A Finite Difference Treatment of Stokes-Type Flows: (Preliminary Report)

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M.E. Rose
Department of Mechanical Engineering
North Carolina A&T State University

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1.Introduction

The equations

$$\nabla^2\omega = 0, \tag{1.1a}$$

$$\omega = -\nabla^2\chi \tag{1.1b}$$

describe, in suitable units, two-dimensional Stokes flow of an incompressible fluid occupying a domain D in which ω is the vorticity and χ is the stream function. The flow is uniquely determined by specifying the velocity on the boundary B of D, a condition which leads to specifying the stream function χ and its normal derivative χ_n on B. A mathematically similar problem arises in describing the equilibrium of a flat plate in structural mechanics where a related one-dimensional problem describes the equilibrium of a clamped beam. A key to treating these simple problems by finite difference or finite element methods is to introduce effective methods for imposing the boundary conditions through which (1.1a) is coupled to (1.1b). These models thus provide a simple starting point for examining the general treatment of boundary conditions for more general time-dependent Navier-Stokes incompressible flows.

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For the purpose of our discussion we may assume D is a square domain. A standard finite difference method to solve (1.1) is to introduce a uniform grid and then employ standard five-point finite difference operators to express each equation in (1.1). At any point on the boundary B a value of χ is specified by the boundary conditions but a value of ω at the same boundary mesh point will also be required to complete the computation. Peyret and Taylor [1] review the use of extrapolation methods to achieve this using Taylor series arguments. However, this technique can be expected to be of limited value when finte-volume methods are used to treat curved boundaries or when geometrical singularities arise when using curvilinear coordinates. As we shall also see, its use can be expected to result in a loss of accuracy even in simple cases. The method discussed in this report can be expected to overcome these difficultites.

2. A Compact Difference Scheme

We first describe a compact finite difference scheme for solving $\phi'' = g$ which has been described in Rose[2]. It is a specialization to one-dimension of a more general finite volume scheme for solving div $\mathbf{u}$ = g, grad $\phi = \mathbf{u}$ on general domains.

The scheme expresses a relationship between certain *primary* variables ϕ_i, $u^-(x_i)$, $u^+(x_i)$ which are associated with the endpoints of the non-overlapping intervals (x_i, x_{i+1}), $i = 0,1,\ldots,M-1$, and *dual* variables ϕ_j which are associated with the interval centerpoints x_j, $j = 1/2, 3/2, \ldots, M-1/2$, which we regard as forming a *dual (or staggared) grid.* (In the following, the index i will be associated with the primary grid, while j will indicate the dual grid.) Indicating the cell endpoints by the symbol x and the cell midpoints by the symbol o, the points form the pattern

x o x o x o x o x o x....x o x

totalling 2M+1 points.

For uniform grids the scheme to solve $\phi'' = g$ can be described as follows: write the equation as the first order system $u' = g, \phi' = u$ and express the first equation by the difference equations

$$[\, u^-(x_{j+1/2}) - u^+(x_{j-1/2})] / \Delta x = g_j \qquad j = 1/2,\ 3/2, \ldots,\ M-1/2 \tag{2.1}$$

in each interval of length $\Delta x = 2h$. We interpret the second equation as relating u^- and u^+ with forward and backward differences involving the variables ϕ at primary and secondary points of the grid:

$$u^-(x_i) = (\phi_i - \phi_{i-1/2})/\,h, \qquad i = 1,2,\ldots,\ M \tag{2.2}$$
$$u^+(x_i) = (\phi_{i+1/2} - \phi_i)/\,h, \qquad i = 0,1,\ldots,\ M-1$$

Introduce the central average and difference operators

$$\mu\, \phi_j \equiv (\, \phi_{j+1/2} + \phi_{j-1/2})/2\ , \quad \Delta\, \phi_j \equiv (\, \phi_{j+1/2} - \phi_{j-1/2}).$$

and impose the condition that the variables $u^-(x_i)$ and $u^+(x_i)$ be continuous at interior endpoints of the primary grid, i.e.,

$$u_i = u^-(x_i) = u^+(x_i), \qquad i = 1,2,\ldots,\ M-1. \tag{2.3}$$

Using the definitions (2.2) in (2.1) we find, with $\kappa = 1/\Delta x^2$,

$$g_j = 4\kappa\ (\mu\phi_j - \phi_j), \qquad j = 1/2,\ 3/2,\ldots,\ M-1/2 \tag{2.4}$$

whike use of the continuity conditions (2.3) leads to

$$0 = \mu\phi_i - \phi_i \qquad i = 1,2,\ldots,\ M-1 \tag{2.5}$$

This tri-diagonal system is to be solved for the variables ϕ with specified boundary conditions of either Dirichlet or Neumann type.

A cyclic (odd-even) reduction technique leads to the system of equations

$$\mu g_i = \kappa\ \Delta^2\phi_i\,, \qquad i = 1,2,\ldots,\ M-1 \tag{2.6}$$

for the primary variables and, separately, to

$$g_j = \kappa\, \Delta^2\phi_j , \qquad j = 3/2,,5/2.., M-3/2 \tag{2.7}$$

for the dual variables. Under Dirichlet-type boundary conditions, the solution of the first set of equations can be solved directly for the primary variables; the values $\phi_{1/2}$, $\phi_{M-1/2}$ for the dual variables can be found in terms of these primary values by solving each of the equations

$$g_j = \kappa\, \Delta^2\phi_j , \qquad j = 1/2,\ M-1/2 \tag{2.8}$$

and these, in turn, can then be used to provide Dirichlet data to solve (2.7). This reduction is less useful when Neumann-type data are imposed; in this case it is best to solve (2.4)-(2.5) directly.

The following table compares numerical results obtaained by solving $\phi'' = g$ by a standard finite difference scheme and by the compact scheme just described.

Error Norm Comparison of Standard and Compact Schemes for the Equation $\phi''=g$

$\phi = x(1-x)$:

(The precision of results for this example is questionable because of machine limitations)

endpoint error norms

	standard		compact	
# intervals	solution	derivative	solution	derivative
6	2.77556e-17	.166667	5.55112e-17	5.55112e-17
12	5.55112e-17	8.33333e-2	8.32667e-17	3.33067e-16
24	1.11022e-16	4.16667e-2	2.49800e-16	9.99201e-16
48	2.63678e-16	2.08333e-2	1.83187e-15	5.77316e-15

$\phi = x^2(1-x)^2$:

endpoint error norms

# intervals	standard solution	standard derivative	compact solution	compact derivative
6	6.17284e-3	6.48148e-2	1.23457e-2	1.85185e-2
12	1.73611e-3	4.16667e-2	3.47222e-3	9.25926e-3
24	4.34028e-4	2.40162e-2	8.68056e-4	2.89352e-3
48	1.08507e-4	1.62399e-2	2.17014e-4	7.95718e-4

3. The Clamped Beam Problem

The deflection $\phi(x)$ of a uniform, straight beam under a load $-f(x)$ per unit length on an interval $[l_-, l_+]$ is, in dimensionless form, governed by the simple fourth order differential equation

$$\phi'''' = -f \tag{3.1}$$

If the slope u, bending moment v, and shear force w are given by

$$u = \phi', \; v = -u', \; w = v' \tag{3.2}$$

then typical well-posed boundary conditions allow one to prescribe pairs of values among (ϕ,u,v,w) at each endpoint of the interval. We may also write (3.1) as a coupled system of second-order equations for ϕ and v

$$\text{(a)} \quad v'' = f, \tag{3.3}$$

$$\text{(b)} \quad \phi'' = -v.$$

For a problem in which the boundary conditions involve the pair of values (ϕ, v) at each endpoint, for example, a simple Green's function technique allows (a) to be solved for v(x) in terms of f(x) together with the boundary conditions $v(l\pm)$, while a similar construction gives $\phi(x)$ in terms of the boundary conditions $\phi(l\pm)$ and v(x). However, for clamped boundary conditions, $\phi = u = 0$ at both endpoints and this Green's function construction fails.

A standard finite difference technique for handling clamped boundary conditions involves the use of a Taylor series expansion at each endpoint in order to express $v(l\pm)$ in terms of $\phi(l\pm)$ and $u(l\pm)$, both values of which are prescribed, as well as one or more values of ϕ at interior mesh points of the interval. We shall describe this technique first, using a rather direct finite difference argument which is suggested by a method described in Peyret and Taylor [1] in connection with a treatment of the Navier-Stokes equations in vorticity-stream function variables. We then discuss and illustrate another, closely related, technique which arises from an application of a compact finite difference method (Rose[2]) to this problem.

4. Some Standard Finite Difference Approaches.

We adopt the standard finite-difference notations $x_i = i\,\Delta x$, $h = \Delta x/2$, $u(x_i) = u_i$. Divide $[l_-, l_+]$ into M non-overlapping intervals $I_j \equiv \{x|\, x_{j-1/2} \le x \le x_{j+1/2}\}$ with centerpoints x_j, j=1/2, 3/2,..., M-1/2. Also, recall the central average and difference operators

$$\mu\,\phi_j \equiv (\phi_{j+1/2} + \phi_{j-1/2})/2\,,\quad \Delta\,\phi_j \equiv (\phi_{j+1/2} - \phi_{j-1/2}). \qquad (4.1)$$

introduced earlier.

With $\kappa = 1/\Delta x^2$, a standard finite difference approach to solving (4.3) is to consider the coupled difference equations

$$\text{(a)}\quad f_i = \kappa\,\Delta^2 v_i \qquad\qquad i = 1, 2, \ldots, M-1 \qquad (4.2)$$

$$\text{(b)}\quad -v_i = \kappa\,\Delta^2 \phi_i$$

each of which may be separately solved with Dirichlet-type data. For the clamped beam, the boundary conditions $\phi(1\pm) = 0$, $u(1\pm) = 0$ translate into

$$\text{(a)} \quad \phi_0 = \phi_M = 0, \tag{4.3}$$

$$\text{(b)} \quad u_0 = u_M = 0.$$

Were v_i known, (4.2b) could easily be solved for ϕ under either pair of these boundary conditions by a standard tri-diagonal solver (in case (b) we can add the condition $\phi_0 = 0$) In order to solve (4.2a) we will specify values of v_0 and v_M Consistency requires that these values be related to values of ϕ and u at points on or near the boundary points and we may write

$$v_0 = B_0(\phi,u), \tag{4.4}$$

$$v_M = B_M(\phi,u)$$

where B is a suitable boundary operator which incorporates the prescribed boundary conditions for ϕ and u.

One method to obtain a boundary operator B is to use a Taylor series approximation as follows: Write

$$\phi_1 = \phi_0 + \Delta x\, u_0 - \tfrac{1}{2}(\Delta x)^2 v_0 + \ldots \tag{4.5}$$

$$\phi_{M-1} = \phi_M - \Delta x\, u_M - \tfrac{1}{2}(\Delta x)^2 v_M + \ldots,$$

so that by imposing the homogeneous clamped beam boundary conditions on ϕ and u we find

$$2\,\phi_1 = -(\Delta x)^2 v_0 + \ldots \tag{4.6}$$

$$2\,\phi_{M-1} = -(\Delta x)^2 v_M + \ldots.$$

Higher order extrapolations are discussed in Peyret and Taylor[1].

5. A Compact Scheme for the Clamped Beam.

Our objective will be to describe a difference scheme based upon these ideas which solves (1) under the clamped beam boundary conditions stated in (4.3), i.e., $\phi_0 = \phi_M = 0$, $u_0 = u_M = 0$. The boundary conditions for v will not require the additional use of a Taylor series other than that which is implicit in the proposed difference equations.

We will consider the coupled system

$$-\mu v_i = \kappa\, \Delta^2 \phi_i, \qquad i = 1,2,\ldots, M-1 \qquad (5.1)$$

$$f_j = \kappa\, \Delta^2 v_j, \qquad j = 3/2,,5/2.., \ M-3/2.$$

The first of these corresponds to (4.5) for the primary variables associated with a compact scheme for solving $\phi'' = -v$ and the second corresponds to (4.6) for the dual variables associated with a compact scheme for solving $v'' = f$.

Using the definitions of u given by (4.2) the values of v at the endpoints of the dual grid are found to satisfy

$$-\Delta x\, v_j = (u_{j+1/2} - u_{j-1/2}) \qquad j = 1/2,\ M-1/2 \qquad (5.2)$$

and imposing the Neumann-type boundary conditions for u gives

$$u_1 = -\Delta x\, v_{1/2}, \quad u_{M-1} = \Delta x\, v_{M-1/2}.$$

Using the definition of u leads to

$$2\kappa\,(\phi_1 - \phi_{1/2}) = -v_{1/2} \qquad (5.3)$$

$$2\kappa\,(\phi_{M-1/2} - \phi_{M-1}) = v_{M-1/2}.$$

One difference between this and the treatment of boundary conditions for the standard finite difference method described earlier lies in the position on the mesh of at which the boundary values assigned to v are imposed. In the standard problem v_i was considered a primary variable and a Taylor expansion was required in order to furnish an additional equation which allowed boundary values of v at the endpoints of the domain to be determined by ϕ_i. In the compact scheme, on the other hand, the v_j are dual variables and the boundary conditions , which determine their values at the boundary of the dual mesh, are a consequence of applying the difference equations $\mu v_i = -\kappa \Delta^2 \phi_i$ when $i = 1/2, M-1/2$. The cyclic reduction technique allowed the dual variables to be solved directly with such data.

This distinction between variables defined on a primary and dual grid is a common feature in treating the Navier-Stokes equations in primitive variables where the pressure term is commonly associated with the dual grid. This connection is worth further exploration.

Numerical Examples for the Clamped Beam.

The following table compares numerical results for the clamped beam problem using the standard finite difference method with extrapolation techniques to set the boundary conditions and the compact scheme described above.

Error Norm Comparison of Standard and Compact Schemes for a Clamped Beam

test solution : $\phi = x^2(1-x)^2$:

Standard scheme using 1st order extrapolated BC:

# intervals	f	u	w
12	1.07692	.213675	1.92901e-2
24	.324713	7.24377e-2	1.17071e-2
48	.124513	5.87587e-2	7.65804e-3
96	5.41195e-2	3.55117e-2	4.37644e-3

Standard scheme using 2nd order extrapolated BC;

# intervals	f	u	w
12	2.80765e-2	.14098	.863799
24	1.77672e-2	9.04946e-2	.313276
48	9.78894e-3	6.49779e-2	.1223
96	5.07409e-3	3.71921e-2	5.17336e-2

Compact scheme ;

# intervals	f	u	w
12	2.82377e-2	3.8407e-5	6.94711e-3
24	6.9977e-3	2.25417e-6	1.73624e-3
48	1.74305e-3	1.58858e-7	4.34032e-4
96	4.35881e-4*	1.39912e-7*	1.08463e-4*

*precision doubtful because of machine limitations.

Note that, as predicted, the compact scheme furnishes second-order accuracy for the variables.

6. A Time-dependent Stokes-type Problem

The time-dependent equations

$$\omega_t = \nabla^2 \omega,$$

$$\omega = -\nabla^2 \chi$$

can serve as a model for studying the effect of handling boundary conditions and differs from the Navier-Stokes equations in that only the convective terms have been omitted. A method for adapting ADI techniques to solve this problem by a compact scheme is outlined in [2].

A code to test the accuracy of solving this problem by a compact scheme has been developed by J. M. Klimkowski and is reported upon separately.

References

[1] Peyret, R. and Taylor, T.D., *Computational Methods for Fluid Flow*, Springer-Verlag, 1983.

[2] Rose, Milton E., *Compact Finite Volume Methods for the Diffusion Equation*, submitted to J. Comp. Phys.