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A Complete Analytical Solution for the Inverse Instantaneous Kinematics of a Spherical-Revolute-Spherical (7R) Redundant Manipulator

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Abstract

Using a method based upon resolving joint velocities using reciprocal screw quantities, compact analytical expressions are generated for the inverse solution of the joint rates of a seven revolute (spherical-revolute-spherical) manipulator. The method uses a sequential decomposition of screw coordinates to identify reciprocal screw quantities used in the resolution of a particular joint rate solution, and also to identify a Jacobian null-space basis used for the direct solution of optimal joint rates. The results of the screw decomposition are used to study special configurations of the manipulator, generating expressions for the inverse velocity solution for all non-singular configurations of the manipulator, and identifying singular configurations and their characteristics.

This paper therefore serves two functions: a new general method for the solution of the inverse velocity problem is presented; and complete analytical expressions are derived for the resolution of the joint rates of a seven degree of freedom manipulator useful for telerobotic and industrial robotic application.

1. Introduction

The inverse velocity problem for a redundant manipulator is underdetermined. That is, an infinite number of joint rate solutions providing a required end effector velocity will exist. A means of resolving the "best" joint rate solution and computation efficiency are requirements of an inverse velocity solution method. To form a complete inverse instantaneous (velocity) kinematic solution for a specific manipulator, special configurations and their characteristics must be identified.

Several approaches for the resolution of "optimal" joint rates for redundant manipulators have been proposed. These techniques can be classified as local (e.g. see Hollerbach and Suh[1] and the references of [1], [2], [3] and [9]), global (e.g. see Kazerounian and Wang[2] and the references of [2], and [3]), kinematic function based (e.g. see Wampler and Baker [3]), and constraint based (e.g. see Baillieul[4]). Global, kinematic function, and constraint techniques (in a local sense), have the advantage of maintaining the same joint displacements during repetitive execution of a task. Local optimizations have the disadvantage of being nonrepetitive, and globally nonoptimal, but remain an important technique where insufficient information or computational time is available for global optimization.

Analytical derivation of expressions for the inverse velocity solution allow a computational efficiency difficult to achieve with numerical solution schemes. Works by Sugimoto[5] ("orthogonal basis" decomposition of screw coordinates), Hunt[6] (direct inversion of a screw coordinate matrix (Jacobian) using convenient frames of reference), and Stanisic et. al.[7] (canonical reference for three parameter motion) are recent examples of techniques for the derivation of analytical expressions for the inverse velocity solution of nonredundant manipulators.

In this work, an inverse velocity solution based on a decomposition of screw coordinates is presented (Sections 2,3 and 4), and is applied to the derivation of analytical results for a seven revolute (7R) manipulator (Section 5). The decomposition identifies reciprocal screw quantities (terminology reviewed in Section 2) used for a particular joint velocity solution, and a basis for the Jacobian null-space useful in joint rate optimization. Optimization for quadratic objective functions, yields direct solutions for the optimum (local) joint rates in terms of pseudo-inverses of a weighting of the null-space basis. These solutions require the inverse of matrices of reduced order, (e.g. a scalar quantity or a seven degree of freedom robot), in comparison to pseudo-inverses of the manipulator Jacobian.

The 7R manipulator analyzed features a spherical base, a revolute elbow, and a spherical wrist. This joint layout was proposed by Hollerbach[8] as an "optimal" seven degree of freedom layout, for which one of the objectives was the elimination of singularities caused by single joint displacement conditions. As such, the robot should be useful for telerobotic and industrial application where a degree of autonomous motion is required (i.e. preplanning for singularity avoidance is not possible). Analytical results are derived for the inverse velocity solution for all non-singular configurations. Singular configurations are examined and characterized in terms of the screw decomposition.

2. Resolving Joint Velocities Using Reciprocal Screws

A *screw* is a line in space having an associated linear *pitch.* [11] As such it represents five independent parameters (four for the line, one for the pitch). Associating an amplitude acting on the screw yields six independent parameters. Screw quantities are natural entities for describing spatial instantaneous motion (velocities) and forces and moments, (i.e. any velocity can be considered to be a rotational velocity about an axis and a translational velocity parallel to the same axis, and any system of forces and moments is equivalent to a force in a direction and a couple in a plane perpendicular to the direction).

A screw can be represented as a dual vector by its screw coordinates, $\{\mathbf{\$}; \mathbf{\$}_o\}^T$,

$$\$ = \{\mathbf{\$}; \mathbf{\$}_o\}^T = \{\mathbf{L}; \mathbf{L}_o + p_L \mathbf{L}\}^T \tag{1}$$

where $\mathbf{L}$ and $\mathbf{L}_o$ are respectively the direction of the line and its moment about a reference origin (Plucker line coordinates), and p_L is the pitch of the screw. A screw quantity is represented by the product of an amplitude and a screw,

$$\mathbf{S} = \alpha\{\mathbf{\$}; \mathbf{\$}_o\}^T = \{\mathbf{s}; \mathbf{s}_o\}^T \tag{2}$$

If S is the velocity of a rigid body, *(a twist about a screw)*, then α is referred to as the *twist amplitude*, $\mathbf{s}$ is the angular velocity vector of the body, and $\mathbf{s}_o$ is the translational velocity of a point on the rigid body (extended to be) coincident with the reference origin. If S represents a system of forces *(a wrench acting on a screw)*, $\mathbf{s}$ is the resultant vector of the forces acting on the body, and $\mathbf{s}_o$ is the resultant vector of the moments acting on the body plus the sum of the moments of all forces about the reference origin.

The reciprocal product of two screw quantities is the inner product,

$$\mathbf{S}_1 \,\chi\, \mathbf{S}_2 = \mathbf{s}_1 \cdot \mathbf{s}_{o_2} + \mathbf{s}_{o_1} \cdot \mathbf{s}_2 \tag{3}$$

The reciprocal product of a "twist" and a "wrench" quantifies a rate of work. Two screws are reciprocal when their reciprocal product is zero, e.g. a body having a motion described by a twist, $\mathbf{S}_i$, subjected to a force system described by a wrench on a screw reciprocal to, $\mathbf{S}_i$ performs no work. A set of r linearly independent screws forms an r-system. Reciprocal to an r-system is a $(6-r)$-system of screws [12].

If a rigid body is acted upon by twist amplitudes about a chain of n screws the resulting velocity, **M**, is

$$\alpha_1 \$_1 + \alpha_2 \$_2 + \cdots + \alpha_n \$_n = \mathbf{M} \tag{4}$$

In **robotics application** the joint axes are the screws, $\$_i$, $i=1,n$, (and the screw coordinates can be shown to be equivalent to the columns of the manipulator Jacobian with respect to the frame of reference, i.e., $[\mathbf{J}] = [\$_1 \cdots \$_n]$). The joint rates, $\dot{q}_i$, $i=1,n$, are the twist amplitudes of equation (4), and **M** is the end effector velocity. In the inverse velocity problem we are concerned with finding $\dot{q}_i$, $i=1,n$ such as to provide a required **M**.

The joint rates can be resolved using reciprocal screw quantities. That is, if a wrench on a screw, **B** is known such that $\mathbf{B} \,\chi\, \$_i = 0$, $i \neq n$, and $\mathbf{B} \,\chi\, \$_n \neq 0$, the nth twist amplitude (joint rate) can be resolved by taking reciprocal products of both sides of equation (4) with **B**.

$$(\alpha_1 \$_1 + \alpha_2 \$_2 + \cdots + \alpha_n \$_n) \,\chi\, \mathbf{B} = \alpha_n \$_n \,\chi\, \mathbf{B} = \mathbf{M} \,\chi\, \mathbf{B} \tag{5}$$

and therefore

$$\alpha_n = \dot{q}_n = (\mathbf{M} \,\chi\, \mathbf{B})/(\$_n \,\chi\, \mathbf{B})$$

Equation (5) represents a virtual work expression, i.e. the rate of work done by the end effector moving at the rate, **M** when subjected to the wrench, **B**, must be equal to the rate of work generated by the joint velocity, $\dot{q}_n$ about $\$_n$ when subjected to the same wrench, since $\mathbf{B} \,\chi\, \$_i = 0, i \neq n$.

3. An Inverse Velocity Solution Based on a Decomposition of Screw Coordinates

After a joint velocity, e.g. $\dot{q}_n$ of equation (5), is resolved, its contribution to the end effector velocity can be removed, e.g. $\mathbf{M}_{eff} = \mathbf{M} - \dot{q}_n \$_n$. Resolution of the next joint velocity, e.g. $\dot{q}_{n-1}$, requires only a screw quantity reciprocal to the remaining screws, e.g. $\$_1 \cdots \$_{n-2}$. These reciprocal screw quantities can be found using the sequential decomposition presented in this section.

Sequentially the jth screw of the Jacobian is decomposed into twist amplitudes, α_{ij}, about the joint screws, $\$_i$, $i<j$, and the complement of a wrench, $\mathbf{B}_j$, having a null reciprocal product (NRP) with the joint screws, $\$_i$, $i<j$. That is,

$$\$_j = \mathbf{B}_j^* + \sum_{i=1}^{j-1} \alpha_{ij}\$_i \tag{6}$$

where $\mathbf{B}_j \chi \$_i = 0$, $i<j$, and $\mathbf{B}^* = \{\mathbf{b}_{o_j}; \mathbf{b}_j\}^T$ is defined as the wrench complement. By sequentially taking reciprocal products with $\mathbf{B}_i$, and noting $\mathbf{B}_j^* \chi \mathbf{B}_i = \mathbf{B}_j \chi \mathbf{B}_i^* = \mathbf{B}_j \chi (\$_i - \alpha_{i-1i}\$_{i-1} - \cdots - \alpha_{1i}\$_1) = 0$ for $i<j$, the required twist amplitudes, α_{ij}, are determined to be

$$\alpha_{ij} = (\$_{ij} \chi \mathbf{B}_i)/(\$_i \chi \mathbf{B}_i), \quad \text{where } \$_{ij} = \$_j - \sum_{k=i+1}^{j-1} \alpha_{kj}\$_k \tag{7}$$

Notice this is a Gram-Schimdt type decomposition [13] where the inner product is the reciprocal product, and dual vectors are being decomposed.

If $\$_j$ is linearly dependent on the previous j-1 linearly independent screws then a set of unique c_i exists such that $c_1\$_1 + c_2\$_2 + \cdots + c_{j-1}\$_{j-1} + \$_j = 0$. In this case the decomposition returns a null $\mathbf{B}_j^*$ value, and the values of α_{ij}, $i<j$, of equation (7) correspond to the negatives of the values of c_i, $i<j$. These values of c_i together with a value of one (1) associated with $\$_j$ form a vector for the null-space basis of [J] ($\equiv [\$_1 \cdots \$_n]$). For the decomposition of the remaining screws, $\$_j$ and its associated null $\mathbf{B}_j$ are not considered.

Let us assume that the first r screws of [J] are linearly independent, where r is the rank of [J], and the remaining $n-r$ screws are linearly dependent on the first r screws, (this is achieved by removing the linearly dependent screws as they are found in the decomposition sequence). The complements of the NRP wrenches may be expressed as

$$[\mathbf{B}^*] = [\mathbf{J}][\mathbf{d}] \tag{8}$$

where $[\mathbf{B}^*] = [\mathbf{B}_1^* \ \mathbf{B}_2^* \cdots \mathbf{B}_r^* \ \mathbf{0} \cdots \mathbf{0}]$, and $d_{ij} = 1$ if $i=j$, or $-\alpha_{ij}$ if $i<j$ and $i \le r$, or 0 otherwise

The last $n-r$ columns of [**d**] form a basis for the null-space of [**J**]. The ordering of the screws when doing the sequential decomposition is arbitrary, but must be maintained throughout the complete solution. The subscripts associated with the screw quantities in the decomposition can be considered to refer to integers of a set, $\$_{ord}$, corresponding to the order of decomposition.

A particular joint velocity solution can be formed by decomposing **M** into joint rates about the "linearly independent" screws of [J], i.e. $\mathbf{M} = \sum_{j=1}^{r} \dot{q}_{j_{part}}\$_j$, where

$$\dot{q}_{j_{part}} = (\mathbf{M}_j \chi \mathbf{B}_j)/(\$_j \chi \mathbf{B}_j), \ j=r, 1, -1, \quad \text{with} \quad \mathbf{M}_j = \mathbf{M} - \sum_{m=j+1}^{r} \dot{q}_{m_{part}}\$_m \tag{9}$$

A general joint velocity solution can be expressed as

$$\{\dot{q}\}_{n\times 1} = \{\dot{q}\}_{part_{n\times 1}} + [a]_{n\times(n-r)}\{\lambda\}_{(n-r)\times 1} \tag{10}$$

where $[a]$ is a null-space for the joint screw coordinates (Jacobian). The particular joint velocity solution of equation (9) corresponds to a solution with $\dot{q}_{j_{part}} = 0$ for $j>r$. If this particular solution is used and $[a]$ is formed from the last $n-r$ columns of [**d**], recalling that [**d**] has unit values on the diagonal and is upper triangular, the $\{\lambda\}$ of equation (10) are seen to correspond to the values of $\dot{q}_j$, $j>r$. These rates shall be referred to as the redundant joint rates.

Optimizing the joint rate solution involves finding the optimal basis multipliers, $\{\lambda\}_{opt}$, (equivalent to the optimal "redundant" joint rates). Substitution into equation (10) then yields the optimal joint rates. For example consider a weighted sum square of the joint velocities, i.e. $f_{obj} = ([\mathbf{W}](\{\dot{q}\}_{part} + [\mathbf{a}]\{\lambda\}))^2$, where [**W**] is a weighting matrix. Differentiating f_{obj} with respect to $\{\lambda\}$ and equating to zero gives

$$\{\lambda\}_{opt} = -([\mathbf{a}]^T[\mathbf{W}]^T[\mathbf{W}][\mathbf{a}])^{-1}[\mathbf{a}]^T[\mathbf{W}]^T[\mathbf{W}]\{\dot{q}\}_{part} \tag{11}$$

Details on using the null-space basis for the optimization of joint rates for obstacle avoidance, joint displacement centering, joint torque minimization, and iterative least squares displacement closure can be found in [10] In each case joint rates are optimized for quadratic objective functions, resulting in direct solutions for the optimal redundant joint rates in terms of a left pseudo-inverse[13] of a weighting of the null-space basis, similar to that of equation (11).

Forming this pseudo-inverse requires the inversion of a $(n-r) \times (n-r)$ matrix, (e.g. for a seven degree of freedom robot with a Jacobian of full rank, this is a scalar quantity).

The results of the screw decomposition characterize the redundancy of the manipulator, indicate the rank of the Jacobian, and allow the solution of the inverse velocity problem. The method is suitable for numerical application, for which the computational costs are discussed in [9]. The method is also useful for the derivation of explicit expressions for the Jacobian null-space basis and null reciprocal product wrenches. These expressions allow analytical solution of the inverse velocity problem, and are useful for the identification of special configurations of the analyzed manipulator, as is demonstrated with an example in Section 5.

4. Multi-arms having Common Redundant Degrees of Freedom

Consider a system comprised of m manipulators (arms) sharing n_c common joint degrees of freedom. The Jacobian of the ith manipulator is composed of the joint screw coordinates individual to the particular manipulator, $[\$]_i$, and the screw coordinates of the common degrees of freedom, $[\$]_c$, i.e., $[\mathbf{J}]_{i\,6\times n_i+n_c} = [[\$]_{i\,6\times n_i}\ \ [\$]_{c\,6\times n_c}]$. The screws $[\$]_i$ are assumed to span the task requirements of the ith manipulator, rendering the $[\$]_c$ joint degrees of freedom "redundant". The individual manipulator joint axes may also have redundancy, (i.e. $n_i > r_i$).

The screw coordinates of $[\mathbf{J}]_i$ can be decomposed yielding $[\mathbf{B}]_i$ and the null-space basis $[\mathbf{a}]_{i\,(n_i+n_c)\times(n_i-r_i+n_c)}$. The null-space bases for each arm can be combined, concatenating the columns for the common degrees of freedom, to yield a null-space basis for the Jacobian of the entire manipulator system. That is,

$$[\mathbf{J}][\mathbf{a}] = \begin{bmatrix} [\$]_{1\,6\times n_1} & \cdot & 0 & [\$]_{c\,6\times n_c} \\ & \cdot & & \vdots \\ 0 & \cdot & [\$]_{m\,6\times n_m} & [\$]_{c\,6\times n_c} \end{bmatrix}_{6m\times(\sum n_i + n_c)} [\mathbf{a}]_{(\sum n_i+n_c)\times(\sum(n_i - r_i)+n_c)} = [\mathbf{0}] \qquad (12)$$

Similarly, particular solutions can be found for each arm and concatenated. The general joint solution solution becomes

$$\begin{Bmatrix} \{\dot q\}_1 \\ \cdot \\ \{\dot q\}_m \\ \{\dot q\}_c \end{Bmatrix}_{(\sum n_i+n_c)\times 1} = \begin{Bmatrix} \{\dot q\}_{1_{part}} \\ \cdot \\ \{\dot q\}_{m_{part}} \\ \{\dot q\}_{c_{part}} \end{Bmatrix}_{(\sum n_i+n_c)\times 1} + [\mathbf{a}]_{(\sum n_i+n_c)\times(\sum(n_i - r_i)+n_c)} \{\lambda\}_{(\sum(n_i-r_i)+n_c)\times 1} \qquad (13)$$

Note that a null-space basis is not a function of frame of reference. That is, a convenient frame of reference can be utilized to form each component of the assembled total system null-space basis. For multi-arm examples the reader is referred to [9] and [10].

5. Analytical Expressions for the Inverse Velocity Solution of a 7R manipulator

Overview

The decomposition of screw coordinates presented in the previous section is used in deriving expressions for $[\mathbf{a}]$ and $[\mathbf{B}]$ for the 7R manipulator illustrated in Figure 1. The manipulator features a spherical group of joints at the base and at the wrist. Hollerbach[8] suggested this joint layout as being the "optimal" for a 7R manipulator, for which one of the objectives of optimality was the elimination of singularities.

Based on the results of the screw decomposition special configurations of the manipulator are identified. These configurations correspond to cases when groups of joint axes become linearly dependent and yet the manipulator retains full motion ability, and to cases of joint dependency leading to motion ability degeneracy (singular configurations). Screw decompositions using two frames of reference are performed to form compact analytical expressions for use in the inverse kinematic solution for all non-singular configurations of the manipulator. Singular configurations are examined and characterized within the context of the screw decomposition.

A screw decomposition

Solution of the inverse kinematic problem can be performed with respect to any frame of reference. A frame of reference was chosen as: $z_{ref\,1}$ aligned in the direction of $\$_5$; $y_{ref\,1}$ in the opposite direction to that of $\$_4$; with the origin of the reference frame located at the intersection of the three wrist axes, see Figure 1. This reference frame was chosen to exploit the decoupling provided by the spherical wrist, and to minimize the complexity of the Jacobian terms. With respect to this reference frame the screw coordinates of the joint axes (columns of the Jacobian) are †

$$
\begin{aligned}
\$_1^{ref\,1} &= \{S_2C_3C_4+C_2S_4,\ -S_2S_3,\ -S_2C_3S_4+C_2C_4;\ -S_2S_3(C_4g+h),\ -S_2C_3g-(S_2C_3C_4+C_2S_4)h,\ S_2S_3S_4g\}^T \\
\$_2^{ref\,1} &= \{-S_3C_4,\ -C_3,\ S_3S_4;\ -C_3(C_4g+h),\ S_3(g+C_4h),\ C_3S_4g\}^T \\
\$_3^{ref\,1} &= \{S_4,\ 0,\ C_4;\ 0,\ -hS_4,\ 0\}^T \\
\$_4^{ref\,1} &= \{0,\ -1,\ 0;\ -h,\ 0,\ 0\}^T \\
\$_5^{ref\,1} &= \{0,\ 0,\ 1;\ 0,\ 0,\ 0\}^T \\
\$_6^{ref\,1} &= \{S_5,\ -C_5,\ 0;\ 0,\ 0,,\ 0\}^T \\
\$_7^{ref\,1} &= \{-C_5S_6,\ -S_5S_6,\ C_6;\ 0,\ 0,\ 0\}^T
\end{aligned}
\tag{14}
$$

Decomposition of the screw set in the order, $\$ord = \{5, 6, 7, 4, 3, 2, 1\}$, yields the null reciprocal wrenches,

$$
\begin{aligned}
\mathbf{B}_5^{ref\,1} &= \{0,\ 0,\ 0;\ 0,\ 0,\ 1\}^T \\
\mathbf{B}_6^{ref\,1} &= \{0,\ 0,\ 0;\ S_5,\ -C_5,\ 0\}^T \\
\mathbf{B}_7^{ref\,1} &= \{0,\ 0,\ 0;\ -C_5S_6,\ -S_5S_6,\ 0\}^T \\
\mathbf{B}_4^{ref\,1} &= \{-h,\ 0,\ 0;\ 0,\ 0,\ 0\}^T \\
\mathbf{B}_3^{ref\,1} &= \{0,\ -S_4h,\ 0;\ 0,\ 0,\ 0\}^T \\
\mathbf{B}_2^{ref\,1} &= \{0,\ 0,\ C_3S_4g;\ 0,\ 0,\ 0\}^T \\
\mathbf{B}_1^{ref\,1} &= \{0,\ 0,\ 0;\ 0,\ 0,\ 0\}^T
\end{aligned}
\tag{15}
$$

A particular joint rate solution can now be resolved. Let $\mathbf{M}^{ref\,1} = \{\omega_x, \omega_y, \omega_z, v_x, v_y, v_z\}^T$ represent the required task space motion. This screw quantity can be explicitly decomposed onto the joint screws yielding,

$$\dot{q}_2 = \mathbf{M}\,\chi\,\mathbf{B}_2^{ref\,1} / \$_2^{ref\,1}\,\chi\,\mathbf{B}_2^{ref\,1} = v_z/(C_3S_4g) \tag{16}$$

$$\dot{q}_3 = \frac{(\mathbf{M} - \dot{q}_2\$_2^{ref\,1})\,\chi\,\mathbf{B}_3^{ref\,1}}{\$_3^{ref\,1}\,\chi\,\mathbf{B}_3^{ref\,1}} = -(v_y - \frac{v_zS_3(g + C_4h)}{C_3S_4g})/S_4h$$

.

etc.

.

where $\dot{q}_i \equiv \dot{\theta}_i$

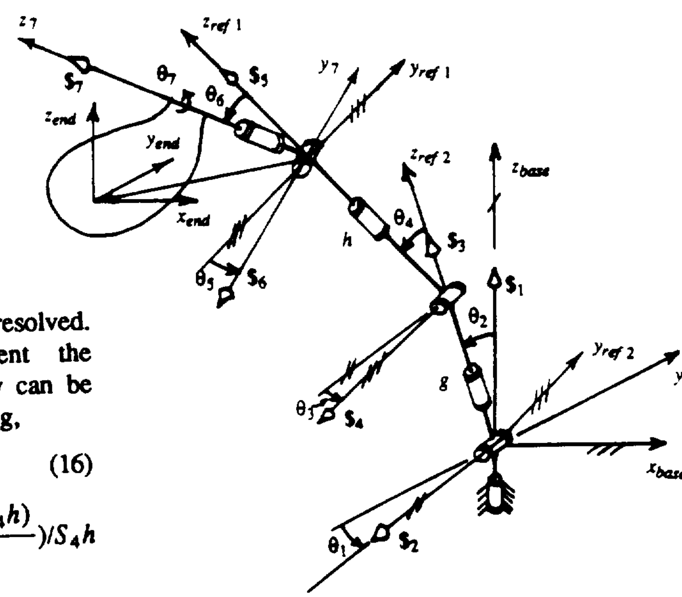

Figure 1 - 7 R Spherical-Revolute-Spherical Manipulator

Alternatively efficient customized code (ignoring zero (0) operations and one (1) multiplications) can be produced at this point for the particular joint rate solution. The operations required for such a solution are; $\dot{q}_2$: 1 × and 0 + , $\dot{q}_3$: 6 × and 5 + , $\dot{q}_4$: 3 × and 2 + , $\dot{q}_7$: 3 × and 2 + , $\dot{q}_6$: 5 × and 4 + , $\dot{q}_5$: 0 × and 0 + , for a total of 18 × and 13 + . No computational costs are involved in finding the reciprocal wrenches once the Jacobian screw coordinates (equation (14)) are known.

The first joint axis for this order of decomposition corresponds to the redundant screw. Decomposing the screw coordinates of this joint yields the null-space basis,

$$
[\mathbf{a}] = d_{j7} = \begin{bmatrix}
1 \\
-S_2S_3/C_3 \\
(-S_2g-S_2C_4h-C_2C_3S_4h)/(C_3S_4h) \\
0 \\
(S_2C_4S_6g+S_2S_4C_5C_6g+S_2S_6h)/(C_3S_4S_6h) \\
S_2S_5g/(C_3h) \\
-S_2C_5g/(C_3S_6h)
\end{bmatrix}
\tag{17}
$$

† Details are included as Appendix 1, $S_i \equiv \mathrm{Sin}(\theta_i)$, $C_i \equiv \mathrm{Cos}(\theta_i)$

The expressions in the null-space basis indicate that $\$_4$ has no component in the null (is linearly independent), and therefore in general (special configurations excepted) joints $\$_1$, $\$_2$, $\$_3$, $\$_5$, $\$_6$ and $\$_7$ as a group have one degree of redundancy. This basis can be utilized in the optimization of the joint rates (e.g. equation (11)).

Special configuration identification

Conditions which cause a normally non-zero wrench, $\mathbf{B}_i$, to become null correspond to special configurations of a manipulator. These special configurations may correspond to a linear dependency within a "redundant group" of joints causing the joint initially chosen to be last (e.g. joint 1 in the above decomposition) to become linearly independent and hence unsuitable as the "redundant" joint. In this case reordering the screws with one of the linearly dependent joints as the redundant joint will yield a complete set of [**B**].

The special configuration may also correspond to further joint linear dependency, (i.e., an increase in the dimension of the null-space of [**J**]). In this case, reordering of the decomposition will find $r<6$ non-zero wrenches, and there will exist a set of linearly independent wrenches, $\mathbf{W}_i$, $i=1,6-r$, reciprocal to the screws of [**J**], where r is the rank of [**J**]. The manipulator will not be able to instantaneously produce motions having non-zero rates of work subject to $\mathbf{W}_i$. This corresponds to a loss of a degree(s) of instantaneous end effector motion capability, and is commonly referred to as a *singularity*. In [9] the authors present a scheme for instantaneously planning "optimal" alternative motion specifications satisfying the required reciprocity with $\mathbf{W}_i$, for manipulators at or near singular configurations.

The above decomposition demonstrated that for the 7R manipulator, typically any one of $\$_1$, $\$_2$, $\$_3$, $\$_5$, $\$_6$ and $\$_7$ could be chosen to represent the redundancy (the "redundant joint") of the manipulator. Furthermore, since six typically non-zero **B** values were found, the Jacobian was seen to normally be of full rank.

Examination of the wrenches of equation (15) reveal null **B** values occur if $C_3=0$, $S_4=0$ or $S_6=0$. If $C_3=0$, then $\$_2$, $\$_3$, $\$_5$, $\$_6$ and $\$_7$ become linearly dependent causing $\$_1$ to be unsuitable choice for redundant joint. Similarly if $S_6=0$ then $\$_5$ and $\$_7$ become linearly dependent, again rendering $\$_1$ as an unsuitable choice for redundant joint. Reordering of the decomposition (performed below) in both of these cases will find six non-zero **B** values, indicating that these configurations do not correspond to singularities. Reordering the decomposition for $S_4=0$, finds only five non-zero wrenches indicating a singular configuration. This case and multiple joint displacement conditions leading to loss of task space freedom are considered later.

A second screw decomposition

A decomposition order having $\$_5$ or $\$_7$ as the final joint axis screw coordinates to be decomposed would be suitable for either $C_3=0$ or $S_6=0$. It is convenient to reference the screws with respect to a frame located at the base spherical group of joints for such a decomposition. Consider the reference frame $ref\,2$ illustrated in Figure 1, where $z_{ref\,2}$ is in the direction of $\$_3$, and $y_{ref\,2}$ is in the opposite direction to that of $\$_4$. The joint screw coordinates with respect to this frame of reference are,

$$\begin{aligned}
\$_1^{ref\,2} &= \{S_2C_3,\ -S_2S_3,\ C_2;\ 0,\ 0,\ 0\}^T \\
\$_2^{ref\,2} &= \{-S_3,\ -C_3,\ 0;\ 0,\ 0,\ 0\}^T \\
\$_3^{ref\,2} &= \{0,\ 0,\ 1;\ 0,\ 0,\ 0\}^T \\
\$_4^{ref\,2} &= \{0,\ -1,\ 0;\ g,\ 0,\ 0\}^T \\
\$_5^{ref\,2} &= \{-S_4,\ 0,\ C_4;\ 0,\ -S_4g,\ 0\}^T \\
\$_6^{ref\,2} &= \{C_4S_5,\ -C_5,\ S_4S_5;\ C_5(g+C_4h),\ S_5(C_4g+h),\ S_4C_5h\}^T \\
\$_7^{ref\,2} &= \{-C_4C_5S_6-S_4C_6,\ -S_5S_6,\ -S_4C_5S_6+C_4C_6;\ S_5S_6(g+C_4h),\ -g(C_4C_5S_6+S_4C_6)-C_5S_6h,\ S_4S_5S_6h\}^T
\end{aligned} \tag{18}$$

Decomposition of the screw set in the order, $\$ord=\{3, 2, 1, 4, 5, 6, 7\}$, yields the null reciprocal wrenches,

$$\begin{aligned}
&\mathbf{B}_3^{ref\,2} = \{0,0,0;0,0,1\}^T,\quad \mathbf{B}_2^{ref\,2} = \{0,0,0;-S_3,-C_3,0\}^T,\quad \mathbf{B}_1^{ref\,2} = \{0,0,0;S_2C_3,-S_2S_3,0\}^T \\
&\mathbf{B}_4^{ref\,2} = \{g,0,0;0,0,0\}^T,\quad \mathbf{B}_5^{ref\,2} = \{0,-S_4g,0;0,0,0\}^T,\quad \mathbf{B}_6^{ref\,2} = \{0,0,S_4C_5h;0,0,0\}^T \\
&\mathbf{B}_7^{ref\,2} = \{0,0,0;0,0,0\}^T
\end{aligned} \tag{19}$$

For this order of decomposition $\$_7$ is the redundant joint, and the null-space basis yielded by the decomposition is the result of equation (17) multiplied by $-C_3S_6h/(S_2C_5g)$.

Examination of the NRP wrenches of equation (19) reveals that null **B** values occur if $S_2 = 0$, $S_4 = 0$ or $C_5 = 0$. For the conditions $S_2 = 0$ or $C_5 = 0$ the results of the first decomposition (equations (14)-(17)) are suitable for use. Hence the above two decompositions of screw coordinates provide expressions for the inverse velocity solution for all non-singular configurations of the manipulator.

Singular configurations

The only single joint displacement condition which causes a loss of motion degree of freedom is $S_4 = 0$, corresponding to a straight arm configuration. In this case decomposition of the screw coordinates, regardless of the order chosen, will yield only five reciprocal wrenches. That is, the rank of the Jacobian is five, the dimension of its null-space is two, and there is a screw, **W**, (a 1-system), reciprocal to all of the joint screws. The manipulator instantaneously cannot produce a motion having a non-zero reciprocal product with **W**. A decomposition is performed below for $S_4 = 0$, generating analytical expressions for [**B**] and $[a]$, and **W** is identified.

Further examination of the wrenches of equations (15) and (19), and the joint screw coordinates of equations (14) and (18), indicate that motion degeneracies (singularities) are also present for multiple joint displacement conditions (e.g.: $S_2 = 0$ and $C_3 = 0$; $S_6 = 0$ and $C_5 = 0$; and $S_2 = 0$ and $S_6 = 0$). Decompositions for these cases are also performed below. The four cases are illustrated in Figure 2.

$\rightarrow S_4 = 0$

Using *ref* 1 as the reference the screw coordinates for $S_4 = 0$ reduce to:

$$\begin{aligned} \$_1^{ref\,1} &= \{S_2C_3,\ -S_2S_3,\ C_2;\ -S_2S_3(g+h),\ -S_2C_3(g+h),\ 0\}^T \\ \$_2^{ref\,1} &= \{-S_3,\ -C_3,\ 0;\ -C_3(g+h),\ S_3(g+h),\ 0\}^T \\ \$_3^{ref\,1} &= \{0, 0, 1; 0, 0, 0\}^T \\ &\$_4^{ref\,1}, \$_5^{ref\,1}, \$_6^{ref\,1}, \$_7^{ref\,1} \text{ as in Equation (14)} \end{aligned} \tag{20}$$

Decomposing the screw coordinates in the order $\$ord = \{5, 6, 7, 4, 2, 1, 3\}$, (shifting 3 to the end when a null $\mathbf{B}_3^{ref\,1}$ is found), yields:

$$\begin{aligned} &\mathbf{B}_5^{ref\,1}, \mathbf{B}_6^{ref\,1}, \mathbf{B}_7^{ref\,1}, \mathbf{B}_4^{ref\,1} \text{ as in Equation (15)} \\ &\mathbf{B}_2^{ref\,1} = \{0, S_3(g+h), 0; 0, 0, 0\}^T \\ &\mathbf{B}_1^{ref\,1} = \{0, 0, 0; 0, 0, 0\}^T, \quad \mathbf{B}_3^{ref\,1} = \{0, 0, 0; 0, 0, 0\}^T \end{aligned} \tag{21}$$

The Jacobian null-space basis generated by the decomposition of $\$_1^{ref\,1}$ and $\$_3^{ref\,1}$ is:

$$[a] = \begin{bmatrix} 1 & S_2C_3/S_3 & 0 & 0 & -(C_2S_3S_6 - S_2S_5C_6)/S_3S_6 & -S_2C_5/S_3 & -S_2S_5/S_3S_6 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}^T \tag{22}$$

A screw reciprocal to $\$_1 \cdots \$_7$ is $\mathbf{W}^{ref\,1} = \{0, 0, 1; 0, 0, 0\}^T$ indicating that a point on the end effector coinciding with the origin of *ref* 1, can have no translational velocity in the $z_{ref\,1}$ direction (the arm direction).

$\rightarrow S_2 = 0$ and $C_3 = 0$

Again using *ref* 1 as the reference the screw coordinates for $S_2 = 0$ and $C_3 = 0$ reduce to:

$$\begin{aligned} \$_1^{ref\,1} &= \{S_4, 0, C_4; 0, -S_4h, 0\}^T \\ \$_2^{ref\,1} &= \{-C_4, 0, S_4; 0, g + C_4h, 0\}^T \\ &\$_3^{ref\,1}, \$_4^{ref\,1}, \$_5^{ref\,1}, \$_6^{ref\,1}, \$_7^{ref\,1} \text{ as in Equation (14)} \end{aligned} \tag{23}$$

Decomposing the screw coordinates in the order $\$ord = \{5, 6, 7, 4, 3, 2, 1\}$, yields:

$$\begin{aligned} &\mathbf{B}_5^{ref\,1}, \mathbf{B}_6^{ref\,1}, \mathbf{B}_7^{ref\,1}, \mathbf{B}_4^{ref\,1}, \mathbf{B}_3^{ref\,1} \text{ as in Equation (15)} \\ &\mathbf{B}_2^{ref\,1} = \{0, 0, 0; 0, 0, 0\}^T, \quad \mathbf{B}_1^{ref\,1} = \{0, 0, 0; 0, 0, 0\}^T \end{aligned} \tag{24}$$

The Jacobian null-space basis generated by the decomposition of $\$_1^{ref\,1}$ and $\$_2^{ref\,1}$ is:

$$[a] = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & (g+C_4h)/S_4h & 0 & -(S_4C_5C_6g+C_4S_6g+S_6h)/S_4S_6h & -S_5g/h & C_5g/S_6h \end{bmatrix}^T \tag{25}$$

A screw reciprocal to $\$_1 \cdots \$_7$ is $\mathbf{W}^{ref\,1} = \{0, 0, 1; 0, 0, 0\}^T$.

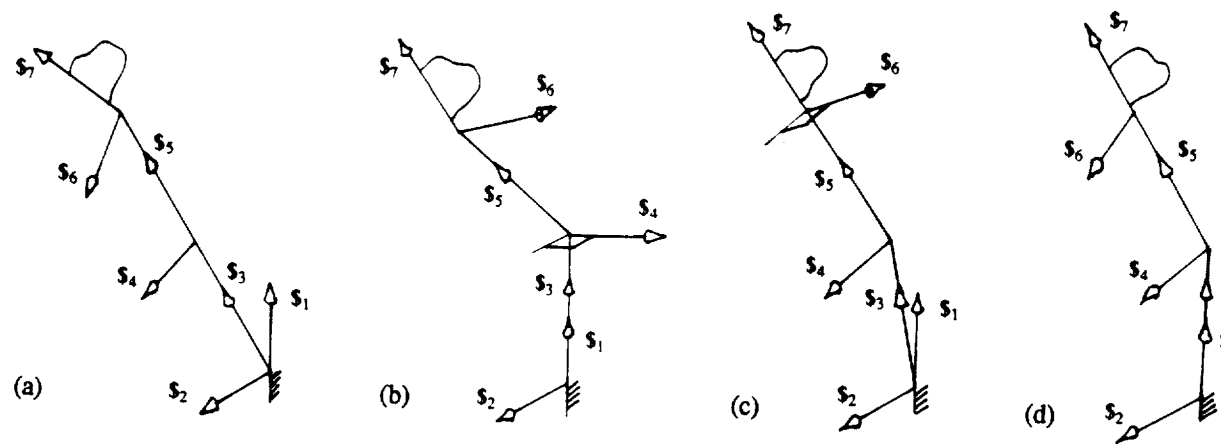

Figure 2 - Singular Configurations, (a) $S_4 = 0$, (b) $S_2 = 0$ and $C_3 = 0$, (c) $S_6 = 0$ and $C_5 = 0$, (d) $S_2 = 0$ and $S_6 = 0$

$\rightarrow S_6 = 0$ and $C_5 = 0$

Using $ref\,2$ as the reference the screw coordinates for $S_6 = 0$ and $C_5 = 0$ reduce to:

$$\begin{aligned} &\$_1^{ref\,2}, \$_2^{ref\,2}, \$_3^{ref\,2}, \$_4^{ref\,2}, \$_5^{ref\,2} \quad \text{as in Equation (18)} \\ &\$_6^{ref\,2} = \{C_4, 0, S_4; 0, C_4g + h, 0\}^T \\ &\$_7^{ref\,2} = \{-S_4, 0, C_4; 0, -S_4g, 0\}^T \end{aligned} \tag{26}$$

Decomposing the screw coordinates in the order $\$ord = \{3, 2, 1, 4, 5, 6, 7\}$, yields:

$$\begin{aligned} &\mathbf{B}_3^{ref\,2}, \mathbf{B}_2^{ref\,2}, \mathbf{B}_1^{ref\,2}, \mathbf{B}_4^{ref\,2}, \mathbf{B}_5^{ref\,2} \quad \text{as in Equation (19)} \\ &\mathbf{B}_6^{ref\,2} = \{0, 0, 0; 0, 0, 0\}^T, \quad \mathbf{B}_7^{ref\,2} = \{0, 0, 0; 0, 0, 0\}^T \end{aligned} \tag{27}$$

The Jacobian null-space basis generated by the decomposition of $\$_6^{ref\,2}$ and $\$_7^{ref\,2}$ is:

$$[a] = \begin{bmatrix} C_3h/S_2g & -S_3h/g & -(S_2S_5g+S_2C_4S_5h+C_2C_3S_4h)/S_2S_4g & 0 & (C_4S_5g + h)/S_4g & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T \tag{28}$$

A screw reciprocal to $\$_1 \cdots \$_7$ is $\mathbf{W}^{ref\,2} = \{0, 0, 1; 0, 0, 0\}^T$.

$\rightarrow S_2 = 0$ and $S_6 = 0$

Using $ref\,1$ as the reference the screw coordinates for $S_6 = 0$ and $C_5 = 0$ reduce to:

$$\begin{aligned} &\$_1^{ref\,1} = \{S_4, 0, C_4; 0, -S_4h, 0\}^T \\ &\$_2^{ref\,1}, \$_3^{ref\,1}, \$_4^{ref\,1}, \$_5^{ref\,1}, \$_6^{ref\,1} \quad \text{as in Equation (14)} \\ &\$_7^{ref\,1} = \{0, 0, 1; 0, 0, 0\}^T \end{aligned} \tag{29}$$

Decomposing the screw coordinates in the order $\$ord = \{5, 6, 4, 3, 2, 1, 7\}$, (where 7 is shifted to the end due to a null $\mathbf{B}_7^{ref\,1}$ being found), yields:

$$\begin{aligned} &\mathbf{B}_5^{ref\,1} = \{0, 0, 0; 0, 0, 1\}^T, \quad \mathbf{B}_6^{ref\,1} = \{0, 0, 0; S_5, -C_5, 0\}^T, \quad \mathbf{B}_4^{ref\,1} = \{-h, 0, 0; -C_5S_5, -S_5^2, 0\}^T \\ &\mathbf{B}_3^{ref\,1} = (1/(h^2 + S_5^2))\{-hS_4C_5S_5, -hS_4(h^2 + S_5^2), 0; S_4C_5^2h^2, S_4C_5S_5h^2, 0\}^T \\ &\mathbf{B}_2^{ref\,1} = \{-C_3C_4g-C_3h+\alpha_{42}h, S_3g+S_3C_4h+\alpha_{32}hS_4, C_3S_4g; -S_3C_4-\alpha_{32}S_4-\alpha_{62}S_5, -C_3+\alpha_{42}+\alpha_{62}C_5, 0\}^T \end{aligned} \tag{30}$$

$\mathbf{B}_1^{ref\,1} = \{0, 0, 0; 0, 0, 0\}^T$, $\mathbf{B}_7^{ref\,1} = \{0, 0, 0; 0, 0, 0\}^T$

where α_{32}, α_{42}, and α_{62} are given in Appendix 2

The Jacobian null-space basis generated by the decomposition of $\$_7^{ref\,1}$ and $\$_1^{ref\,1}$ is:

$$[a] = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T \qquad (31)$$

A screw quantity (not normalized) reciprocal to $\$_1 \cdots \$_7$ is $\mathbf{W}^{ref\,1} = \{ -S_5, C_5, -C_4S_5/S_4 - S_3C_5/C_3S_4; C_5h, S_5h, 0\}^T$.

Further Conditions of Degeneracy

Each of the above conditions correspond to a single loss of task space motion freedom. Further multiple joint displacement conditions leading to the loss of more than one motion degree of freedom can be observed by examination of the reciprocal wrenches of equations (21), (24), (27), and (30). These multiple conditions include: $S_4 = 0$, $S_3 = 0$, and $S_2 = 0$; and $S_4 = 0$, $S_5 = 0$, and $S_6 = 0$; both leading to a loss of two degrees of task space freedom. The multiple condition $S_2 = 0$, $S_3 = 0$, $S_4 = 0$, $S_5 = 0$, and $S_6 = 0$ results in a loss of three task space motion degrees of freedom. The manipulator can never lose more than three degrees of freedom.

6. Conclusions

The general method for the inverse solution of manipulator joint velocities based on a decomposition of screw coordinates presented in this work, has the advantage of inherently identifying a basis for the null-space of the Jacobian. The null-space basis has been shown to be useful for the resolution of locally optimum joint velocities, generating direct solutions for the optimal joint rates in terms of a pseudo-inverse of a weighting of the basis, for quadratic joint rate objective functions. The inverse velocity solution method has been demonstrated to be suitable for the derivation of analytical expressions for manipulators by application to a specific example.

Efficient resolution of the joint velocities for the spherical-revolute-spherical (7R) manipulator is possible using the analytical expressions derived in this work. The identification of the special configurations of the manipulator, and the characterization of singular configurations, make this a complete solution.

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Appendix 1 - Generation of the Screw Coordinate Models

In terms of Denevit and Hartenberg parameters[14] the 7R manipulator links can be described as tabulated in Table 1. The following rotation matrices can be found,

$$[\mathbf{R}]_1 = \begin{bmatrix} C_1 & 0 & S_1 \\ S_1 & 0 & -C_1 \\ 0 & 1 & 0 \end{bmatrix}, \quad [\mathbf{R}]_2 = \begin{bmatrix} C_2 & 0 & -S_2 \\ S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix}, \quad [\mathbf{R}]_3 = \begin{bmatrix} C_3 & 0 & S_3 \\ S_3 & 0 & -C_3 \\ 0 & 1 & 0 \end{bmatrix},$$

$$[\mathbf{R}]_4 = \begin{bmatrix} C_4 & 0 & -S_4 \\ S_4 & 0 & C_4 \\ 0 & -1 & 0 \end{bmatrix}, \quad [\mathbf{R}]_5 = \begin{bmatrix} C_5 & 0 & S_5 \\ S_5 & 0 & -C_5 \\ 0 & 1 & 0 \end{bmatrix}, \quad [\mathbf{R}]_6 = \begin{bmatrix} C_6 & 0 & -S_6 \\ S_6 & 0 & C_6 \\ 0 & -1 & 0 \end{bmatrix},$$

$$[\mathbf{R}]_7 = \begin{bmatrix} C_7 & -S_7 & 0 \\ S_7 & C_7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Link	Variable	Twist	a	d
1	θ_1	90.	0.	0.
2	θ_2	−90.	0.	0.
3	θ_3	90.	0.	g
4	θ_4	−90.	0.	0.
5	θ_5	90.	0.	h
6	θ_6	−90.	0.	0.
7	θ_7	0.	0.	0.

Table 1 - D & H Parameters

The screw coordinates of the joint axes are expressed with respect to the reference frames, *ref* 1 and *ref* 2, in terms of the rotation matrices in Tables 2 and 3 respectively. The operator $z([\mathbf{A}])$ is defined as $z([\mathbf{A}]) = [\mathbf{A}]\{0, 0, 1\}^T$.

A screw quantity, $\mathbf{S}^s$, known in frame s, can be transformed and expressed in frame, f, by

$$\mathbf{S}^f = \{[\mathbf{R}]_{fs}\mathbf{s}^s;\ [\tilde{p}^f]_{fs}[\mathbf{R}]_{fs}\mathbf{s}^s + [\mathbf{R}]_{fs}\mathbf{s}_o^s\} \equiv [\mathbf{T}]_{fs}\mathbf{S}^s \quad \text{where} \quad [\mathbf{T}]_{fs} = \begin{bmatrix} [\mathbf{R}]_{fs} & [\mathbf{0}] \\ [\tilde{p}^f]_{fs}[\mathbf{R}]_{fs} & [\mathbf{R}]_{fs} \end{bmatrix}$$

with $[\mathbf{R}]_{fs}$ the 3×3 rotation matrix describing the orientation of the frame s with respect to the frame f, $[\tilde{p}^f]_{fs}$ a 3×3 skew symmetric (cross product matrix) of the location of the s reference origin with respect to the f origin expressed in the f reference coordinates, and $[\mathbf{0}]$ a 3×3 null matrix. Rotation matrices and displacement vectors required for transformation of an end effector velocity screw known with respect to an inertially oriented, end effector tip located frame, to the reference frames are

$$[\mathbf{R}]_{ref\,1\rightarrow end} = ([\mathbf{R}]_1[\mathbf{R}]_2[\mathbf{R}]_3[\mathbf{R}]_4)^T, \quad \{p^{ref\,1}_{ref\,1\rightarrow end}\} = [\mathbf{R}]_5[\mathbf{R}]_6[\mathbf{R}]_7\{x\}^{end^T}$$

$$[\mathbf{R}]_{ref\,2\rightarrow end} = ([\mathbf{R}]_1[\mathbf{R}]_2[\mathbf{R}(\theta_3)]_z)^T,$$

$$\{p^{ref\,2}_{ref\,2\rightarrow end}\} = \{0, 0, g\}^T + [\mathbf{R}(90)]_x[\mathbf{R}]_4\{0, 0, h\}^T + [\mathbf{R}(90)][\mathbf{R}]_4[\mathbf{R}]_5[\mathbf{R}]_6[\mathbf{R}]_7\{x\}^{end^T}$$

where $\{x\}^{end^T}$ is the tip location with respect to an end effector oriented wrist located frame.

i	$\$_i^{ref\,1}$	
	$\{\$_i^{ref\,1}\}^T$	$\{\$_o^{ref\,1}{}_i\}^T$
5	$\{0, 0, 1\}$	$\{0, 0, 0\}$
6	$z([\mathbf{R}]_5)$	$\{0, 0, 0\}$
7	$z([\mathbf{R}]_5[\mathbf{R}]_6)$	$\{0, 0, 0\}$
4	$z([\mathbf{R}]_4^T)$	$\{x^{ref\,1}_{elbow}\} \times \{\$_4^{ref\,1}\}$
3	$z([\mathbf{R}]_4^T[\mathbf{R}]_3^T)$	$\{x^{ref\,1}_{base}\} \times \{\$_3^{ref\,1}\}$
2	$z([\mathbf{R}]_4^T[\mathbf{R}]_3^T[\mathbf{R}]_2^T)$	$\{x^{ref\,1}_{base}\} \times \{\$_2^{ref\,1}\}$
1	$z([\mathbf{R}]_4^T[\mathbf{R}]_3^T[\mathbf{R}]_2^T[\mathbf{R}]_1^T)$	$\{x^{ref\,1}_{base}\} \times \{\$_1^{ref\,1}\}$

where $\{x^{ref\,1}_{elbow}\} = \{0, 0, -h\}^T$
and $\{x^{ref\,1}_{base}\} = \{x^{ref\,1}_{elbow}\} + [\mathbf{R}]_4^T[\mathbf{R}]_3^T\{0, 0, -g\}^T$

Table 2 - 7R Joint Screw Coordinates (*ref* 1 reference)

i	$\$_i^{ref\,2}$	
	$\{\$_i^{ref\,2}\}^T$	$\{\$_o^{ref\,2}{}_i\}^T$
3	$\{0, 0, 1\}$	$\{0, 0, 0\}$
2	$z([\mathbf{R}(\theta_3)]_z^T[\mathbf{R}]_2^T)$	$\{0, 0, 0\}$
1	$z([\mathbf{R}(\theta_3)]_z^T[\mathbf{R}]_2^T[\mathbf{R}]_1^T)$	$\{0, 0, 0\}$
4	$z([\mathbf{R}(90)]_x)$	$\{x^{ref\,2}_{elbow}\} \times \{\$_4^{ref\,2}\}$
5	$z([\mathbf{R}(90)]_x[\mathbf{R}]_4)$	$\{x^{ref\,2}_{wrist}\} \times \{\$_6^{ref\,2}\}$
6	$z([\mathbf{R}(90)]_x[\mathbf{R}]_4[\mathbf{R}]_5)$	$\{x^{ref\,2}_{wrist}\} \times \{\$_7^{ref\,2}\}$
7	$z([\mathbf{R}(90)]_x[\mathbf{R}]_4[\mathbf{R}]_5[\mathbf{R}]_6)$	$\{x^{ref\,2}_{wrist}\} \times \{\$_7^{ref\,2}\}$

where $\{x^{ref\,2}_{elbow}\} = \{0, 0, g\}^T$
and $\{x^{ref\,2}_{base}\} = \{x^{ref\,2}_{elbow}\} + [\mathbf{R}(90)]_x[\mathbf{R}]_4\{0, 0, h\}^T$

Table 3 - 7R Joint Screw Coordinates (*ref* 2 reference)

Appendix 2 - α_{32}, α_{42}, α_{62} for $S_2 = 0, S_6 = 0$

$$\alpha_{32} = \frac{-hS_3C_4S_4(1 + h^2) - g(S_3S_4S_5{}^2 - C_3C_4S_4C_5S_5 + h^2S_3S_4)}{h(S_4{}^2 + h^2)}$$

$$\alpha_{42} = \frac{C_3C_4gh + C_3h^2 + S_3C_4C_5S_5 + \alpha_{32}S_4C_5S_5 + C_3S_5{}^2}{S_5{}^2 + h^2}, \quad \alpha_{62} = C_3C_5 - S_3C_4S_5 - \alpha_{32}S_4S_5 - \alpha_{42}C_5$$

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