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Galerkin Finite Difference Laplacian Operators on Isolated Unstructured Triangular Meshes by Linear Combinations

Kenneth J. Baumeister
Lewis Research Center
Cleveland, Ohio

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Kenneth J. Baumeister
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

ABSTRACT

The Galerkin weighted residual technique using linear triangular weight functions is employed to develop finite difference formulae in Cartesian coordinates for the Laplacian operator on isolated unstructured triangular grids. The weighted residual coefficients associated with the weak formulation of the Laplacian operator along with linear combinations of the residual equations are used to develop the algorithm. The algorithm was tested for a wide variety of unstructured meshes and found to give satisfactory results.

NOMENCLATURE

A	area
A_e	element area, see Fig. 1
A_T	total cell area, see Fig. 1
A_V	Voronoi (finite difference) area
B	bias constant, Eq. (13)
f	linear combination factor, Eq. (22)
h	coordinate spacing distance
M	number of elements in cell
N	linear interpolation function
W	weight function, see Eq. (2)
x	axial distance coordinate
y	transverse distance coordinate
α	weighted residual coefficient, Eq. (2)
β	property constant, Eq. (1)
ϕ	dependent variable, Eq. (1)

Subscripts

$e+$	index, Eq. (8)
fd	finite difference or Taylor series
j	element index, Eq. (10) and Fig. 1
k	element index, Eq. (10) and Fig. 1
M	total number of elements
n	global node index, Eqs. (6) and (7), Fig. 1
nm	index, Eq. (9)
o	central node index
P	number of outer nodal points, see Fig. 1
α	index, Eq. (11)
μ	index, Eq. (5)

Superscripts

$\sim$	numerical estimate
$-$	average value, Eq. (B3)
(e)	element value
n	global element index, Eq. (3)
nm	global element index, Eqs. (3) and (8)

INTRODUCTION

Linear triangular finite element theories (Baumeister and Horowitz, 1984) have been successfully applied to the Fourier transformed linearized gas dynamics equations ("steady" state equations) in the study of noise propagation in variable area ducts and

the acoustic design of turbofan engine nacelles. Similarly, transient solutions to the acoustic wave equation have used conventional finite difference operators and elliptic mapping methods (Raad and White, 1986) to successful model noise propagation in ducts with variable geometry. Advantageously, the transient solutions eliminate the storage of the very large global matrix associated with the "steady" state solution of wave propagation problems. However, at the present time the transient technique employing elliptic mapping has been unsuccessful in predicting the acoustic far field radiation pattern from aircraft inlets. To overcome this problem, the present study is concerned with adapting previously developed "steady" state triangular finite element mesh generators (Baumeister and Horowitz, 1984) for use in transient variable geometry acoustics or electromagnetics in the form of unstructured finite difference theory.

In developing unstructured finite difference equations, Jameson (1987) (Eq. (4.10)) has applied the standard weighted residual Galerkin method to obtain a time dependent discretization of the Euler equations for an unstructured triangular mesh. Erlebacher (1985) has utilized variational methods on a pointwise basis to establish a difference algorithm for the Laplacian operator for a central difference cell similar to that shown in Fig. 1. Unfortunately, from a pointwise perspective, Erlebacher has shown that the difference algorithms that result from a variational approach are only zero order accurate (Erlebacher, 1985, p. 42) for the Laplacian operator on nonuniform unstructured grids as well as many uniform grids. Baumeister (1988) obtained similar results using the Galerkin weighted residual techniques. The Galerkin algorithms in Baumeister (1988) for the Laplacian operator were considered valid for a general unstructured grid when used only in conjunction with a complete global grid system, as commonly employed in conventional finite element analysis. Baumeister (1988) also showed that finite element and finite difference (Taylor series) algorithms are related by an area rule and that the finite element algorithm will agree with a Taylor series approach on a global average.

At present, the difference expression for the Laplacian operator based on variational theory is valid on a local grid only under restricted conditions, namely when the area rule is satisfied (Baumeister, 1988). To generate finite difference approximations for the Laplacian operator which are valid for local single cells such as shown in Fig. 1, the present paper will utilize the global Galerkin weighted residual technique in conjunction with linear combinations of the weighted residual equations.

GALERKIN WEIGHTED RESIDUAL FORMULATION

The Laplacian operator plays a major role in the study of acoustic and electromagnetic wave propagation. Consider the following expression for the Laplacian operator

$$\nabla \cdot \beta \nabla \phi \tag{1}$$

where ϕ is a scalar or potential quantity and β is a variable property coefficient that depends on its spatial orientation. To obtain the finite difference expressions for this Laplacian operator at some point labeled o, a finite difference cell is defined by connecting the P nodal (grid) points surrounding the central grid point o as shown in Fig. 1. This difference cell of total area A_T is then divided into discrete triangular areas A_e staked out by nodal (grid) points P as shown in Fig. 1. The number of

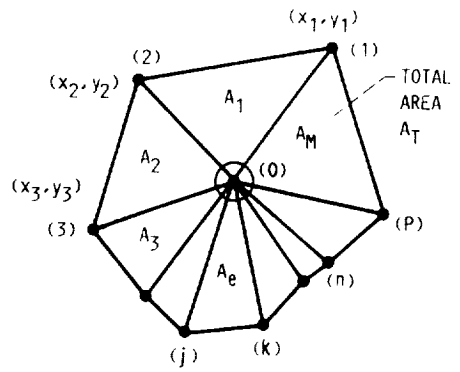


FIGURE 1. - FINITE DIFFERENCE CELL CONFIGURATION.

connected areas (called an element of the cell) needed to define the difference equation for node o is labeled M.

Using the weak formulation of the method of weighted residuals (Lapidus and Pinder, 1982, p. 443), the Laplacian operator in Eq. (1) can be integrated to obtain (see Appendix A for brief derivation)

$$\int\int_T \left[-\nabla W_0 \cdot \beta \nabla \tilde{\phi}\right] dA = \alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \ldots + \alpha_n\phi_n + \ldots + \alpha_p\phi_p \tag{2}$$

Using the Galerkin approach, the spatial weight W is approximated by the standard linear interpolation functions and the hat over the potential ϕ indicates that it is the approximate numerical solution for the potential. ϕ_n represents the potential at any general nodal point n. The value of the general ϕ_n coefficients depends only on the location of the grid points (x_n, y_n) and the property coefficient β and are of the form

$$\alpha_n = -\left[\beta^{(nm)}/4A^{(nm)}\right]\left[(y_{n-} - y_n)(y_o - y_{n-}) + (x_n - x_{n-})(x_{n-} - x_o)\right] - \left[\beta^{(n)}/4A^{(n)}\right]\left[(y_n - y_{n+}) \times (y_{n+} - y_o) + (x_{n+} - x_n)(x_o - x_{n+})\right] \tag{3}$$

$$\alpha_o = -\sum_{e=1}^{M}\left[\beta^{(e)}/4A^{(e)}\right]\left[(y_e - y_{e+})^2 + (x_{e+} - x_e)^2\right] \tag{4}$$

The indices are defined as

$$\delta_{\alpha,\mu} = 0 \quad \alpha \neq \mu \quad \text{or} \quad \delta_{\alpha,\mu} = 1 \quad \alpha = \mu \tag{5}$$

$$n- = n - 1 + P\delta_{n,1} \tag{6}$$

$$n+ = n + 1 + P\delta_{n,P} \tag{7}$$

$$e+ = e + 1 + P\delta_{e,P} \tag{8}$$

$$nm = n - 1 + M\delta_{n,1} \tag{9}$$

with element areas given by Lapidus, and Pinder, 1982, p. 110 (Eq. (3.1.1))

$$A^{(e)} = \left(\frac{1}{2}\right)\left[x_o(y_j - y_k) + x_j(y_k - y_0) + x_k(y_0 - y_j)\right] \tag{10}$$

LAPLACIAN OPERATOR

The present paper will establish a relationship between the Laplacian operator and the α coefficients from the method of weighted residuals as given by Eqs. (3) and (4). An expression for the Laplacian of the form

$$\nabla^2\phi = f(\alpha_1, \alpha_2, \ldots) \tag{11}$$

is desired. Using the result from Appendix B for a linear shape function, the Laplacian can be expressed as

$$\nabla^2\phi = \frac{\alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \ldots \alpha_p\phi_p}{\frac{A_T}{3}} \tag{12}$$

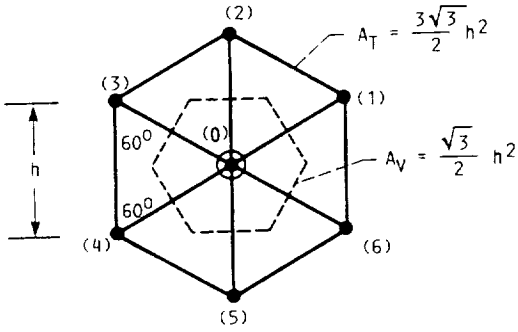
where A_T is the total area of the cell.

For the six point hexagon difference grid systems as shown in Figs. 2 and 3, the difference equations for the Laplacian as calculated from Eq. (12) are in agreement with the standard Taylor series finite difference operators that appear in the literature (Rektorys, 1969, p. 1114). However, this will not generally be the case, as discussed earlier in relation to Erlbacher and Baumeister's works and also discussed in Lapidus and Pinder (1982), p. 105. For uniform grids, of the type displayed in Figs. 4 to 5, the finite difference and finite element expressions for the Laplacian operator will differ with the conventional Taylor series difference equation. Only when the bias constant B defined by the following area rule is identical to zero will the finite element algorithm and Taylor series approximations be equal.

$$B = \frac{3A_V}{A_T} - 1 \tag{13}$$

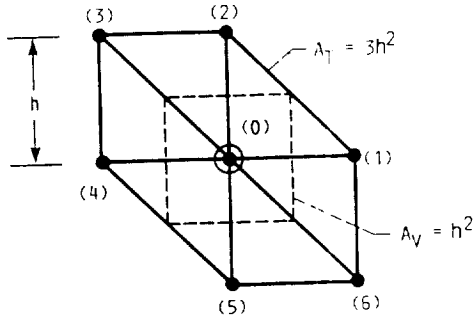
A_V is the Voronoi neighborhood (Erlbacher, 1985) or the area normally employed in the finite difference analysis and is the area enclosed by the dashed line in Figs. 2 to 5. Baumeister (1988) presents a simple algorithm to determine the Voronoi (finite difference) area for an unstructured grid based on the α coefficients in Eqs. (3) and (4).

If B is not zero in Eq. (13), the coefficients in front of the weighted residual expression for the



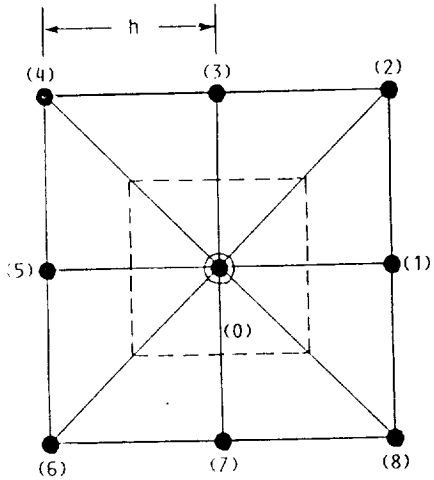
$$\nabla^2\Phi = \frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 - 6\Phi_0}{\frac{3}{2} h2}$$

FIGURE 2. - UNIFORM 6 NODE HEXAGON GRID.

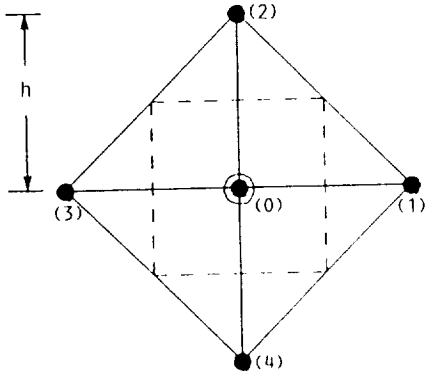


$$\nabla^2\Phi = \frac{\Phi_1 + \Phi_2 + \Phi_4 + \Phi_5 - 4\Phi_0}{h^2}$$

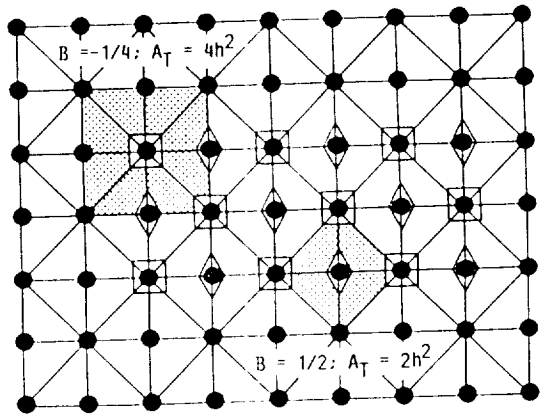
FIGURE 3. - UNIFORM 6 NODE SLANTED HEXAGON GRID.



$$\nabla^2\Phi = \frac{3}{4}\left[\frac{\Phi_1 + \Phi_3 + \Phi_5 + \Phi_7 - 4\Phi_0}{h^2}\right]$$

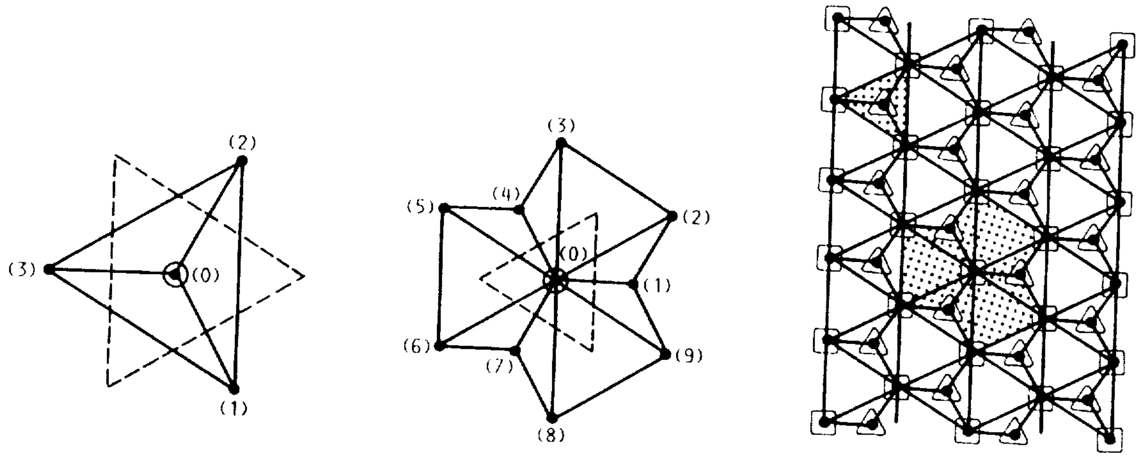


$$\nabla^2\Phi = \frac{3}{2}\left[\frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_0}{h^2}\right]$$



$$B_{GLOBAL} = \frac{1}{2} \times 2h^2 + (-1/4)\, 4h^2 = 0$$

FIGURE 4. - GLOBAL CONVERGENCE EXAMPLE WITH RECTANGULAR ELEMENTS.



$$\nabla^2\Phi = 3\left[\frac{\Phi_1 + \Phi_2 + \Phi_3 - 3\Phi_0}{3/4h^2}\right]$$

$B = 2 \quad A_T = 1 \quad A_V = 1$

$$\nabla^2\Phi = 0.6\left[\frac{\Phi_1 + \Phi_4 + \Phi_7 - 3\Phi_0}{3/4h^2}\right]$$

$B = -0.4 \quad A_T = 5 \quad A_V = 1$

$B_{GLOBAL} = 2 \times 1 + (-0.4) \times 5 = 0$

FIGURE 5. - GLOBAL CONVERGENCE WITH TRIANGULAR GRIDS.

Laplacian operator will differ from the expression for the Taylor series expression by a factor of 1 plus B. If B is zero, a single pattern of uniform elements can normally be placed in a regular domain. For the uniform grid systems in Figs. 2 to 5, the area rule leads to the following simple relationship between the weighted residual expression for the Laplacian operator (Eq. (12)) and the standard Taylor series (finite difference) representation

$$\nabla^2\phi = \frac{3A_V}{A_T}\,\nabla^2\phi\big|_{fd} \tag{14}$$

For the conventional four point square difference grid, as shown in the upper right of Fig. 4, the difference equation for the Laplacian is different by a factor of 3/2 from the conventional Taylor series representation of the Laplacian operator (Rektorys, 1969, p. 1114), as predicted by Eq. (14). Similarly, the eight-node system shown in the upper right in Fig. 4 differs by a factor of 3/4. However, since the method of weighted residual (as applied in finite element theory) is guaranteed to converge, these are acceptable formulae provided they are applied to a consistent global mesh and not a single arbitrary cell. The purpose of the present paper is to adapt Eq. (12) so that it can be applied on a local grid independent of the surrounding grid.

Although the weighted residual difference equation for the Laplacian operator in Fig. 4 differ in each cell from the standard Taylor series difference equation, from a global point of view the Taylor series and weighted residual equations will average out to the same value. Since B is not zero, the global domain will be filled with different element types, as shown in the combined mesh of Fig. 4. In the case shown in Fig. 4, the area average of the eight-node element combined with the four-node element will be such that the Taylor series will be valid. As shown in Fig. 4, the average global value of B is equal to zero. (B is a measure of the difference between the finite element and Taylor Series as just defined.) A similar situation occurs for triangular grids as shown in Fig. 5.

Figure 6 displays the difference equations developed from Eq. (12) for a nonuniform six-node cell. As discussed previously, the Laplacian operator shown in Fig. 6 is valid when taken in conjunction with the other grid systems that surround it in a given finite element type domain. However, since the bias of -0.1367 is different from zero, the difference equation displayed in Fig. 6 will not be valid for a local isolated cell. For nonuniform grids, the following test functions will be used to check the validity of the difference formula on a local unstructured nonuniform grid.

$$1 \quad x \quad y \quad xy \quad x^2 \quad y^2 \tag{15}$$

The difference equation in Fig. 6 satisfies the 1, x, and y test functions; however, it does not satisfy the xy, x^2, and the y^2 test functions. It does however satisfy an $x^2 + y^2$ test function which leads to a simple formula for the Voronoi area in terms of the α coefficients (Baumeister, 1988, Eq. (10)).

Equation (12) will now be adapted to produce difference equations valid on a local grid. That is, a difference equation which satisfies the six constraints given in Eq. (15).

MODIFIED LAPLACIAN OPERATORS

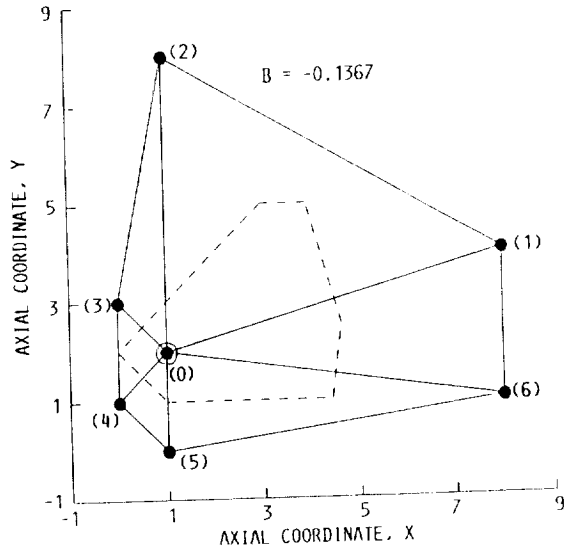
The method of linear combinations will be used in the next section of this paper to adapt the method of residuals to obtain local convergence for uniform grids and unstructured grids of general shape. For convenience, a modified version of Eq. (12) will be used in the combination process.

As shown by the area rule, if the bias constant is equal to zero (B = 0), then Eq. (13) becomes

$$\frac{A_T}{3} = A_V \tag{16}$$

Equation (16) combined with Eq. (12) yields

$$\nabla^2\phi = \frac{\alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots \alpha_p\phi_p}{A_V} \qquad (B = 0) \tag{17}$$



$$\nabla^2\Phi = 0.864\left[\frac{2.307\Phi_1 + \Phi_2 + 19.385\Phi_3 + 6.461\Phi_4 + 11.077\Phi_5 + 1.384\Phi_6 - 41.615\Phi_0}{80.884}\right]$$

FIGURE 6. - NONUNIFORM 6 NODE GRID SYSTEM.

where A_V is the Voronoi area of the cell and the α values are given by Eqs. (3) and (4). Surprisingly, Eq. (17) was checked against a number of uniform grids such as shown in Figs. 4 and 5 where B was not equal to zero and was found to be identical to the Taylor series representation which satisfies the conditions given by Eq. (15).

In the linear combination process to follow, arbitrary constants will be assigned to the Laplacian operators. Therefore, either Eq. (12) or (17) could be used in the weighting process for unstructured grids since they only differ by a constant for a given difference cell. However, since Eq. (17) fortuitously seems to work for all uniform grids system, Eq. (17) will be employed in the superposition process.

LINEAR COMBINATION FOR UNSTRUCTURED GRIDS

The method of linear combinations has been employed in the past to obtain higher order difference equation for finite difference grid systems. For example, Salvadori and Baron (1962), p. 235, used linear combinations of a four-node system to obtain the difference equation for the eight-node system shown in Fig. 7. In this case the difference equation for the eight-node system can be written as

$$\nabla^2\Phi = f_1\nabla_1^2\Phi + f_2\nabla_2^2\Phi \tag{18}$$

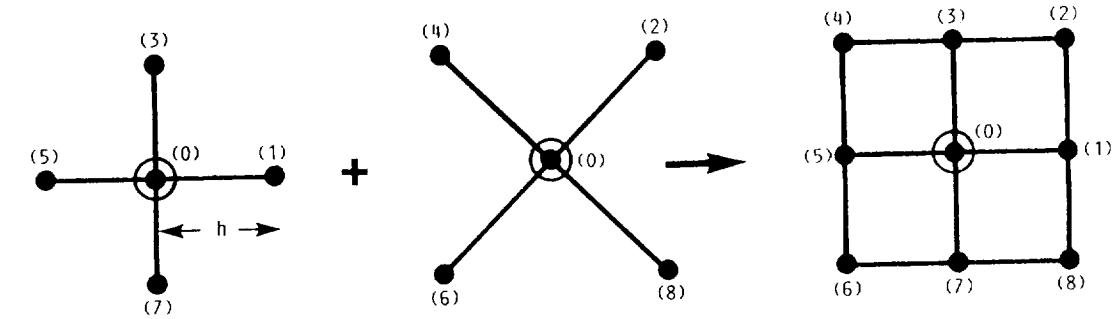
Equation (18) automatically satisfies the first four conditions of Eq. (15) since operators 1 and 2 individually meet these conditions of a zero valued Laplacian. Substituting either the x^2 or the y^2 term into Eq. (18) requires a Laplacian value of 2 or that

$$1 = f_1 + f_2 \tag{19}$$

Thus, there are an infinite number of combinations of difference equations that can represent the eight-node grid system in Fig. 7. For example if $f_1 = 2/3$ and $f_2 = 1/3$ the difference equation for the Laplacian becomes (Rektorys, 1969, p. 1114)

$$\nabla^2\Phi = \frac{4(\Phi_1 + \Phi_3 + \Phi_5 + \Phi_7) + (\Phi_2 + \Phi_4 + \Phi_6 + \Phi_8) - 20\Phi_0}{6h^2} \tag{20}$$

or if $f_1 = 1/3$ and $f_2 = 2/3$ the difference equation for the Laplacian takes on the form



$$\nabla_1^2\Phi = \frac{\Phi_1 + \Phi_3 + \Phi_5 + \Phi_7 - 4\Phi_0}{h^2} \qquad \nabla_2^2\Phi = \frac{\Phi_2 + \Phi_4 + \Phi_6 + \Phi_8 - 4\Phi_0}{2h^2} \qquad \nabla^2\Phi$$

FIGURE 7. - LINEAR COMBINATION OF OPERATORS USING FINITE DIFFERENCE THEORY.

$$\nabla^2\phi = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7 + \phi_8 - 8\phi_0}{3h^2} \qquad (21)$$

This is the identical expression obtained from finite element theory using bilinear weight functions (Lapidus and Pinder, 1982, p. 104). Although both equations are equally valid, the round-off errors associated with each equation are different. An evaluation of the round-off error can be found in reference (Salvadori and Baron, 1962, p. 236).

The first example of the linear combination method will be to determine the Laplacian operator for the left-hand cell shown in Fig. 8. This main cell is labeled f_1 while the f_2 and f_3 cells are subdivisions of the main cell using only a portion of the grid points to define the cell. The Laplacian operator for cell 1 can be expressed as the linear combination of these cells in the following form with f's representing the unknown linear combination coefficients:

$$\nabla^2\phi = f_1\nabla_1^2\phi + f_2\nabla_2^2\phi + f_3\nabla_3^2\phi \qquad (22)$$

Equation (22) satisfies identically the first three conditions in Eq. (15) for any values of the f constants since Eq. (17) satisfies these conditions. The later three conditions in Eq. (15) will be satisfied by a simultaneous solution of the following three constraint equations with unknown f_1, f_2, and f_3 coefficients:

($\phi = xy$ constraint)

$$f_1\nabla_1^2\phi\Big|_{\phi=xy} + f_2\nabla_2^2\phi\Big|_{\phi=xy} + f_3\nabla_3^2\phi\Big|_{\phi=xy} = 0 \qquad (23)$$

($\phi = x^2$ constraint)

$$f_1\nabla_1^2\phi\Big|_{\phi=x^2} + f_2\nabla_2^2\phi\Big|_{\phi=x^2} + f_3\nabla_3^2\phi\Big|_{\phi=x^2} = 2 \qquad (24)$$

($\phi = y^2$ constraint)

$$f_1\nabla_1^2\phi\Big|_{\phi=y^2} + f_2\nabla_2^2\phi\Big|_{\phi=y^2} + f_3\nabla_3^2\phi\Big|_{\phi=y^2} = 2 \qquad (25)$$

The known values of the Laplacian terms in Eqs. (23) to (25) are easily determined from Eq. (17) with the x coefficients evaluated from the known position of the grid points as required in Eqs. (3) and (4). The ϕ_n values used to evaluate the Laplacian terms in Eqs. (23) to (26) are determine directly from the trial functions xy, x^2, and y^2.

In matrix form, Eqs. (23) to (25) can be written in compact form as

$$\begin{vmatrix} \nabla_1^2\phi|_{\phi=xy} & \nabla_2^2\phi|_{\phi=xy} & \nabla_3^2\phi|_{\phi=xy} \\ \nabla_1^2\phi|_{\phi=x^2} & \nabla_2^2\phi|_{\phi=x^2} & \nabla_3^2\phi|_{\phi=x^2} \\ \nabla_1^2\phi|_{\phi=y^2} & \nabla_2^2\phi|_{\phi=y^2} & \nabla_3^2\phi|_{\phi=y^2} \end{vmatrix} \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 2 \\ 2 \end{vmatrix} \qquad (26)$$

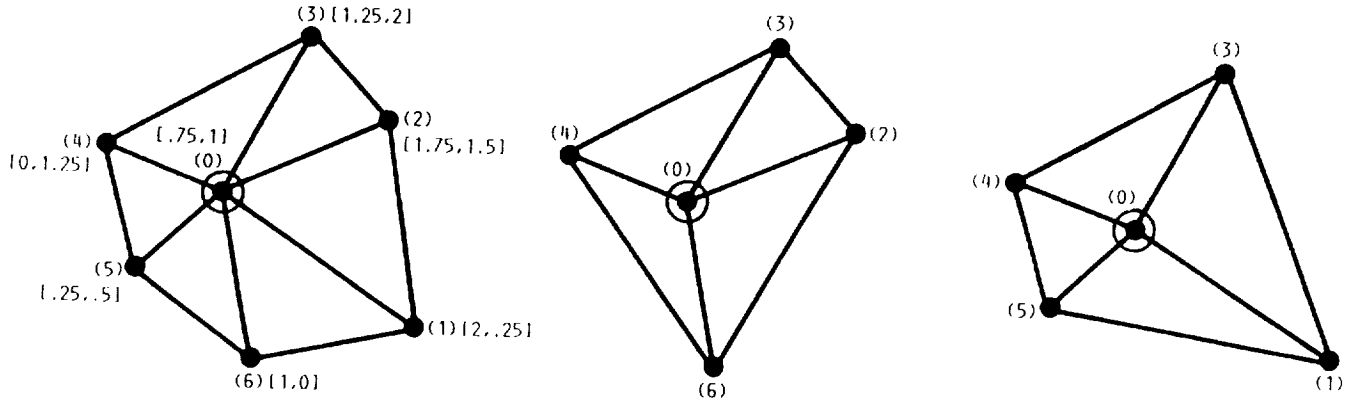
Equation (26) is a very simple system easily solved by third-order determinants for the values of f_1, f_2, and f_3.

The Laplacian difference equation for the six-node grid in Fig. 8 is shown by the formula in the lower portion of the figure. This equation meets all the test conditions given by Eq. (15). Other subdivisions of cell 1 could have been performed to obtain a different form of the difference equation. Similar to Eqs. (20) and (21), it is expected that these equations would also have different values of the round-off error.

Similar results for the five-node cell system are shown in Fig. 9. An attempt to adapt the approach to the four-node system in Fig. 10 proved unsuccessful. Apparently, the matrix equations are linearly dependent in this case and a solution is not possible.

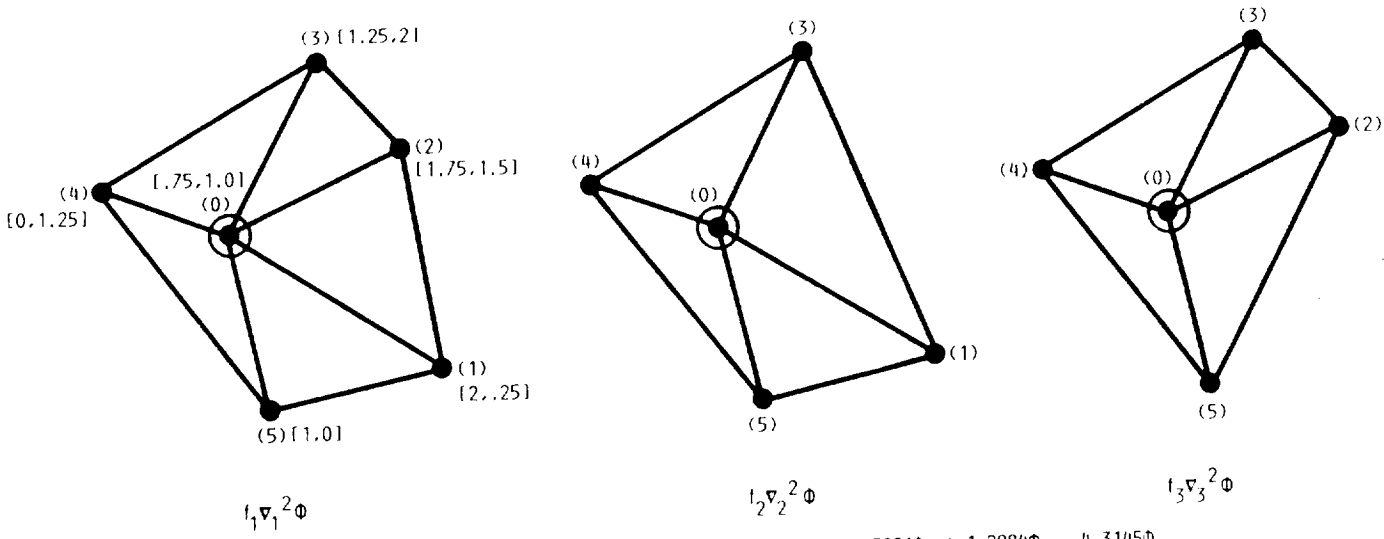
HIGHER ACCURACY

Increasing the number of grid points in the difference operator generally improves the accuracy of finite difference algorithms. A similar improvement in the accuracy could be obtain from Eq. (26) by extending the relationships shown in Eq. (15) to include higher order terms in the Pascal polynomial triangle. The Pascal's triangle for complete polynomials in terms of two variables x and y can be written as:



$$\nabla^2\Phi = 0.26444\Phi_1 + 0.43499\Phi_2 + 0.71219\Phi_3 + 1.3970\Phi_4 + 0.55045\Phi_5 + 0.80539\Phi_6 - 4.1645\Phi_0$$

FIGURE 8. - SIX NODE LOCAL LAPLACIAN OPERATOR.



$$\nabla^2\Phi = -0.033813\Phi_1 + 0.8656\Phi_2 + 0.39222\Phi_3 + 1.7921\Phi_4 + 1.2984\Phi_5 - 4.3145\Phi_0$$

FIGURE 9. - FIVE NODE LOCAL LAPLACIAN OPERATOR.

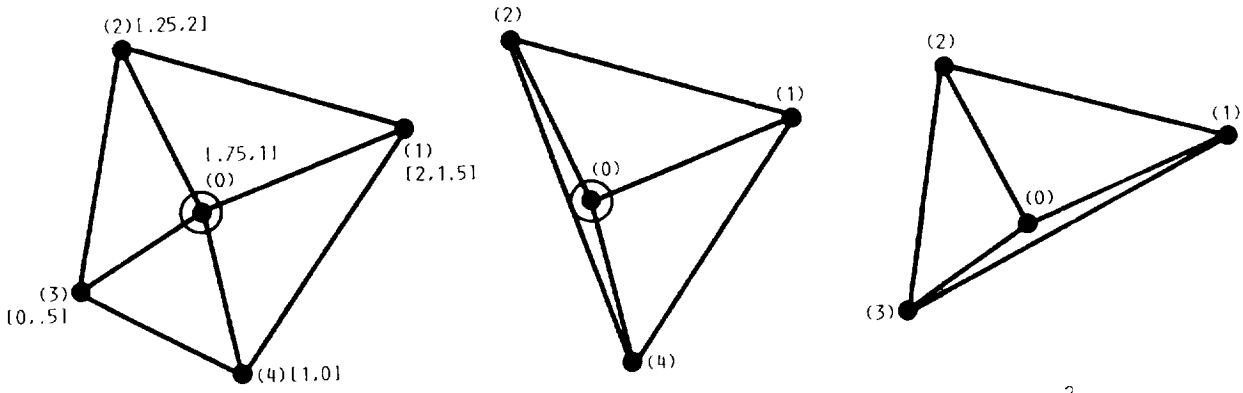


FIGURE 10. - FOUR NODE LOCAL LAPLACIAN OPERATOR.

$$\begin{array}{ccccc} & & 1 & & \\ & & x & y & \\ & & x^2 & xy & y^2 \\ & x^3 & x^2y & xy^2 & y^3 \\ x^4 & x^3y & x^2y^2 & xy^3 & y^4 \end{array} \qquad (27)$$

If a sufficient number of external nodes exist, Eq. (26) could be extended to include the cubic or even the quadratic terms in Eq. (27). This should improve the accuracy to the difference algorithm by decreasing the truncation error.

CONCLUSIONS

The Galerkin weighted residual technique using linear triangular weight functions and the method of linear combinations are employed to develop finite difference formulae in Cartesian coordinates for the Laplacian operator on local unstructured triangular grids. The algorithm was tested for a wide variety of unstructured meshes and found to give satisfactory results.

APPENDIX A - DERIVATION GALERKIN WEIGHTED RESIDUAL EQUATIONS

Let $\tilde{\Phi}$ be an approximate weighted residual solution for the potential Φ defined by Laplace's equation. Next, the total cell area A_T is divided into M elements as shown in Fig. 1. The spatial dependence of $\tilde{\Phi}$ in each element can be approximated by

$$\tilde{\Phi}^{(e)}(x,y) = \sum_{i=1}^{i=3} N_i^{(e)}(x,y)\Phi_i^{(e)} = \left[N^{(e)}\right]\left\{\Phi^{(e)}\right\} \qquad (A1)$$

(in Fig. 1; $0 = 1$, $j = 2$, $k = 3$)

The standard linear local element shape function $N^{(e)}$ is employed here. These common linear pyramid like interpolation functions N are presented in most finite element texts (Lapidus and Pinder, 1982, p. 111, Eq. (3.1.4)). The superscript (e) is used to designate the element value. Likewise the superscript (e) on the Φ_i designates the i nodes associated with a particular element.

The weak formulation of the weighted residual approach is used (Lapidus and Pinder, 1982, p. 443 - integration by parts plus the divergence theorem) along with the Galerkin approximation to obtain the standard text book weighted residual approximation for the Laplacian:

$$\sum_{e=1}^{M} \beta^{(e)} \int\int_{A_e} \left[-\nabla N_o^{(e)} \cdot \nabla\left[N^{(e)}\right]\left\{\phi^{(e)}\right\}\right] dA = 0 \qquad (A2)$$

Equation (A2) can now be easily evaluated using conventional finite element theory to obtain the difference coefficients listed in Eq. (2) in the body of this report for the central cell node labeled o.

APPENDIX B - AVERAGE OPERATORS

Consider the partial differential equation of the form

$$\nabla^2\phi + \frac{\delta\phi}{\delta x} = 0 \qquad (B1)$$

Employing the Galerkin weight residual solution of Eq. (B1) for the four-node cell in the upper right of Fig. 4 yields

$$[\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4 + \alpha_0\phi_0] + h\left[\frac{\phi_1 - \phi_3}{3}\right] = 0 \qquad (B2)$$

The coefficients associated for the Laplacian have been left in general form while the coefficient of the first derivative have been written for the specific grid associated with the four-node grid of Fig. 4.

It is desired to put Eq. (B2) in a more familiar form which coincides with the standard Taylor Series finite difference approach. The average difference value of the first derivative using the method of residual can be written as (Baumeister, 1988, Appendix B)

$$\overline{\frac{\delta\phi}{\delta x}} \int\int_{A_T} N_o \, dA = h\left[(\phi_1 - \phi_3)/3\right] \qquad (B3)$$

In this case, the integral of the weight N_o over the total cell area is equal to $A_T/3$, which was determined with the aid of the standard area integration formula (Lapidus and Pinder, 1982, p. 112) or as shown explicitly in Allaire (1985), p. 45). Therefore, in Eq. (B2), division by the area factor $A_T/3$ will convert the weighted residual coefficients into the Taylor series difference approximations for the first derivative. Furthermore, for the specific four-node grid shown in Fig. 4, A_T corresponds to $2h^2$, thus Eq. (B2) becomes

$$\frac{[\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4 + \alpha_0\phi_0]}{\frac{A_T}{3}} + \left[\frac{\phi_1 - \phi_3}{2h}\right] = 0 \qquad (B4)$$

The right-hand term is immediately recognized as the Taylor series difference approximation for the first derivative term in Eq. (B1) for the four-node grid system shown in Fig. 4. Likewise, the left-hand term can be taken to be the Laplacian difference equation associated with the first term in Eq. (B1) and is the basis for Eq. (12) in the body of this report. A similar derivation would apply to more general linear second-order partial differential such as the Helmholtz equation.

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16. Abstract

The Galerkin weighted residual technique using linear triangular weight functions is employed to develop finite difference formulae in Cartesian coordinates for the Laplacian operator on isolated unstructured triangular grids. The weighted residual coefficients associated with the weak formulation of the Laplacian operator along with linear combinations of the residual equations are used to develop the algorithm. The algorithm was tested for a wide variety of unstructured meshes and found to give satisfactory results.

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