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Asymptotic Analysis of Dissipative Waves With Applications to Their Numerical Simulation

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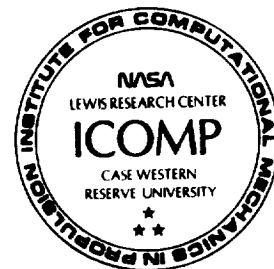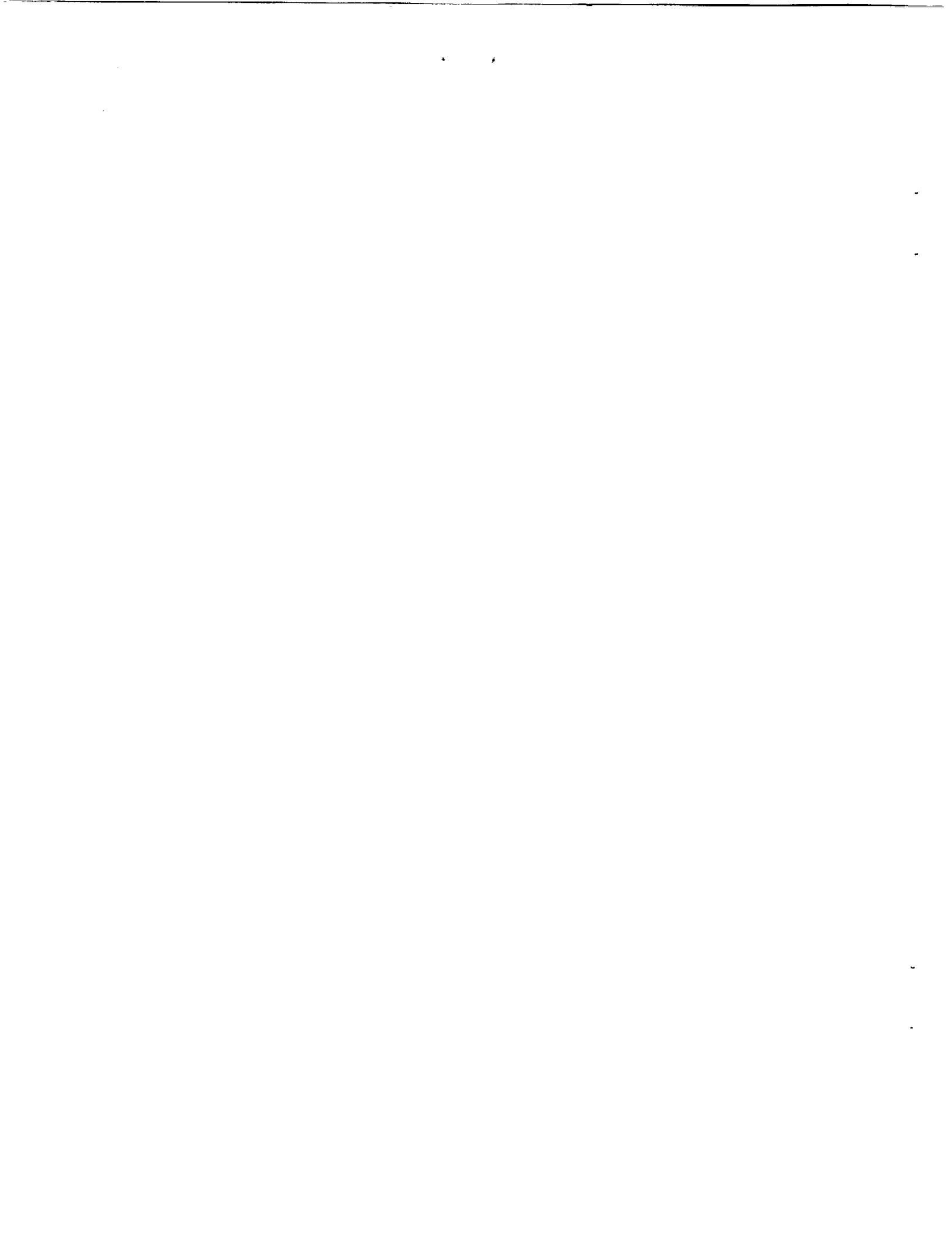Lewis Research Center
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Abstract

This paper is concerned with various problems involving the interplay of asymptotics and numerics in the analysis of wave propagation in dissipative systems. A general approach to the asymptotic analysis of linear, dissipative waves is developed. We apply it to the derivation of asymptotic boundary conditions for numerical solutions on unbounded domains. Applications include the Navier-Stokes equations. Multidimensional traveling wave solutions to reaction-diffusion equations are also considered. We present a preliminary numerical investigation of a thermo-diffusive model of flame propagation in a channel with heat loss at the walls.

1 Introduction

The mathematical theory of wave propagation in dissipative systems is rich both in applications and challenges. Two important classes of problems are those involving convection dominated but viscous fluid flows and those involving traveling wave solutions of nonlinear equations of parabolic or incompletely parabolic type. (The two classes overlap, of course, in the study of viscous shock profiles.) In this paper we shall discuss a variety of problems involving the asymptotic and numerical analysis of dissipative waves with an emphasis on the connection between the two.

The construction of numerical radiation boundary conditions at artificial boundaries is an issue of great importance in many branches of fluid mechanics including acoustics, aerodynamics, meteorology and oceanography. It is also a prime example of an area where asymptotic analysis can contribute to the design of numerical methods. Our approach is to study the far-field asymptotics of propagating waves and use it to design accurate boundary operators. In Section 2 we develop a general asymptotic analysis of the long range propagation of waves in linear, dissipative systems. This is based on the identification of dominant wave groups associated with minimal decay rates. In Section 3 we show how to use these to construct asymptotic boundary conditions and also give some consideration to

the small dissipation limit. An early version of these ideas was applied to an advection-diffusion equation in [11]. Error estimates are given in [13] while their application to the incompressible Navier-Stokes equations is discussed in [12].

Traveling wave solutions to reaction-diffusion equations in one space dimension have been extensively studied. In Section 4 we consider a natural multidimensional generalization of these: traveling wave solutions in cylindrical domains. Far less is known about such solutions. It is particularly difficult to generalize the existence theory as it typically makes use of a phase space analysis. We consider the numerical computation of such waves. As an example we take a thermo-diffusive model of flame propagation in a channel with heat loss at the walls. Some preliminary numerical results are presented. All the work in this section has been carried out jointly with Steve Buonincontri of Lawrence Livermore National Laboratories.

2   Asymptotic Expansions for Dissipative Waves

We consider the following system of partially separable second order differential equations in an n-dimensional cylindrical spatial domain, $(x, y) \in [0, \infty) \times \Omega$:

$$M\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + \sum_{j=1}^{n-1} B_j\frac{\partial u}{\partial y_j} + Cu = \mathcal{D}\sum_{i,j=0}^{n-1} \frac{\partial}{\partial y_i}D_{ij}\frac{\partial u}{\partial y_j}. \tag{1}$$

(Here we identify x with y_0.) These are supplemented by boundary and initial conditions defining a signalling problem:

$$u(x, y, 0) = 0, \tag{2}$$

$$E_0 u(0, y, t) = g(y, t), \tag{3}$$

$$T_0\frac{\partial u}{\partial n} + T_1 u = 0, \quad y \in \partial\Omega, \tag{4}$$

where $\frac{\partial}{\partial n}$ denotes the conormal derivative. For now we simply assume that the $m \times m$ matrices M, A, B_j, C, D_{ij}, E_0 and T_i are functions only of the cross-sectional coordinate y and that (1-4) is well-posed. A formal representation of the solution may be obtained using Laplace transformations in x and t. This leads to the eigenvalue problem in Ω:

$$-\mathcal{D}\sum_{i,j=1}^{n-1} \frac{\partial}{\partial y_i}D_{ij}\frac{\partial v}{\partial y_j} + \sum_{j=1}^{n-1} P_j\frac{\partial v}{\partial y_j} + Qv = 0, \tag{5}$$

$$Q = sM + \lambda A + C - \mathcal{D}(\lambda^2 D_{00} + \lambda\sum_{j=1}^{n-1} \frac{\partial D_{j0}}{\partial y_j}), \tag{6}$$

$$P_j = B_j - \lambda\mathcal{D}(D_{0j} + D_{j0}). \tag{7}$$

An expression for u then is:

$$u(x, y, t) = \sum_{l \in \mathcal{N}} u_l(x, y, t), \tag{8}$$

$$u_l(x, y, t) = \int_0^t c_l(p)q_l(x, y, t - p)dp, \tag{9}$$

where

$$q_l(x, y, t) = \frac{1}{2\pi i}\int_C e^{st + \lambda_l(s)x}v_l(y; s)ds. \tag{10}$$

(Here $l \in \mathcal{N}$ if $\Re(\lambda_l) < 0$, $\Re(s) \to \infty$.)

2.1 Asymptotic Analysis

We now use the method of steepest descent to compute asymptotic expansions of q_l, valid for x large. Evaluating (10) along rays $t = \gamma x$, $x \gg 1$, we seek points s^* such that:

$$\lambda_l'(s^*) = -\gamma, \tag{11}$$

$$\Re(\gamma) > 0, \tag{12}$$

$$\Im(\gamma) = 0. \tag{13}$$

Assuming that for $0 \leq \gamma_{min} \leq \gamma \leq \gamma_{max} \leq \infty$ a solution to (11) exists we have:

$$q_l(x, y, t) \sim e^{x\left(s^*\left(\frac{t}{x}\right)\left(\frac{t}{x}\right) + \lambda_l\left(s^*\left(\frac{t}{x}\right)\right)\right)} \frac{v_l(y; s^*(\frac{t}{x}))}{\sqrt{2\pi \lambda_l''(s^*(\frac{t}{x}))x}} \equiv \phi_l(x, y, t), \tag{14}$$

$$\gamma_{min} \leq \frac{t}{x} \leq \gamma_{max}. \tag{15}$$

Substituting these into (9) formally yields an approximation of u_l for $t > \gamma_{min}x$;

$$u_l(x, y, t) \sim \int_{max(0, t - \gamma_{max}x)}^{t - \gamma_{min}x} c_l(p)\phi_l(x, y, t - p)dp. \tag{16}$$

In (16) u_l is represented as a superposition of wave groups propagating at their group velocity. Further simplifications may be obtained by noting that, for problems with dissipation, each wave group decays exponentially with the rate:

$$-\Re(\gamma s^* + \lambda_l(s^*)) \neq 0. \tag{17}$$

For general signal data we expect that the large x behavior will be dominated by the wave group with least decay, so we seek to minimize the decay rate as a function of γ. Differentiating with respect to γ yields:

$$\Re(s^* + \frac{ds^*}{d\gamma}(\gamma + \lambda_l')) = 0, \tag{18}$$

which by (11) reduces to:

$$\Re(s^*(\gamma)) = 0. \tag{19}$$

Assuming (11) defines a curve in s space, we have shown that critical points of the decay rate occur as the curve crosses the imaginary axis. This may occur for $s^* = 0$ or for complex conjugate pairs $s^* = \pm\kappa i$. In the spirit of Laplace's method a crossing at $\gamma = \tilde{\gamma}_l$ will lead to a contribution to the asymptotic approximation of u_l computed by replacing ϕ_l in (16) by:

$$e^{\lambda_l^0 x} F(x + \frac{t - p}{\lambda_l^1}, t - p)v_l^0(y), \tag{20}$$

$$F(z, \tau) = e^{s^* z \lambda_l^1 + \frac{(\lambda_l^1)^3 z^2}{4\lambda_l^2 \tau}} \frac{1}{\sqrt{\frac{-4\pi\lambda_l^2 \tau}{\lambda_l^1}}}. \tag{21}$$

Here we have:

$$\lambda_l^0 = \tilde{\gamma}_l s^* + \lambda_l(s^*), \tag{22}$$

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$$\lambda_l^1 = \frac{d\lambda_l}{ds}(s^*) = -\tilde{\gamma}_l, \tag{23}$$

$$\lambda_l^2 = \frac{1}{2}\frac{d^2\lambda_l}{ds^2}(s^*), \tag{24}$$

where s^* is evaluated at $\tilde{\gamma}_l$. For $s^* = 0$ this is indeed Laplace's method, and the approximation will give the leading order asymptotics. For imaginary s^* this is not true pointwise due to the oscillatory nature of ϕ_l. The results of [13] imply, however, that it provides an asymptotic approximation in an appropriate norm.

2.2 Sufficient Conditions for $s^*(\tilde{\gamma}_l) = 0$

Although our analysis allows for $s^* \neq 0$, we are then faced with the potentially difficult problem of locating the points on the imaginary s-axis where the group velocity is real. Numerical methods for this are outlined in [13] and have been carried out for the linearized Navier-Stokes equations. They are, however, expensive and cannot be guaranteed to locate all critical points. It is, therefore, of interest to identify a priori problems for which $s^* = 0$ is a solution of (11-13). We find that sufficient conditions can be given if we make the following partial symmetrizability assumption.

Assumption 1 *There exists a smooth matrix $S(y)$ with smooth inverse such that $\tilde{u} = Su$ satisfies:*

$$\tilde{M}\frac{\partial \tilde{u}}{\partial t} + \tilde{A}\frac{\partial \tilde{u}}{\partial x} + \tilde{C}\tilde{u} = \mathcal{D}\nabla \cdot \tilde{D}\nabla\tilde{u}, \tag{25}$$

where the matrices $\tilde{M}$, $\tilde{A}$, $\tilde{C}$ and $\tilde{D}$ are symmetric and:

$$\tilde{D} > 0, \quad \tilde{M} \geq 0, \quad \tilde{D}_{0j} = 0, \quad j = 1, \ldots n - 1. \tag{26}$$

Furthermore, the boundary conditions on $\partial\Omega$ either take the form:

$$\tilde{T}\frac{\partial \tilde{u}}{\partial n} + \tilde{u} = 0, \tag{27}$$

or

$$\frac{\partial \tilde{u}}{\partial n} + \tilde{T}\tilde{u} = 0, \tag{28}$$

with $\tilde{T}$ symmetric and positive semidefinite.

We note that Assumption 1 will hold for a number of interesting physical problems including some advection-diffusion equations as well as linearized advection-reaction-diffusion equations with parallel base flows. It does not, however, hold for the Navier-Stokes equations linearized about a parallel flow. In fact $s^* \neq 0$ may be important in our asymptotic analysis of linearized viscous flow both for very low and very high Reynolds numbers [12].

We now turn to the analysis of (5). In what follows we'll drop the tildes and assume that (1-4) has been given in the partially symmetrized form above. If we can show that $\lambda_l(s)$ is real for real s in an interval containing zero, than we will have shown that the real s-axis is locally a curve of real group velocities. Note that the eigenvalue problem is not self-adjoint due to the dependence of Q on λ. Nonetheless, by a simple adaptation of an argument due to Berestycki and Nirenberg [3], we are able to prove:

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Theorem 1 *Suppose Assumption 1 holds and, if the boundary condition is given by (28), the matrix $\int_{\partial\Omega} T + \int_{\Omega} C$ is positive definite. Then there exists $\delta < 0$ such that solutions, $\lambda(s)$, of (5) are real when s is real and $s > \delta$.*

Proof: Following [3] we write $\lambda = \mu + i\nu$ and by an integration by parts obtain:

$$\mathcal{B}(v,\lambda) + s \int_{\Omega} v^H M v = \lambda \mathcal{G}(v,\mu), \tag{29}$$

$$\mathcal{B}(v,\lambda) = \int_{\Omega} \left(\mathcal{D}(\sum_{i,j} \frac{\partial v^H}{\partial y_i} D_{ij} \frac{\partial v}{\partial y_j} + |\lambda|^2 v^H D_{00} v) + v^H C v \right) + \int_{\partial\Omega} \left\{ \begin{array}{c} v^H T v \\ \text{or} \\ \frac{\partial v^H}{\partial n} T \frac{\partial v}{\partial n} \end{array} \right\}, \tag{30}$$

$$\mathcal{G}(v,\mu) = \int_{\Omega} v^H (2\mu \mathcal{D} D_{00} - A) v. \tag{31}$$

As $\mathcal{G}$ and $\mathcal{B}$ are real, if s is real λ can have a nonzero imaginary part only if both sides of (29) are zero. The hypothesis of the theorem, moreover, implies the existence of $\delta < 0$ such that for $s > \delta$ the left-hand side is positive, guaranteeing that λ is real and of the same sign as $\mathcal{G}$.

Equation (29) can further be used to establish a connection between the sign of $\lambda_l(0)$ and both its large s behavior and the direction of propagation of the corresponding wave group:

Theorem 2 *Suppose the conditions of Theorem 1 hold and $M > 0$. Then $l \in \mathcal{N}$ and $\lambda'_l(0) < 0$ if and only if $\lambda_l(0) < 0$.*

Proof: Consider any nonsingular solution branch $\lambda_l(s)$ as s varies along the nonnegative real axis. As the left-hand side of (29) is strictly positive, the right-hand side can never be zero, so λ_l must be of one sign. Moreover, if $\lambda_l < 0$ then $\mathcal{G} < 0$. Differentiating (5) with respect to s, multiplying by v^T and integrating we obtain a general formula for λ'_l:

$$\int_{\Omega} v^T M v = \lambda'_l \int_{\Omega} v^T (2\lambda_l \mathcal{D} D_{00} - A) v. \tag{32}$$

For λ_l and v real the factor multiplying λ'_l is simply $\mathcal{G}$ so that the sign of λ'_l is the same as that of $\mathcal{G}$ and, hence, that of λ_l.

Note that the conditions be relaxed to allow $\lambda_l(0) = 0$ so long as $\mathcal{G}(v(y;0),0) \neq 0$. Then the sign of $\mathcal{G}$ determines the sign of λ'_l and λ_l. It should also be noted that these theorems, though establishing the contribution of the neighborhood of $s = 0$ to the asymptotics, do *not* rule out the existence of imaginary s contributions. That is, we have been unable to prove that real group velocities cannot occur for imaginary s even with the assumptions made above.

3 Construction of Asymptotic Boundary Conditions

Now suppose we are interested in the numerical solution of (1-4), or, more generally, of a problem for which these equations represent a far-field approximation. In order to restrict the computational domain to a finite region, an artificial boundary is typically introduced.

say at $x = \tau$. In order to close the system, additional boundary conditions must be imposed. That is, we approximate u by u_f which satisfies (1-4) for $(x, y) \in [0, \tau] \times \Omega$ as well as:

$$Bu_f(\tau, y, t) = 0. \tag{33}$$

In the past fifteen years an extensive literature has developed discussing the problem of choosing the operator B, though mainly in the difficult case of hyperbolic problems. There are three basic criteria which it should ideally satisfy:

1. The finite domain problem is well-posed.

2. Bu is small in an appropriate norm.

3. The efficient numerical solution of the finite domain problem is possible.

The first two criteria are the specialization of the usual notions of stability and consistency to the problem of numerical boundary conditions, and can be used to derive error estimates [13]. Techniques for establishing the first criterion are well-known [7], [17], but the second is far more difficult to satisfy. In particular, exact conditions, i.e. those for which $Bu = 0$ whenever u solves (1) and (4), are invariably nonlocal in both space and time and, hence, violate the third criterion. Asymptotic analysis can provide a means for constructing local, or at least local in time, operators which still satisfy the 'consistency' criterion.

The Laplace transforms of $\frac{\partial q_l}{\partial x}$ and q_l are related by:

$$\frac{\partial \hat{q}_l}{\partial x}(x, y; s) = \lambda_l(s)\hat{q}_l(x, y; s). \tag{34}$$

If the assumptions of the previous section are satisfied, the asymptotic analysis involves the restriction of the transforms to a neighborhood of $s = 0$. An asymptotic expansion of the x derivative may then be obtained by replacing $\lambda_l(s)$ by its Taylor series about 0:

$$\lambda_l(s) \approx \lambda_l(0) + \lambda_l'(0)s + \ldots. \tag{35}$$

This leads to the local relationship:

$$\frac{\partial q_l}{\partial x} \sim \left(\lambda_l(0) + \lambda_l'(0)\frac{\partial}{\partial t}\right) q_l. \tag{36}$$

These may be substituted into (9) to finally obtain a condition on u_l. The time derivative is brought outside the integral to further simplify the expression. This involves the neglect of terms from the limits of integration which should be exponentially small. The asymptotic boundary condition we propose is, then, given by:

$$\frac{\partial u_l}{\partial x} = \left(\lambda_l(0) + \lambda_l'(0)\frac{\partial}{\partial t}\right) u_l. \tag{37}$$

The construction of B now hinges on the number of modes, u_l, which are important in the asymptotic development of u. In the simplest case there is a single mode, say $l = 1$, satisfying:

$$e^{\tau\lambda_1(0)} \gg e^{\tau\lambda_l(0)}, \quad l \neq 1. \tag{38}$$

Then we may take:

$$B \equiv \frac{\partial}{\partial x} - \lambda_1(0) - \lambda_1'(0)\frac{\partial}{\partial t}. \tag{39}$$

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In [13] it is shown that this procedure leads to an error estimate of the form:

$$\|u - u_f\| \leq K \frac{e^{\tau \lambda_1(0)}}{\tau} \|u\|, \tag{40}$$

where K is a constant. In the most interesting cases $\lambda_1(0)$ is small so that the $\frac{1}{\tau}$ decay, which is a direct result of the use of asymptotic conditions, is important.

More generally we may assume that $\lambda_l(0) \leq \lambda_{l+1}(0)$ and, for some J:

$$e^{\tau \lambda_J(0)} \gg e^{\tau \lambda_{J+1}(0)}. \tag{41}$$

If J is small, for example 2 or 3, a Jth order operator B may be constructed as the product of first order conditions:

$$B \equiv \prod_{l=1}^{J} \left(\frac{\partial}{\partial x} - \lambda_l(0) - \lambda_l'(0) \frac{\partial}{\partial t} \right). \tag{42}$$

For J larger, a nonlocal in space condition is likely to be more useful. Let $\mathcal{K}_i$, $i = 1, \ldots, J$ be linear functionals on the appropriate space of functions on Ω with the property:

$$\mathcal{K}_i v_l(\cdot; 0) = \delta_{il}, \quad i, l = 1, \ldots, J. \tag{43}$$

The existence of such functionals is guaranteed by the linear independence of the eigenfunctions $v_l(y; 0)$. An asymptotic boundary operator is then given by:

$$B \equiv \frac{\partial}{\partial x} - \Lambda_0 - \Lambda_1 \frac{\partial}{\partial t}, \tag{44}$$

where

$$\Lambda_0 u = \sum_{i=1}^{J} \lambda_i(0) \left(\mathcal{K}_i u \right) v_i(y; 0), \tag{45}$$

$$\Lambda_1 u = \sum_{i=1}^{J} \lambda_i'(0) \left(\mathcal{K}_i u \right) v_i(y; 0). \tag{46}$$

Product conditions with $J = 2$ have been used in [11] and [12], but the nonlocal condition (44) is as yet untried.

3.1 Behavior for $\mathcal{D}$ Small

The study of waves in dissipative systems with small dissipation, in our case for small $\mathcal{D}$, is of particular interest. For the problem of boundary conditions it provides a potential link between our results and the hyperbolic theory. In a recent paper Halpern [15] has studied this question for incompletely parabolic perturbations of linear hyperbolic systems with constant coefficients. The conditions derived there involve viscous corrections to the hyperbolic conditions of Engquist and Majda [8].

Relaxing the the assumptions of the previous section to allow $\lambda(0) = 0$, we seek solutions of (5) with bounded decay rates. Thus the term proportional to $\mathcal{D}\lambda^2$ can be ignored to leading order. We distinguish between two separate cases:

Case 1: There are no smooth functions $v(y)$ satisfying (4) and $Cv = 0$.

Case 2: $C \equiv O$.

(We are excluding the more complicated case of $C \neq O$ but violating the case 1 assumption.)

Under case 1 we must consider solutions of the singular perturbation problem:

$$-\mathcal{D}\sum_{i,j=1}^{n-1}\frac{\partial}{\partial y_i}D_{ij}\frac{\partial v}{\partial y_j} + Cv = -\lambda Av. \tag{47}$$

This is rather difficult to analyze, and we won't discuss it further. Note, however, that we expect the decay rates to be bounded away from zero independent of $\mathcal{D}$.

Under case 2, on the other hand, we find many eigenvalues which approach 0 with $\mathcal{D}$. Introducing the scaling $\lambda = \mathcal{D}\hat{\lambda}$ we consider the nonsingular reduced problem:

$$-\sum_{i,j=1}^{n-1}\frac{\partial}{\partial y_i}D_{ij}\frac{\partial v}{\partial y_j} = -\hat{\lambda}Av. \tag{48}$$

All eigenvalues of the reduced problem which are negative are of importance in the asymptotic boundary conditions for $\mathcal{D}$ sufficiently small, as their decay rates are approaching zero. This suggests the use of the nonlocal conditions of the preceding section. If $A > O$, which we may associate with all characteristics pointing out of the computational domain, all eigenvalues are negative and the reduced equation itself may be used as a boundary condition:

$$M\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \mathcal{D}\sum_{i,j=1}^{n-1}\frac{\partial}{\partial y_i}D_{ij}\frac{\partial u}{\partial y_j}. \tag{49}$$

If $A < O$, corresponding to all characteristics pointing in, then there are no negative eigenvalues bounded independent of $\mathcal{D}$. Dirichlet conditions may then be imposed on u. The most interesting case occurs when A has eigenvalues of both sign. Then the boundary condition will involve a coupling between incoming and outgoing characteristics of the related hyperbolic problem.

It should be noted that we do not expect the error estimates to be uniformly valid as $\mathcal{D}$ approaches 0. For hyperbolic problems curves of real group velocity will coincide with the imaginary s axis, which indicates nonuniformity in the steepest descent calculations. That is, we cannot expect only the low frequencies to be present for moderate τ. It would be of interest in this regard to extend our method to viscous perturbations of the wave equation and study its connections with the recent work of Engquist and Halpern [9]. They consider low frequency corrections to the Engquist-Majda conditions and study the long time behavior of the error.

3.2 Applications to the Navier-Stokes Equations

The Navier-Stokes equations provide a most important example of a dissipative system with propagating solutions and were the prime motivation for the work described here. The detailed application of our technique to the derivation of asymptotic boundary conditions for the incompressible equations linearized about plane parallel flow is described in [12]. The eigenvalue problem (5) is then given by the well-known Orr-Sommerfeld equation of hydrodynamic stability. As the equations do not satisfy Assumption 1, the results guaranteeing $s^* = 0$ cannot be applied. Nonetheless, for large Reynolds number, we find that an analogue to (48) can be constructed which does lead to slowly decaying propagating modes near $s^* = 0$. A boundary condition based on two of these modes was utilized in the simulation of vortical disturbances to Poiseuille flow and was quite successful for Reynolds numbers in the hundreds. For higher Reynolds numbers a condition such as (49) is likely to

be effective. For flows with a critical Reynolds number, however, solutions with imaginary s^* will eventually become dominant. Our analysis suggests that the nonlocal condition (44) based on solutions of the full Orr-Sommerfeld equation may then be useful, but they have not yet been tried.

Presently under consideration are applications to compressible flows and to nonisothermal incompressible flows. Also of note are the detailed studies by Halpern and coworkers [16], [15] of boundary conditions for both the compressible and incompressible Navier-Stokes equations linearized about a uniform flow.

4 Traveling Wave Solutions in Cylindrical Domains

Among the most active areas in the study of wave propagation in dissipative systems has been the analysis of planar traveling wave solutions to reaction-diffusion equations. Applications have been made to models of flame propagation, nerve conduction and the internal structure of shocks in viscous conservation laws to name a few. Simple examples include Fisher's equation, which is the diffusion equation plus the nonlinearity $F = u(1 - u)$, where a continuum of waves is found connecting a linearly stable and linearly unstable homogeneous state, and the bistable equation, $F = u(u - \alpha)(1 - u)$, where a unique wave connects two stable states.

A natural, nontrivial generalization of these solutions to higher space dimensions comes from the consideration of waves propagating in cylindrical domains. In general we consider:

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n-1} \frac{\partial}{\partial y_i} D_{ij} \frac{\partial u}{\partial y_j} + F(u, \nabla u, y), \quad (x, y) \in (-\infty, \infty) \times \Omega. \tag{50}$$

Here again we identify y_0 with x and suppose that neither D_{ij} nor F depend explicitly on x. Boundary conditions (4) are also imposed. We further suppose that (at least) two solutions independent of x and t, $u_{\pm}(y)$ exist. A traveling wave solution of (50) takes the form $u = w(x - ct, y)$ with:

$$lim_{z \to \pm\infty} w(z, y) = u_{\pm}(y). \tag{51}$$

It satisfies the elliptic equation:

$$\sum_{i,j=0}^{n-1} \frac{\partial}{\partial y_i} D_{ij} \frac{\partial w}{\partial y_j} + c \frac{\partial w}{\partial z} + F(w, \nabla w, y) = 0. \tag{52}$$

Only a few studies of solutions to (52) have appeared in the literature. Berestycki and Nirenberg [3], [2] have established monotonicity results for general scalar problems. Gardner [10] proves the existence of a nonplanar wave for the bistable equation in a channel. In joint work with Buonincontri [5], we have numerically computed traveling wave solutions to the Frank-Kamenetski equations in a channel and also developed some stability results. As the waves connect stable and unstable states, this may be thought of as a multidimensional problem of Fisher type.

4.1   A Model of Flame Propagation in a Channel with Cold Walls

Consider the following system of reaction-diffusion equations modeling combustion in the channel $(x,y) \in (-\infty,\infty) \times [0,1]$:

$$\rho\frac{\partial T}{\partial t} - \nabla^2 T = \mathcal{D}\rho Y e^{\frac{-\theta}{T}}, \tag{53}$$

$$\rho\frac{\partial Y}{\partial t} - \mathcal{L}^{-1}\nabla^2 Y = -\mathcal{D}\rho Y e^{\frac{-\theta}{T}}, \tag{54}$$

$$\rho T = k, \quad \text{constant}, \tag{55}$$

$$\frac{\partial T}{\partial n} = -\alpha(T - T_o), \qquad \frac{\partial Y}{\partial n} = 0, \quad \text{at } y = 0,1. \tag{56}$$

Here T is the temperature, Y is the mass fraction of the reactant, $\mathcal{L}$ is the Lewis number and θ is the activation energy. The model has heat loss at the channel walls, which precludes the existence of planar traveling wave solutions.

This thermo-diffusive model ignores convection by the fluid flow induced by thermal expansion. Most authors, in using such a model, take ρ constant rather than proportional to T^{-1}. In order to define the traveling wave we must find unburnt and burnt limiting states at $\pm\infty$. The burnt state solution has no reactant present and is at the ambient temperature:

$$Y = 0, \quad T = T_o. \tag{57}$$

Note that in the study of plane flames the burnt boundary is hot, $T > T_o$. The structure of the flame we compute will be much different, exhibiting a nonmonotonic temperature profile rising to a maximum in the reaction zone then slowly decaying due to the heat loss. There is no unburnt equilibrium due to the form we've chosen for the reaction term. However, supposing the dimensionless ratio of ambient temperature to activation energy to be small, $\epsilon = \frac{T_o}{\theta} \ll 1$, we can compute a slowly varying unburnt state:

$$Y = Y_o(\epsilon t) + \epsilon Y_1(\epsilon t, y) + O(\epsilon^2), \quad T = T_o + \epsilon T_1(\epsilon t, y) + O(\epsilon^2). \tag{58}$$

Among the questions one would like to answer about the propagating flames are:

1. For what values of the parameters do traveling waves exist? Is there extinction for sufficiently large heat loss? (Evidence for extinction will be nonexistence of propagating solutions.)

2. How does the shape and speed of the waves vary with the parameters and compare with planar results?

A detailed discussion of the derivation of the model equations can be found in the doctoral dissertation of Buonincontri [4]. In a recent paper, Benkhaldoun, Larrouturou and Denet [1] present a numerical study of essentially the same problem using a slightly different thermo-diffusive system. Rather than directly solving the equations defining the traveling wave, they solve an initial-boundary value problem.

4.2 Numerical Methods and Preliminary Results

In this section we shall describe our numerical methods for computing the traveling wave, which are of general interest, and also present some preliminary results. All the computations are due to Buonincontri. A more detailed and definitive study will be published elsewhere when complete.

The fundamental numerical difficulty is the proper imposition of boundary conditions at artificial boundaries, $z = z_{\pm}$, introduced to limit the computational domain, and the determination of the unknown wave speed c. We are guided by the general theory of asymptotic boundary conditions for nonlinear elliptic boundary value problems given in [14]. For ordinary differential equations the numerical problem has also been considered by Doedel and Friedman [6]. They obtain impressive theoretical results, but the application of their technique to partial differential equations would be too expensive.

An asymptotic analysis of the traveling wave as $z \to \pm\infty$ yields:

$$\begin{pmatrix} T - T_o - \epsilon T_1 \\ Y - Y_o - \epsilon Y_1 \end{pmatrix} \sim \gamma_+ e^{-\lambda_+ z} \begin{pmatrix} v_+(y) \\ w_+(y) \end{pmatrix}, \quad z \to \infty, \tag{59}$$

$$\begin{pmatrix} T - T_o \\ Y \end{pmatrix} \sim \gamma_- e^{\lambda_- z} \begin{pmatrix} v_-(y) \\ w_-(y) \end{pmatrix}, \quad z \to -\infty. \tag{60}$$

The exponential decay rates, $\pm\lambda$, and corresponding eigenfunctions are defined by the eigenvalue problem (5) for the equations linearized about the asymptotic states. It should be noted that they depend on the unknown wave speed c. From these expansions we derive the asymptotic boundary conditions:

$$\left(\frac{\partial}{\partial z} + \lambda_+ \right) \begin{pmatrix} T - T_o - \epsilon T_1 \\ Y - Y_o - \epsilon Y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad z = z_+, \tag{61}$$

$$\left(\frac{\partial}{\partial z} - \lambda_- \right) \begin{pmatrix} T - T_o \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad z = z_-. \tag{62}$$

The traveling wave problem (52) clearly cannot possess unique solutions due to the translation invariance of the wave. To get uniqueness, we must impose an additional phase condition which we do by fixing γ_+:

$$\int_0^1 \{ v_+(T - (T_o + \epsilon T_1)) + w_+(Y - (Y_o + \epsilon Y_1)) \} dy = \bar{k}. \tag{63}$$

This additional equation balances the additional unknown, c. In [5], where a stable-unstable connection is computed, the use of the theory of [14] leads to one less boundary condition at z_+ corresponding to the fact that waves exist for a range of speeds.

We approximate (52) using centered finite differences and solve for the wave and c simultaneously using Newton's method. Note that the ϵt dependence of the wave only comes in, at leading order, through the boundary condition. In Figure 4.2 we plot a typical profile. At present we have only investigated a small part of parameter space with $\mathcal{L} > 1$. We have found a variety of interesting phenomena including extinction for α large enough as well as an apparent limit point as $\mathcal{L}$ is varied.

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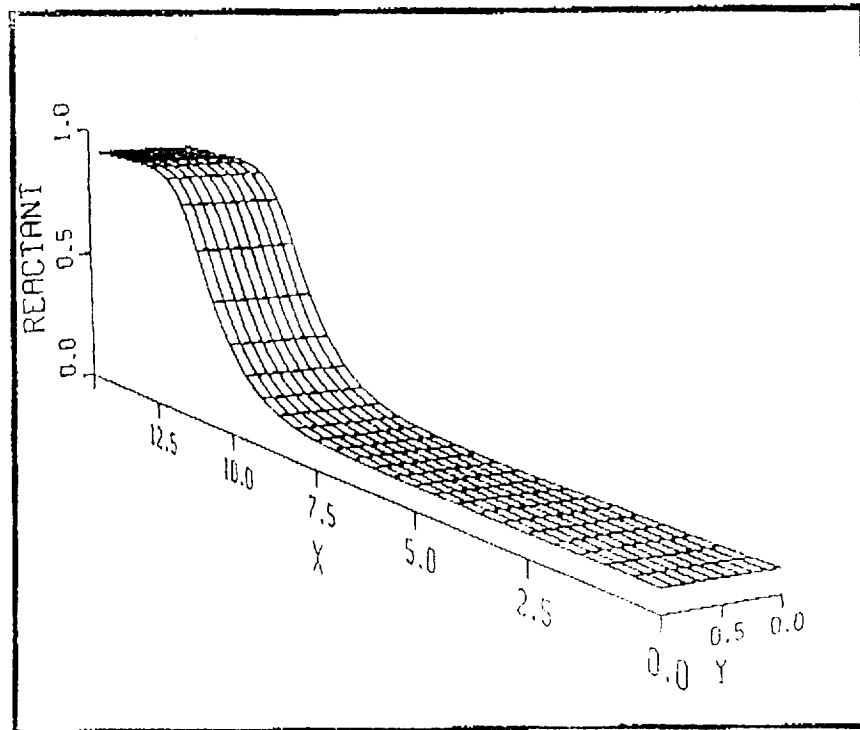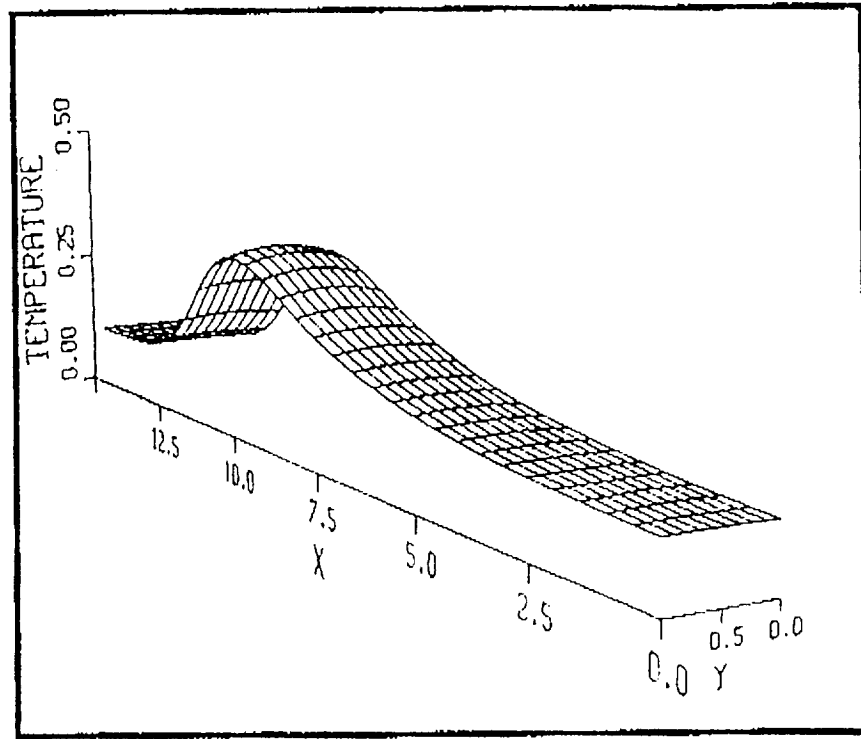
Figure 1: Flame with $\mathcal{L} = 1.5$, $\theta = 1$, $T_o = .1$, $Y_o = .92$, $\mathcal{D} = 6$, $k = 1$, $\alpha = .37$ and $c = .23$

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15. Supplementary Notes

16. Abstract

This paper is concerned with various problems involving the interplay of asymptotics and numerics in the analysis of wave propagation in dissipative systems. A general approach to the asymptotic analysis of linear, dissipative waves is developed. We apply it to the derivation of asymptotic boundary conditions for numerical solutions on unbounded domains. Applications include the Navier-Stokes equations. Multidimensional traveling wave solutions to reaction-diffusion equations are also considered. We present a preliminary numerical investigation of a thermo-diffusive model of flame propagation in a channel with heat loss at the walls.

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