{x\theta}}{\partial x} - \frac{\partial N_\theta}{R\partial\theta} \right\} v_1 \\
&+ \Bigg\{ -\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{x\theta}}{Rd\theta\partial x} - \frac{\partial^2 M_\theta}{R^2\partial\theta^2} - \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) - \frac{\partial}{R\partial\theta}\left(N_\theta \frac{\partial w}{R\partial\theta}\right) \\
&- \frac{\partial}{R\partial\theta}\left(N_{x\theta}\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial x}\left(N_{x\theta}\frac{\partial w}{R\partial\theta}\right) + \frac{N_\theta}{R} - q \Bigg\} w_1 \Bigg] Rd\theta dx \\
&\int_\theta \Bigg\{ \left(N_x\left(+\frac{L}{2}, \theta\right) - N_x^+(\theta) \right) u_1\left(+\frac{L}{2}, \theta\right) \\
&- \left(N_x\left(-\frac{L}{2}, \theta\right) - N_x^-(\theta) \right) u_1\left(-\frac{L}{2}, \theta\right) \\
&+ \left(N_{x\theta}\left(+\frac{L}{2}, \theta\right) - N_{x\theta}^+(\theta) \right) v_1\left(+\frac{L}{2}, \theta\right) \\
&- \left(N_{x\theta}\left(-\frac{L}{2}, \theta\right) - N_{x\theta}^-\left(-\frac{L}{2}\right) \right) v_1\left(-\frac{L}{2}, \theta\right) \\
&+ \left(\left(\frac{\partial M_x}{\partial x} + N_x \frac{\partial w}{\partial x} + N_{x\theta}\frac{\partial w}{R\partial\theta} + 2\frac{\partial M_{x\theta}}{R\partial\theta} \right) w_1 \right) \Bigg|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} \\
&+ \left(-M_x \frac{\partial w_1}{\partial x} \right) \Bigg|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} \Bigg\} Rd\theta \, .
\end{aligned}
\tag{91}
$$

For the first variation to be zero, each individual term in each of the integrals must be zero, the Euler equilibrium equations coming from the two-dimensional integral, and the boundary conditions coming from the one dimensional integral with respect to θ. Thus, the three governing equilibrium equations are

$$
\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\theta}}{R\partial\theta} = 0 \tag{92a}
$$

$$
\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial\theta} = 0 \tag{92b}
$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{x\theta}}{R\partial\theta\partial x} + \frac{\partial^2 M_\theta}{R^2\partial\theta^2} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right)$$
$$+ \frac{\partial}{R\partial\theta}\left(N_\theta \frac{\partial w}{R\partial\theta}\right) + \frac{\partial}{R\partial\theta}\left(N_{x\theta} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{x\theta} \frac{\partial w}{R\partial\theta}\right) \tag{92c}$$
$$- \frac{N_\theta}{R} + q = 0$$

Using the first two equilibrium equations in the third one, the three equations can be written as

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\theta}}{R\partial\theta} = 0 \tag{93a}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial\theta} = 0 \tag{93b}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{x\theta}}{R\partial\theta\partial x} + \frac{\partial^2 M_\theta}{R^2\partial\theta^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{x\theta}\frac{\partial^2 w}{R\partial\theta\partial x}$$
$$+ N_\theta \frac{\partial^2 w}{R^2\partial\theta^2} - \frac{N_\theta}{R} + q = 0 \tag{93c}$$

The variationally consistent boundary conditions at the ends of the cylinder are:

at x = - L/2

i) $N_x = N_x^-$ or u must be specified ,
ii) $N_{x\theta} = N_{x\theta}^-$ or v must be specified ,
iii) $\frac{\partial M_x}{\partial x} + N_x \frac{\partial w}{\partial x} + N_{x\theta}\frac{\partial w}{R\partial\theta} + 2\frac{\partial M_{x\theta}}{R\partial\theta} = 0$ or w must be specified , (94a)
iv) $M_x = 0$ or $\frac{\partial w}{\partial x}$ must be specified .

at x = + L/2

i) $N_x = N_x^+$ or u must be specified ,

ii) $N_{x\theta} = N_{x\theta}^+$ or v must be specified ,

iii) $\frac{\partial M_x}{\partial x} + N_x \frac{\partial w}{\partial x} + N_{x\theta} \frac{\partial w}{R\partial\theta} + 2\frac{\partial M_{x\theta}}{R\partial\theta} = 0$ or w must be specified , (94b)

iv) $M_x = 0$ or $\frac{\partial w}{\partial x}$ must be specified .

Second Variation

With the equilibrium conditions established, attention now turns to the stability of these conditions. As stated by the Trefftz criterion, stability information can be obtained by examining the first variation of the second variation, π_2. The second variation can be isolated from $\Delta\pi$ of eq. 57 and is given by

$$\begin{aligned}\pi_2 = \frac{1}{2}\int_x\int_\theta \Big[&(N_x - N_x^P)\varepsilon_{x_2} + N_{x_2}\varepsilon_x + N_{x_1}\varepsilon_{x_1} + (N_\theta - N_\theta^P)\varepsilon_{\theta_2} \\ &+ N_{\theta_2}\varepsilon_\theta + N_{\theta_1}\varepsilon_{\theta_1} + (N_{x\theta} - N_{x\theta}^P)\gamma_{x\theta_2} + N_{x\theta_2}\gamma_{x\theta} + N_{x\theta_1}\gamma_{x\theta_1} \\ &+ M_{x_2}\kappa_x + M_{x_1}\kappa_{x_1} + M_{\theta_2}\kappa_\theta + M_{\theta_1}\kappa_{\theta_1} + M_{x\theta_2}\kappa_{x\theta} \\ &+ M_{x\theta_1}\kappa_{x\theta_1}\Big] R\,d\theta\,dx\,. \end{aligned} \tag{95}$$

A more useful form of π_2 can be obtained by substituting for N_{x_2}, N_{θ_2}, $N_{x\theta_2}$, M_{x_2}, M_{θ_2}, and $M_{x\theta_2}$ from eqs. 50f through 55f. Doing this, and regrouping terms, results in

$$\begin{aligned}\pi_2 = \int_x\int_\theta \Big[&N_x\varepsilon_{x_2} + N_\theta\varepsilon_{\theta_2} + N_{x\theta}\gamma_{x\theta_2} \\ &+ \frac{1}{2}\big(N_{x_1}\varepsilon_{x_1} + N_{\theta_1}\varepsilon_{\theta_1} + N_{x\theta_1}\gamma_{x\theta_1} + M_{x_1}\kappa_{x_1} + M_{\theta_1}\kappa_{\theta_1} \\ &+ M_{x\theta_1}\kappa_{x\theta_1}\big)\Big] R\,d\theta\,dx \end{aligned} \tag{96}$$

Again note the disappearance of N_x^p, N_θ^p, and $N_{x\theta}^p$ and the algebra with the factor of 1/2. Since the variational process involves kinematic variables, here u_1, v_1, and w_1, the increments in the strains, curvatures, and stress resultants should be written in terms of these kinematic variables before the variational process begins. To that end, using the expressions for $N_{x_1}, \ldots, M_{x\theta_1}$ from eqs. 50e through 55e, $\varepsilon_{x_1}, \ldots, \kappa_{x\theta_1}$ from eqs. 32, 37, 42, and 45-47, and ε_{x_2}, ε_{θ_2}, and $\gamma_{x\theta_2}$ from eqs. 33, 38, and 43,

$$
\begin{aligned}
\pi_2 = \int_x \int_\theta \Big[& \frac{1}{2} N_x \beta_{x_1}^2 + \frac{1}{2} N_\theta \beta_{\theta_1}^2 + N_{x\theta} \beta_{x_1} \beta_{\theta_1} \\
& + \frac{1}{2} \Bigg[A_{11}\left(\frac{\partial u_1}{\partial x} + \beta_x \beta_{x_1} \right) + A_{12}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta \beta_{\theta_1} \right) \\
& + A_{16}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x \beta_{\theta_1} + \beta_\theta \beta_{x_1} \right) + B_{11} \frac{\partial \beta_{x_1}}{\partial x} + B_{12} \frac{\partial \beta_{\theta_1}}{R\partial\theta} \\
& + B_{16}\left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta} \right) \Bigg\} \left\{ \frac{\partial u_1}{\partial x} + \beta_x \beta_{x_1} \right\} \\
& + \frac{1}{2} \Bigg\{ A_{12}\left(\frac{\partial u_1}{\partial x} + \beta_x \beta_{x_1} \right) + A_{22}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta \beta_{\theta_1} \right) \\
& + A_{26}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x \beta_{\theta_1} + \beta_\theta \beta_{x_1} \right) + B_{12} \frac{\partial \beta_{x_1}}{\partial x} + B_{22} \frac{\partial \beta_{\theta_1}}{R\partial\theta} \\
& + B_{26}\left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta} \right) \Bigg\} \left\{ \frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta \beta_{\theta_1} \right\} \\
& + \Bigg\{ A_{16}\left(\frac{\partial u_1}{\partial x} + \beta_x \beta_{x_1} \right) + A_{26}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta \beta_{\theta_1} \right) \\
& + A_{66}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x \beta_{\theta_1} + \beta_\theta \beta_{x_1} \right) + B_{16} \frac{\partial \beta_{x_1}}{\partial x} + B_{26} \frac{\partial \beta_{\theta_1}}{R\partial\theta} \\
& + B_{66}\left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta} \right) \Bigg\} \left\{ \frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x \beta_{\theta_1} + \beta_\theta \beta_{x_1} \right\} \\
& + \frac{1}{2} \Bigg\{ B_{11}\left(\frac{\partial u_1}{\partial x} + \beta_x \beta_{x_1} \right) + B_{12}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta \beta_{\theta_1} \right) \\
& + B_{16}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x \beta_{\theta_1} + \beta_\theta \beta_{x_1} \right) + D_{11} \frac{\partial \beta_{x_1}}{\partial x} + D_{12} \frac{\partial \beta_{\theta_1}}{R\partial\theta} \\
& + D_{16}\left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta} \right) \Bigg\} \frac{\partial \beta_{x_1}}{\partial x}
\end{aligned}
\qquad (97)
$$

$$+\frac{1}{2}\left\{B_{12}\left(\frac{\partial u_1}{\partial x}+\beta_x\beta_{x_1}\right)+B_{22}\left(\frac{\partial v_1}{R\partial\theta}+\frac{w_1}{R}+\beta_\theta\beta_{\theta_1}\right)\right.$$

$$+B_{26}\left(\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{R\partial\theta}+\beta_x\beta_{\theta_1}+\beta_\theta\beta_{x_1}\right)+D_{12}\frac{\partial\beta_{x_1}}{\partial x}+D_{22}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$\left.+D_{26}\left(\frac{\partial\beta_{\theta_1}}{\partial x}+\frac{\partial\beta_{x_1}}{R\partial\theta}\right)\right\}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$+\frac{1}{2}\left[B_{16}\left(\frac{\partial u_1}{\partial x}+\beta_x\beta_{x_1}\right)+B_{26}\left(\frac{\partial v_1}{R\partial\theta}+\frac{w_1}{R}+\beta_\theta\beta_{\theta_1}\right)\right.$$

$$+B_{66}\left(\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{R\partial\theta}+\beta_x\beta_{\theta_1}+\beta_\theta\beta_{x_1}\right)+D_{16}\frac{\partial\beta_{x_1}}{\partial x}+D_{26}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$\left.+D_{66}\left(\frac{\partial\beta_{\theta_1}}{\partial x}+\frac{\partial\beta_{x_1}}{R\partial\theta}\right)\right\}\left\{\frac{\partial\beta_{x_1}}{R\partial\theta}+\frac{\partial\beta_{\theta_1}}{\partial x}\right\}\Bigg]R\,d\theta\,dx$$

Taking the variation of eq. 97 with respect to u_1, v_1, and w_1 results in

$$\delta\pi_2=\frac{1}{2}\int_x\int_\theta[2N_x\beta_{x_1}\delta\beta_{x_1}+2N_\theta\beta_{\theta_1}\delta\beta_{\theta_1}+2N_{x\theta}(\beta_{x_1}\delta\beta_{\theta_1}+\beta_{\theta_1}\delta\beta_{x_1})$$

$$+\left\{A_{11}\left(\frac{\partial\delta u_1}{\partial x}+\beta_x\delta\beta_{x_1}\right)+A_{12}\left(\frac{\partial\delta v_1}{R\partial\theta}+\frac{\delta w_1}{R}+\beta_\theta\delta\beta_{\theta_1}\right)\right.$$

$$+A_{16}\left(\frac{\partial\delta v_1}{\partial x}+\frac{\partial\delta u_1}{R\partial\theta}+\beta_x\delta\beta_{\theta_1}+\beta_\theta\delta\beta_{x_1}\right)+B_{11}\frac{\partial\delta\beta_{x_1}}{\partial x}+B_{12}\frac{\partial\delta\beta_{\theta_1}}{R\partial\theta}$$

$$\left.+B_{16}\left(\frac{\partial\delta\beta_{\theta_1}}{\partial x}+\frac{\partial\delta\beta_{x_1}}{R\partial\theta}\right)\right\}\left\{\frac{\partial u_1}{\partial x}+\beta_x\beta_{x_1}\right\}$$

$$+\left\{A_{11}\left(\frac{\partial u_1}{\partial x}+\beta_x\beta_{x_1}\right)+A_{12}\left(\frac{\partial v_1}{R\partial\theta}+\frac{w_1}{R}+\beta_\theta\beta_{\theta_1}\right)\right. \tag{98}$$

$$+A_{16}\left(\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{R\partial\theta}+\beta_x\beta_{\theta_1}+\beta_\theta\beta_{x_1}\right)+B_{11}\frac{\partial\beta_{x_1}}{\partial x}+B_{12}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$\left.+B_{16}\left(\frac{\partial\beta_{\theta_1}}{\partial x}+\frac{\partial\beta_{x_1}}{R\partial\theta}\right)\right\}\left\{\frac{\partial\delta u_1}{\partial x}+\beta_x\delta\beta_{x_1}\right\}$$

$$+\left\{A_{12}\left(\frac{\partial\delta u_1}{\partial x}+\beta_x\delta\beta_{x_1}\right)+A_{22}\left(\frac{\partial\delta v_1}{R\partial\theta}+\frac{\delta w_1}{R}+\beta_\theta\delta\beta_{\theta_1}\right)\right.$$

$$+A_{26}\left(\frac{\partial\delta v_1}{\partial x}+\frac{\partial\delta u_1}{R\partial\theta}+\beta_x\delta\beta_{\theta_1}+\beta_\theta\delta\beta_{x_1}\right)+B_{12}\frac{\partial\delta\beta_{x_1}}{\partial x}+B_{22}\frac{\partial\delta\beta_{\theta_1}}{R\partial\theta}$$

$$+ B_{26}\left(\frac{\partial\delta\beta_{\theta_1}}{\partial x} + \frac{\partial\delta\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\left\{\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right\}$$

$$+ \Bigg\{A_{12}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + A_{22}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)$$

$$+ A_{26}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) + B_{12}\frac{\partial\beta_{x_1}}{\partial x} + B_{22}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$+ B_{26}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\left\{\frac{\partial\delta v_1}{R\partial\theta} + \frac{\delta w_1}{R} + \beta_\theta\delta\beta_{\theta_1}\right\}$$

$$+ \Bigg\{A_{16}\left(\frac{\partial\delta u_1}{\partial x} + \beta_x\delta\beta_{x_1}\right) + A_{26}\left(\frac{\partial\delta v_1}{R\partial\theta} + \frac{\delta w_1}{R} + \beta_\theta\delta\beta_{\theta_1}\right)$$

$$+ A_{66}\left(\frac{\partial\delta v_1}{\partial x} + \frac{\partial\delta u_1}{R\partial\theta} + \beta_x\delta\beta_{\theta_1} + \beta_\theta\delta\beta_{x_1}\right) + B_{16}\frac{\partial\delta\beta_{x_1}}{\partial x} + B_{26}\frac{\partial\delta\beta_{\theta_1}}{R\partial\theta}$$

$$+ B_{66}\left(\frac{\partial\delta\beta_{\theta_1}}{\partial x} + \frac{\partial\delta\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\left\{\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right\}$$

$$+ \Bigg\{A_{16}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + A_{26}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)$$

$$+ A_{66}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) + B_{16}\frac{\partial\beta_{x_1}}{\partial x} + B_{26}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$+ B_{66}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\left\{\frac{\partial\delta v_1}{\partial x} + \frac{\partial\delta u_1}{R\partial\theta} + \beta_x\delta\beta_{\theta_1} + \beta_\theta\delta\beta_{x_1}\right\}$$

$$+ \Bigg\{B_{11}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + B_{12}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)$$

$$+ B_{16}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) + D_{11}\frac{\partial\beta_{x_1}}{\partial x} + D_{12}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$+ D_{16}\left(\frac{\partial\beta_{\theta_1}}{\partial x} + \frac{\partial\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\frac{\partial\delta\beta_{x_1}}{\partial x} + \Bigg\{B_{11}\left(\frac{\partial\delta u_1}{\partial x} + \beta_x\delta\beta_{x_1}\right)$$

$$+ B_{12}\left(\frac{\partial\delta v_1}{R\partial\theta} + \frac{\delta w_1}{R} + \beta_\theta\delta\beta_{\theta_1}\right) + B_{16}\left(\frac{\partial\delta v_1}{\partial x} + \frac{\partial\delta u_1}{R\partial\theta} + \beta_x\delta\beta_{\theta_1} + \beta_\theta\delta\beta_{x_1}\right)$$

$$+ D_{11}\frac{\partial\delta\beta_{x_1}}{\partial x} + D_{12}\frac{\partial\delta\beta_{\theta_1}}{R\partial\theta} + D_{16}\left(\frac{\partial\delta\beta_{\theta_1}}{\partial x} + \frac{\partial\delta\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\frac{\partial\beta_{x_1}}{\partial x}$$

$$+ \Bigg\{B_{12}\left(\frac{\partial u_1}{\partial x} + \beta_x\beta_{x_1}\right) + B_{22}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta\beta_{\theta_1}\right)$$

$$+ B_{26}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x\beta_{\theta_1} + \beta_\theta\beta_{x_1}\right) + D_{12}\frac{\partial\beta_{x_1}}{\partial x} + D_{22}\frac{\partial\beta_{\theta_1}}{R\partial\theta}$$

$$+ D_{26}\left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta}\right)\Bigg\} \frac{\partial \delta\beta_{\theta_1}}{R\partial\theta} + \Bigg\{ B_{12}\left(\frac{\partial \delta u_1}{\partial x} + \beta_x \delta\beta_{x_1}\right) + B_{22}\left(\frac{\partial \delta v_1}{R\partial\theta} + \frac{\delta w_1}{R} + \beta_\theta \delta\beta_{\theta_1}\right)$$

$$+ B_{26}\left(\frac{\partial \delta v_1}{\partial x} + \frac{\partial \delta u_1}{R\partial\theta} + \beta_x \delta\beta_{\theta_1} + \beta_\theta \delta\beta_{x_1}\right) + D_{12}\frac{\partial \delta\beta_{x_1}}{\partial x} + D_{22}\frac{\partial \delta\beta_{\theta_1}}{R\partial\theta}$$

$$+ D_{26}\left(\frac{\partial \delta\beta_{\theta_1}}{\partial x} + \frac{\partial \delta\beta_{x_1}}{R\partial\theta}\right)\Bigg\} \frac{\partial \beta_{\theta_1}}{R\partial\theta} + \Bigg\{ B_{16}\left(\frac{\partial \delta u_1}{\partial x} + \beta_x \delta\beta_{x_1}\right) + B_{26}\left(\frac{\partial \delta v_1}{R\partial\theta} + \frac{\delta w_1}{R} + \beta_\theta \delta\beta_\theta\right)$$

$$+ B_{66}\left(\frac{\partial \delta v_1}{\partial x} + \frac{\partial \delta u_1}{R\partial\theta} + \beta_x \delta\beta_{\theta_1} + \beta_\theta \delta\beta_{x_1}\right) + D_{16}\frac{\partial \delta\beta_{x_1}}{\partial x} + D_{26}\frac{\partial \delta\beta_{\theta_1}}{R\partial\theta}$$

$$+ D_{66}\left(\frac{\partial \delta\beta_{\theta_1}}{\partial x} + \frac{\partial \delta\beta_{x_1}}{R\partial\theta}\right)\Bigg\}\left\{\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta}\right\} + \Bigg\{ B_{16}\left(\frac{\partial u_1}{\partial x} + \beta_x \beta_{x_1}\right)$$

$$+ B_{26}\left(\frac{\partial v_1}{R\partial\theta} + \frac{w_1}{R} + \beta_\theta \beta_{\theta_1}\right) + B_{66}\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{R\partial\theta} + \beta_x \beta_{\theta_1} + \beta_\theta \beta_{x_1}\right)$$

$$+ D_{16}\frac{\partial \beta_{x_1}}{\partial x} + D_{26}\frac{\partial \beta_{\theta_1}}{R\partial\theta} + D_{66}\left(\frac{\partial \beta_{\theta_1}}{\partial x} + \frac{\partial \beta_{x_1}}{R\partial\theta}\right)\Bigg\}\left\{\frac{\partial \delta\beta_{\theta_1}}{\partial x} + \frac{\partial \delta\beta_{x_1}}{R\partial\theta}\right\}\Bigg\} R\,d\theta\,dx$$

Using the definition of N_{x_1}, N_{θ_1}, ..., $M_{x\theta_1}$, and combining terms leads to an important form of eq. 98, namely,

$$\delta\pi_2 = \int_x \int_\theta \Bigg\{ N_x \beta_{x_1} \delta\beta_{x_1} + N_\theta \beta_{\theta_1} \delta\beta_{\theta_1} + N_{x\theta}(\beta_{x_1}\delta\beta_{\theta_1} + \beta_{\theta_1}\delta\beta_{x_1}) + N_{x_1}\left(\frac{\partial \delta u_1}{\partial x} + \beta_x \delta\beta_{x_1}\right)$$
$$+ N_{\theta_1}\left(\frac{\partial \delta v_1}{R\partial\theta} + \frac{\delta w_1}{R} + \beta_\theta \delta\beta_{\theta_1}\right) + N_{x\theta_1}\left(\frac{\partial \delta v_1}{\partial x} + \frac{\partial \delta u_1}{R\partial\theta} + \beta_x \delta\beta_{\theta_1} + \beta_\theta \delta\beta_{x_1}\right) \tag{99}$$
$$+ M_{x_1}\frac{\partial \delta\beta_{x_1}}{\partial x} + M_{\theta_1}\frac{\partial \delta\beta_{\theta_1}}{R\partial\theta} + M_{x\theta_1}\left(\frac{\partial \delta\beta_{\theta_1}}{\partial x} + \frac{\partial \delta\beta_{x_1}}{R\partial\theta}\right)\Bigg\} R\,d\theta\,dx\,.$$

Since stability is studied by the condition $\delta\pi_2 = 0$, the above integral is equated zero. Then this form of $\delta\pi_2 = 0$ can be used for approximate solutions to the stability conditions. A Rayleigh-Ritz formulation can start from this form of $\delta\pi_2 = 0$.

To determine the differential equations, and associated boundary conditions from which to study stability, integration by parts is used to eliminate differentiation of the variables of

u_1, v_1, and w_1 with respect to the spatial variables x and θ This is done, as before, on a term by term basis as follows:

first term

$$\int_x \int_\theta N_x \beta_{x_1} \delta \beta_{x_1} R d\theta dx = \int_x \int_\theta N_x \frac{\partial w_1}{\partial x} \frac{\partial \delta w_1}{\partial x} R d\theta dx \tag{100}$$

$$\begin{aligned}\int_x \int_\theta N_x \beta_{x_1} \delta \beta_{x_1} R d\theta dx = & \int_\theta \left(N_x \frac{\partial w_1}{\partial x} \delta w_1 \right) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ & - \int_x \int_\theta \frac{\partial}{\partial x} \left(N_x \frac{\partial w_1}{\partial x} \right) \delta w_1 R d\theta dx\end{aligned} \tag{101}$$

second term

$$\int_x \int_\theta N_\theta \beta_{\theta_1} \delta \beta_{\theta_1} R d\theta dx = \int_x \int_\theta N_\theta \frac{\partial w_1}{R \partial \theta} \frac{\partial \delta w_1}{R \partial \theta} R d\theta dx \tag{102}$$

$$\int_x \int_\theta N_\theta \beta_{\theta_1} \delta \beta_{\theta_1} R d\theta dx = - \int_x \int_\theta \frac{\partial}{R \partial \theta} \left(N_\theta \frac{\partial w_1}{R \partial \theta} \right) \delta w_1 R d\theta dx\,, \tag{103}$$

where continuity of the cylinder has been used.

third term

$$\int_x \int_\theta N_{x\theta} \beta_{x_1} \delta \beta_{\theta_1} R d\theta dx = \int_x \int_\theta N_{x\theta} \frac{\partial w_1}{\partial x} \frac{\partial \delta w_1}{R \partial \theta} R d\theta dx \tag{104}$$

$$\int_x \int_\theta N_{x\theta} \beta_{x_1} \delta \beta_{\theta_1} R d\theta dx = -\int_x \int_\theta \frac{\partial}{R \partial \theta} \left(N_{x\theta} \frac{\partial w_1}{\partial x} \right) \delta w_1 R d\theta dx \tag{105}$$

fourth term

$$\int_x \int_\theta N_{x\theta} \beta_{\theta_1} \delta \beta_{x_1} R d\theta dx = \int_x \int_\theta N_{x\theta} \frac{\partial w_1}{R \partial \theta} \frac{\partial \delta w_1}{\partial x} R d\theta dx \tag{106}$$

$$\begin{aligned} \int_x \int_\theta N_{x\theta} \beta_{\theta_1} \delta \beta_{x_1} R d\theta dx = & \int_\theta \left(N_{x\theta} \frac{\partial w_1}{R \partial \theta} \delta w_1 \right) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ & - \int_x \int_\theta \frac{\partial}{\partial x} \left(N_{x\theta} \frac{\partial w_1}{R \partial \theta} \right) \delta w_1 R d\theta dx \end{aligned} \tag{107}$$

fifth term

$$\int_x \int_\theta N_{x_1} \frac{\partial \delta u_1}{\partial x} R d\theta dx = \int_\theta (N_{x_1} \delta u_1) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta - \int_x \int_\theta \frac{\partial N_{x_1}}{\partial x} \delta u_1 R d\theta dx \tag{108}$$

sixth term

$$\int_x \int_\theta N_{x_1} \beta_x \delta \beta_{x_1} R d\theta dx = \int_x \int_\theta N_{x_1} \frac{\partial w}{\partial x} \frac{\partial \delta w_1}{\partial x} R d\theta dx \tag{109}$$

$$\int_x \int_\theta N_{x_1} \beta_x \delta\beta_{x_1} R d\theta dx = \int_\theta \left(N_{x_1} \frac{\partial w}{\partial x} \delta w_1 \right) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta - \int_x \int_\theta \frac{\partial}{\partial x} \left(N_{x_1} \frac{\partial w}{\partial x} \right) \delta w_1 R d\theta dx \tag{110}$$

<u>seventh term</u>

$$\int_x \int_\theta N_{\theta_1} \frac{\partial \delta v_1}{R \partial \theta} R d\theta dx = - \int_x \int_\theta \frac{\partial N_{\theta_1}}{R \partial \theta} \delta v_1 R d\theta dx \tag{111}$$

<u>ninth term</u>

$$\int_x \int_\theta N_{\theta_1} \beta_\theta \delta\beta_{\theta_1} R d\theta dx = \int_x \int_\theta N_{\theta_1} \frac{\partial w}{R \partial \theta} \frac{\partial \delta w_1}{R \partial \theta} R d\theta dx \tag{112}$$

$$\int_x \int_\theta N_{\theta_1} \beta_\theta \delta\beta_{\theta_1} R d\theta dx = \int_x \int_\theta \frac{\partial}{R \partial \theta} \left(N_{\theta_1} \frac{\partial w}{R \partial \theta} \right) \delta w_1 R d\theta dx \tag{113}$$

<u>tenth term</u>

$$\int_x \int_\theta N_{x\theta_1} \frac{\partial \delta v_1}{\partial x} R d\theta dx = \int_\theta (N_{x\theta_1} \delta v_1) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta - \int_x \int_\theta \frac{\partial N_{x\theta_1}}{\partial x} \delta v_1 R d\theta dx \tag{114}$$

eleventh term

$$\int_x \int_\theta N_{x\theta_1} \frac{\partial \delta u_1}{R\partial\theta} R d\theta dx = -\int_x \int_\theta \frac{\partial N_{x\theta_1}}{R\partial\theta} \delta u_1 R d\theta dx \tag{115}$$

twelfth term

$$\int_x \int_\theta N_{x\theta_1} \beta_x \delta\beta_{\theta_1} R d\theta dx = \int_x \int_\theta N_{x\theta_1} \frac{\partial w}{\partial x} \frac{\partial \delta w_1}{R\partial\theta} R d\theta dx \tag{116}$$

$$\int_x \int_\theta N_{x\theta_1} \beta_x \delta\beta_{\theta_1} R d\theta dx = -\int_x \int_\theta \frac{\partial}{R\partial\theta} \left(N_{x\theta_1} \frac{\partial w}{\partial x} \right) \delta w_1 R d\theta dx \tag{117}$$

thirteenth term

$$\int_x \int_\theta N_{x\theta_1} \beta_\theta \delta\beta_{x_1} R d\theta dx = \int_x \int_\theta N_{x\theta_1} \frac{\partial w}{R\partial\theta} \frac{\partial \delta w_1}{\partial x} R d\theta dx \tag{118}$$

$$\begin{aligned} \int_x \int_\theta N_{x\theta_1} \beta_\theta \delta\beta_{x_1} R d\theta dx = & \int_\theta \left(N_{x\theta_1} \frac{\partial w}{R\partial\theta} \delta w_1 \right) \Bigg|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ & - \int_x \int_\theta \frac{\partial}{\partial x} \left(N_{x\theta_1} \frac{\partial w}{R\partial\theta} \right) \delta w_1 R d\theta dx \end{aligned} \tag{119}$$

fourteenth term

$$\int_x \int_\theta M_{x_1} \frac{\partial \delta\beta_{x_1}}{\partial x} R d\theta dx = \int_\theta (M_{x_1} \delta\beta_{x_1}) \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta - \int_x \int_\theta \frac{\partial M_{x_1}}{\partial x} \delta\beta_{x_1} R d\theta dx \tag{120}$$

Using the definition of β_{x_1}, and interchanging the δ and differential operators, the second term on the right hand side becomes

$$-\int_x\int_\theta \frac{\partial M_{x_1}}{\partial x}\delta\beta_{x_1}Rd\theta dx = \int_x\int_\theta \frac{\partial M_{x_1}}{\partial x}\frac{\partial \delta w_1}{\partial x}Rd\theta dx \tag{121}$$

or

$$\begin{aligned}-\int_x\int_\theta \frac{\partial M_{x_1}}{\partial x}\delta\beta_{x_1}Rd\theta dx = \int_\theta\left(\frac{\partial M_{x_1}}{\partial x}\delta w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ -\int_x\int_\theta \frac{\partial^2 M_{x_1}}{\partial x^2}\delta w_1 Rd\theta dx\end{aligned} \tag{122}$$

With this the fourteenth term becomes

$$\begin{aligned}\int_x\int_\theta M_{x_1}\frac{\partial \delta\beta_{x_1}}{\partial x}Rd\theta dx = -\int_\theta\left(M_{x_1}\delta\left(\frac{\partial w_1}{\partial x}\right)\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta \\ +\int_\theta\left(\frac{\partial M_{x_1}}{\partial x}\delta w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} Rd\theta - \int_x\int_\theta \frac{\partial^2 M_{x_1}}{\partial x^2}\delta w_1 Rd\theta dx\end{aligned} \tag{123}$$

<u>fifteenth term</u>

$$\int_x\int_\theta M_{\theta_1}\frac{\partial \delta\beta_{\theta_1}}{R\partial\theta}Rd\theta dx = -\int_x\int_\theta \frac{\partial M_{\theta_1}}{R\partial\theta}\delta\beta_{\theta_1}Rd\theta dx \tag{124}$$

Using the definition of β_{θ_1}, and interchanging operators, the right hand side becomes

$$-\int_x\int_\theta \frac{\partial M_{\theta_1}}{R\partial\theta}\delta\beta_{\theta_1}Rd\theta dx = \int_x\int_\theta \frac{\partial M_{\theta_1}}{R\partial\theta}\frac{\partial \delta w_1}{R\partial\theta}Rd\theta dx \tag{125}$$

or

$$-\int_x\int_\theta \frac{\partial M_{\theta_1}}{R\partial\theta}\,\delta\beta_{\theta_1} R d\theta dx = -\int_x\int_\theta \frac{\partial^2 M_{\theta_1}}{R^2\partial\theta^2}\,\delta w_1 R d\theta dx \tag{126}$$

Therefore, the fifteenth term becomes

$$\int_x\int_\theta M_{\theta_1}\frac{\partial\delta\beta_{\theta_1}}{R\partial\theta} R d\theta dx = -\int_x\int_\theta \frac{\partial^2 M_{\theta_1}}{R^2\partial\theta^2}\,\delta w_1 R d\theta dx \tag{127}$$

<u>sixteenth term</u>

$$\begin{aligned}\int_x\int_\theta M_{x\theta_1}\frac{\partial\delta\beta_{\theta_1}}{\partial x} R d\theta dx = \int_\theta (M_{x\theta_1}\delta\beta_{\theta_1})\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \\ -\int_x\int_\theta \frac{\partial M_{x\theta_1}}{\partial x}\,\delta\beta_{\theta_1} R d\theta dx\end{aligned} \tag{128}$$

Both of these integrals can be expanded by using the definition β_{θ_1}. The first integral becomes

$$\int_\theta (M_{x\theta_1}\delta\beta_{\theta_1})\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta = -\int_\theta\left(M_{x\theta_1}\frac{\partial\delta w_1}{R\partial\theta}\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta \tag{129}$$

$$\int_\theta (M_{x\theta_1}\delta\beta_{\theta_1})\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta = \int_\theta\left(\frac{\partial M_{x\theta_1}}{R\partial\theta}\,\delta w_1\right)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta\,, \tag{130}$$

where continuity of the cylinder has been used once again. The second integral on the right hand side of eq. 128 becomes

$$-\int_x\int_\theta \frac{\partial M_{x\theta_1}}{\partial x}\,\delta\beta_{\theta_1} R d\theta dx = \int_x\int_\theta \frac{\partial M_{x\theta_1}}{\partial x}\frac{\partial\delta w_1}{R\partial\theta} R d\theta dx \tag{131}$$

or

$$-\int_x\int_\theta \frac{\partial M_{x\theta_1}}{\partial x}\delta\beta_{\theta_1} R d\theta dx = -\int_x\int_\theta \frac{\partial^2 M_{x\theta_1}}{R\partial x\partial\theta}\delta w_1 R d\theta dx \tag{132}$$

Hence the sixteenth term becomes

$$\int_x\int_\theta M_{x\theta_1}\frac{\partial\delta\beta_{\theta_1}}{\partial x} R d\theta dx = \int_\theta\left(\frac{\partial M_{x\theta_1}}{R\partial\theta}\delta w_1\right)\Bigg|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta - \int_x\int_\theta \frac{\partial^2 M_{x\theta_1}}{R\partial x\partial\theta}\delta w_1 R d\theta dx \tag{133}$$

seventeenth term

$$\int_x\int_\theta M_{x\theta_1}\frac{\partial\delta\beta_{x_1}}{R\partial\theta} R d\theta dx = -\int_x\int_\theta \frac{\partial M_{x\theta_1}}{R\partial\theta}\delta\beta_{x_1} R d\theta dx \tag{134}$$

Rewritting β_{x_1},

$$\int_x\int_\theta M_{x\theta_1}\frac{\partial\delta\beta_{x_1}}{R\partial\theta} R d\theta dx = \int_x\int_\theta \frac{\partial M_{x\theta_1}}{R\partial\theta}\frac{\partial\delta w_1}{\partial x} R d\theta dx\,, \tag{135}$$

the seventeenth term becomes

$$\int_x\int_\theta M_{x\theta_1}\frac{\partial\delta\beta_{x_1}}{R\partial\theta} R d\theta dx = \int_\theta\left(\frac{\partial M_{x\theta_1}}{R\partial\theta}\delta w_1\right)\Bigg|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta - \int_x\int_\theta \frac{\partial^2 M_{x\theta_1}}{R\partial\theta dx}\delta w_1 R d\theta dx \tag{136}$$

Using the results from eqs. 101, 103, 105, 107, 108, 110, 111, 113, 114, 115, 117, 119, 123, 127, 133, and 136, the variation of π_2 becomes

$$\delta\pi_2 = \int_x \int_\theta \Bigg\{ \left\{ -\frac{\partial N_{x_1}}{\partial x} - \frac{\partial N_{x\theta_1}}{R\partial\theta} \right\} \delta u_1 + \left\{ -\frac{\partial N_{x\theta_1}}{\partial x} - \frac{\partial N_{\theta_1}}{R\partial\theta} \right\} \delta v_1$$

$$+ \Bigg\{ -\frac{\partial}{\partial x}\left(N_x \frac{\partial w_1}{\partial x}\right) - \frac{\partial}{R\partial\theta}\left(N_\theta \frac{\partial w_1}{R\partial\theta}\right) - \frac{\partial}{R\partial\theta}\left(N_{x\theta} \frac{\partial w_1}{\partial x}\right)$$

$$- \frac{\partial}{\partial x}\left(N_{x\theta} \frac{\partial w_1}{R\partial\theta}\right) - \frac{\partial}{\partial x}\left(N_{x_1} \frac{\partial w}{\partial x}\right) - \frac{\partial}{R\partial\theta}\left(N_{\theta_1} \frac{\partial w}{R\partial\theta}\right) - \frac{\partial}{R\partial\theta}\left(N_{x\theta_1} \frac{\partial w}{\partial x}\right)$$

$$- \frac{\partial}{\partial x}\left(N_{x\theta_1} \frac{\partial w}{R\partial\theta}\right) - \frac{\partial^2 M_{x_1}}{\partial x^2} - \frac{\partial^2 M_{\theta_1}}{R^2 \partial\theta^2} - 2\frac{\partial^2 M_{x\theta_1}}{R\partial\theta\partial x} + \frac{N_{\theta_1}}{R} \Bigg\} \delta w_1 \Bigg\} R d\theta dx \tag{137}$$

$$+ \int_\theta (N_{x_1}\delta u_1)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} + (N_{x\theta_1}\delta v_1)\Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}}$$

$$+ \int_\theta \left\{ \left\{ N_x \frac{\partial w_1}{\partial x} + N_{x\theta}\frac{\partial w_1}{R\partial\theta} + N_{x_1}\frac{\partial w}{\partial x} + N_{x\theta_1}\frac{\partial w}{R\partial\theta} + \frac{\partial M_{x_1}}{\partial x} + 2\frac{\partial M_{x\theta_1}}{R\partial\theta} \right\} \delta w_1 \right\} \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta$$

$$- \int_\theta \left\{ M_{x_1} \delta\left(\frac{\partial w_1}{\partial x}\right) \right\} \Big|_{x=-\frac{L}{2}}^{x=+\frac{L}{2}} R d\theta = 0$$

Based on the above variation, the Euler equations are

$$\frac{\partial N_{x_1}}{\partial x} + \frac{\partial N_{x\theta_1}}{R\partial\theta} = 0 \tag{138a}$$

$$\frac{\partial N_{x\theta_1}}{\partial x} + \frac{\partial N_{\theta_1}}{R\partial\theta} = 0 \tag{138b}$$

$$\frac{\partial^2 M_{x_1}}{\partial x^2} + 2\frac{\partial^2 M_{x\theta_1}}{R\partial\theta\partial x} + \frac{\partial^2 M_{\theta_1}}{R^2\partial\theta^2}$$
$$+ \frac{\partial}{\partial x}\left(N_x \frac{\partial w_1}{\partial x} + N_{x\theta}\frac{\partial w_1}{R\partial\theta} + N_{x_1}\frac{\partial w}{\partial x} + N_{x\theta_1}\frac{\partial w}{R\partial\theta}\right) \tag{138c}$$
$$+ \frac{\partial}{R\partial\theta}\left(N_\theta \frac{\partial w_1}{R\partial\theta} + N_{x\theta}\frac{\partial w_1}{\partial x} + N_{\theta_1}\frac{\partial w}{R\partial\theta} + N_{x\theta_1}\frac{\partial w}{\partial x}\right) - \frac{N_{\theta_1}}{R} = 0$$

Using the first two equilibrium equations from setting $\pi_1 = 0$, eqs. 93a and b, and the first two of the above in the third equation above, the three Euler equations from $\delta\pi_2 = 0$ can be written as

$$\frac{\partial N_{x_1}}{\partial x}+\frac{\partial N_{x\theta_1}}{R\partial\theta}=0 \tag{139a}$$

$$\frac{\partial N_{x\theta_1}}{\partial x}+\frac{\partial N_{\theta_1}}{R\partial\theta}=0 \tag{139b}$$

$$\begin{aligned}\frac{\partial^2 M_{x_1}}{\partial x^2}&+2\frac{\partial^2 M_{x\theta_1}}{Rd\theta dx}+\frac{\partial^2 M_{\theta_1}}{R^2\partial\theta^2}\\&+N_x\frac{\partial^2 w_1}{\partial x^2}+2N_{x\theta}\frac{\partial^2 w_1}{R\partial\theta\partial x}+N_\theta\frac{\partial^2 w_1}{R^2\partial\theta^2}\\&+N_{x_1}\frac{\partial^2 w}{\partial x^2}+2N_{x\theta_1}\frac{\partial^2 w}{R\partial\theta\partial x}+N_{\theta_1}\frac{\partial^2 w}{R^2\partial\theta^2}-\frac{N_{\theta_1}}{R}=0\end{aligned} \tag{139c}$$

The variationally consistent boundary conditions are:

<u>at $x=\pm L/2$</u>

i) $N_{x_1}=0$ or u_1 must be specified

ii) $N_{x\theta_1}=0$ or v_1 must be specified

iii) $\left(N_x\frac{\partial w_1}{\partial x}+N_{x\theta}\frac{\partial w_1}{R\partial\theta}+N_{x_1}\frac{\partial w}{\partial x}+N_{x\theta_1}\frac{\partial w}{R\partial\theta}+\frac{\partial M_{x_1}}{\partial x}+2\frac{\partial M_{x\theta_1}}{R\partial\theta}\right)=0$ (140)

or w_1 must be specified

vi) $M_{x_1}=0$ or $\frac{\partial w_1}{\partial x}$ must be specified

Equations 139, or the alternative form eq. 138, are referred to as the buckling equations. They along with the boundary conditions of eq. 140 provide the conditions that must prevail when the cylinder passes from a stable equilibrium configuration to an unstable configuration. Solution of eqs. 139 satisfying eqs. 140 leads to the value(s) of the applied load(s) that cause instability. Solution of these equations also gives the buckling shapes relative to the equilibrium configuration.

SIMPLIFICATION OF THE EQUILIBRIUM EQUATIONS DUE TO THE CONDITIONS OF AXISYMMETRY

The studies here will focus on the case of axisymmetric end loads and axisymmetric response. For this situation

$$\frac{\partial(\ )}{\partial\theta}=0 \quad \text{and} \quad \frac{\partial(\ )}{\partial x}=\frac{d(\ )}{dx}\ , \tag{141}$$

( ) being any response quantity. With this condition, the kinematic relations simplify considerably. Specifically, using eqs. 3 and 6, and reintroducing the superscript 'o' notation where applicable,

$$\begin{gathered}
\beta_x^o=-\frac{dw^o}{dx} \qquad \beta_\theta^o=0 \\
\varepsilon_x^o=\frac{du^o}{dx}+\frac{1}{2}\,\beta_x^{o^2}\,; \quad \varepsilon_\theta=\frac{w^o}{R}\ ; \quad \gamma_{x\theta}^o=\frac{dv^o}{dx} \\
\kappa_x^o=\frac{d\beta_x^o}{dx}\ ; \quad \kappa_\theta^o=0\,; \quad \kappa_{x\theta}^o=0\,.
\end{gathered} \tag{142}$$

As a result of eq. 141, the equilibrium equations, eq. 93, simplify to

$$\frac{dN_x}{dx}=0 \tag{143a}$$

$$\frac{dN_{x\theta}}{dx}=0 \tag{143b}$$

$$\frac{d^2M_x}{dx^2}+N_x\frac{d^2w}{dx^2}-\frac{N_\theta}{R}+q=0\,. \tag{143c}$$

The accompanying boundary conditions are at x = - L/2

$$\text{i)}\ N_x = N_x^- \text{ or } u^o \text{ must be specified ,}$$
$$\text{ii)}\ N_{x\theta} = N_{x\theta}^- \text{ or } v^o \text{ must be specified ,}$$
$$\text{iii)}\ \frac{dM_x}{dx} + N_x \frac{dw^o}{dx} = 0 \text{ or } w^o \text{ must be specified ,} \qquad (144a)$$
$$\text{iv)}\ M_x = 0 \text{ or } \frac{dw^o}{dx} \text{ must be specified .}$$

at x = + L/2

$$\text{i)}\ N_x = N_x^+ \text{ or } u^o \text{ must be specified ,}$$
$$\text{ii)}\ N_{x\theta} = N_{x\theta}^+ \text{ or } v^o \text{ must be specified ,}$$
$$\text{iii)}\ \frac{dM_x}{dx} + N_x \frac{dw^o}{dx} = 0 \text{ or } w^o \text{ must be specified ,} \qquad (144b)$$
$$\text{iv)}\ M_x = 0 \text{ or } \frac{dw^o}{dx} \text{ must be specified .}$$

Equations 143 and 144 will be the focus of the remainder of the study. In the next section the solution of these equations for the case of a known axial end load will be derived.

SOLUTION OF EQUATIONS FOR THE CASE OF AN AXIAL END LOAD

The first equilibrium equation of eq. 143 integrates to become

$$N_x = \text{constant .} \qquad (145)$$

Since the axial load is known at the end of the cylinder, this constant is the applied end load. It will be referred to simply as N. The second equation of eq. 143 integrates to

$$N_{x\theta} = \text{another constant .} \qquad (146)$$

Since in the present work there will be no torsional loading on the end of the cylinder, the shear $N_{x\theta}$ is zero there and thus this constant must be zero, i.e.,

$$N_{x\theta} = 0. \tag{147}$$

To solve the third equation of eq. 143 it is most convenient to express all quantities in that equation in terms of w^o. This is accomplished by using the definitions of the stress resultants, eq. 20. Here the preloading condition is thermal. Thus eq. 22 is used in conjunction with eq. 20. In using these equations it convenient to consider only cylinders that will not experience a global twist due to the application of axial end loads, i.e., consider the case of A_{16} and A_{26} equal to zero. This is a valid assumption for most applications of cylinders. With this situation, using the definitions of the stress resultants and the simplified kinematic relations,

$$\begin{aligned}
N_x &= A_{11}\varepsilon_x^o + A_{12}\frac{w^o}{R} - B_{11}\frac{d^2w^o}{dx^2} - N_x^T = N \\
N_\theta &= A_{12}\varepsilon_x^o + A_{22}\frac{w^o}{R} - B_{12}\frac{d^2w^o}{dx^2} - N_\theta^T \\
N_{x\theta} &= A_{66}\gamma_{x\theta}^o - B_{16}\frac{d^2w^o}{dx^2} - N_{x\theta}^T = 0 \\
M_x &= B_{11}\varepsilon_x^o + B_{12}\frac{w^o}{R} + B_{16}\gamma_{x\theta}^o - D_{11}\frac{d^2w^o}{dx^2} - M_x^T .
\end{aligned} \tag{148}$$

Solving the first equation for ε_x^o,

$$\varepsilon_x^o = \frac{1}{A_{11}}\left(N + N_x^T - A_{12}\frac{w^o}{R} + B_{11}\frac{d^2w^o}{dx^2}\right), \tag{149a}$$

and the third for $\gamma_{x\theta}^o$,

$$\gamma_{x\theta}^o = \left(\frac{B_{16}}{A_{66}}\frac{d^2w^o}{dx^2} + \frac{N_{x\theta}^T}{A_{66}}\right). \tag{149b}$$

Substituting these into the expressions for N_θ and M_x yields

$$
\begin{aligned}
N_\theta = &\left(A_{22} - \frac{A_{12}^2}{A_{11}}\right)\frac{w^o}{R} - \left(B_{12} - \frac{B_{11}A_{12}}{A_{11}}\right)\frac{d^2w^o}{dx^2} \\
&+ \frac{A_{12}}{A_{11}}\left(N + N_x^T\right) - N_\theta^T
\end{aligned}
\tag{150a}
$$

$$
\begin{aligned}
M_x = &\left(B_{12} - \frac{B_{11}A_{12}}{A_{11}}\right)\frac{w^o}{R} - \left(D_{11} - \frac{B_{16}^2}{A_{66}} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w^o}{dx^2} \\
&+ \frac{B_{11}}{A_{11}}\left(N + N_x^T\right) - M_x^T + \frac{B_{16}}{A_{66}}N_{x\theta}^T
\end{aligned}
\tag{150b}
$$

For the three laminates of interest, i.e., the $(0_8/90_8)_T$, the $(90_8/0_8)_T$, and the $(0/90)_{4S}$ laminates,

$$
B_{22} = -B_{11}; \quad B_{12} = B_{16} = B_{26} = B_{66} = N_{x\theta}^T = 0. \tag{151}
$$

The expressions for N_θ and M_x simplify to

$$
\begin{aligned}
N_\theta = &\left(A_{22} - \frac{A_{12}^2}{A_{11}}\right)\frac{w^o}{R} + \left(\frac{A_{12}B_{11}}{A_{11}}\right)\frac{d^2w^o}{dx^2} \\
&+ \frac{A_{12}}{A_{11}}\left(N + N_x^T\right) - N_\theta^T
\end{aligned}
\tag{152a}
$$

$$
\begin{aligned}
M_x = &\left(-\frac{A_{12}B_{11}}{A_{11}}\right)\frac{w^o}{R} - \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w^o}{dx^2} \\
&+ \frac{B_{11}}{A_{11}}\left(N + N_x^T\right) - M_x^T .
\end{aligned}
\tag{152b}
$$

Substituting these into the third equilibrium equation, and considering the case of $q = 0$, leads to the equation governing w^o. This equation is

$$
\begin{aligned}
&\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^4w^o}{dx^4} + \left(\frac{2A_{12}B_{11}}{A_{11}R} - N\right)\frac{d^2w^o}{dx^2} \\
&+ \left(A_{22} - \frac{A_{12}^2}{A_{11}}\right)\frac{w^o}{R^2} = \frac{N_\theta^T}{R} - \frac{A_{12}}{RA_{11}}\left(N + N_x^T\right).
\end{aligned}
\tag{153}
$$

This equation is a linear differential equation with constant and known coefficients. The applied axial load N is known, as are the material properties, equivalent thermal loads, and geometry. This structurally nonlinear problem results in a mathematically linear problem.

Having addressed the three equilibrium equations, attention focuses on the boundary conditions. Two of the four on each end have been specified. In particular, statements have been made regarding N_x and $N_{x\theta}$. These satisfy eq. 144a and b, i and ii. To be considered are eq. 144a and b, iii and iv. Regarding these two remaining conditions, within the context of the admissible conditions, three physically plausible boundary conditions can be imposed on the ends of the cylinder. These are:

1 - lubricated boundaries;

2 - simply supported boundaries; and,

3 - clamped boundaries.

For lubricated boundaries the shear force and moment at the ends are zero. For simply supported boundaries the radial displacement and the moment are zero. For the clamped boundaries the radial displacement and the slope are zero. The terminology 'lubricated' comes from the fact that the radial displacement has no resistance, as if a highly lubricated flat plate were pushing axially against the ends of the cylinder. This lubricated flat plate would not resist rotation of the ends either.

For a lubricated boundary, the conditions are formally, from eq. 144,

$$\frac{dM_x}{dx} + N\frac{dw^o}{dx} = 0$$

and (154)

$$M_x = 0,$$

while for a simple support the conditions are

$$w^o = 0$$
$$M_x = 0\,. \tag{155}$$

For a clamped support the conditions are

$$w^o = 0$$
$$\frac{dw^o}{dx} = 0\,. \tag{156}$$

In terms of w^o, for the three laminates of interest the lubricated support conditions are

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^3w^o}{dx^3} + \left(\frac{A_{12}B_{11}}{RA_{11}} - N\right)\frac{dw^o}{dx} = 0$$

and

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w^o}{dx^2} + \left(\frac{A_{12}B_{11}}{RA_{11}}\right)w^o - \frac{B_{11}}{A_{11}}\left(N + N_x^T\right) + M_x^T = 0\,. \tag{157}$$

For a simply supported boundary, the conditions are:

$$w^o = 0$$

and

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w^o}{dx^2} + \left(\frac{A_{12}B_{11}}{RA_{11}}\right)w^o - \frac{B_{11}}{A_{11}}\left(N + N_x^T\right) + M_x^T = 0. \tag{158}$$

For a clamped boundary the conditions are as given in eq. 156. Here interest will focus on simply supported and clamped boundaries.

PRELOADING RESPONSE DUE TO THERMAL EFFECTS

Cylinders are generally fabricated on a mandrel and are consolidated at an elevated temperature. After consolidation the temperature is lowered to the ambient temperature and the finished cylinder is removed from the mandrel. Assuming that the fabrication and consolidation are axisymmetric, the shape of the cylinder can be determined using the governing conditions from the previous section. Specifically, since there are no loads applied to a cylinder that has been removed from the mandrel, eq. 153 with N equated to zero governs the response. That is,

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^4w^o}{dx^4} + \frac{2A_{12}B_{11}}{A_{11}R}\frac{d^2w^o}{dx^2} + \left(A_{22} - \frac{A_{12}^2}{A_{11}}\right)\frac{w^o}{R^2} = \frac{N_\theta^T}{R} - \frac{A_{12}}{RA_{11}}N_x^T . \tag{159}$$

With the cylinder simply in the ambient environment, the ends of the cylinder are traction free. Thus the two other boundary conditions, $N_x = 0 = N_{x\theta}$ being the first two, are

$$\frac{dM_x}{dx} = M_x = 0 . \tag{160}$$

In terms of displacements, the boundary conditions at $x = \pm L/2$ are

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^3w^o}{dx^3} + \left(\frac{A_{12}B_{11}}{RA_{11}}\right)\frac{dw^o}{dx} = 0$$

and

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w^o}{dx^2} + \left(\frac{A_{12}B_{11}}{RA_{11}}\right)w^o - \frac{B_{11}}{A_{11}}N_x^T - M_x^T = 0 . \tag{161}$$

The complete solution to eq. 159 consists of homogeneous and particular parts, i.e.,

$$w^o(x) = w^o_{homo}(x) + w_{part}(x) . \tag{162}$$

By inspection, the particular solution is

$$w^o_{part}(x) = \frac{R\left(A_{11}N^T_\theta - A_{12}N^T_x\right)}{A_{11}A_{22} - A^2_{12}} = w^T_{part}\,, \tag{163}$$

where the notation w^T_{part} denotes the particular solution to the thermal problem. Note that it is not a function of x.

The homogeneous solution is of the form

$$w^o_{homo}(x) = A\,e^{\lambda x}\,. \tag{164}$$

Substituting this assumed form into eq. 159 results in the characteristic equation, namely

$$\left(D_{11}A_{11} - B^2_{11}\right)\lambda^4 + \left(\frac{2A_{12}B_{11}}{R}\right)\lambda^2 + \left(\frac{A_{11}A_{22} - A^2_{12}}{R^2}\right) = 0\,. \tag{165}$$

The four roots to this equation are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{-A_{12}B_{11} + \sqrt{A_{11}\left(D_{11}A^2_{12} + A_{22}B^2_{11} - A_{11}A_{22}D_{11}\right)}}{R\left(D_{11}A_{11} - B^2_{11}\right)}} \tag{166}$$

These four roots are of the form

$$\lambda_{1,2,3,4} = \pm\,\alpha\,\pm\,i\beta\,, \tag{167}$$

α and β real and positive. It should be noted that the characteristic roots are strictly functions of the elastic properties and geometry. Thermal effects, in terms of material expansion coefficients or temperature, are not involved. At this point in the analysis α and β are known.

Because of eq. 167, the complete solution for the thermally-induced response is given by

$$w_T^o(x) = A_1 e^{(\alpha+i\beta)x} + A_2 e^{(\alpha-i\beta)x} + A_3 e^{(-\alpha+i\beta)x} + A_4 e^{(-\alpha-i\beta)x} + \frac{R\left(A_{11}N_\theta^T - A_{12}N_x^T\right)}{A_{11}A_{22} - A_{12}^2} , \tag{168}$$

where A_i are the four unknown constants of integration and the subscript 'T' with $w^o(x)$ is used to denote the fact the solution is for the case of thermally-induced preloading.

Since the boundary conditions, eq. 161, are the same at each end of the cylinder, the thermally-induced response is expected to be an even function of x. Using this fact, the odd functions of x of solution eq. 168 can be eliminated to yield

$$w_T^o(x) = F \cosh(\alpha x) \cos(\beta x) + G \sinh(\alpha x) \sin(\beta x) + \frac{R\left(A_{11}N_\theta^T - A_{12}N_x^T\right)}{A_{11}A_{22} - A_{12}^2} . \tag{169}$$

The unknown A_i are combined to form unknown constants F and G. By substituting the solution form eq. 169 into the boundary conditions, constants F and G can be solved for. With F and G known, the displacement $w_T^o(x)$ is known. (Obviously with evenness of the solution being enforced, the boundary conditions are enforced at one end of the cylinder only.)

Numerical results for the three laminates of interest are illustrated in figs. 2-4. The cylinder geometry and material properties used to obtain numerical results, in these figures and throughout, are as follows: The cylinder radius R is 10 in., the cylinder length is 30 in. ($L/R = 3$, a short to intermediate length cylinder), and the cylinder thickness H is 0.080 in. ($R/H = 125$, a thin laminate). The material properties are

$$\begin{gathered} E_1 = 20\ \text{Msi}; \quad E_2 = 1.3\ \text{Msi}; \quad G_{12} = 1.03\ \text{Msi} \\ \nu_{12} = 0.3; \quad h\ \text{(lamina thickness)} = 0.005\ \text{in.} \\ \alpha_1 = -0.167 \times 10^{-6}/^\circ\text{F}; \quad \alpha_2 = 15.6 \times 10^{-6}/^\circ\text{F}. \end{gathered} \tag{170}$$

In terms of laminate properties, for the $(90_8/0_8)_T$ laminate

$$A_{11} = A_{22} = 857{,}000; \quad A_{12} = 31{,}400 \text{ lb/in.}$$
$$B_{11} = 15{,}100 \text{ lb/in.}; \quad D_{11} = 457 \text{ lb/in.}^2 \tag{171a}$$
$$N_x^T = N_\theta^T = -259 \text{ lb/in./°F}; \quad M_x^T = 3.94 \text{ lb/in.}^2\text{/°F} .$$

For the $(0_8/90_8)_T$ laminate,

$$A_{11} = A_{22} = 857{,}000; \quad A_{12} = 31{,}400 \text{ lb/in.}$$
$$B_{11} = -15{,}100 \text{ lb/in.}; \quad D_{11} = 457 \text{ lb/in.}^2 \tag{171b}$$
$$N_x^T = N_\theta^T = -259 \text{ lb/in./°F}; \quad M_x^T = -3.94 \text{ lb/in.}^2\text{/°F} .$$

For the $(0/90)_{4s}$ laminate,

$$A_{11} = A_{22} = 857{,}000; \quad A_{12} = 31{,}400 \text{ lb/in.}$$
$$B_{11} = 0; \quad D_{11} = 532 \text{ lb/in.}^2 \tag{171c}$$
$$N_x^T = N_\theta^T = -259 \text{ lb/in./°F}; \quad M_x^T = 0 .$$

To neglect thermally-induced preloading effects, N_x^T, N_θ^T and M_x^T are set to zero. Note that for all three laminates the elements of the A matrix are identical. The quantity B_{11} is of opposite sign for the two unsymmetric laminates. Of course $B_{11} = 0$ for the symmetric laminate. Since the 0° layers are on the extreme outside and inside of the cylinder wall, the value of D_{11} is greater for the symmetric laminate. The quantities N_x^T and N_θ^T are the same for all three laminates, whereas M_x^T is opposite in sign for the two unsymmetric laminates and zero for the symmetric case. The rearrangement of the 0° and 90° layers in the three laminates only influence the out-of-plane, or bending, properties of the laminate. A temperature change of $\Delta T = -280$°F is used to represent the temperature change from a 350°F consolidation temperature to an ambient temperature of 70°F.

The radial displacement, $w_T^o(x)$, as a function of length along the cylinder for the $(90_8/0_8)_T$ laminate is shown in fig. 2. The displacements have been normalized by the laminate thickness while the axial coordinate has been normalized by the cylinder length. Several interesting characteristics of the thermally-induced response are illustrated in the figure. These will be particularly interesting when compared with results for the $(0_8/90_8)_T$ laminate, to be discussed shortly. From the figure it is seen that over the majority of the length of the cylinder the displacement is uniform and inward, i.e., $w < 0$. Near the end, $x/L > 0.4$, there is a rapid change in the radial displacement, the displacement becoming more inward at the end of the cylinder ($x/L = 0.5$). This behavior is due to the unsymmetric nature of the laminate, in particular, the thermally-induced moment generated within the laminate. Away from the ends of the cylinder the moment has no effect on the displacement. Thermal expansion in the x and θ directions, as well as the elastic properties in these directions, control the displacements away from the ends. At the ends of the cylinder there is no material to react the thermally-induced moment. The cylinder responds as if the ends were being subjected to an applied moment. The ends of the cylinder 'curl', producing a boundary layer effect.

If the stacking arrangement is reversed and a $(0_8/90_8)_T$ laminate is considered, the cylinder deforms as shown in fig. 3. Interestingly enough, the central portion of the cylinder, away from the ends, moves radially inward the exact same amount as the $(90_8/0_8)_T$ cylinder. Near the ends there is the rapid change in radial displacement. In this case, however, at the end the cylinder deforms outward. This is opposite the situation for the $(90_8/0_8)_T$ laminate. The direction the ends of the cylinder 'curl' is a function of the sign of the thermally-induced moment. The sign of the thermally-induced moment is a function of the stacking arrangement. The sign of the thermally-induced moment is opposite in the two cases, resulting in opposite bending deformations. The point to be made at this time is that before any loading is applied to the cylinder, there is a nonuniform radial displacement along the length of the cylinder. The lack of a uniform displacement, particularly since it is concentrated in a boundary layer near the ends, where the loads are applied, and particularly since the specific displacement is laminate-dependent and sign-sensitive, is an important point.

For completeness, the thermal deformations of a $(0/90)_{4s}$ laminate are illustrated in fig. 4. For this symmetric laminate the radial deformation due to the temperature change is uniform along the length. The inward radial displacement is identical to the inward radial in the central portion of the $(90_8/0_8)_T$ and $(0_8/90_8)_T$ laminates.

A note of interest: The deformation in the central portion of the cylinders, for any lamination sequence, is given by the particular solution, $w^o_{part}(x)$. The response near the ends is controlled by the homogeneous solution. For the symmetric $(0/90)_{4s}$ laminate the homogeneous solution is zero. For the other two cases the homogeneous solution is a major influence. Since the particular solution is the same for each of the three laminates, the inward radial displacement is the same away from the ends for each of the three laminates.

Of interest is the response that results when an axial compressive load is applied to the $(90_8/0_8)_T$ and $(0_8/90)_T$ thermally-deformed cylinders. Is inclusion or exclusion of this thermally-induced preloading deformation important for predicting the load-induced response of the cylinder? These issues are discussed next. With or without the thermal preloading condition, calculating the response to an applied load is a more complicated problem.

SOLUTION OF EQUATIONS FOR THE CASE OF THERMAL EFFECTS AND AN AXIAL END LOAD

The combined problem of thermal and mechanical loads is governed by eq. 153 and boundary conditions eqs. 154, 155, or 156. As with the thermal-only problem, the solution consists of a homogeneous part and a particular part, i.e.,

$$w^o(x) = w^o_{homo}(x) + w^o_{part}(x)\,. \tag{172}$$

The particular solution is again determined by inspection to be

$$w^o_{part}(x) = \frac{R\left(A_{11}N^T_\theta - A_{12}\left(N^T_x + N\right)\right)}{A_{11}A_{22} - A^2_{12}} = w_{part}\,. \tag{173}$$

Note that this particular solution is also independent of x. The homogeneous solution to this problem is taken to be of the form given by eq. 164. For this case, the characteristic equation is given by

$$(D_{11}A_{11} - B_{11}^2)\lambda^4 + \left(\frac{2A_{12}B_{11} - A_{11}RN}{R} \right)\lambda^2 + \left(\frac{A_{11}A_{22} - A_{12}^2}{R^2} \right) = 0 . \qquad (174)$$

Note the applied axial load appears in the characteristic equation. The level of applied load will be specified and hence all coefficients of λ in the above equation are known. The four values of λ are thus known at this point.

The roots to the characteristic equation are given by

$$\lambda_{1,2,3,4} = \pm \sqrt{\frac{(A_{11}RN - 2A_{12}B_{11}) \pm \sqrt{(A_{11}RN - 2A_{12}B_{11})^2 - 4(A_{11}A_{22} - A_{12}^2)(D_{11}A_{11} - B_{11}^2)}}{2(D_{11}A_{11} - B_{11}^2)R}} . \qquad (175)$$

Though it is not totally obvious, there is an interesting character to the roots to eq. 175. This is due to the dependence of the roots on the level of applied load, N. The character of the roots can be examined by studying λ^2 instead of λ, i.e.,

$$\lambda^2 = \frac{(A_{11}RN - 2A_{12}B_{11}) \pm \sqrt{(A_{11}RN - 2A_{12}B_{11})^2 - 4(A_{11}A_{22} - A_{12}^2)(D_{11}A_{11} - B_{11}^2)}}{2(D_{11}A_{11} - B_{11}^2)R} . \qquad (176)$$

The first important character to observe is that at a certain level of applied load the discriminant will be zero. In particular, the discriminant is zero when $N = N^*$, N^* being given by

$$N^* = \frac{2}{A_{11}R}\left\{ \pm \sqrt{\left(A_{11}A_{22} - A_{12}^2\right)\left(D_{11}A_{11} - B_{11}^2\right)} + A_{12}B_{11}\right\}. \tag{177}$$

For a symmetric laminate $B_{11} = 0$ and the determinant only vanishes for negative N^*, a compressive load. For unsymmetric laminates, though it is unlikely, the determinant could vanish for positive N^*, depending on the magnitude of A_{11}, A_{12}, A_{22}, B_{11}, and D_{11}. Here the negative sign will be assumed in eq. 177 and interest will focus on negative N^*.

Returning to eq. 175, for $N = 0$, the roots to λ^2 are, in general, complex. The roots to λ are thus of the form

$$\lambda_{1,2,3,4} = \pm\ \alpha\ \pm i\beta\,. \tag{178}$$

If the laminate is symmetric, $N = 0$ leads to two purely imaginary roots for λ^2. The roots for λ are still of the form of eq. 178.

For N increasing compressive the roots to λ^2 remain complex, the roots for λ being given by eq. 178. When $N = N^*$, λ^2 has negative real repeating roots. In particular,

$$\lambda^2 = -\sqrt{\frac{A_{11}A_{22} - A_{12}^2}{\left(D_{11}A_{11} - B_{11}^2\right)R^2}}\,. \tag{179}$$

The four roots for λ are then two repeating pure imaginary roots of the form

$$\lambda_{1,2,3,4} = \pm\ i\beta,\ \pm\ i\beta\,. \tag{180}$$

For N increasing in compression beyond N^*, the roots for λ^2 are distinct and negative real. The roots for λ are of the form

$$\lambda_{1,2,3,4} = \pm\ i\beta_1,\ \pm\ i\beta_2\,. \tag{181}$$

Because of these three different forms for the roots, depending on the value of N relative to N*, the functional form of the x dependence of the homogeneous solution depends on the value of N. This fact leads to a dependence on N of the deformed shape along the length of the cylinder. The amplitude of the deformed shape also depends on N. For the linear problem, shape of the deformation does not depend on N. Only the amplitude of the deformation depends on N. This difference in the dependence on N between the linear and nonlinear cases is important.

The functional dependence of homogeneous solution on x is as follows:

For $|N| < |N^*|$, from eq. 178

$$w^o_{homo}(x) = A_1 e^{(\alpha+i\beta)x} + A_2 e^{(\alpha-i\beta)x} + A_3 e^{(-\alpha+i\beta)x} + A_4 e^{(-\alpha-i\beta)x} . \tag{182}$$

This is the same form as for the thermally-induced preloading deformations.

For $|N| = |N^*|$, from eq. 180

$$w^o_{homo}(x) = (A_1 + A_2 x)e^{i\beta x} + (A_3 + A_4 x)e^{-i\beta x} . \tag{183}$$

For $|N| > |N^*|$, from eq. 181,

$$\begin{aligned} w^o_{homo}(x) = {} & A_1 \cos(\beta_1 x) + A_2 \sin(\beta_1 x) + A_3 \cos(\beta_2 x) \\ & + A_4 \sin(\beta_2 x) . \end{aligned} \tag{184}$$

Combining these solutions with the particular solution, eq. 173, and considering only the portion of the solution that is symmetric in x, the three forms are:

For $|N| < |N^*|$,

$$w^o(x) = F \cosh(\alpha x)\cos(\beta x) + G \sinh(\alpha x)\sin(\beta x) + \frac{R\left(A_{11}N_\theta^T - A_{12}\left(N_x^T + N\right)\right)}{A_{11}A_{22} - A_{12}^2} . \tag{185}$$

For $|N| = |N^*|$,

$$w^o(x) = F \cos(\beta x) + G\, x \sin(\beta x) + \frac{R\left(A_{11}N_\theta^T - A_{12}\left(N_x^T + N\right)\right)}{A_{11}A_{22} - A_{12}^2} . \tag{186}$$

For $|N| > |N^*|$,

$$w^o(x) = F \cos(\beta_1 x) + G \cos(\beta_2 x) + \frac{R\left(A_{11}N_\theta^T - A_{12}\left(N_x^T + N\right)\right)}{A_{11}A_{22} - A_{12}^2} . \tag{187}$$

The constants F and G can be determined by the application of boundary conditions. The case of thermally-induced preloading deformations ($N = 0$), eq. 179, is a subcase of the solution given by eq. 185.

With the above solution, the response of a cylinder to an axial end load in the presence of the thermally-induced preloading deformation can be determined as a function of N^*. With the thermally-induced preloading the boundary conditions must be carefully interpreted. In particular, since the ends of the cylinder are already deformed due to the thermal preloading, the simple support conditions become

$$w^o\left(\frac{L}{2}\right) = w_T^o\left(\frac{L}{2}\right)$$

$$\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w^o}{dx^2} + \left(\frac{A_{12}B_{11}}{RA_{11}}\right)w^o - \frac{B_{11}}{A_{11}}\left(N + N_x^T\right) + M_x^T = 0 . \tag{188}$$

For a clamped boundary the conditions are

$$w^o\left(\frac{L}{2}\right) = w_T^o\left(\frac{L}{2}\right)$$
$$\frac{dw^o\left(\frac{L}{2}\right)}{dx} = \frac{dw_T^o\left(\frac{L}{2}\right)}{dx} \,. \tag{189}$$

Any experimental set-up that is simulating simple support or clamped conditions would resist any end displacement beyond to the preloading value. Clamped conditions would prevent any rotation beyond the preloading value. Simple support and clamped conditions for the $(90_8/0_8)_T$ cylinder are depicted in figs. 5a and b, respectively. The situation for the $(0/90)_{4S}$ cylinder is shown in figs. 6a and b. For unsymmetric laminates the load N^* is applied eccentrically relative to the midsurface of the cylinder wall. This could have a large influence on the response of the cylinder near the ends.

NUMERICAL RESULTS FOR THE CASE OF THERMAL EFFECTS AND A COMPRESSIVE AXIAL END LOAD

The primary reason for the current study is summarized in the figures to follow. Results from the closed-form solution of the radial displacement as a function of length along the cylinder for increasing compressive axial load levels are presented. The load levels examined are normalized by N^*. Note that N^* is independent of boundary conditions and thermal preloading. The value of N equal to N^* has been interpreted as the value of N to cause buckling. Flugge [4] discusses this in connection with isotropic cylinders but the point is not pursued here. As will be seen, the response of the cylinder when $N = N^*$ suggests that N^* may be associated with an instability.

Simply-supported boundary conditions

The deformed shape of a simply-supported $(90_8/0_8)_T$ cylinder for $N = 0$ is shown in fig. 7. As before, in this and subsequent figures the radial displacement has been normalized by the thickness of the cylinder wall, H . The axial coordinate is normalized by cylinder length, L . Figure 7 is identical to fig. 2 except the radial displacement at the end of the cylinder is defined to be the location of zero displacement. In reality the radial displacement of the end of the cylinder is $w_T(L/2)$. This shift of coordinates is strictly cosmetic. If thermally-induced preloading deformations are neglected, the radial displacement is represented by the line $w/H = 0$. That the cylinder is assumed to be simply supported is irrelevant for this figure. The figure would be identical if the supports were considered clamped. Experimentally the cylinder would be mounted in the fixturing so as to effect the simple supports or the clamped supports. Since, in theory, no loads are involved in mounting the cylinder in the fixture, the preloading shape of the cylinder would not be influenced by the support conditions. Obviously, when loads are applied to the cylinder, the support conditions become very important.

The radial response of the simply-supported cylinder when the load is 10% of N^* is shown in fig. 8. The response when the thermally-induced preloading deformations are incorrectly neglected in the analysis is indicated by the dashed line. This situation will be discussed. For the $(90_8/0_8)_T$

$$N^* = -2460 \text{ lb/in.} \tag{190}$$

At the 10% level, the response due to the applied load is not too different than the thermally-induced preloading response, fig. 7. When the compressive load reaches 50% level, fig. 9, the applied load causes considerable response (note the vertical scale compared to fig. 8). The central portion of the cylinder has moved outward some, but the major effect is at the end of the cylinder. At the end of the cylinder the magnitude of the radial deformations has increased considerably, and though not detectable from the figure, the length of the boundary region has increased somewhat. The rapid change in magnitude at the cylinder end could clearly generate significant bending strain. When the load is increased to 90% of N^*, fig. 10, the boundary layer has clearly lengthened. The oscillatory nature of the deformations encompass one-half the cylinder. In addition, the magnitude of the deformations has become severe. In particular, the cylinder wall has displaced one wall thickness. Since there are high localized curvatures associated with this level of deflection. The level should cause concern. The oscillatory nature of the displacements is such that at points (e.g., $x/L = 0.44$) the cylinder radius is less than the undeformed radius, despite the fact that axial compression causes the cylinder, in general, to become larger in diameter. The stresses and strain that accompany these displacements are not reported on here.

As mentioned above, the dashed lines in figs. 8-10 represent the predicted response if the thermally-induced preloading deformations are not included in the analysis. As can be seen by examining the figures, neglecting the thermally-induced deformations results in a predicted displacement that is less than is predicted when the thermally-induced displacements are included. At $N = 0.5N^*$ the predicted peak displacements at $x/L = 0.47$ are different by a factor of two. At $N = 0.9N^*$ the peak displacements computed by neglecting the preloading thermal effects are about 40% lower than if preloading effects are included.

The response of the simply supported $(90_8/0_8)_T$ cylinder when the applied load equals N^* is shown in fig. 11. It is clear that the magnitude of the deformation is essentially unbounded at this load level. The deformations are oscillatory, the spatial frequency being given by eq. 179. As stated previously, the present work will not investigate the relationship of this load level with the buckling load level. With the load increasing to $1.1N^*$ the response of fig. 12 results. The amplitude of the deformations has decreased considerably relative to the $N = N^*$ level. There are two basic frequencies to the oscillatory character of the response, these two frequencies being given by eq. 181.

That the boundary layer increases in length as the load is increased is an important nonlinear effect. For a linear analysis, the length of the boundary layer does not depend on the level of applied load. For this nonlinear problem, the length of the boundary layer and the level of applied load are related through the characteristic roots λ, of eq. 179. In particular, the length of the boundary layer is related to the recriprocal of the real part of λ, namely $1/\alpha$. The values of $1/\alpha$ as a function of the load level for the $(90_8/0_8)_S$ laminate of figs. 7-10, and for the other laminates to be discussed shortly, are presented in Table 1.

Table 1

Boundary Layer Length, $1/\alpha$, for the Three Laminates

N	$(90_8/0_8)_T$	$(0_8/90_8)_T$	$(0/90)_{4S}$
0	0.56 1/in.	0.54	0.71
0.1N*	0.59	0.56	0.74
0.5N*	0.79	0.76	1.00
0.9N*	1.77	1.70	2.23

As can be seen, for the $(90_8/0_8)_T$ laminate, the boundary layer triples in length as the load level changes from zero to 0.9N*.

As a contrast to the response of the $(90_8/0_8)_T$ cylinder, consider the response of a simply-supported $(0_8/90_8)_T$ cylinder. The characteristics for this cylinder are illustrated in figs. 13-18. The radial displacement at the condition of no axial load is shown in fig. 13, the displacement at the end of the cylinder again being taken as the zero displacement location. Figures 13 and 3 differ only by a shift of origin. For the $(0_8/90_8)_T$ cylinder

$$N^* = -2681 \text{ lb/in.} \tag{191}$$

The sign of B_{11} is responsible for the difference between N^* for the $(90_8/0_8)_T$ laminate and N^* for the $(0_8/90_8)_T$ laminate.

As the load is applied to the $(0_8/90_8)_T$ cylinder, the central portion of the cylinder moves outward, as it did with the $(90_8/0_8)_T$ laminate. However, the boundary layer region develops large inward deflections. At the location $x/L = 0.47$, the cylinder wall has moved in about one wall thickness when the load level reaches 90% of N^*. Again the boundary layer region has grown considerably and developed rapid oscillations which could lead to large stresses. The length of the boundary layer, given in Table 1, is similar to that of the $(90_8/0_8)_T$ laminate. It is interesting to compare figs. 16 and 10. In the boundary region, the rapid changes in the displacement of the $(0_8/90_8)_T$ laminate are exactly opposite in sign to the rapid changes in the displacement of the $(90_8/0_8)$ laminate. The sign of B_{11}, which is opposite for the two laminates, is directly responsible for this feature. The response at $N = N^*$ is illustrated in fig. 17. The response is essentially unbounded at this load level. Comparing the response of this cylinder at $N = N^*$ with the response of the $(90_8/0_8)_T$ cylinder at $N = N^*$, fig. 11, it is clear that the responses of the two cylinders are exactly out of phase. This again is due to the sign of B_{11}. Since the values of N^* are not the same for these two cases, the load is not the same in these two cases. The response for the case of $N = 1.1N^*$ is shown in fig. 18. The amplitude of the deformations has decreased relative to the $N = N^*$ level, and two basic frequencies are again evident.

Since the unsymmetric laminates exhibit considerable radial deformation when compressed axially, it is of interest to study the radial deformations of a symmetric laminate. The response of the $(0/90)_{4S}$ laminate is shown in figs. 19-24. The preloading radial deformations due to thermal effects are simply a spatially uniform displacement inward, as was shown in fig. 4. This uniform radial displacement is shown again in fig. 19 with the reference for zero displacement shifted, as has been done with the other two laminates. For this symmetric laminate the shifted preloading radial displacement coincides with the $w = 0$ line for all x. For the $(0/90)_{4S}$ laminate

$$N^* = -4270 \text{ lb/in.} \tag{192}$$

This is significantly higher than the values of N^* for the other two laminates.

The radial displacement for a load of N equal 10% of N* is illustrated in fig. 20. The deflection is everywhere outward. For the 50% and 90% levels, figs. 21 and 22, respectively, the deflections increase, and the boundary layer increases in length. The deflections at N* and 1.1N* are shown in figs. 23 and 24, respectively. The deflections, though increasing with increasing N, do not reach the magnitudes that were observed for the two unsymmetric laminates. In addition, by comparing the values of N* for the three laminates, the loading on the symmetric laminate at 90% of N* is considerably higher than the loading on the unsymmetric laminate at 90%. Yet the displacements are a factor of 20 less with the symmetric laminate. Referring to Table 1, the boundary layer for this symmetric laminate is greater than for either unsymmetric laminate. When the thermally-induced preloading deformations are ignored in the analysis for the symmetric laminate, the response is predicted to be the same as when they are included. The lack of difference between the two situations can be attributed to the fact that the preloading deformations are spatially uniform for the symmetric laminate. There is no initial eccentricity at the ends of the cylinder where the load is applied.

Clamped boundary conditions

Since the simply supported unsymmetric laminates exhibited large, rapidly changing displacements near the end of the cylinder as the load level increased, it is legitimate to ask if similar behavior would be observed for other boundary conditions. It would seem that clamping the boundary would restrain the deformations considerably. The response of the $(90_8/0_8)_T$ laminate with clamped ends is illustrated in figs. 25-30. The case of N = 0 is shown in fig. 25, this being a repeat of figs. 2 and 7. This figure is repeated here for convenience. When a load of N equal 10% of N* is applied, fig. 26, the central portion of the cylinder moves outward and the deformations near the boundary begin to grow. As the load level is increased to 50% and 90% of N* , figs. 27 and 28, respectively, the boundary layer grows in length and the amplitude of the deformations in the boundary layer increase significantly, as they did for the simply-supported conditions. However, compared to the simply-supported case, figs. 8-12, the clamped conditions sharply restrict the boundary layer deformations. At $N = 0.9N^*$, the deformations in the boundary layer for the clamped condition are one-fifth the deformations in the boundary layer for the simply-supported condition. The lengths of the boundary layers for the two support conditions are the same. The lengths of the boundary layers, given in Table 1, do not depend on the boundary conditions. At $N = N^*$, the magnitude of the response of the clamped cylinder, fig. 29, is much less than for the simply supported case. If N^* is associated with buckling, based on a comparison of figs. 11 and 29, it would have to be more closely associated with buckling for the simply-supported conditions than for the clamped conditions. For the clamped cylinder, at $N = 1.1N^*$, fig. 30, the amplitude of the deformation is greater than for $N = N^*$. This was not the case for the simply supported cylinders, the response decreasing rapidly for $N > N^*$.

The dashed line in figs. 25-30 represent the predicted deformations when the thermally-induced preloading deformations are neglected for the clamped case. Neglecting the preloading deformations for the clamped case seriously under predicts the amplitude of the oscillating behavior in the boundary layer. At the 90% level the peak-to-peak amplitude of the oscillations of w/H is predicted to be over 0.1 if the preloading deformations are included, as they should be. If the preloading deformations are neglected, the peak-to-peak amplitude of the oscillations is quite small in comparison.

Clamping the boundaries of the $(0_8/90_8)_T$ laminate, figs. 31-36, produces a similar reduction in the amplitude of the oscillations in the boundary layer when compared to the simply-supported case. At the 90% load level the peak-to-peak amplitude of the oscillations is again about one-fifth the magnitude of the peak-to-peak amplitude of the oscillations for the simply-supported case. As with the comparison between the $(90_8/0_8)_T$ and the $(0_8/90_8)_T$ laminates with simply supported boundaries, the sense of the boundary oscillations of the $(0_8/90_8)_T$ laminate are opposite those of the $(90_8/0_8)$ laminate. At the x location where the radial displacement is most outward with the $(90_8/0_8)_T$, the radial displacement of the $(0_8/90_8)_T$ is most inward. Also, for the $(0_8/90_8)_T$ laminate, neglecting the preloading deformations leads to considerably different predictions for the response due to load. In fact, if the preloading deformations are neglected, the response of the $(0_8/90_8)_T$ and the $(90_8/0_8)_T$ are practically identical, as can be seen by comparing the dashed lines in figs. 25-30 with those in figs. 31-36.

The effects of clamping the boundaries of the $(0/90)_{4S}$ are shown in figs. 37-42, fig. 37 being trivial and a repeat of previous figures. As with the two unsymmetric laminates, clamping greatly reduces the amplitude of the oscillations in the boundary layer. At the 90% level and for simple supports, fig. 22, the peak-to-peak amplitude of the oscillations is about 0.05. Clamping, fig. 40, reduces the peak-to-peak level to less than 0.015. In addition, compared to the clamped unsymmetric laminates, the magnitude of the peak radial displacement of the clamped symmetric laminate is considerably less. Hence symmetric construction of the laminate reduces the displacements in the boundary layer independent of boundary conditions. Boundary conditions, however, have a significant influence on the displacements in the boundary layer. In this regard, it is important to note that boundary conditions have no influence on the radial deformations in the central portion of the cylinder, away from the boundary layer. A close look at all the figures just discussed will show this to be the case. As mentioned before, the deformations in the central region are determined primarily by the particular solution to the problem.

Before concluding this study, a comment should be made regarding the axial displacement. While attention has been focused on the radial displacement, the axial displacement can be easily determined. Using eq. 142 in eq. 149a and integrating, an expression for $u^o(x)$ can be obtained. By applying the condition that, for example $u^o(0) = 0$, a complete expression for $u^o(x)$ results. The functional form of the integration depends, of course, on the value of N relative to N^*.

CONCLUSIONS

From the results presented, several important conclusions can be drawn. First, the inclusion of the thermally-induced preloading deformations in the analysis of response due to axial load has a significant influence on the predicted response. In general, if the thermally-induced preloading deformations are not included in the analysis, the predicted deformations are less. The specific amount less depends on the laminate, the boundary conditions, and the load level. A second conclusion is that unsymmetric lamination sequences result in much larger radial deformations than with symmetric laminates. The differences can be as much as an order of magnitude. Third, the sense of the lack of symmetry in the lamination sequence (i.e., the sign of the B_{ij} components) determines to a large degree the sign of the radial deformations. This is particularly true for the sign of the oscillatory deformation, near the boundary layer. Forth, for this nonlinear problem, the length of the boundary layer is a function of the applied load level. In addition, the length of the boundary layer for the symmetric laminate is greater than the length of the boundary layer of either unsymmetric laminate. Finally, clamped boundary conditions suppress the radial deformation considerably relative to the simply supported case. The effects of asymmetry and thermally-induced preloading are, for the most part, independent of the support conditions.

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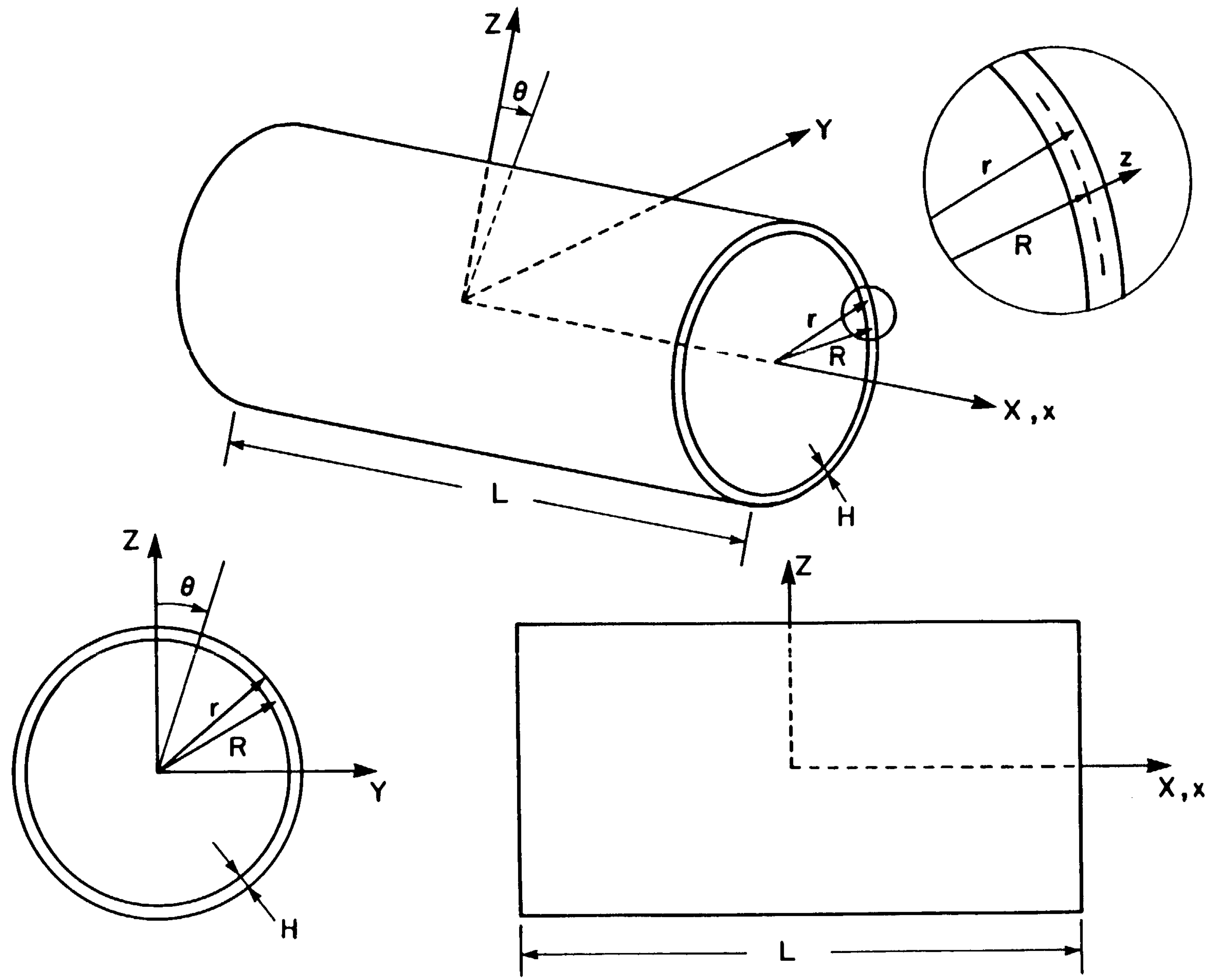

Fig. 1 - Geometry and Nomenclature for Cylinder.

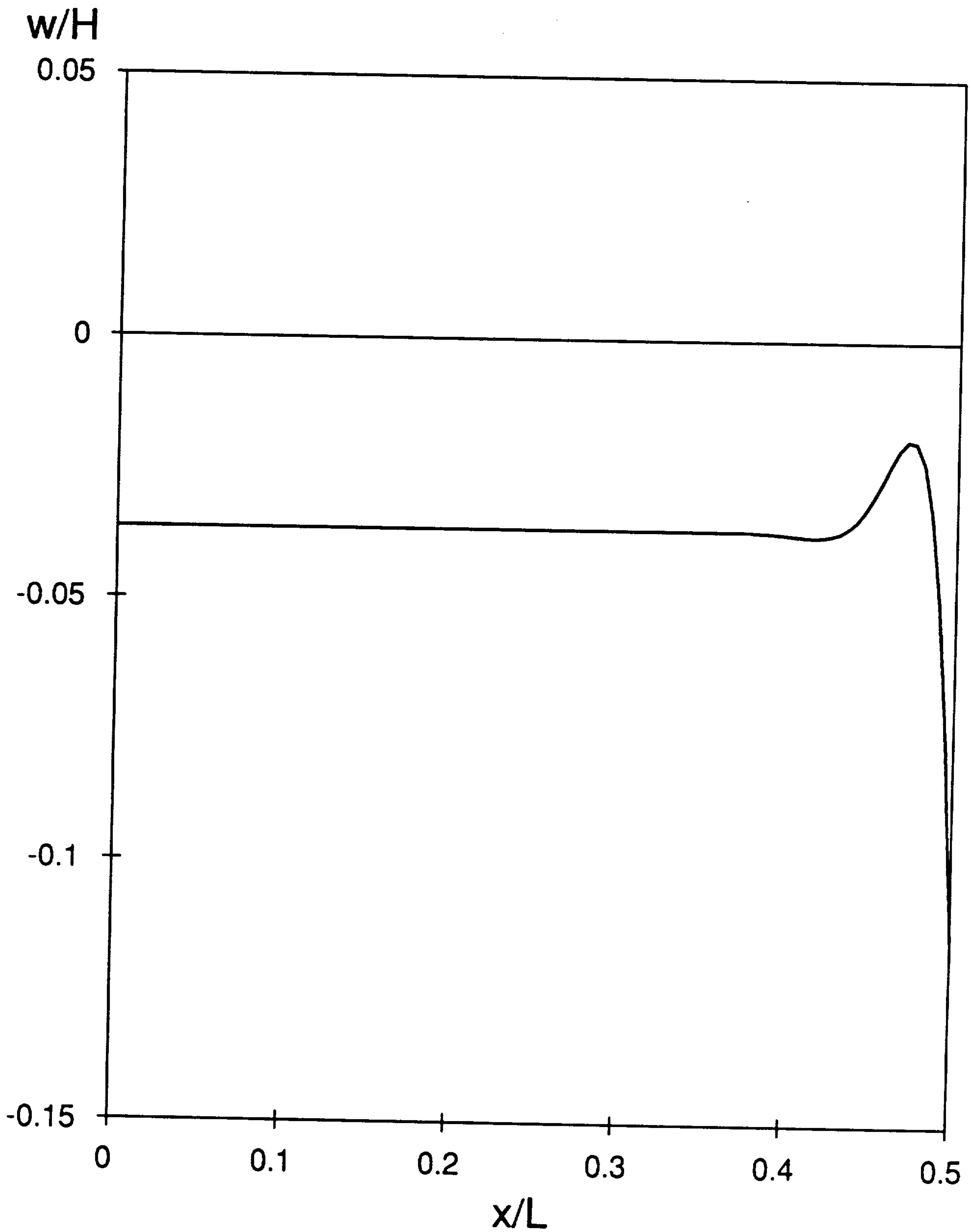

Fig. 2 - Room-Temperature Shape of the $(90_8/0_8)_T$ Cylinder ($\Delta T = -280°F$).

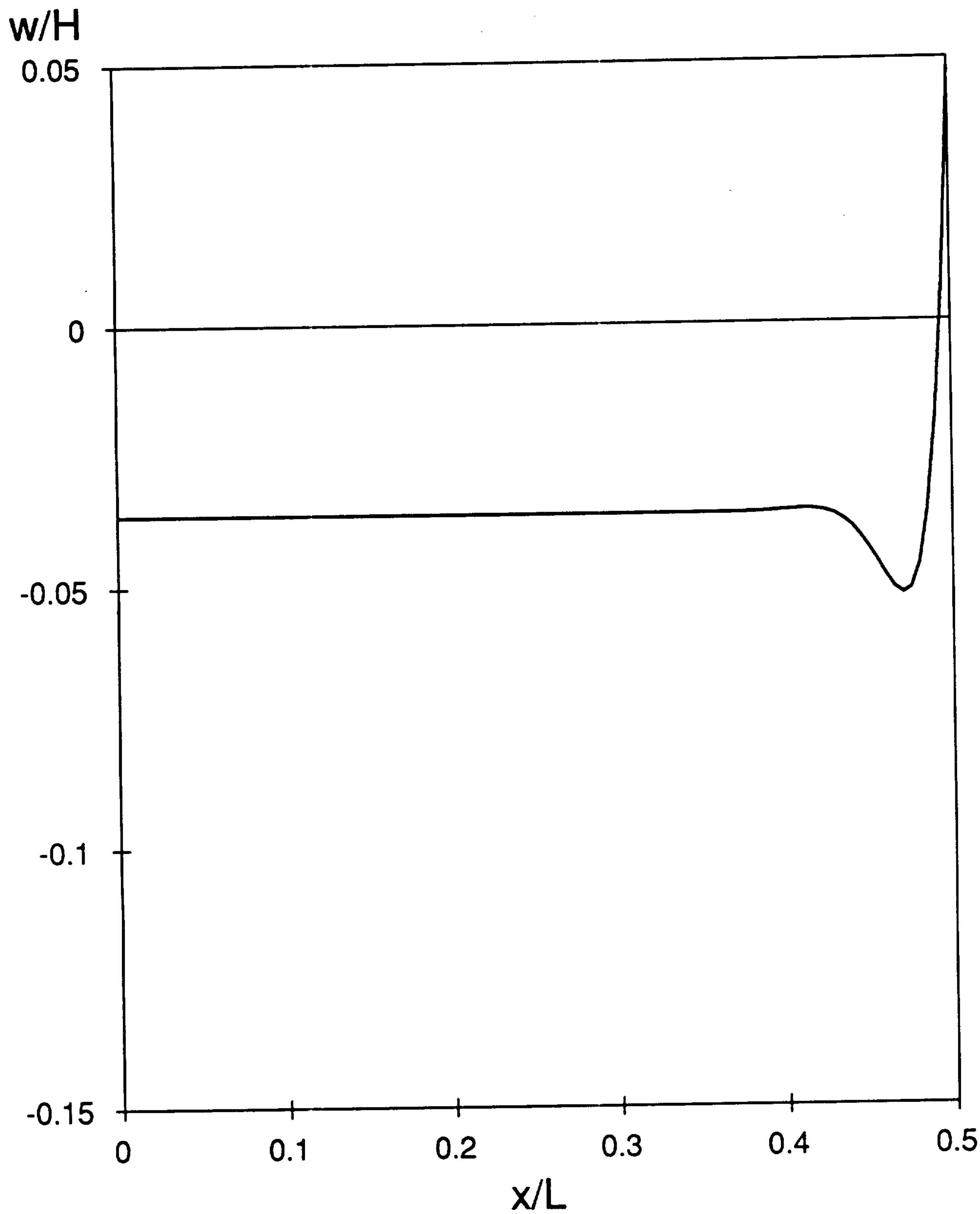

Fig. 3 - Room-Temperature Shape of the $(0_s/90_s)_T$ Cylinder (ΔT = −280°F).

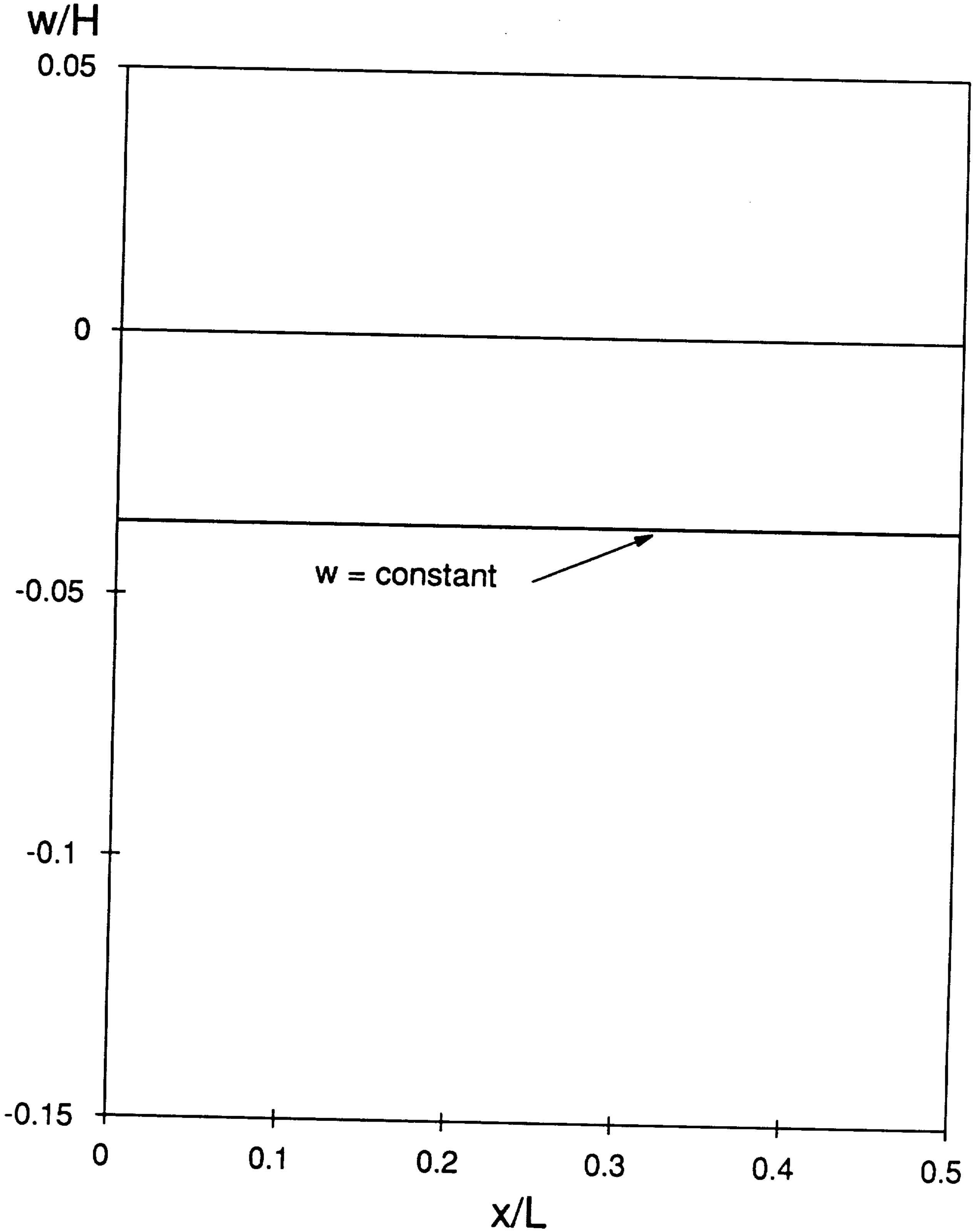

Fig. 4 - Room-Temperature Shape of the $(0/90)_{4s}$ Cylinder (ΔT = −280°F).

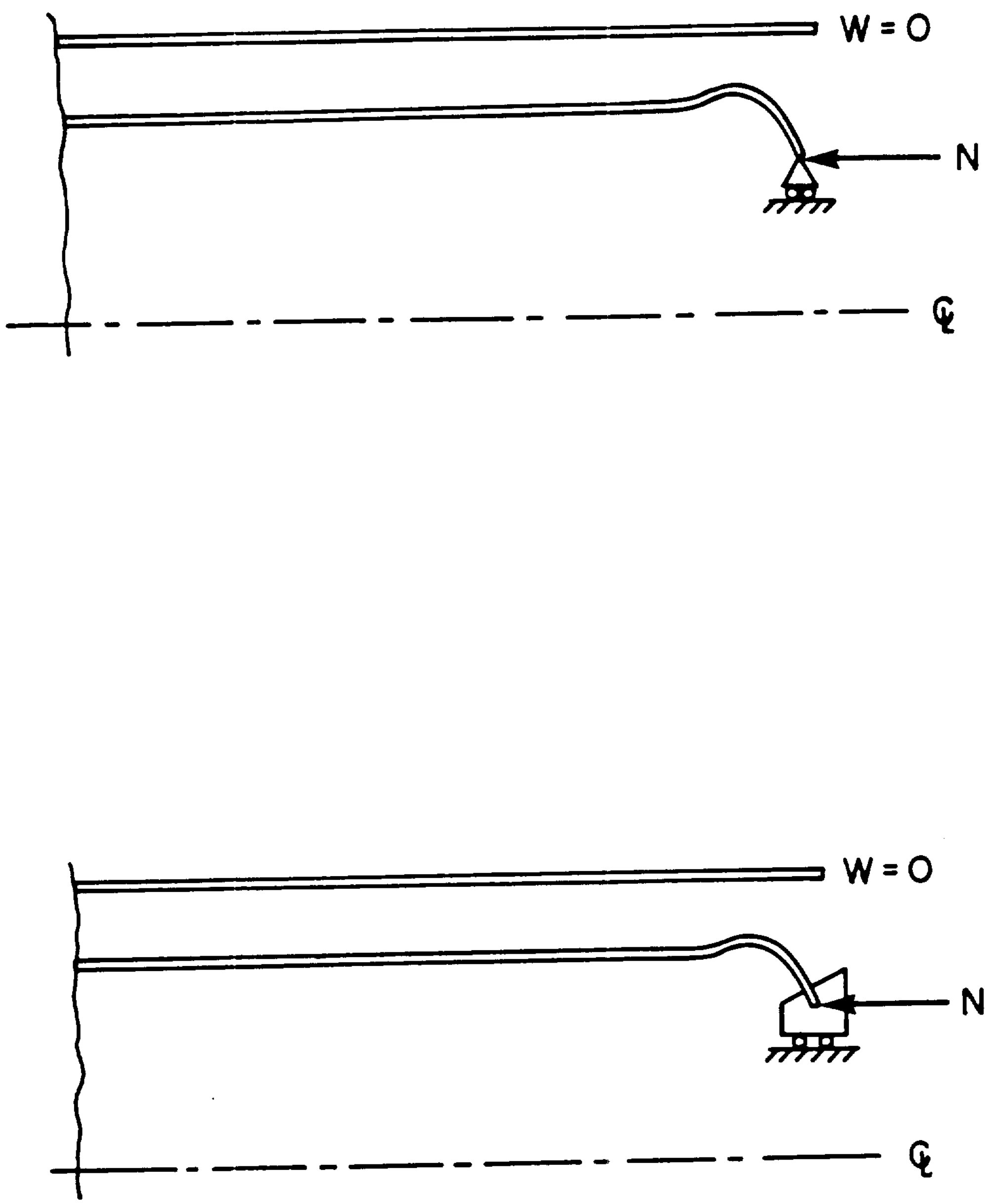

Fig. 5 - Support Conditions for the $(90_8/0_8)_T$ Cylinder in the Presence of Thermally-Induced Preloading Deformations: (a) - Simple Supports; (b) - Clamped Supports.

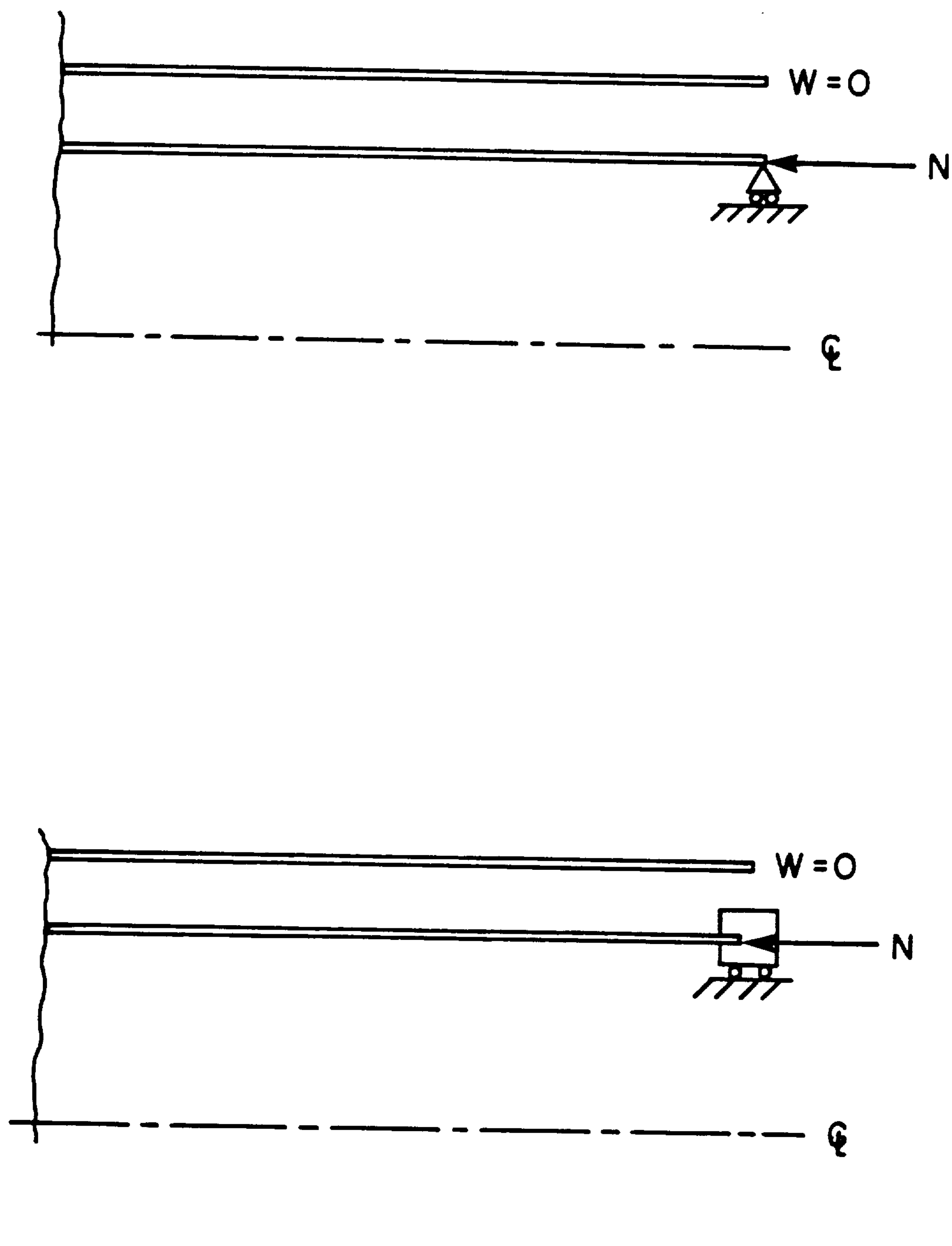

Fig. 6 - Support Conditions for the $(0/90)_{4s}$ Cylinder in the Presence of Thermally-Induced Preloading Deformations: (a) - Simple Supports; (b) - Clamped Supports.

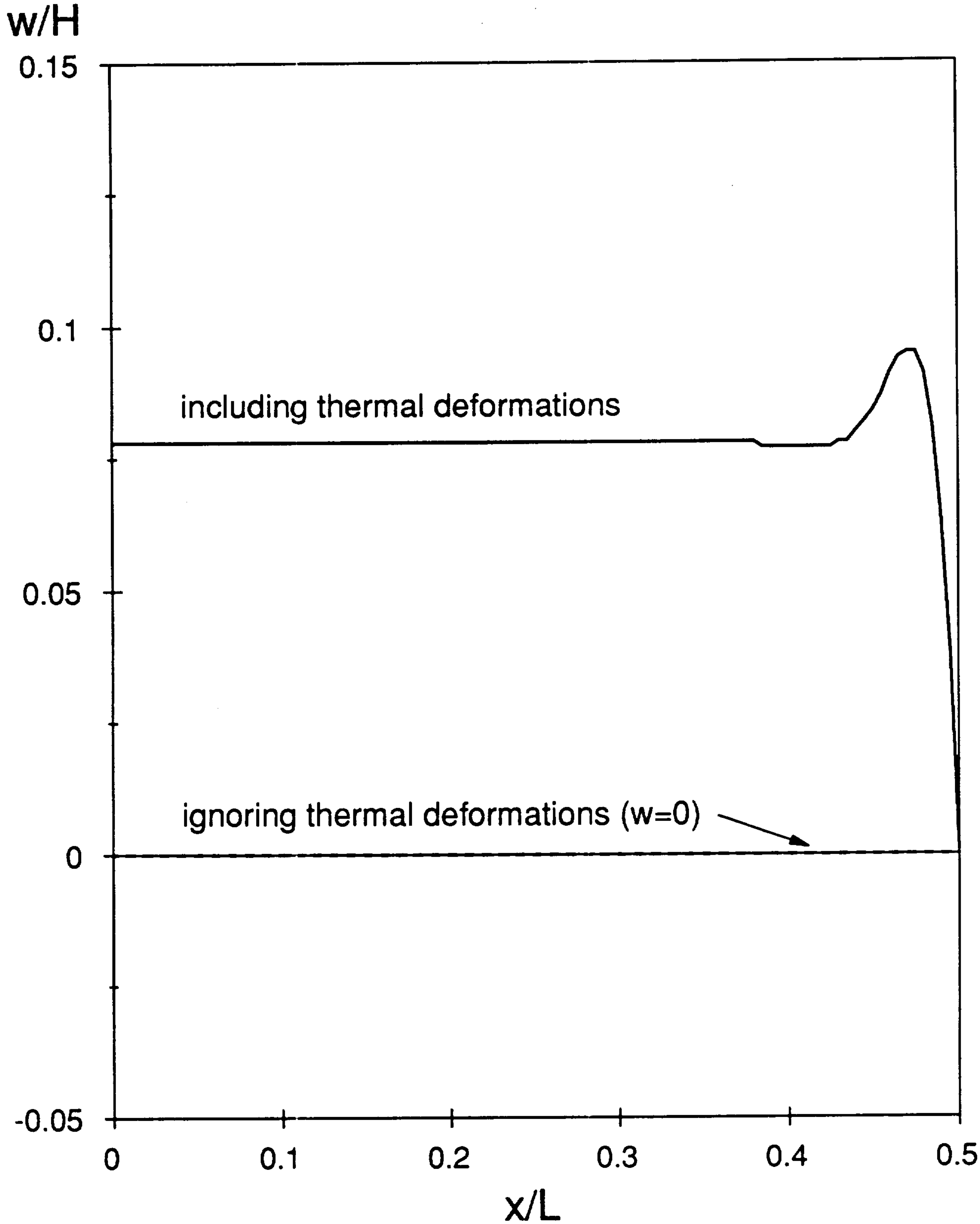

Fig. 7 - Radial Deformations of the Simply-Supported $(90_8/0_8)_T$ Cylinder, N = 0.

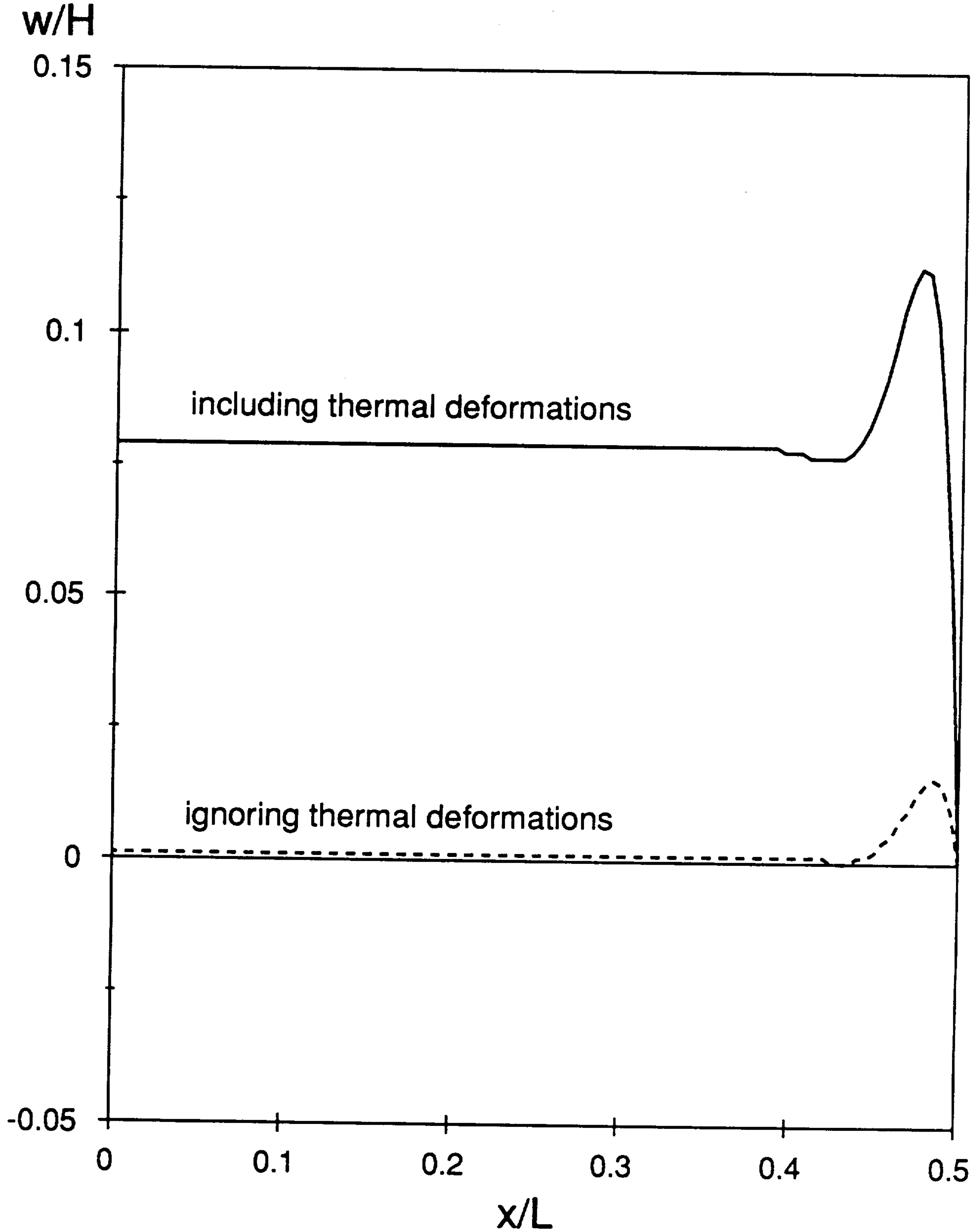

Fig. 8 - Radial Deformations of the Simply-Supported $(90_8/0_8)_T$ Cylinder, N = 0.1N*.

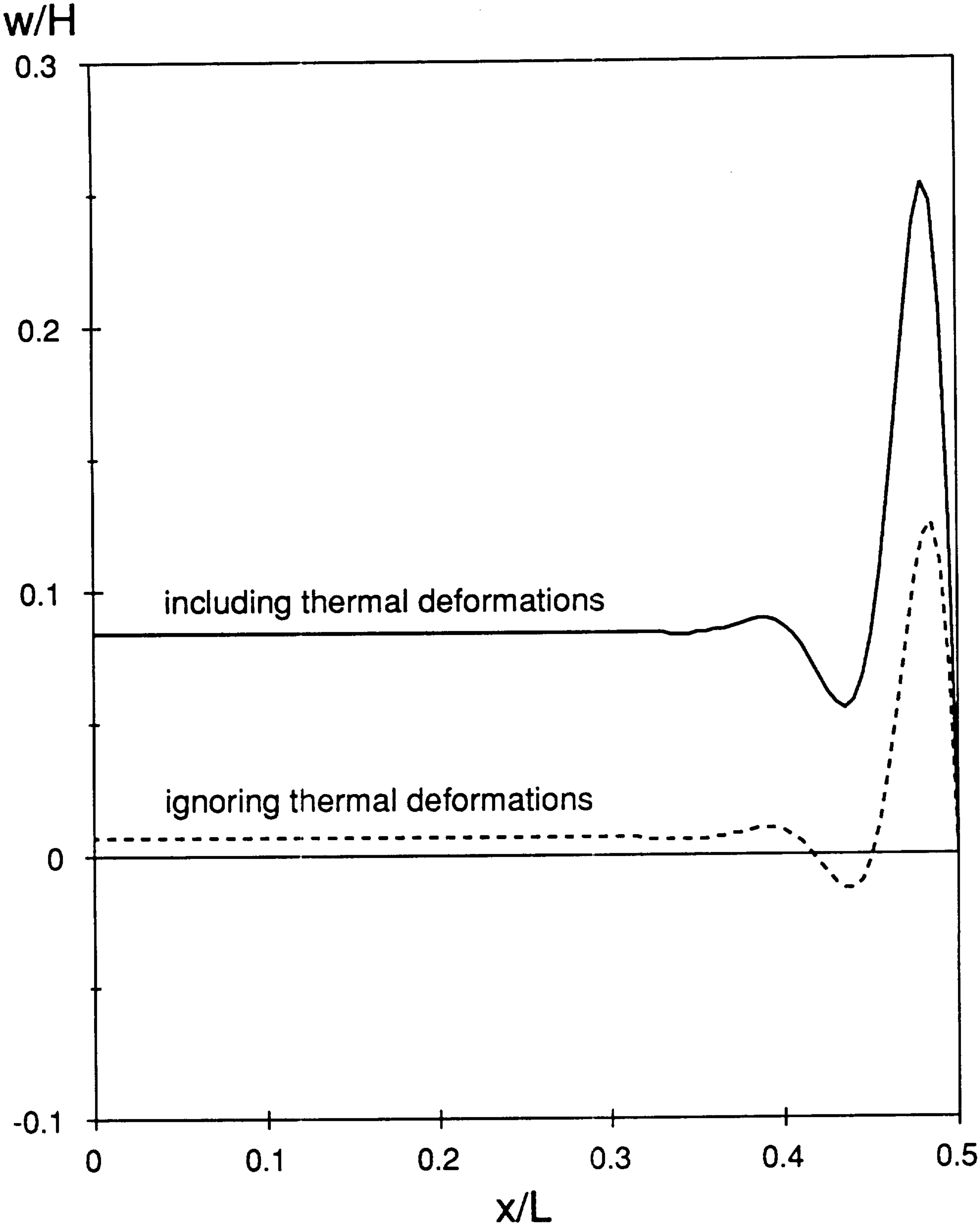

Fig. 9 - Radial Deformations of the Simply-Supported $(90_8/0_8)_T$ Cylinder, N = 0.5N*.

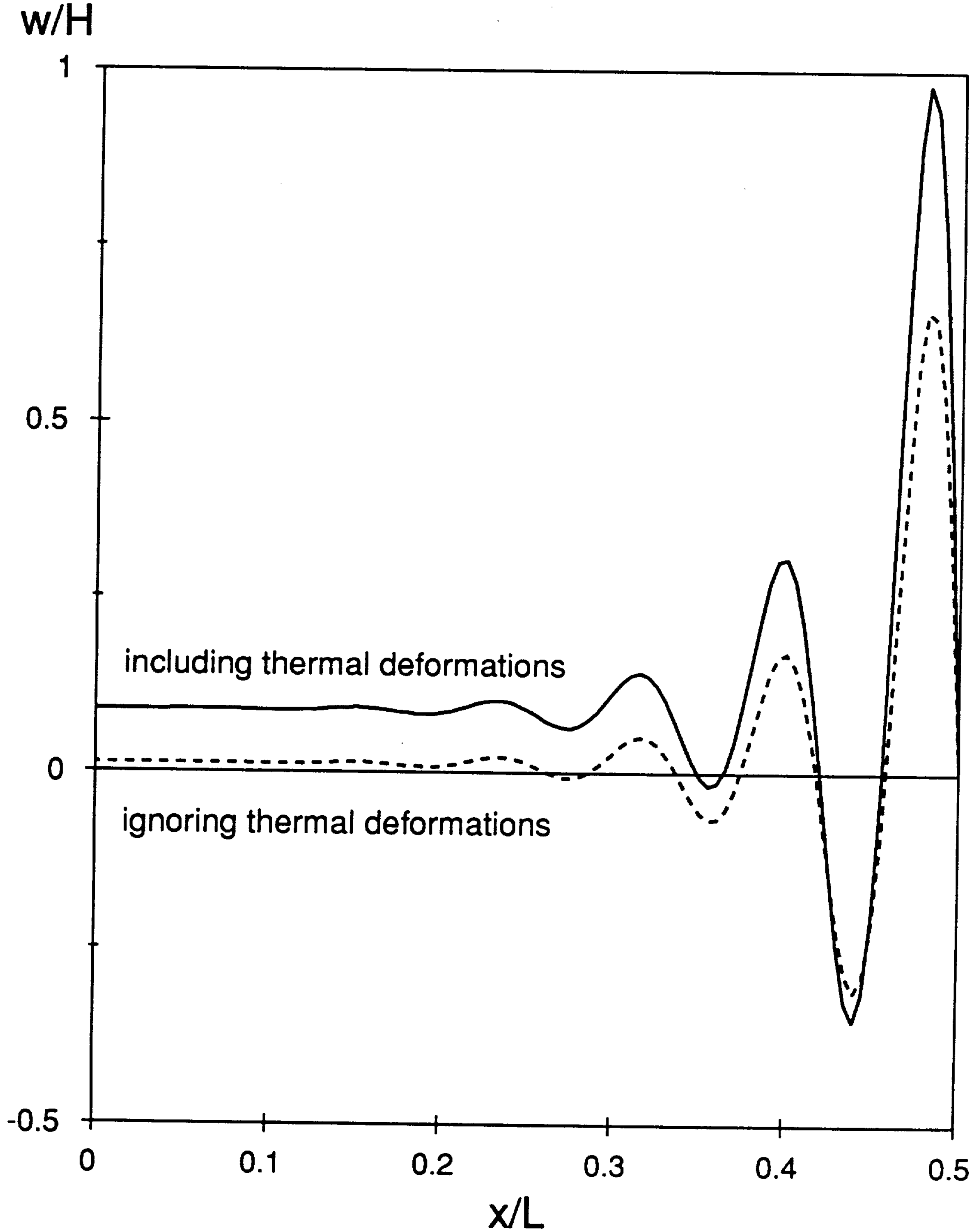

Fig. 10 - Radial Deformations of the Simply-Supported $(90_8/0_8)_T$ Cylinder, $N = 0.9N^*$.

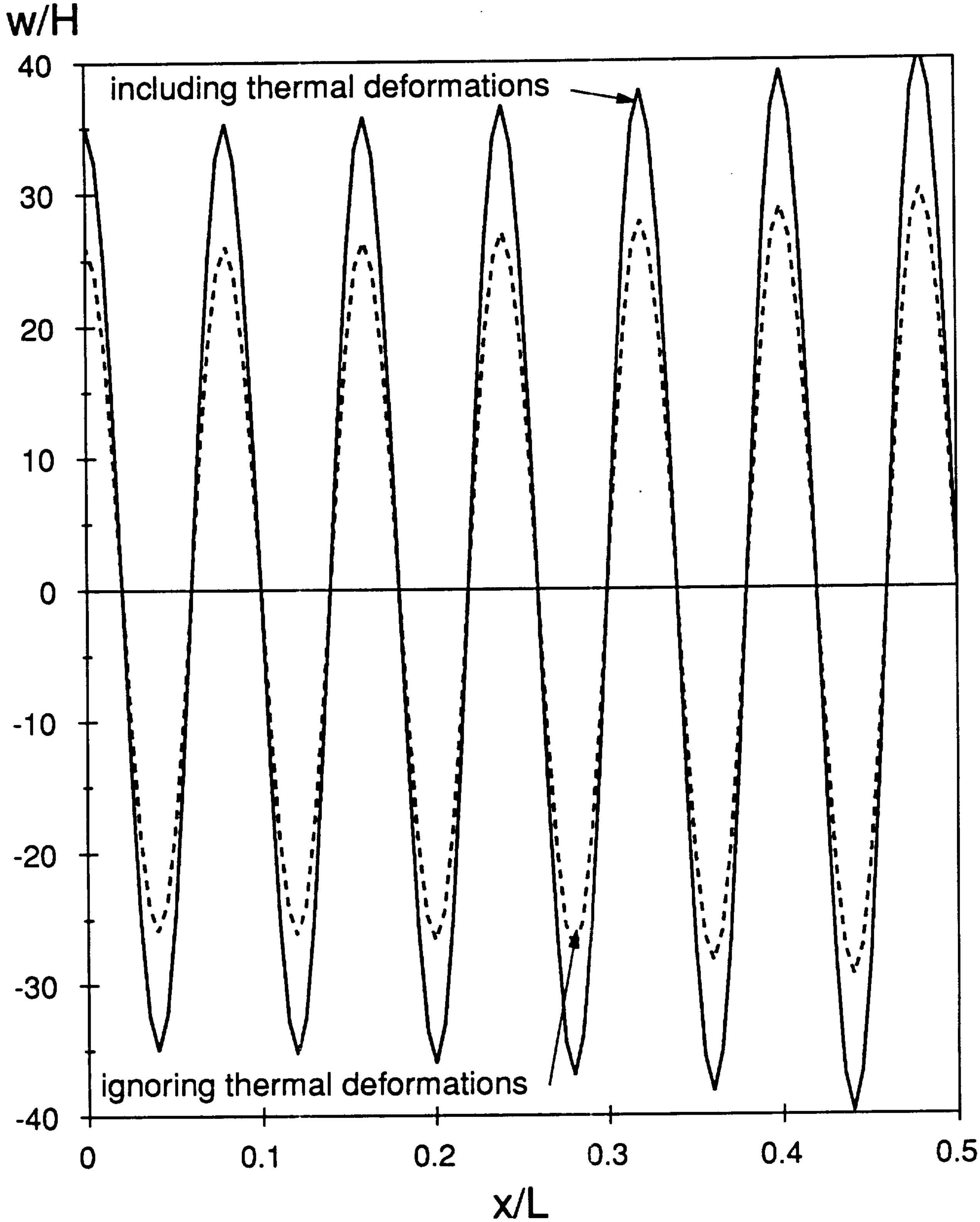

Fig. 11 - Radial Deformations of the Simply-Supported $(90_s/0_s)_T$ Cylinder, N = 1.0N*.

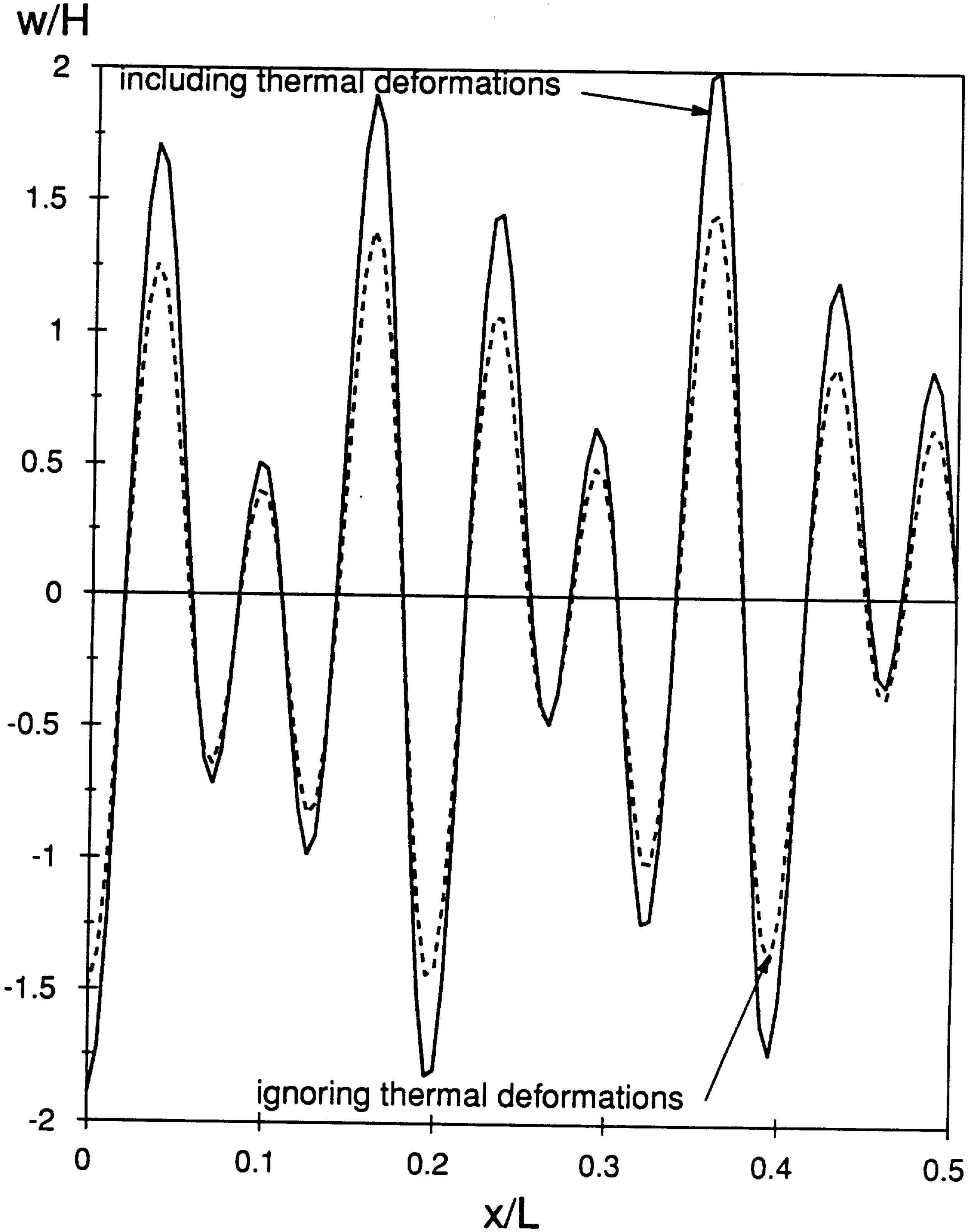

Fig. 12 - Radial Deformations of the Simply-Supported $(90_8/0_8)_T$ Cylinder, N = 1.1N*.

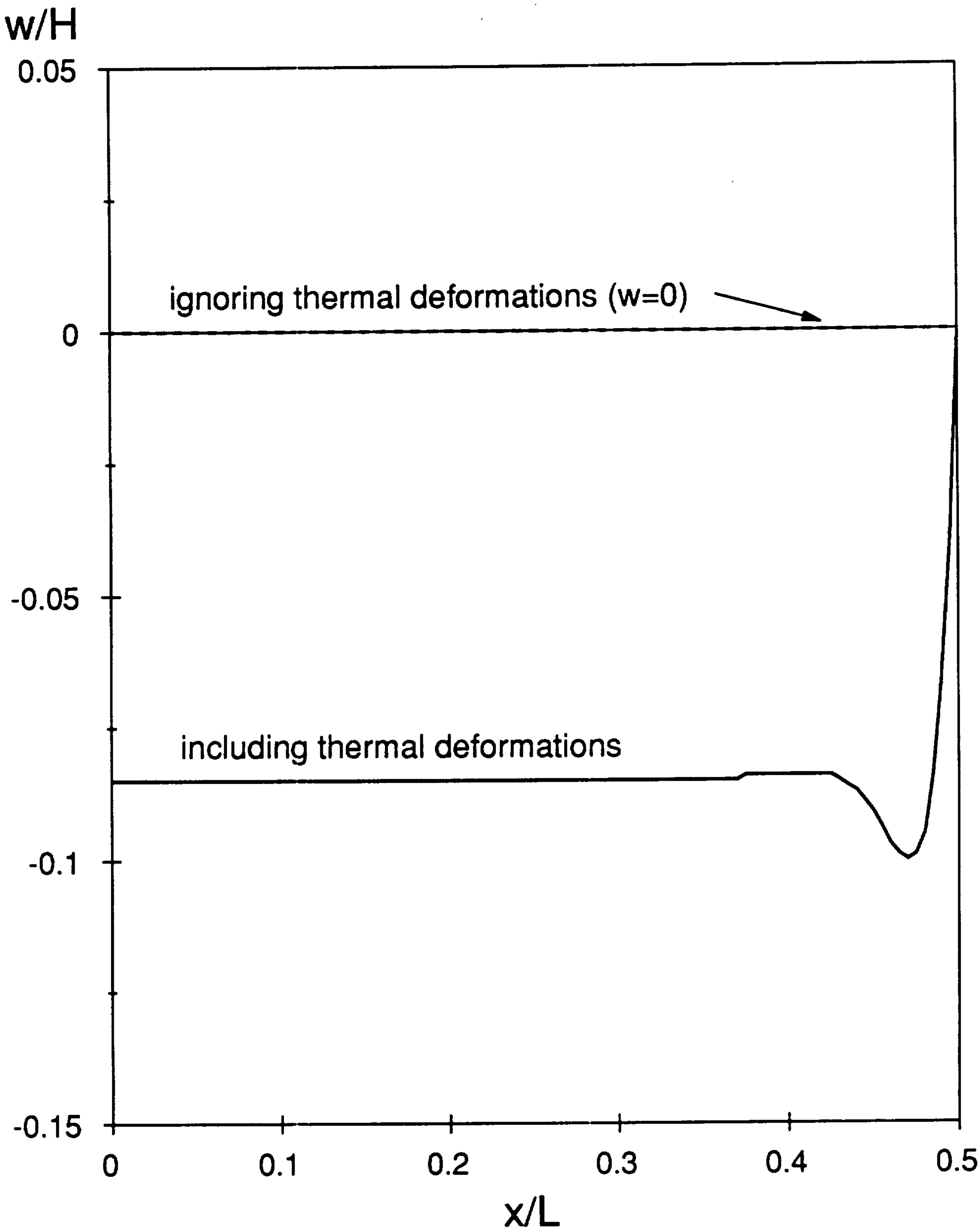

Fig. 13 - Radial Deformations of the Simply-Supported $(0_8/90_8)_T$ Cylinder, N = 0.

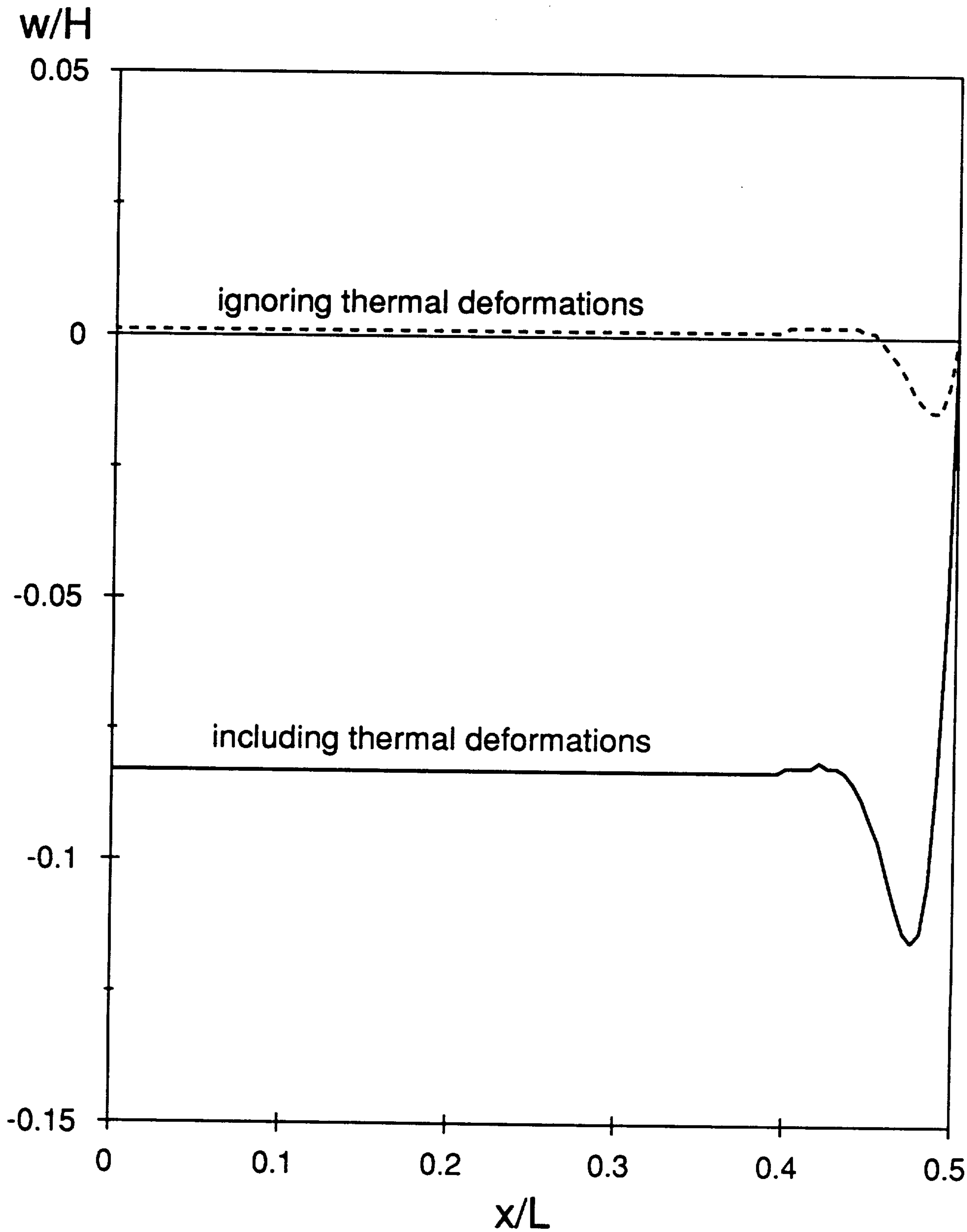

Fig. 14 - Radial Deformations of the Simply-Supported $(0_8/90_8)_T$ Cylinder, N = 0.1N*.

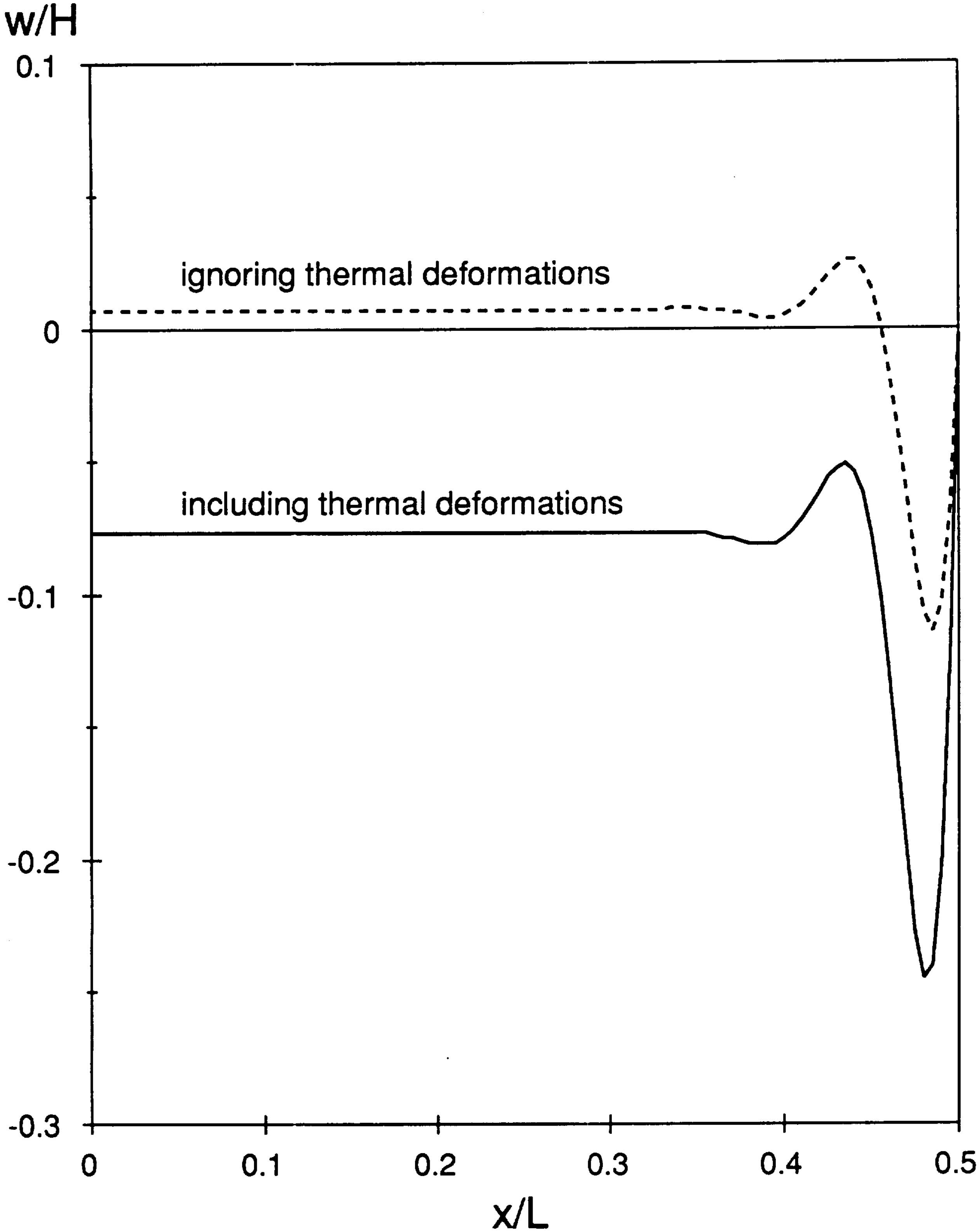

Fig. 15 - Radial Deformations of the Simply-Supported $(0_s/90_s)_T$ Cylinder, N = 0.5N*.

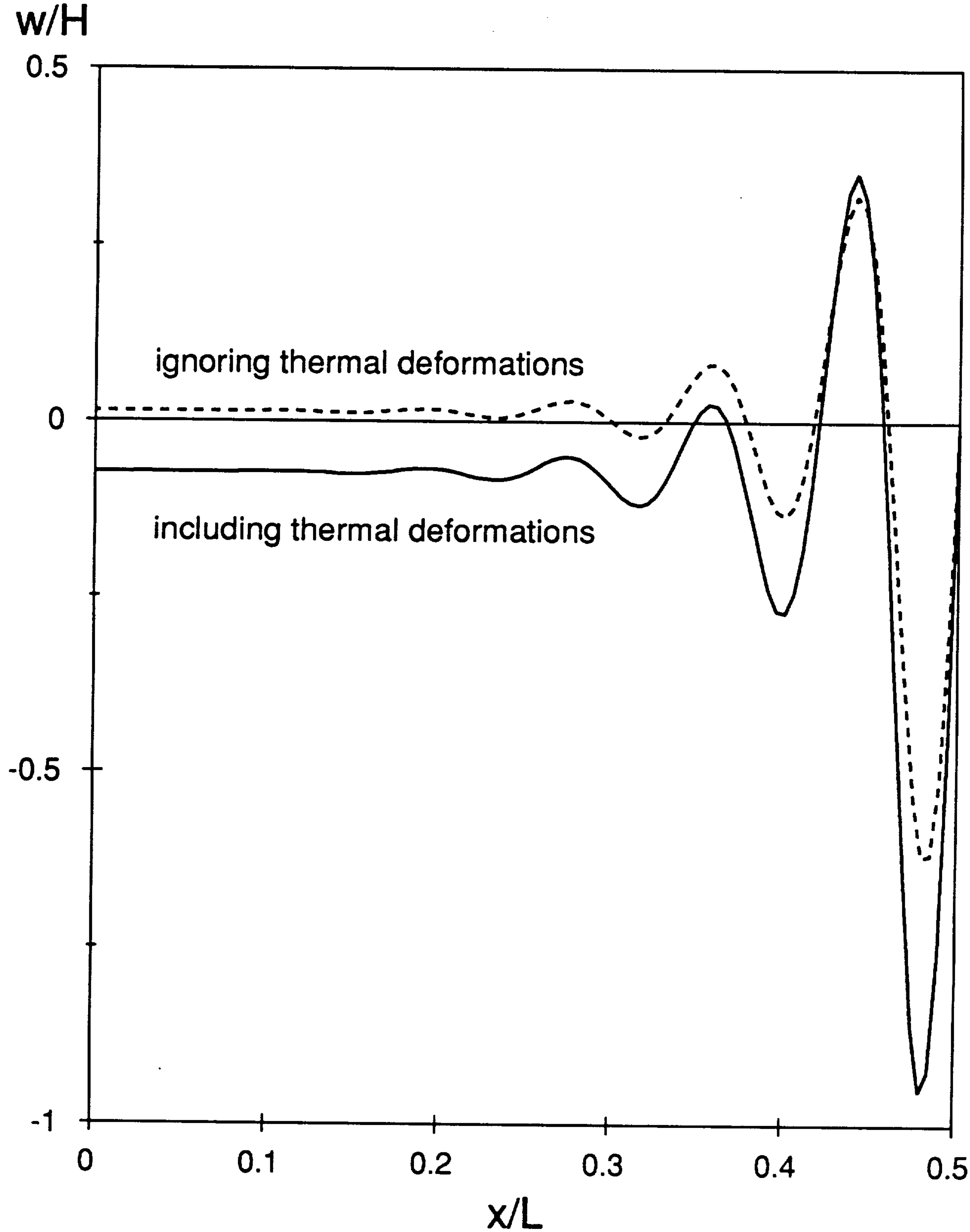

Fig. 16 - Radial Deformations of the Simply-Supported $(0_8/90_8)_T$ Cylinder, N = 0.9N*.

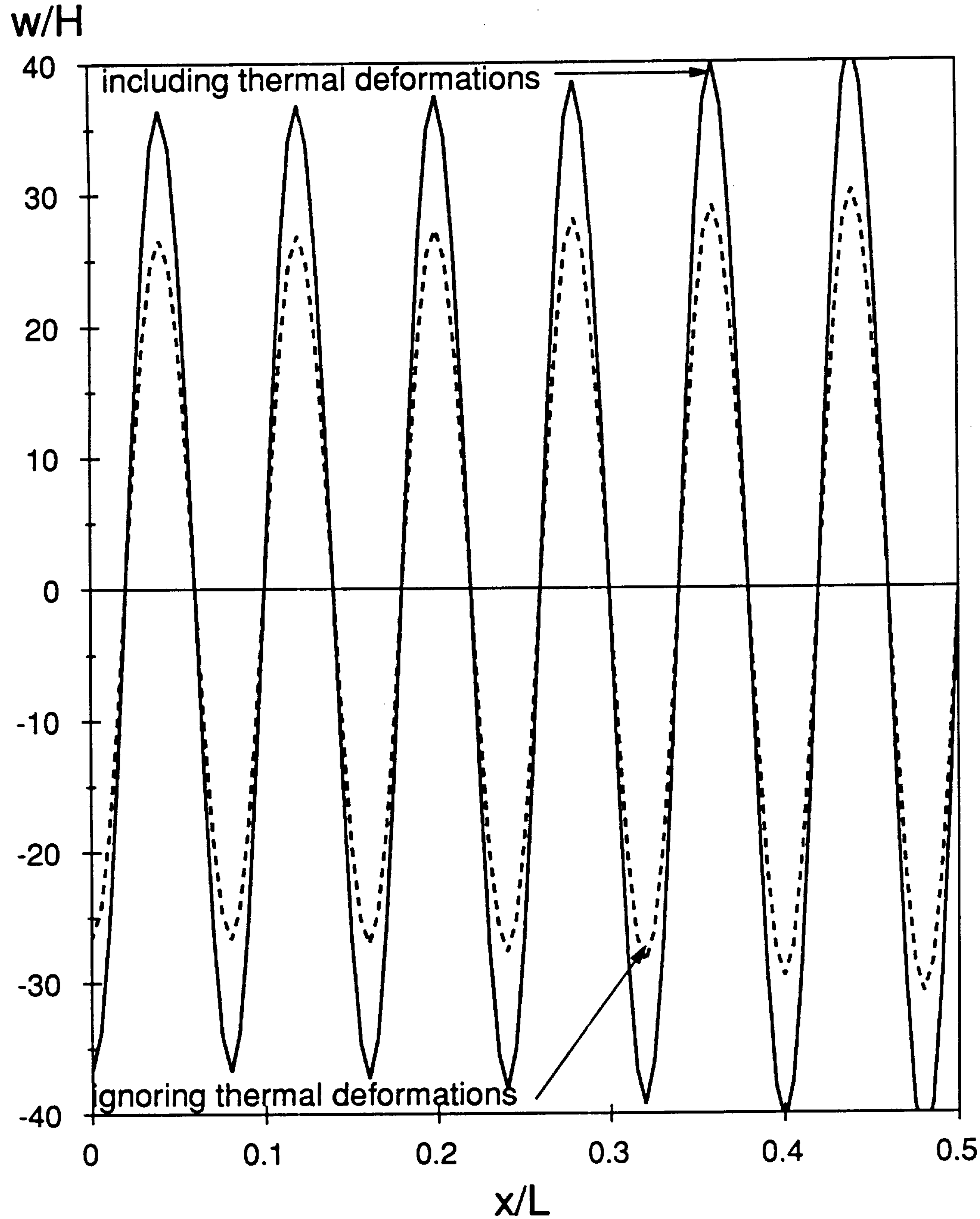

Fig. 17 - Radial Deformations of the Simply-Supported $(0_s/90_s)_T$ Cylinder, N = 1.0N*.

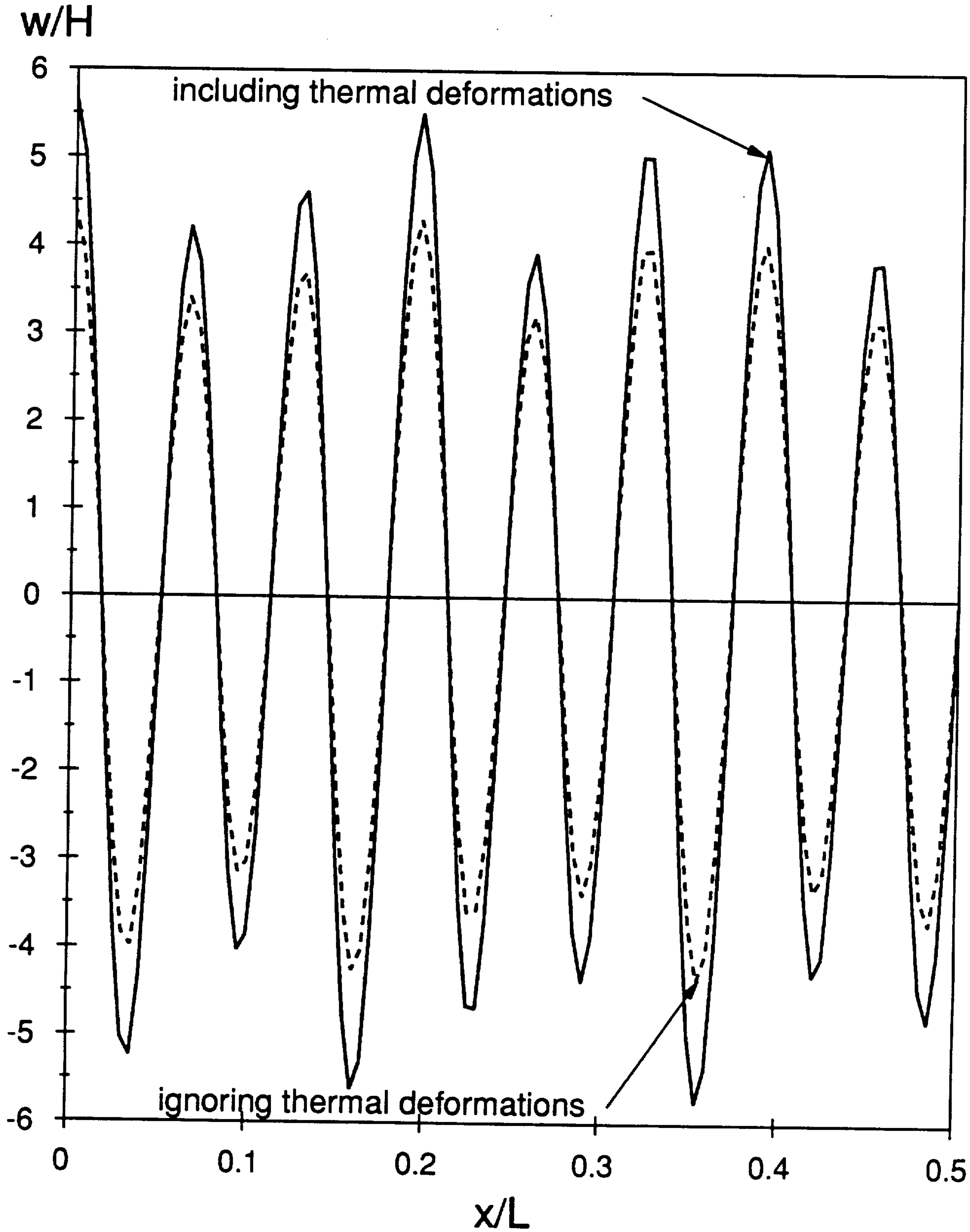

Fig. 18 - Radial Deformations of the Simply-Supported $(0_8/90_8)_T$ Cylinder, N = 1.1N*.

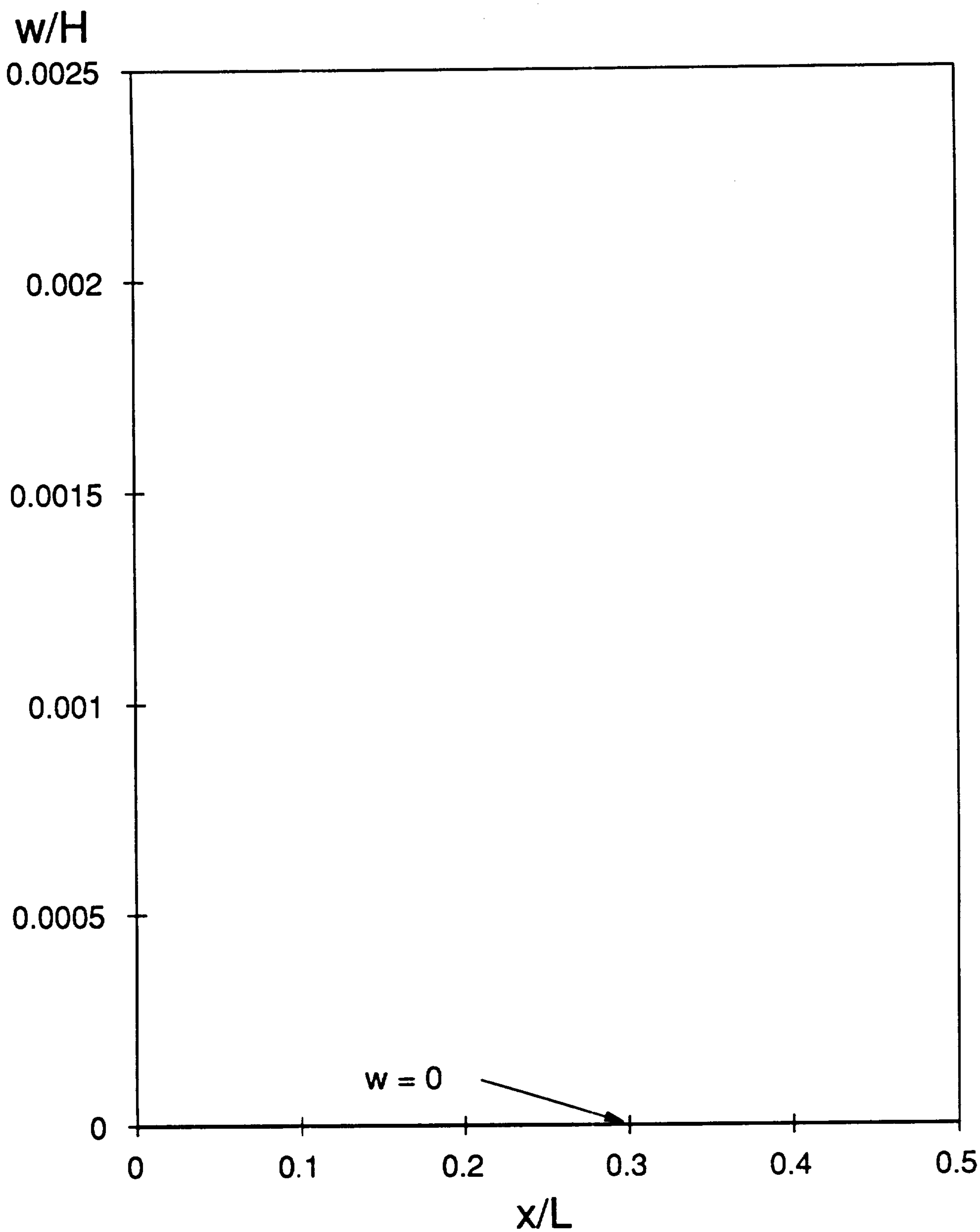

Fig. 19 - Radial Deformations of the Simply-Supported $(0/90)_{4S}$ Cylinder, N = 0.

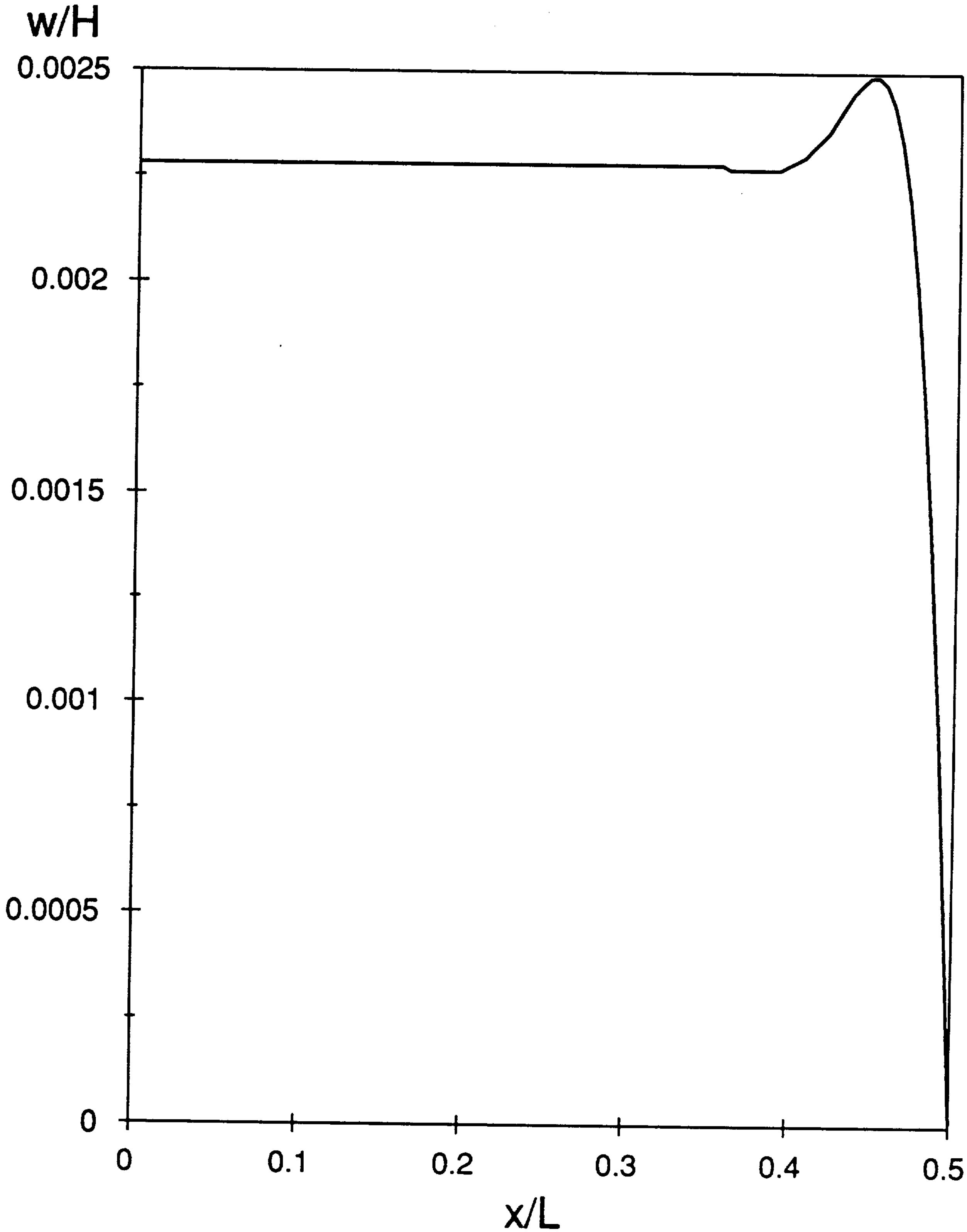

Fig. 20 - Radial Deformations of the Simply-Supported $(0/90)_{4s}$ Cylinder, $N = 0.1N^*$.

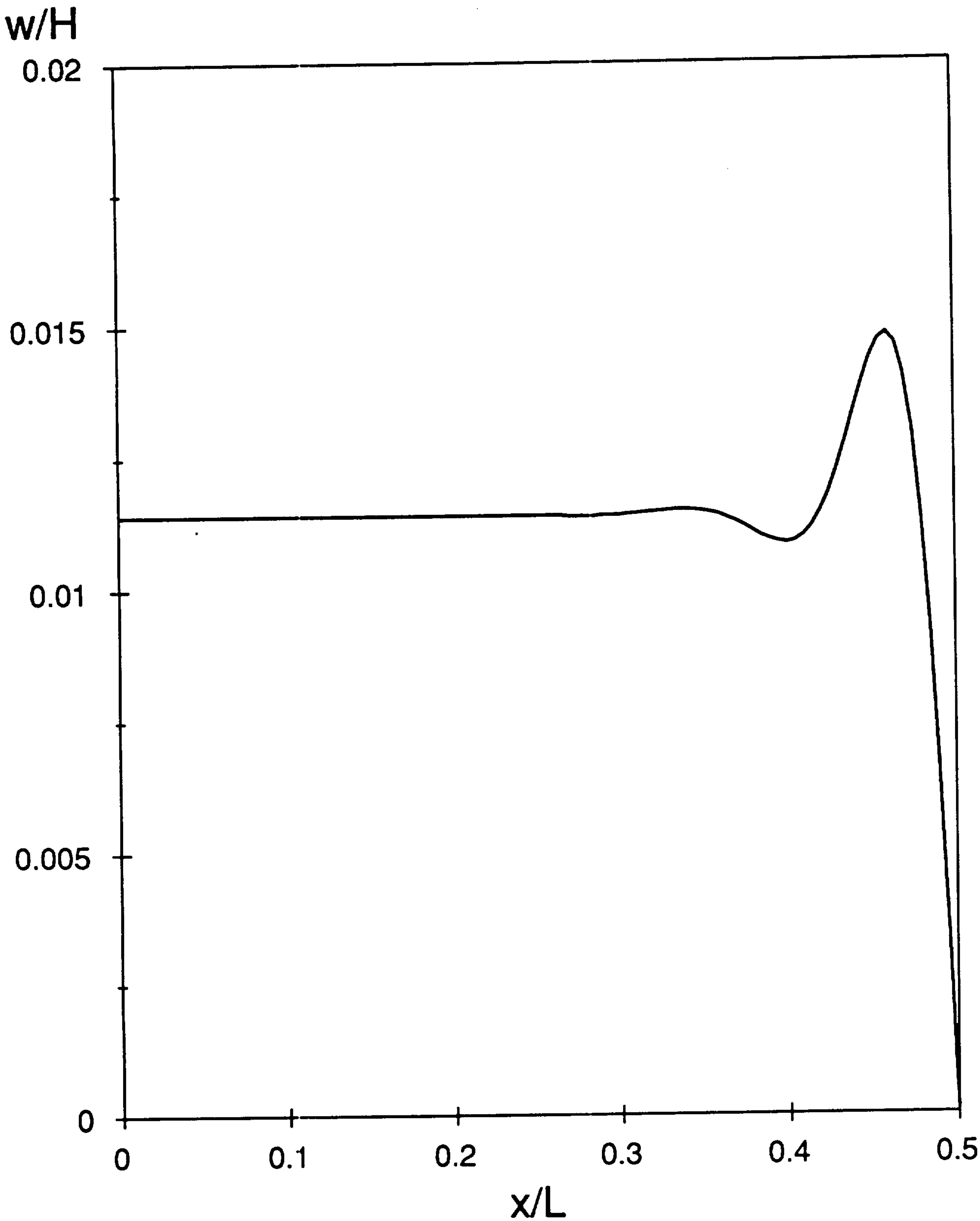

Fig. 21 - Radial Deformations of the Simply-Supported $(0/90)_{48}$ Cylinder, N = 0.5N*.

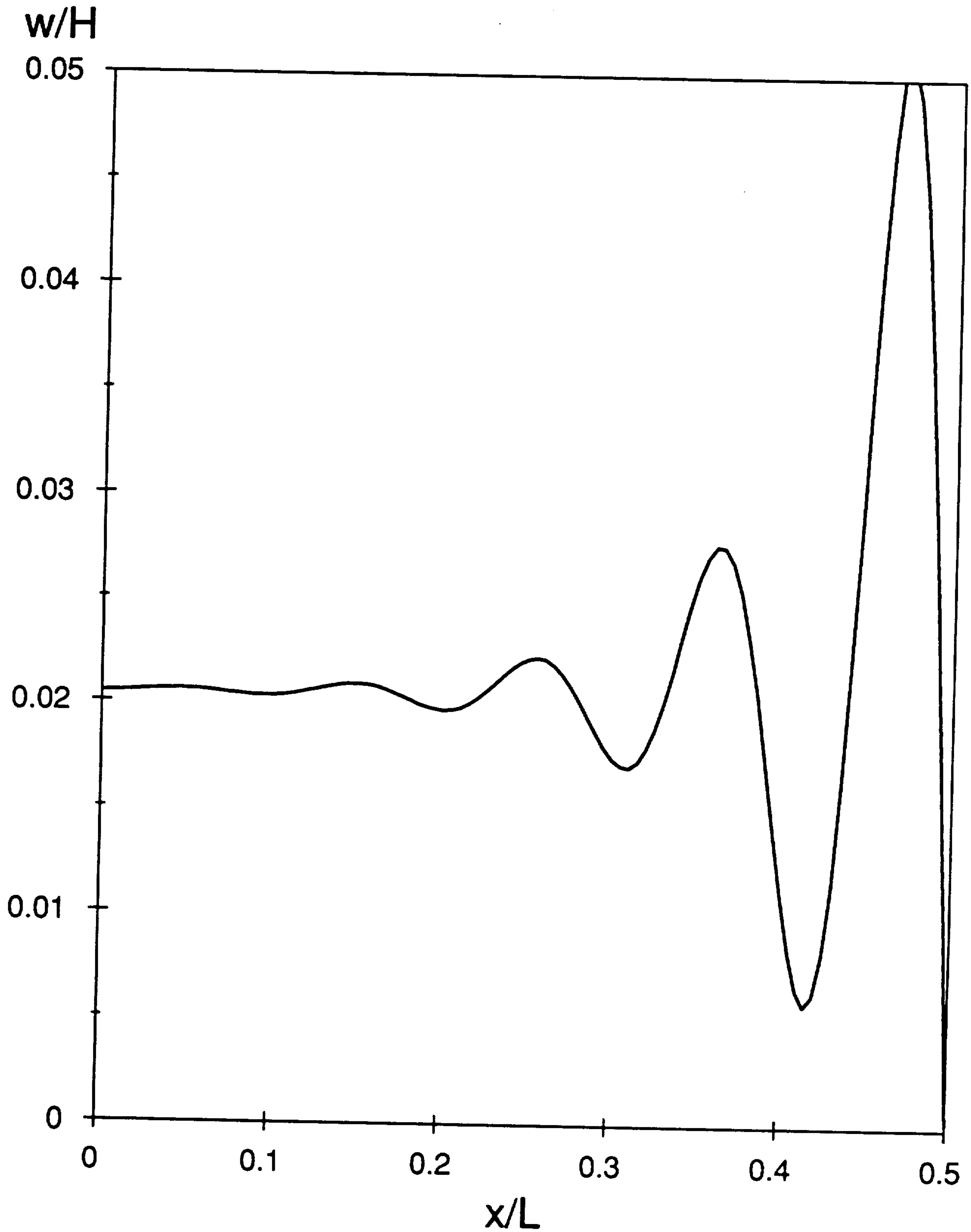

Fig. 22 - Radial Deformations of the Simply-Supported $(0/90)_{4s}$ Cylinder, N = 0.9N*.

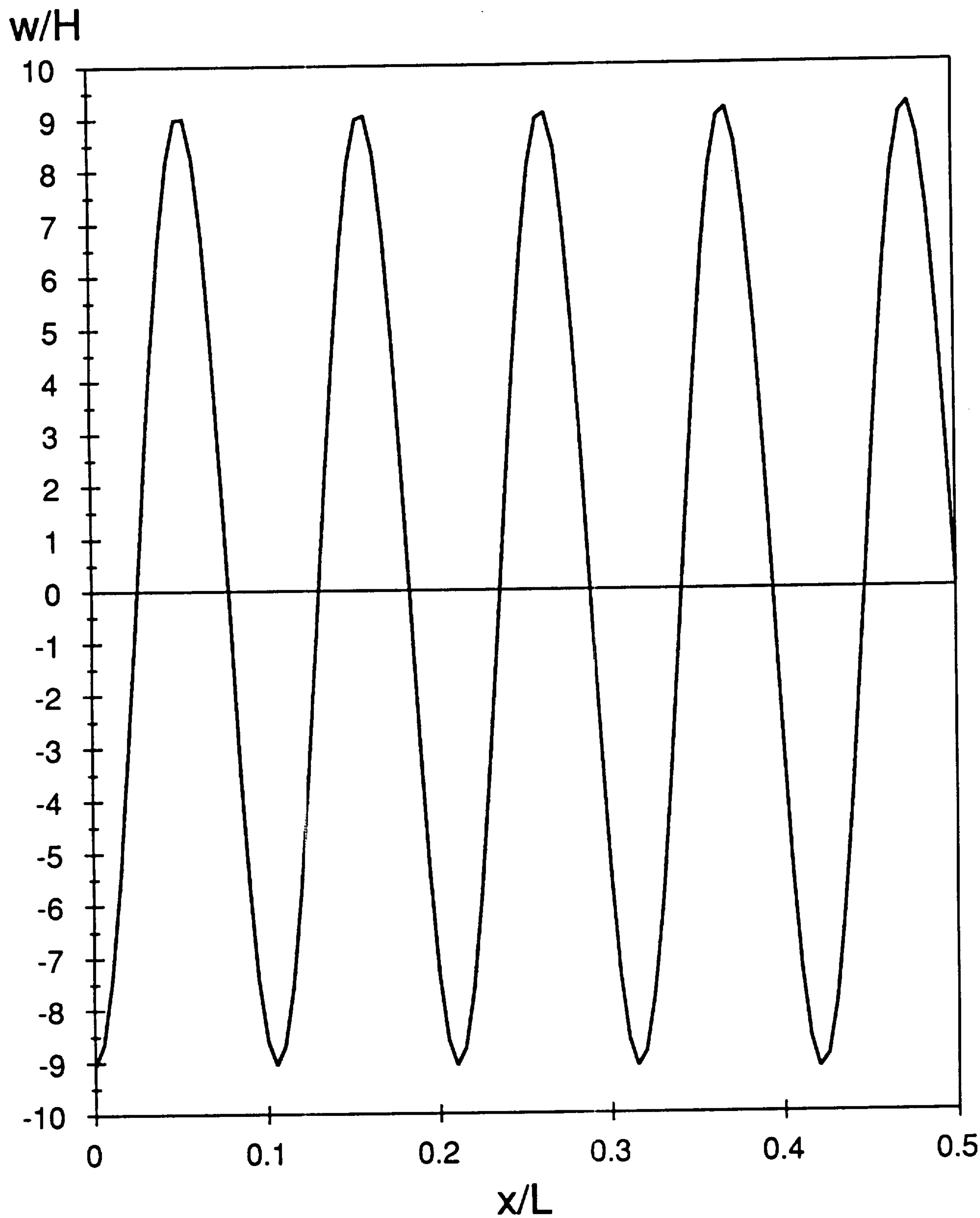

Fig. 23 - Radial Deformations of the Simply-Supported $(0/90)_{4s}$ Cylinder, N = 1.0N*.

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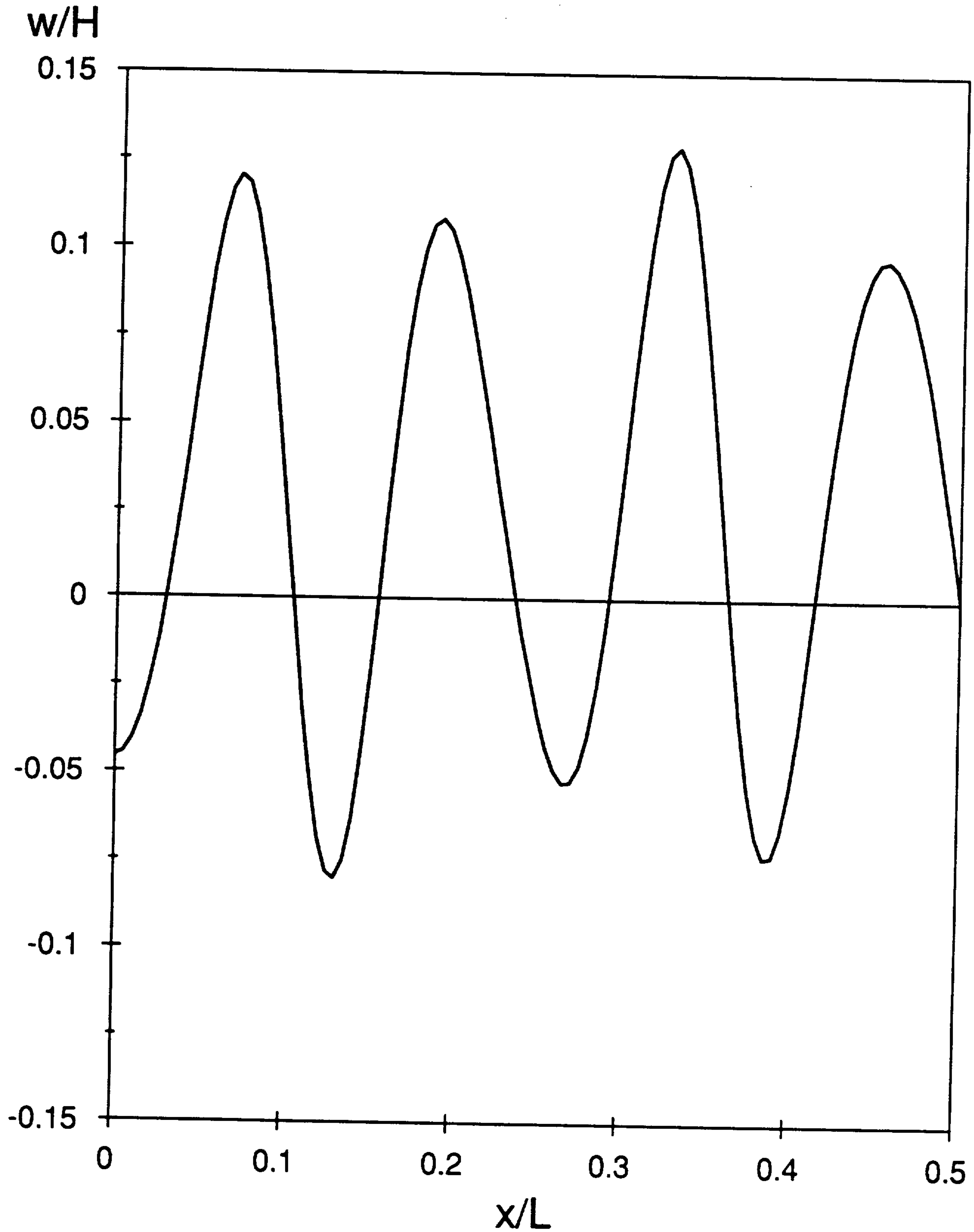

Fig. 24 - Radial Deformations of the Simply-Supported $(0/90)_{4s}$ Cylinder, $N = 1.1N^{*}$.

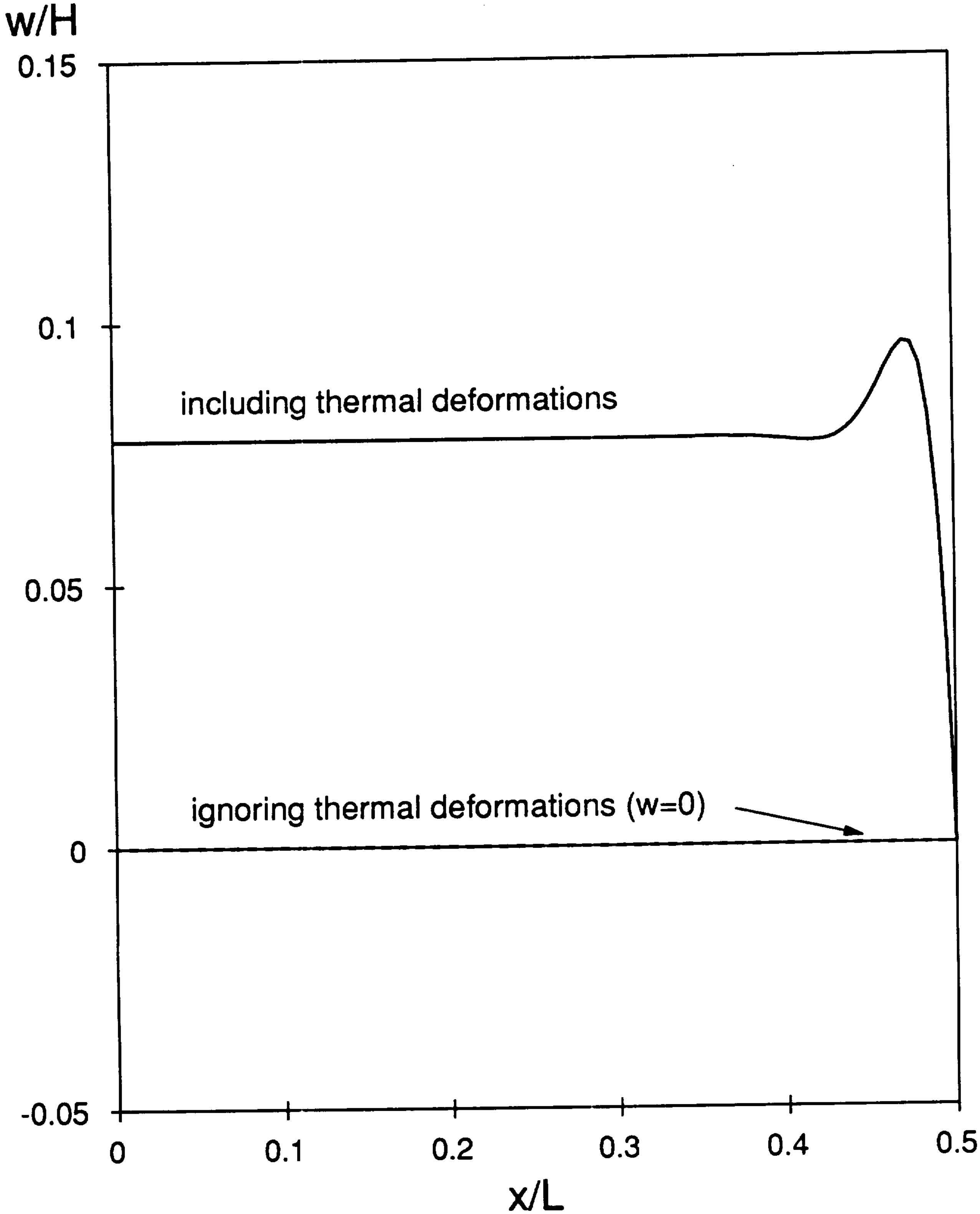

Fig. 25 - Radial Deformations of the Clamped $(90_s/0_s)_T$ Cylinder, N = 0.

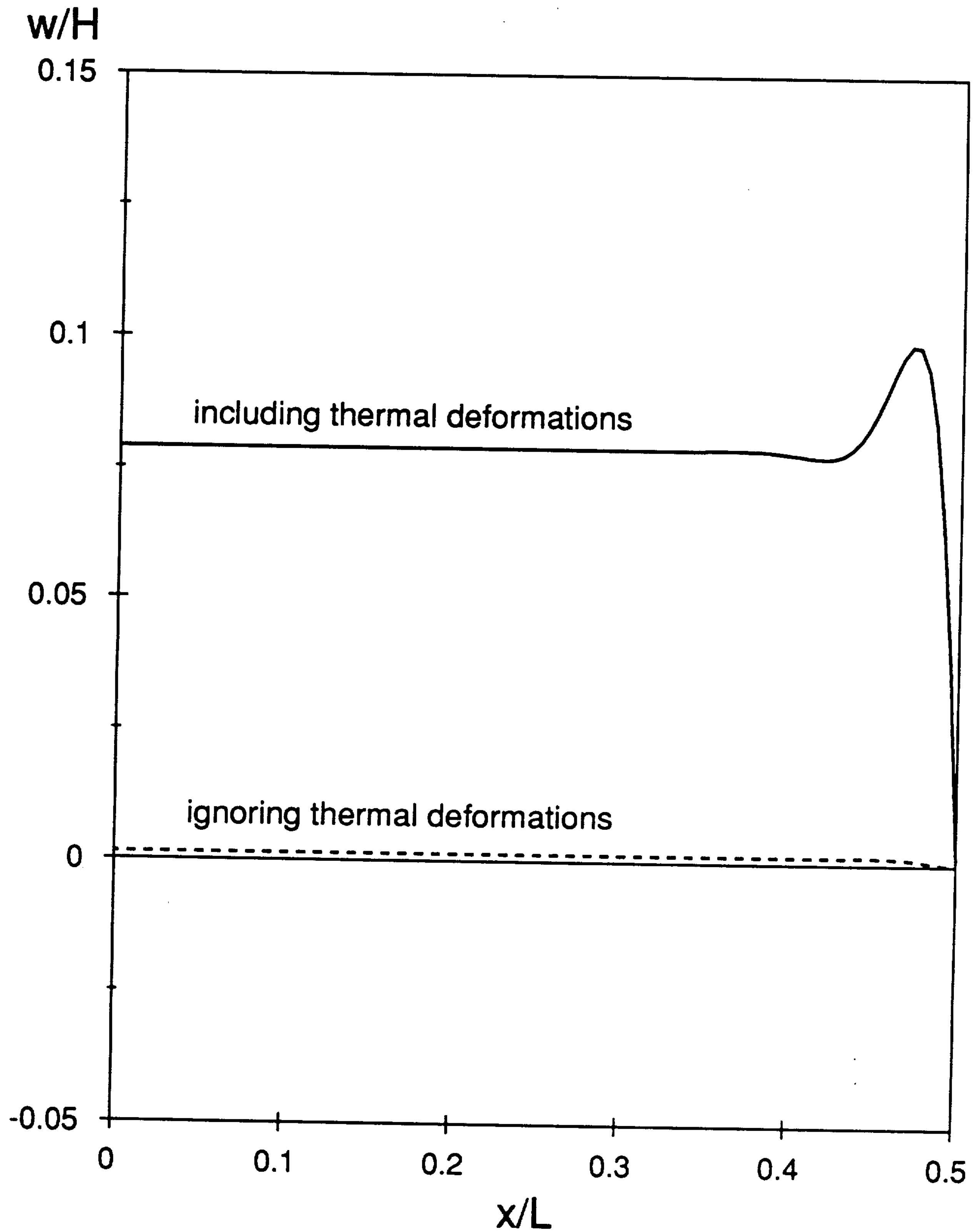

Fig. 26 - Radial Deformations of the Clamped $(90_s/0_s)_T$ Cylinder, N = 0.1N*.

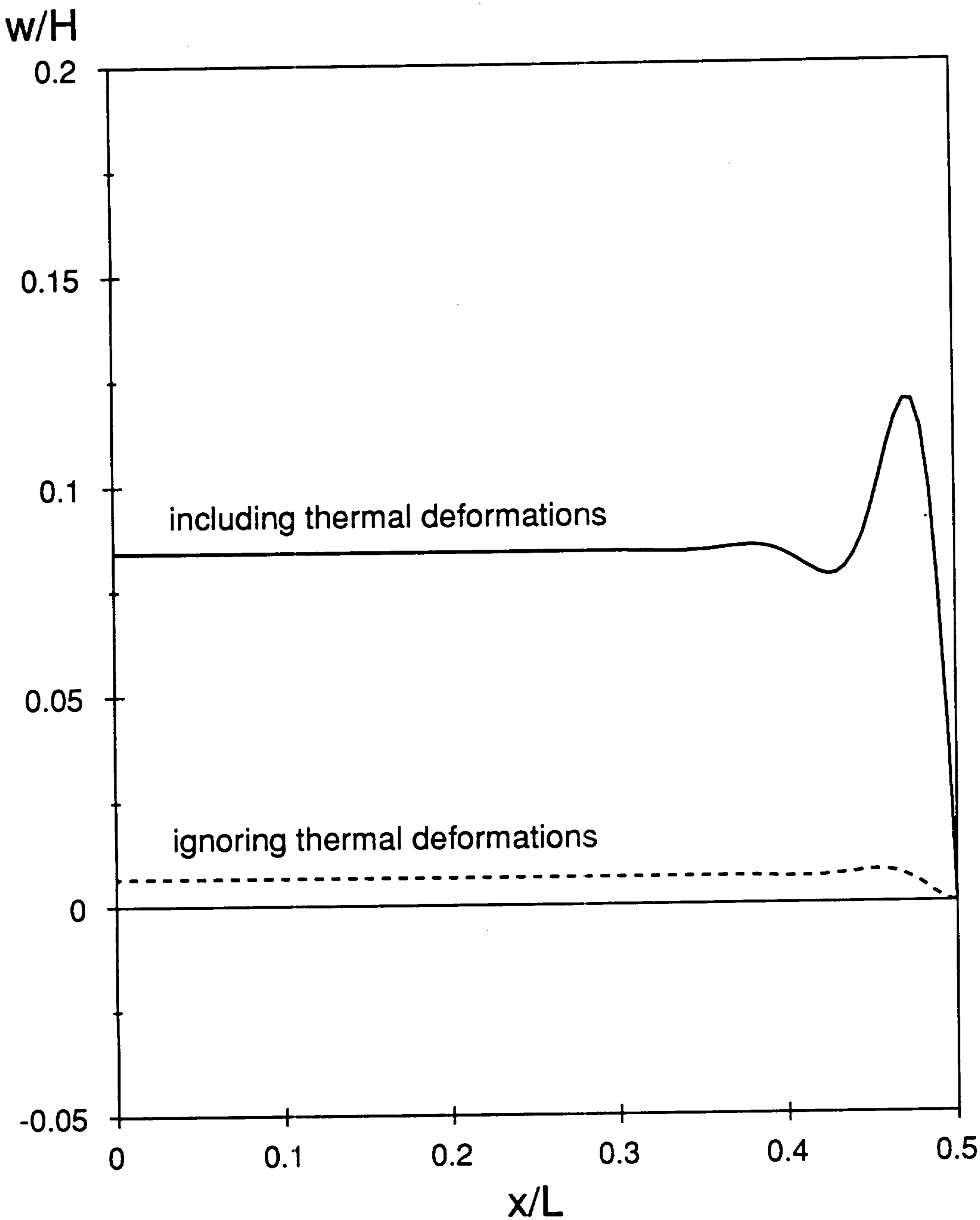

Fig. 27 - Radial Deformations of the Clamped $(90_s/0_s)_T$ Cylinder, N = 0.5N*.

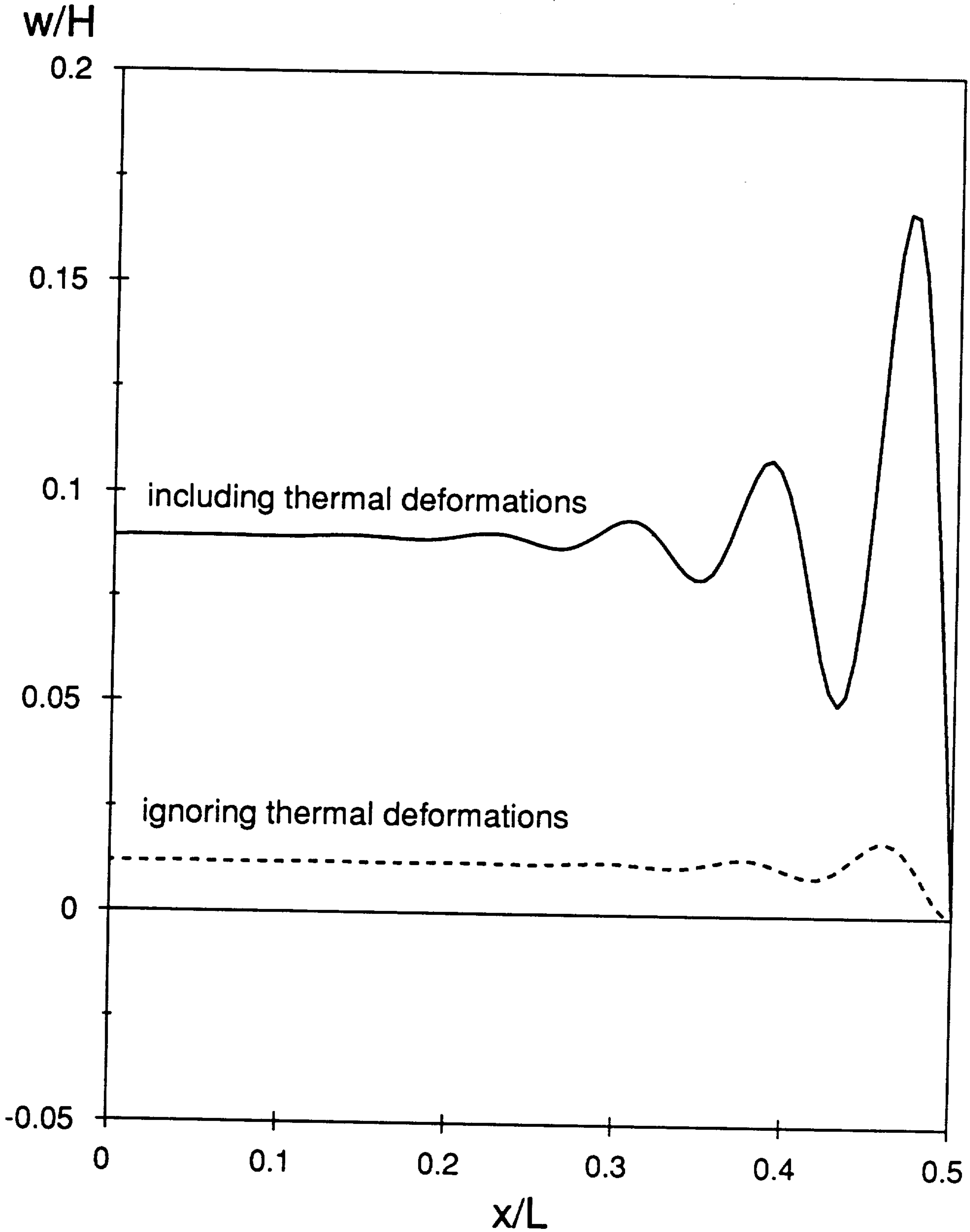

Fig. 28 - Radial Deformations of the Clamped $(90_8/0_8)_T$ Cylinder, N = 0.9N*.

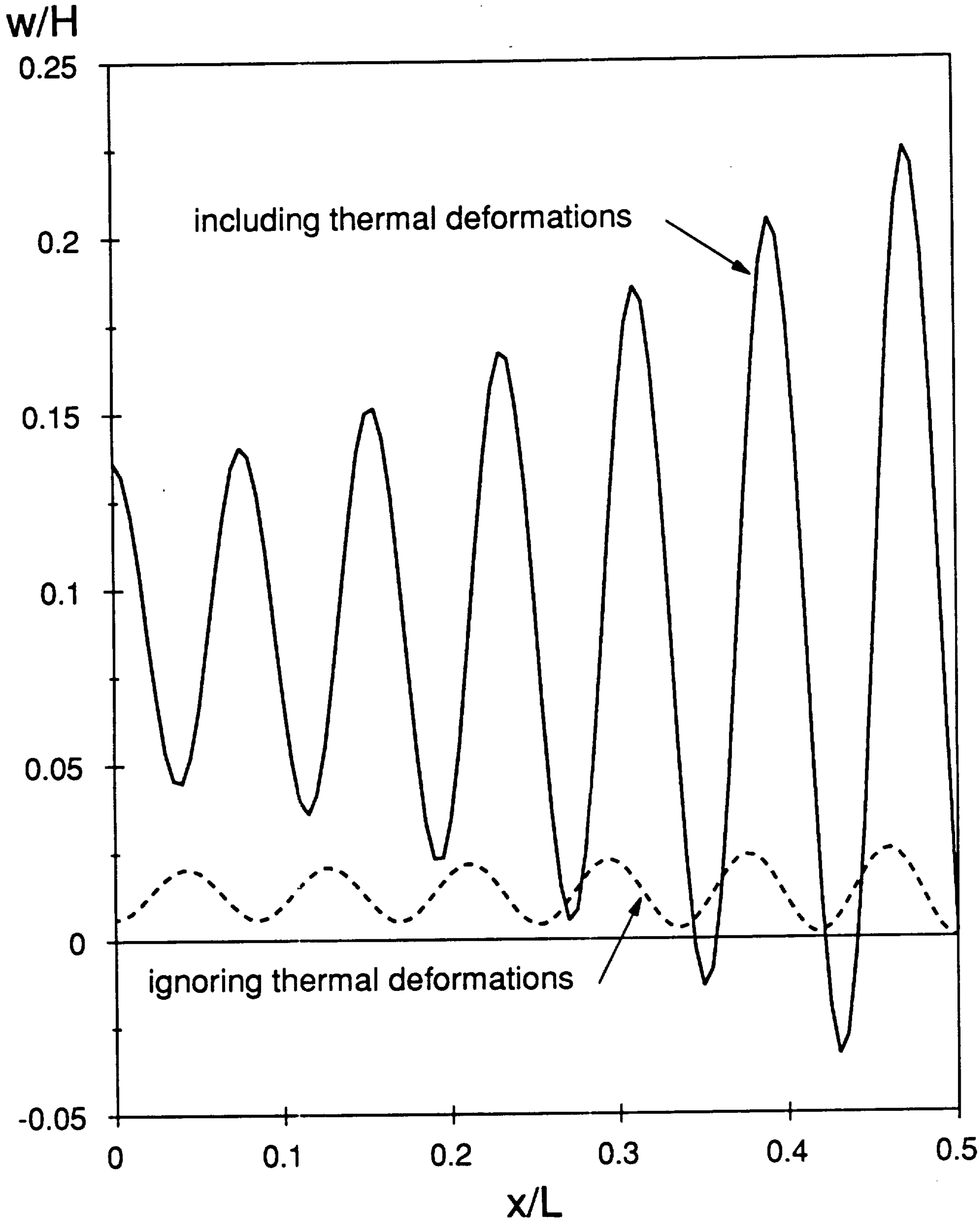

Fig. 29 - Radial Deformations of the Clamped $(90_s/0_s)_T$ Cylinder, N = 1.0N*.

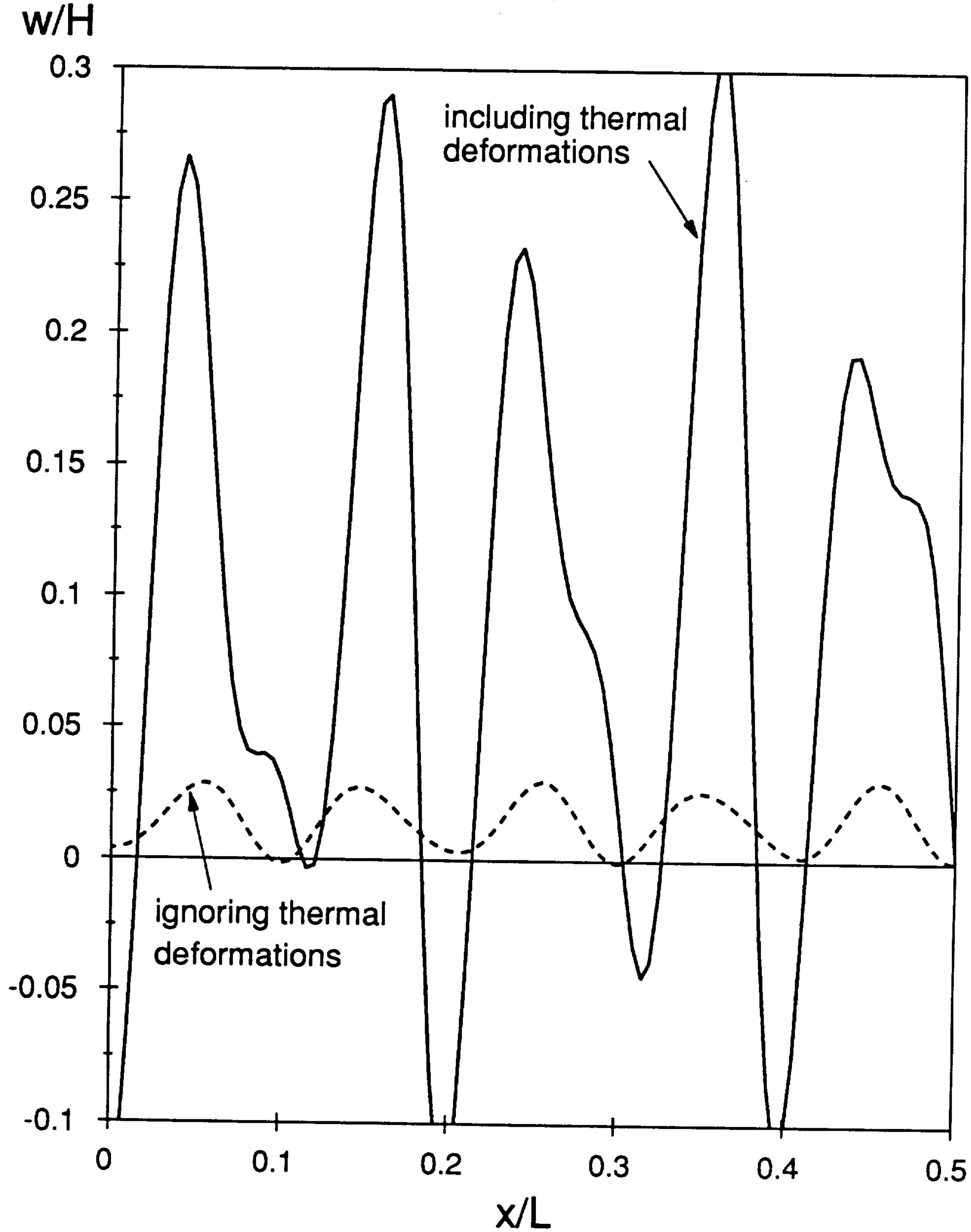

Fig. 30 - Radial Deformations of the Clamped $(90_s/0_s)_T$ Cylinder, N = 1.1N*.

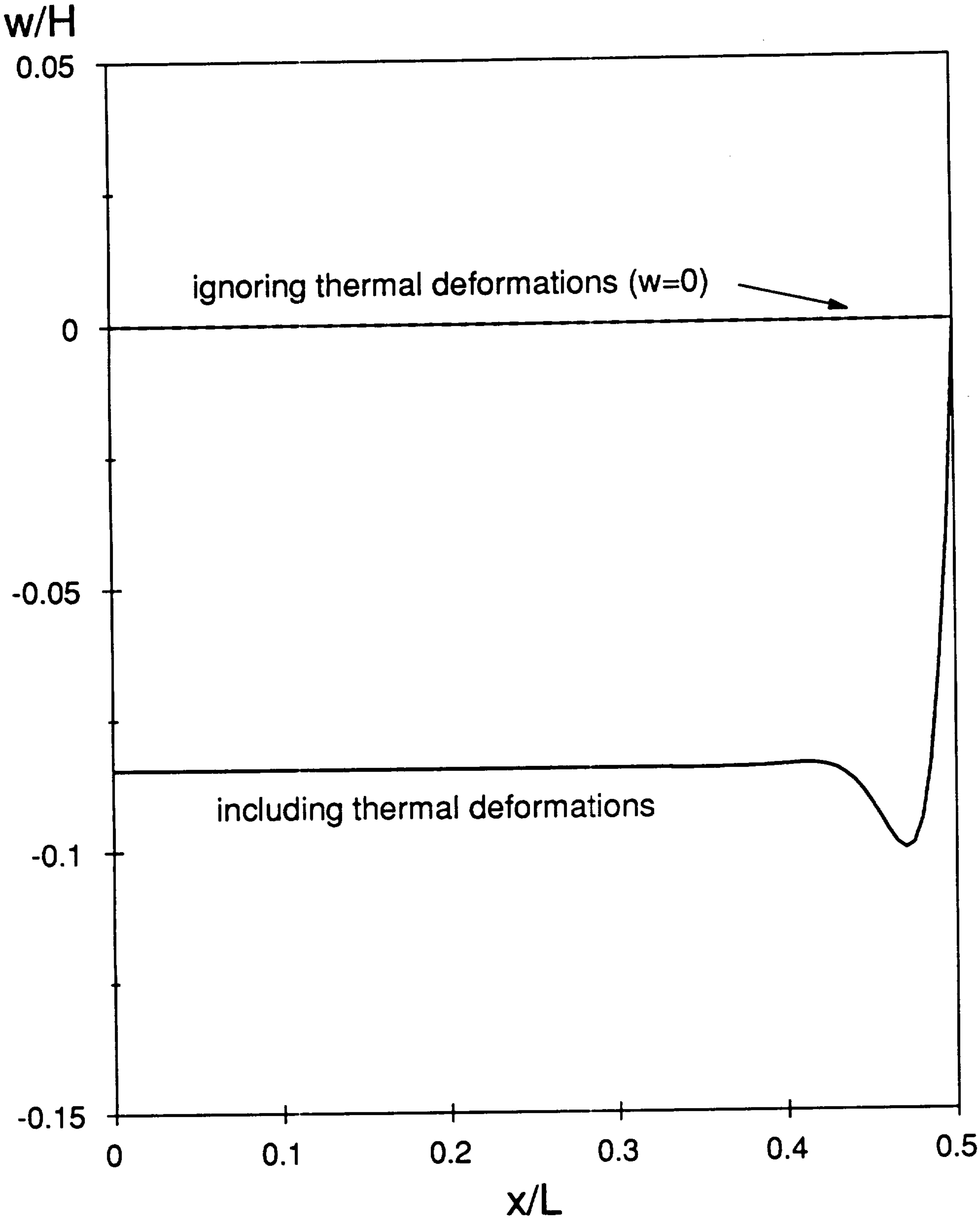

Fig. 31 - Radial Deformations of the Clamped $(0_s/90_s)_T$ Cylinder, N = 0.

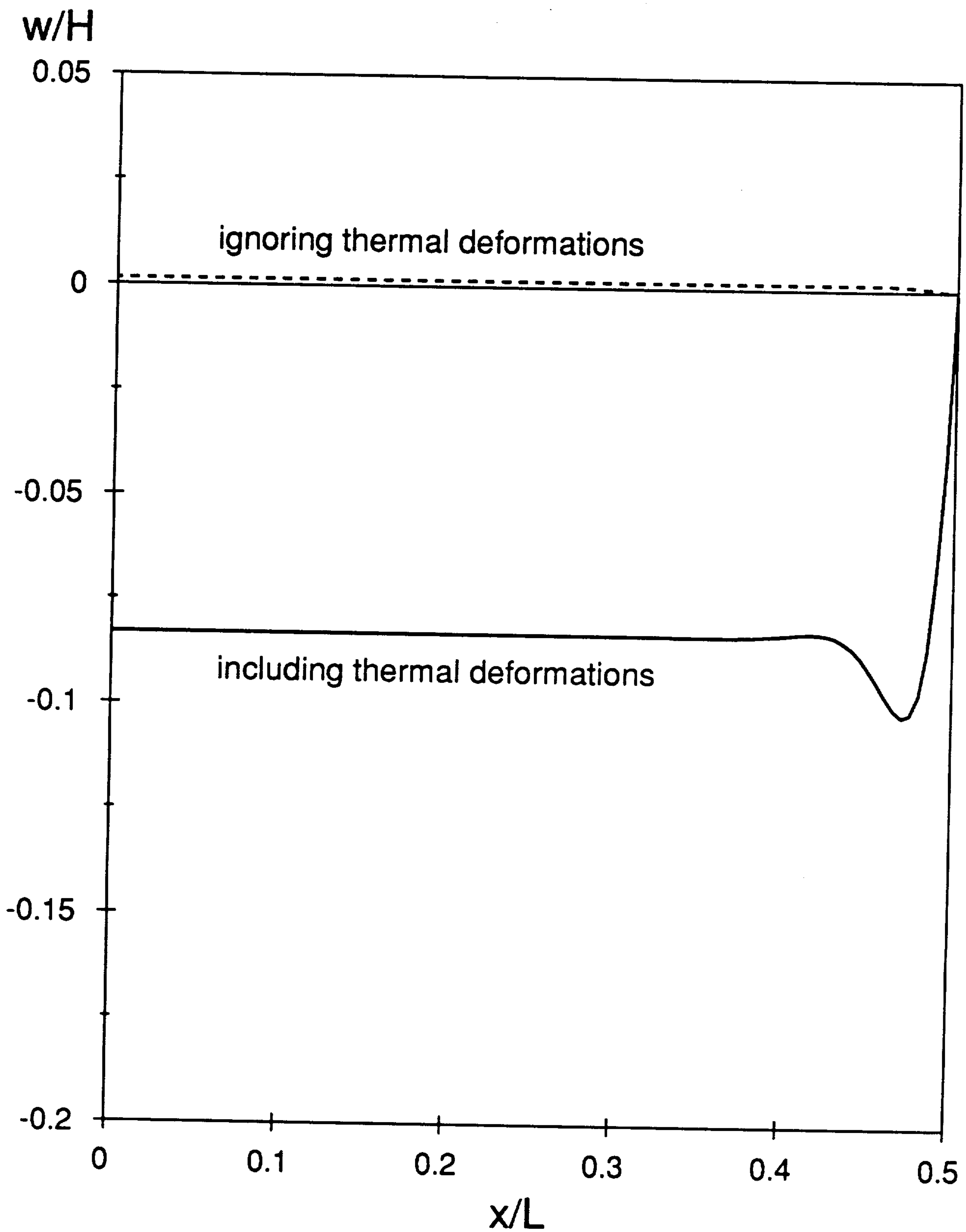

Fig. 32 - Radial Deformations of the Clamped $(0_8/90_8)_T$ Cylinder, N = 0.1N*.

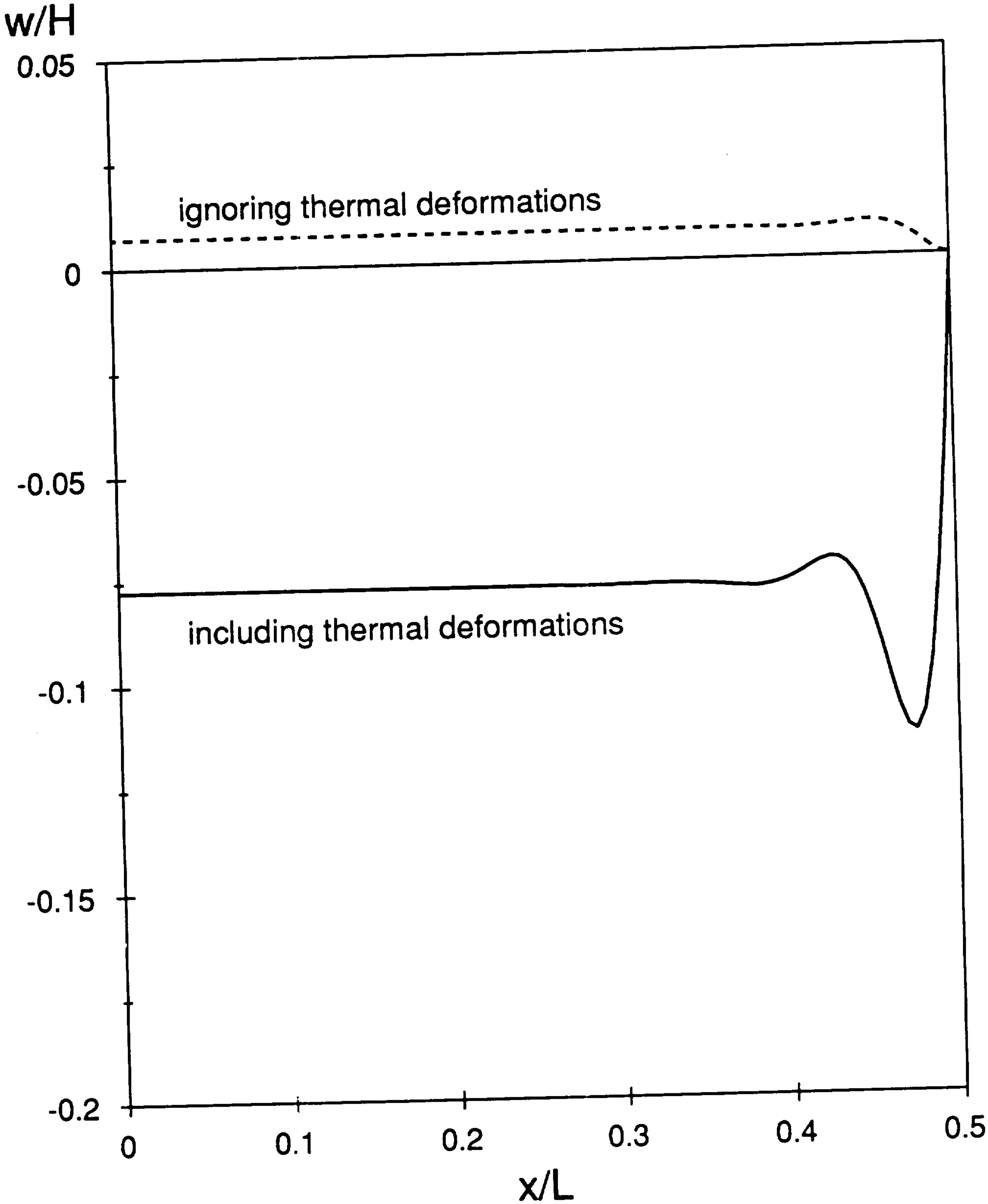

Fig. 33 - Radial Deformations of the Clamped $(0_s/90_s)_T$ Cylinder, N = 0.5N*.

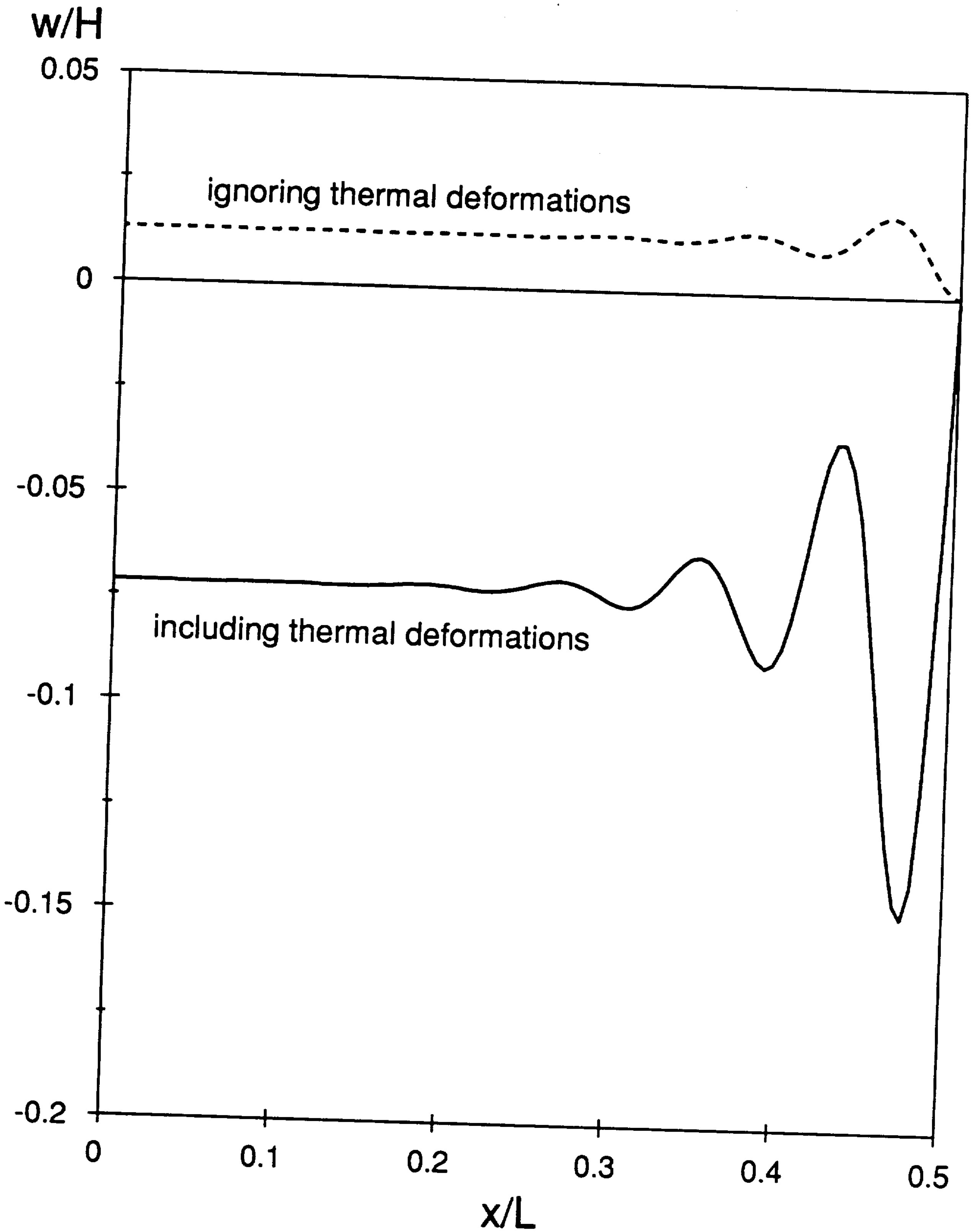

Fig. 34 - Radial Deformations of the Clamped $(0_s/90_s)_T$ Cylinder, N = 0.9N*.

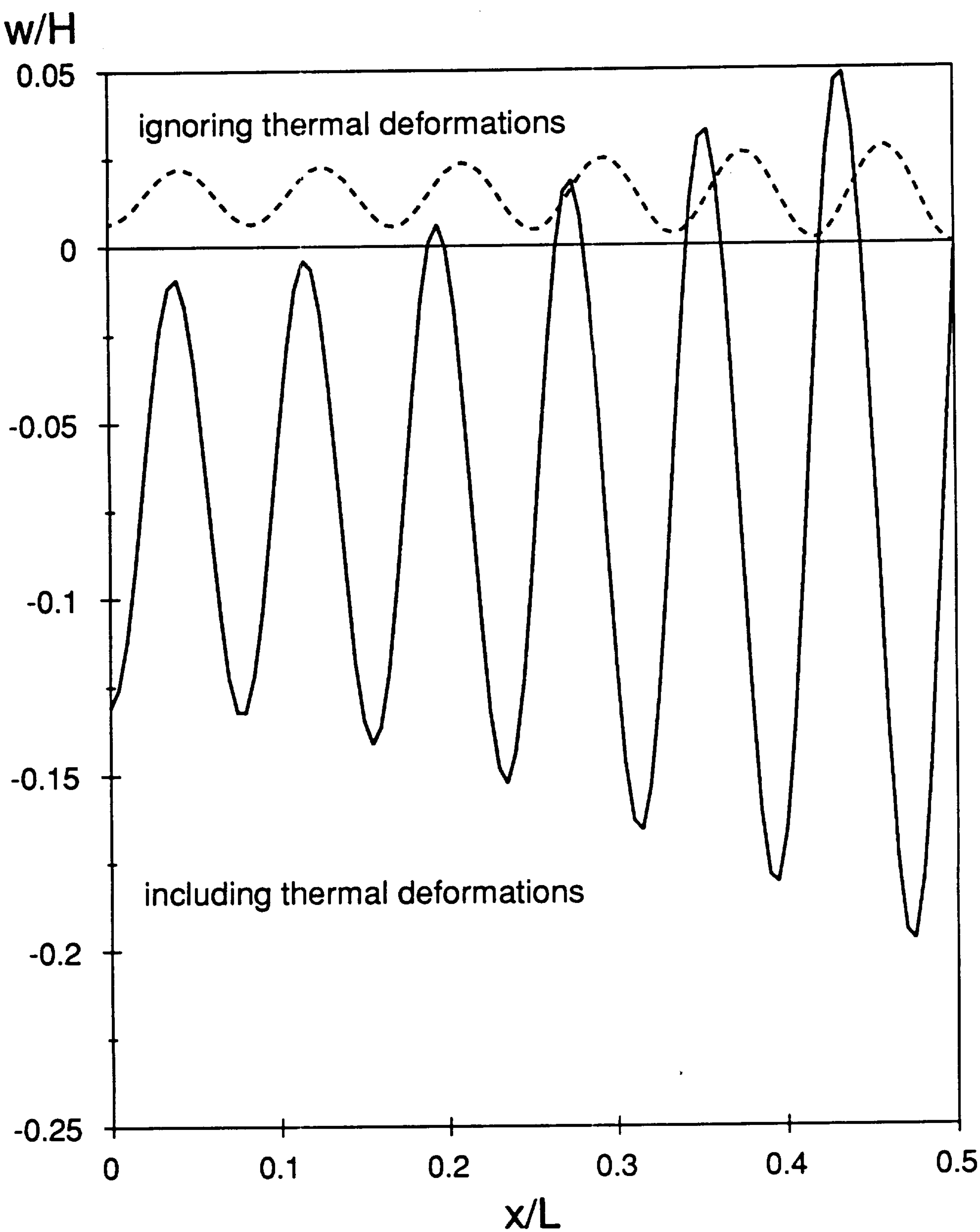

Fig. 35 - Radial Deformations of the Clamped $(0_s/90_s)_T$ Cylinder, N = 1.0N*.

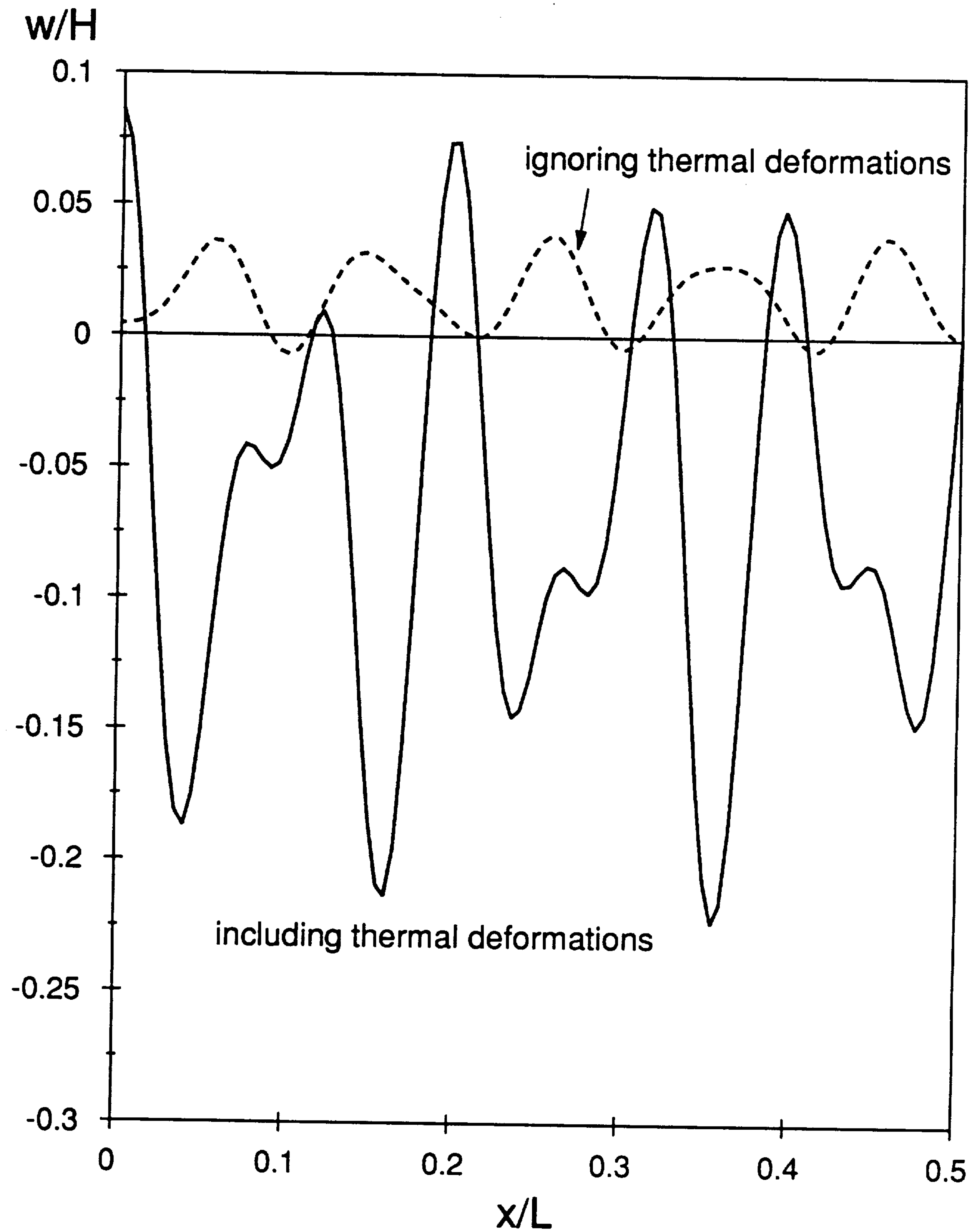

Fig. 36 - Radial Deformations of the Clamped $(0_8/90_8)_T$ Cylinder, N = 1.1N*.

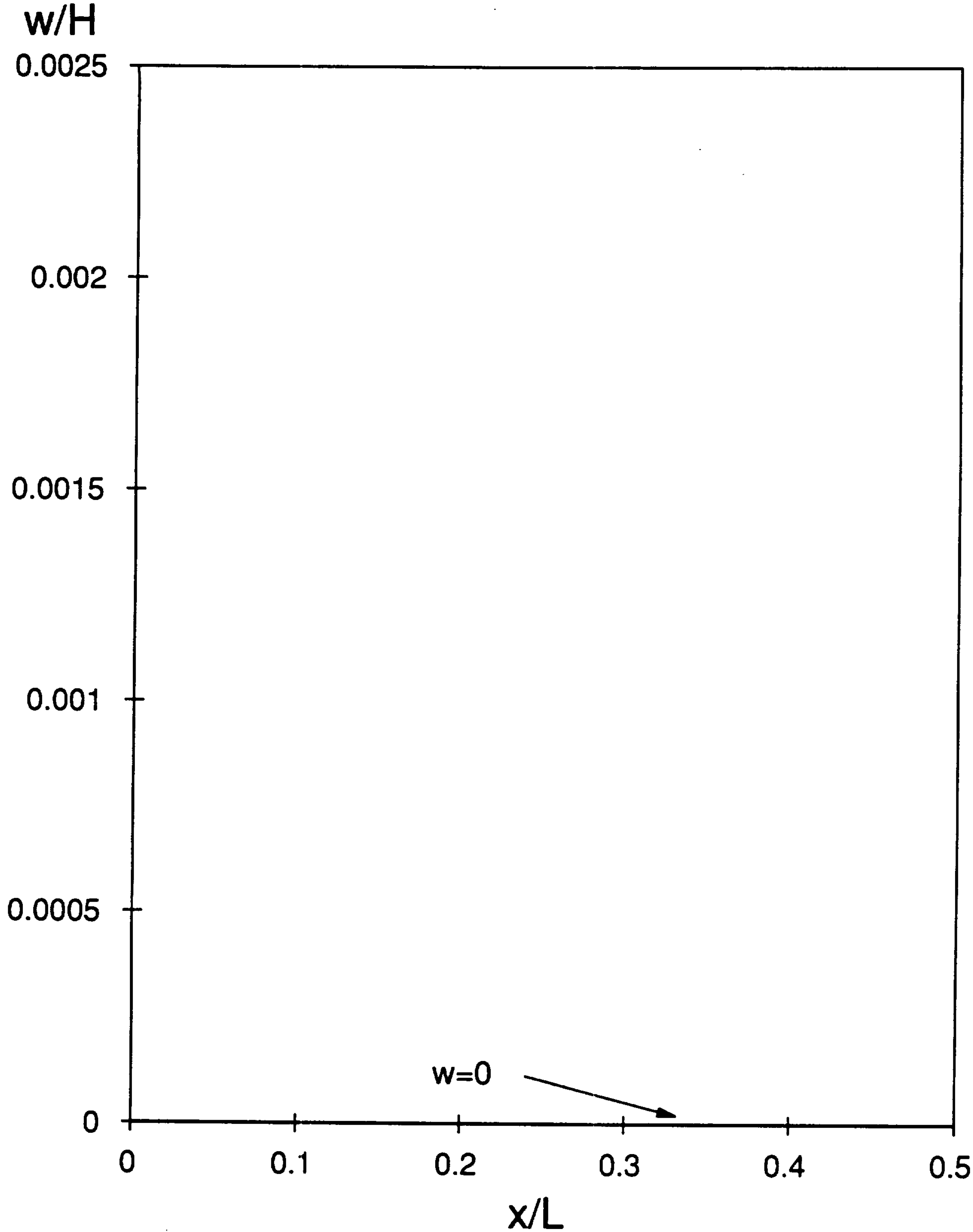

Fig. 37 - Radial Deformations of the Clamped $(0/90)_{4s}$ Cylinder, N = 0.

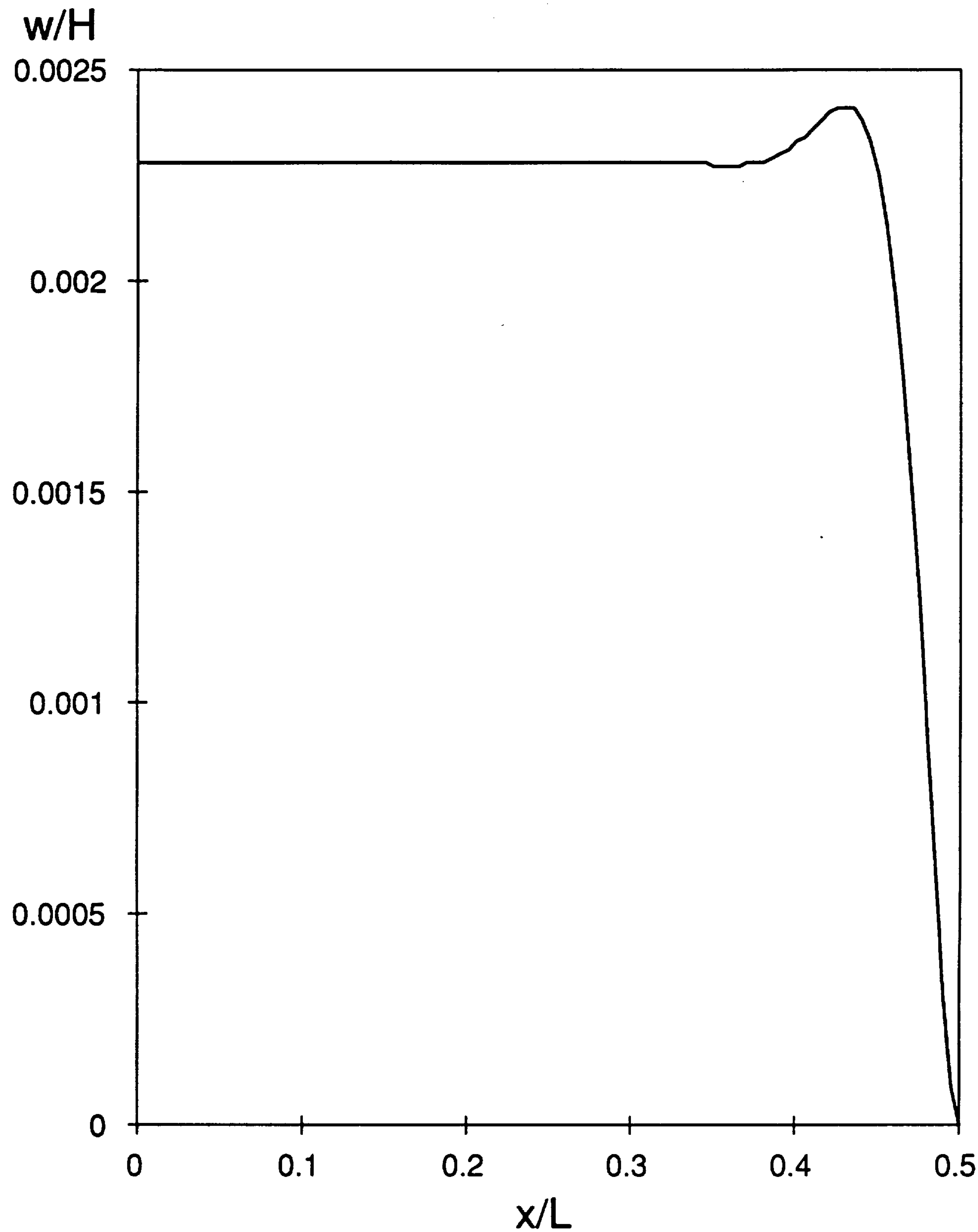

Fig. 38 - Radial Deformations of the Clamped $(0/90)_{4S}$ Cylinder, N = 0.1N*.

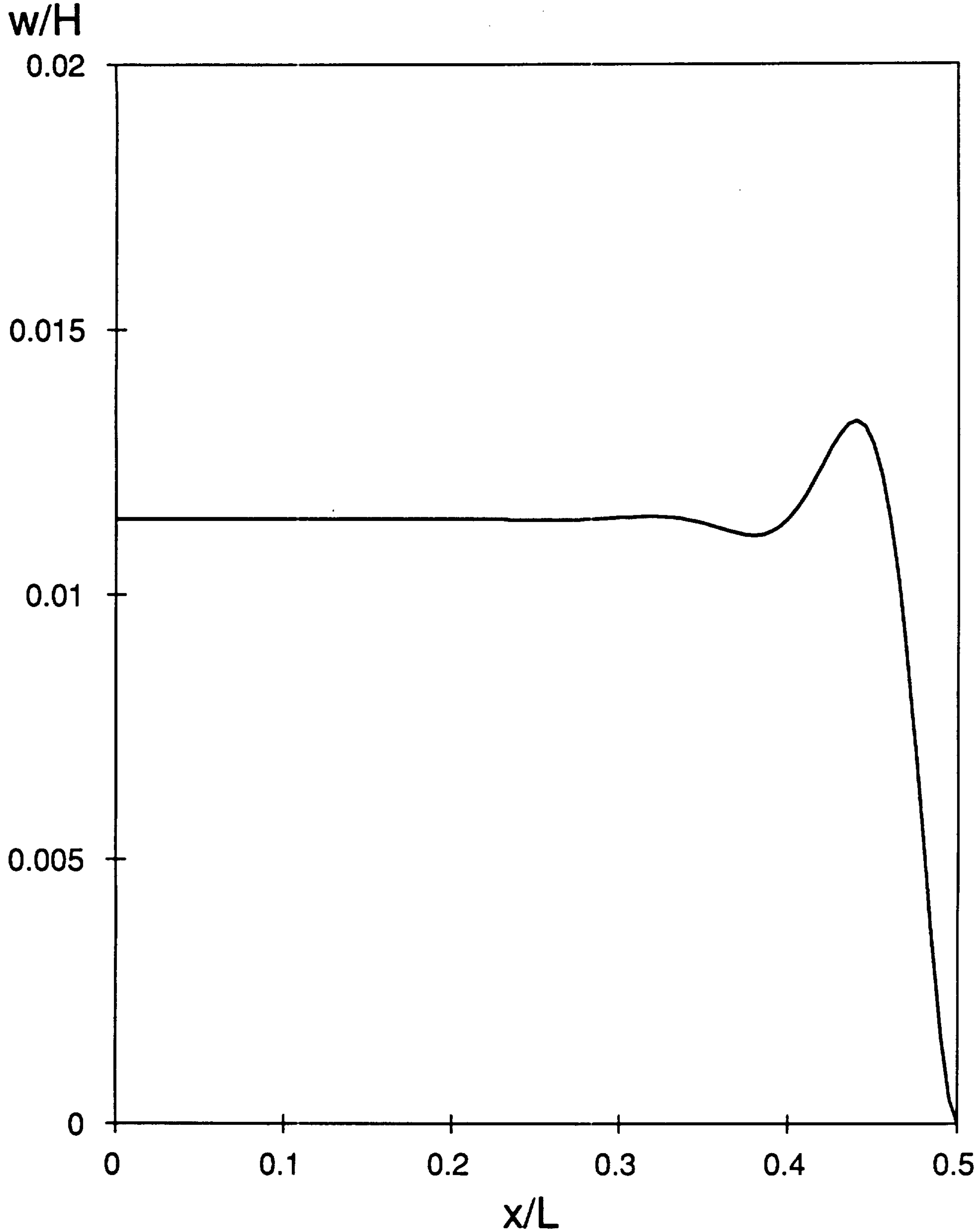

Fig. 39 - Radial Deformations of the Clamped $(0/90)_{4s}$ Cylinder, $N = 0.5N^*$.

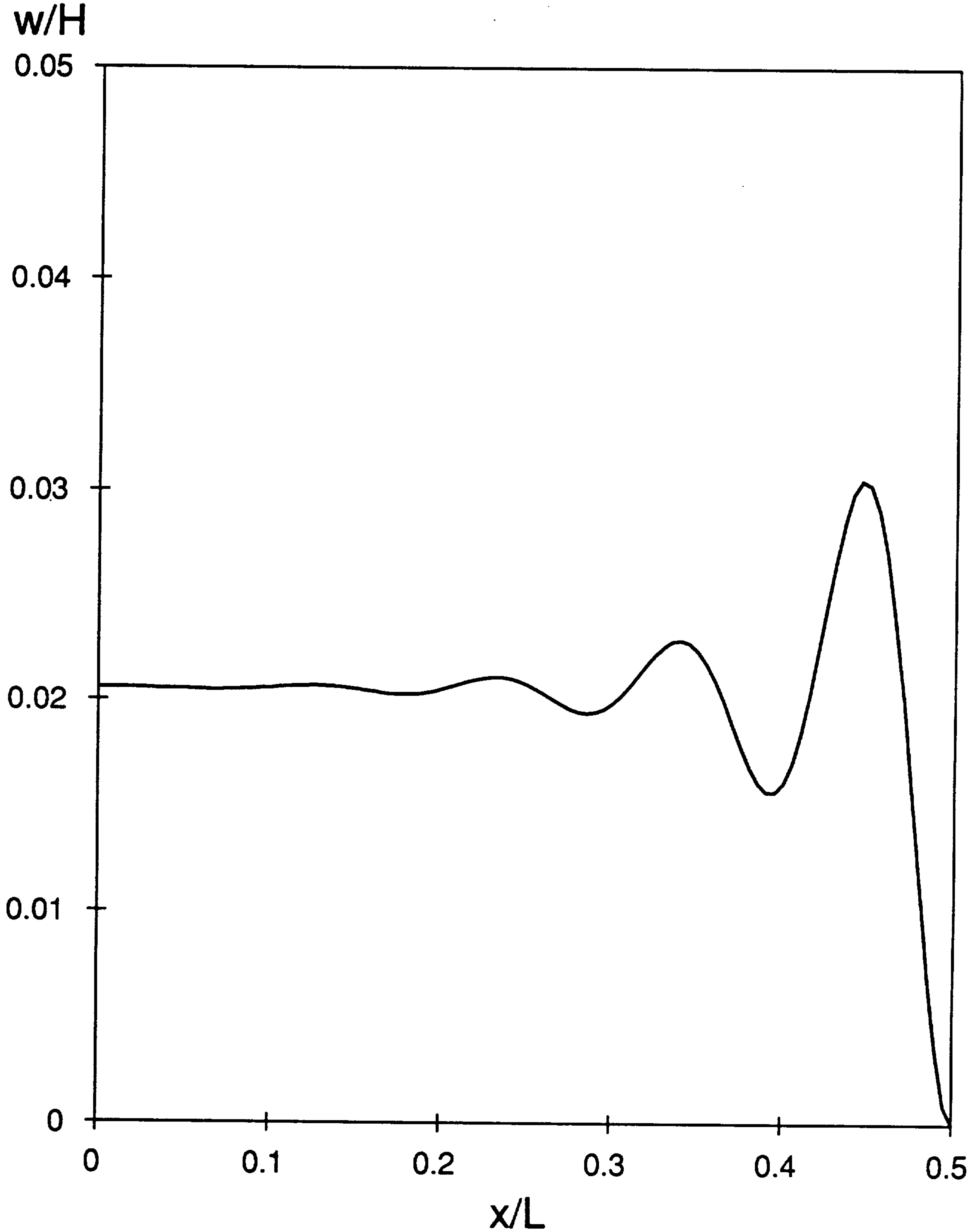

Fig. 40 - Radial Deformations of the Clamped $(0/90)_{4s}$ Cylinder, N = 0.9N*.

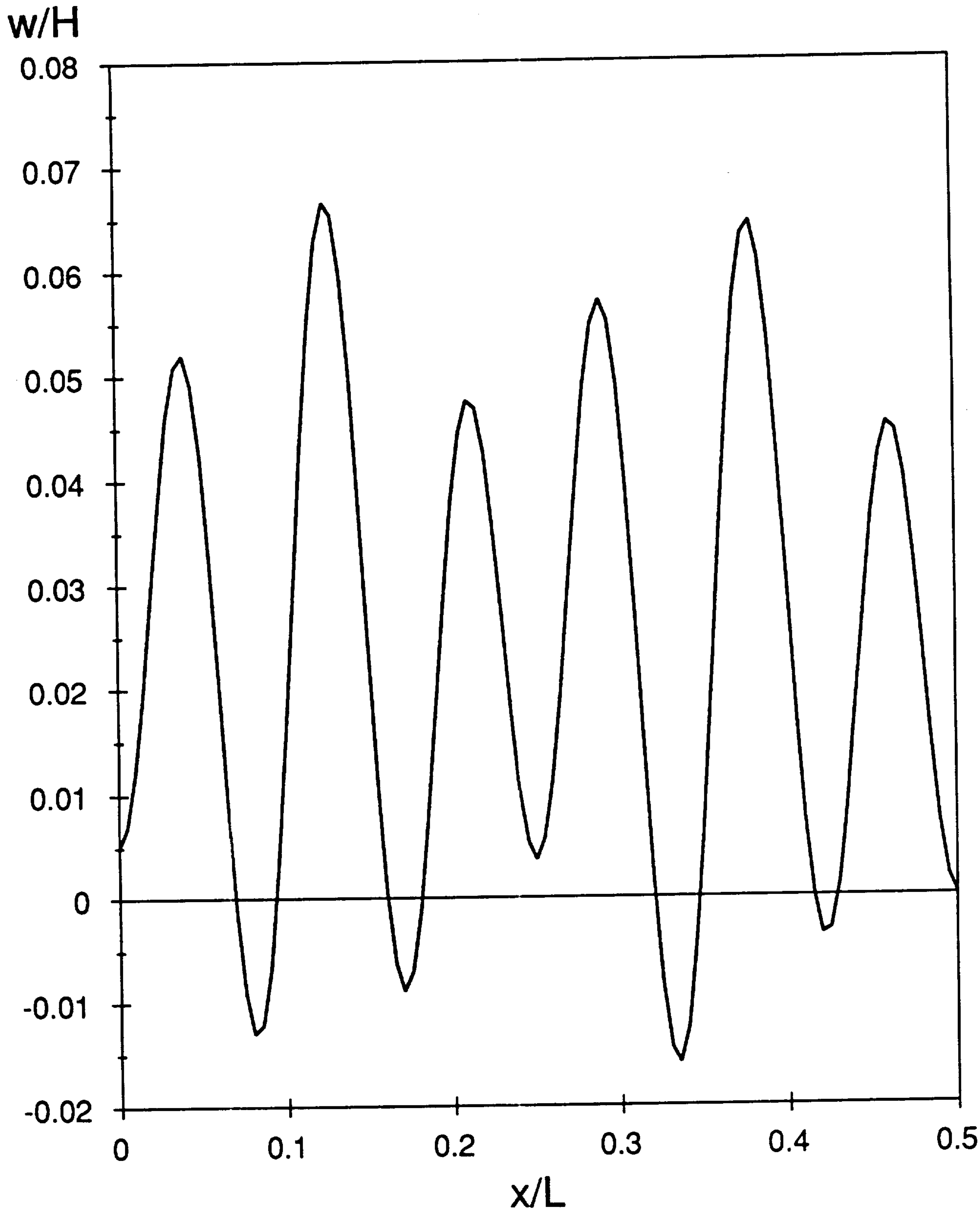

Fig. 42 - Radial Deformations of the Clamped $(0/90)_{4s}$ Cylinder, N = 1.1N*.

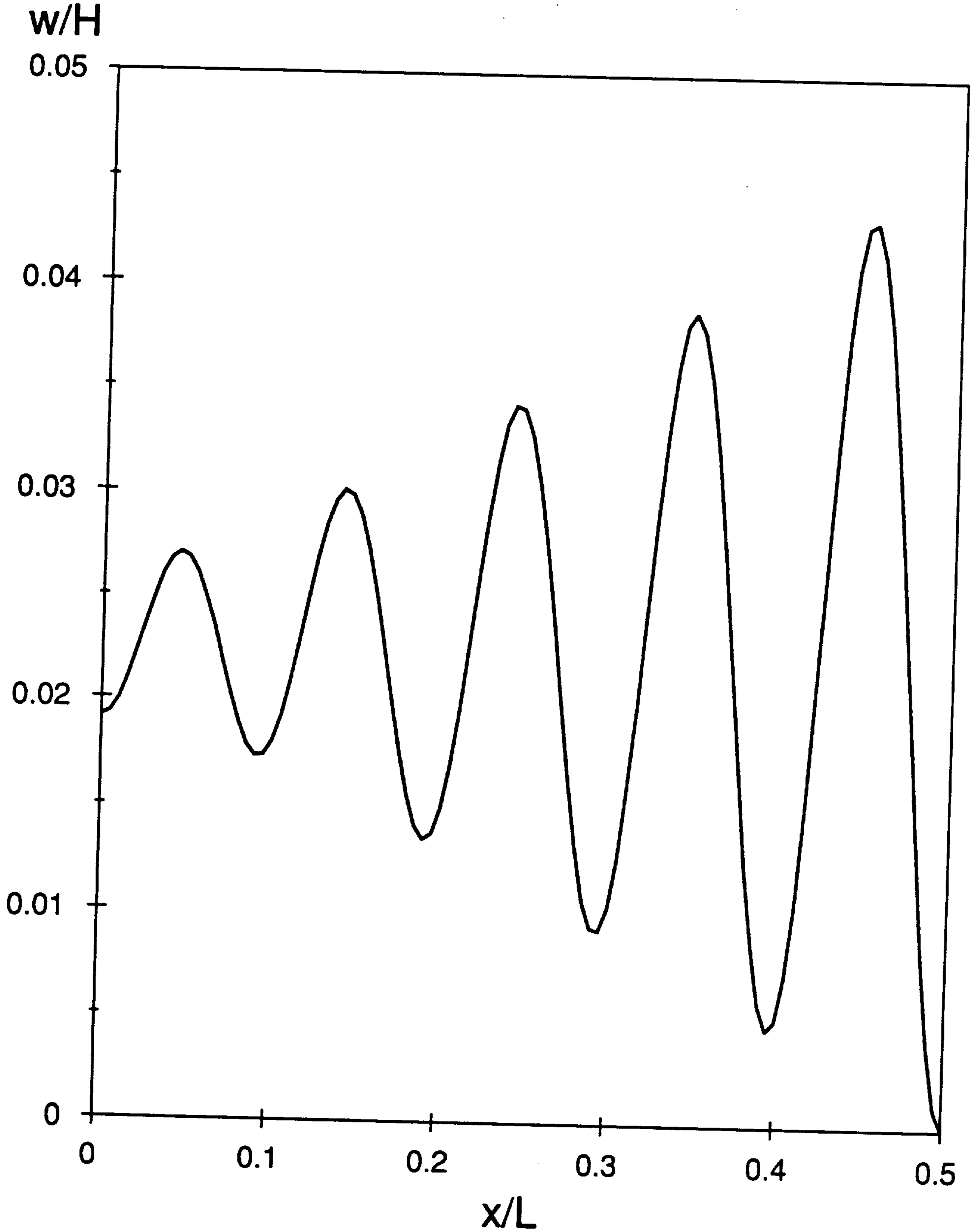

Fig. 41 - Radial Deformations of the Clamped $(0/90)_{4s}$ Cylinder, N = 1.0N*.

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16. Abstracts

The study focuses on the axisymmetric deformation response of unsymmetrically laminate cylinders loaded in axial compression by known loads. A geometrically nonlinear analysis is used. Though buckling is not studied, the deformations can be considered to be the prebuckling response. Attention is directed at three 16 layer laminates: a $(90_8/0_8)_T$; a $(0_8/90_8)_T$; and a $(0/90)_{4s}$. The symmetric laminate is used as a basis for comparison, while the two unsymmetric laminates were chosen because they have equal but opposite bending-stretching effects. Particular attention is given to the influence of the thermally-induced preloading deformations that accompany the cool-down of any unsymmetric laminate from the consolidation temperature. Simple support and clamped boundary conditions are considered. It is concluded that: (1) The radial deformations of an unsymmetric laminate are significantly larger than the radial deformations of a symmetric laminate. For both symmetric and unsymmetric laminates the large deformations are confined to a boundary layer near the ends of the cylinder; (2) For this nonlinear problem the length of the boundary layer is a function of the applied load; (3) The sign of the radial deformations near the supported end of the cylinder depends strongly on the sense (sign) of the laminate asymmetry; (4) For unsymmetric laminates, ignoring the thermally-induced preloading deformations that accompany cool-down results in load-induced deformations that are under predicted; and (5) The support conditions strongly influence the response but the influence of the sense of asymmetry and the influence of the thermally-induced preloading deformations are independent of the support conditions.

17. Key Words and Document Analysis. 17a. Descriptors

unsymmetric laminate, composite cylinders, nonlinear response

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

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