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"Physical Interpretation and Application of Principles of Ultrasonic Nondestructive Evaluation of High-Performance Materials"

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Physical Interpretation and Application of Principles of Ultrasonic Nondestructive Evaluation of High-Performance Materials

I. Introduction:

In order to meet the need for more advanced ultrasonic nondestructive testing systems, capable of ascertaining the strengths and limitations of high-performance materials, a more fundamental understanding of the anisotropic properties of these materials is necessary. The basic knowledge of how the material responds to ultrasonic interrogation will permit the optimization of the measurement system for the extraction of information needed for making material integrity decisions. To aid in the development of ultrasonic measurement systems, improved visualization techniques for the physical interpretation of the elastic properties of materials and their inter-relationships are beneficial. During the current grant period valuable insight has been gained through the production of 3-dimensional representations of the anisotropic nature of the ultrasonic group velocity and the engineering parameters (Young's and shear moduli) for graphite/epoxy composites. Video-taped animations of these surfaces have been delivered by Dr. James G. Miller, principal investigator for this grant, during a visit with Drs. Heyman, Madaras, and Johnston at NASA Langley Research Center in August, 1990. Visualization of the anisotropic properties of composite materials along with experimental verification provides necessary information for the design of advanced measurement systems.

In Section II we discuss an ultrasonic measurement system employed in the experimental interrogation of the anisotropic properties (through the measurement of the elastic stiffness constants) of the uniaxial graphite/epoxy composites received from NASA Langley Research Center. Section III discusses our continuing effort for the development of improved visualization techniques for physical parameters. In this Section we set the background for the understanding and visualization of the relationship between the phase and energy/group velocity for propagation in high-performance anisotropic materials by investigating the general requirements imposed by the classical wave equation. Section IV considers the consequences when the physical parameters of the anisotropic material are inserted into the classical wave equation by a linear elastic model. Section V describes the relationship between the phase velocity and the energy/group velocity 3-dimensional surfaces through graphical techniques.

II. Elastic Stiffness Coefficient Measurements:

In this Section we describe the measurement system employed for the determination of the anisotropic elastic properties of a set of uniaxial graphite/epoxy composites received from NASA Langley Research Center. The structure of uniaxial composites can be approximated by hexagonal symmetry. This implies that five elastic coefficients are required to describe the structure of the material. Table 1 displays the functional relationship between the ultrasonic phase velocities and the elastic stiffness coefficients for measurements in a meridian plane (a plane that completely contains the fibers). The velocity notation is defined as

$$V^{\text{Propagation w.r.t. fibers}}_{\text{Mode}_{\text{Polarization w.r.t. fibers}}} \quad (1)$$

Measurement of Stiffness Coefficients - Meridian Plane (Fiber axis aligned along the x axis)	
$c_{11} = \rho \, (V^{\parallel}_{L_{\parallel}})^2$	**Longitudinal Mode:** Propagation parallel to fiber axis
$c_{55} = \rho \, (V^{\parallel}_{S_{\text{arbitrary}}})^2$	**Shear Mode:** arbitrary polarization Propagation parallel to fiber axis
$c_{55} = \rho \, (V^{\perp}_{S_{\parallel}})^2$	**Shear Mode:** polarized along fiber axis - Propagation perpendicular to fiber axis
$c_{22} = \rho \, (V^{\perp}_{L_{\perp}})^2$	**Longitudinal Mode:** Propagation perpendicular to fiber axis
$c_{23} = c_{22} - 2 \rho \, (V^{\perp}_{S_{\perp}})^2$	**Shear Mode:** polarized perpendicular to fiber axis - Propagation perpendicular to fiber axis
$c_{12} = \dfrac{\sqrt{b - a \rho V^2 + \rho^2 V^4}}{\lvert\sin\Psi\rvert \; \lvert\cos\Psi\rvert} - c_{55}$	V represents either V^2_{qL} or V^2_{qS}. The angle between the propagation direction and the fiber direction is given by Ψ.
where $a \equiv c_{22} \sin^2\Psi + c_{11} \cos^2\Psi + c_{55}$ and $b \equiv (c_{11} \cos^2\Psi + c_{55} \sin^2\Psi)(c_{22} \sin^2\Psi + c_{55} \cos^2\Psi)$	

Table 1

Sample Preparation:

As described in the March 1990 Progress Report we have prepared three uniaxial samples for the measurements. Two of the samples were surface-ground so that their sides were parallel and perpendicular to the fiber orientation as illustrated in Figure 1. The final dimensions of the samples are 21.4 X 30.4 X 28.9 mm.

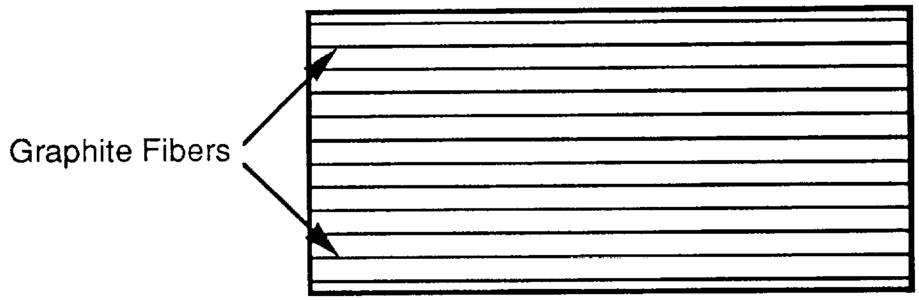

Figure 1: Two samples prepared for the propagation of longitudinal and shear waves parallel and perpendicular to the fiber orientation.

The third sample was prepared so that insonification normal to the surface will produce ultrasonic waves whose phase velocity direction will be at an angle of 77° inside the sample with respect to the fiber orientation (see Figure 2).

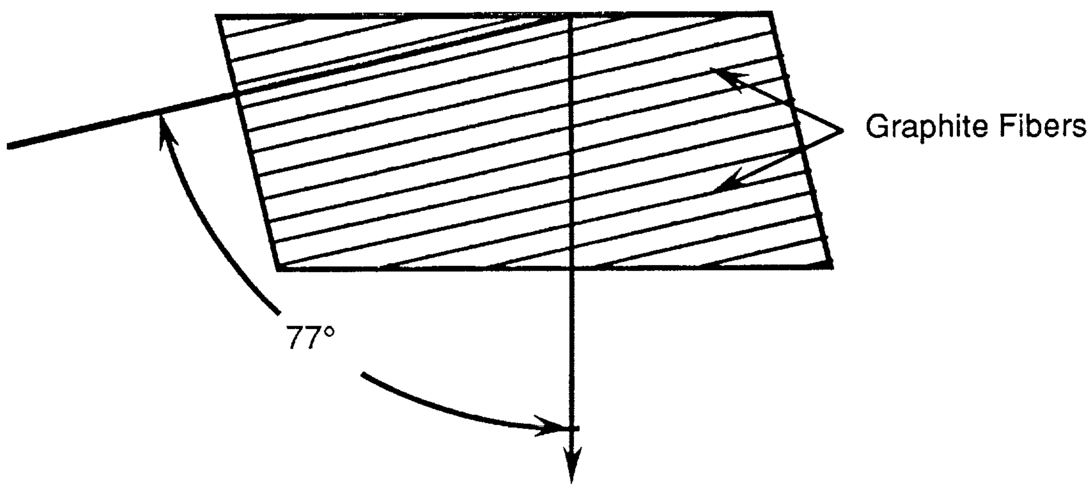

Figure 2: In the third sample the direction of the phase velocities for longitudinal and shear waves will be at an angle of 77° with respect to the fiber orientation.

The final dimensions of this sample are 14.5 X 30.4 X 28.9 mm.

Experimental Measurement System and Protocol:

The time-of-flights were measured in a reflection mode system using 2.25 MHz, 0.5 inch diameter longitudinal and shear wave contact transducers, as illustrated in Figure 3. A HP8112A pulse generator provided the master clock signal to which all subsequent timing measurements were referenced. The trigger output of the pulse generator is the initial timing event which triggers the digital oscilloscope. The output port of the pulse generator is delayed with respect to the trigger output by 100 nsec. This signal is used as the external trigger input to the ultrasonic pulse generator. RF attenuators were incorporated into the system to ensure that the electronics were operating in a linear fashion. The returned RF pulse is routed to a wideband RF receiver for amplification before being fed to the digital oscilloscope.

Reflection Mode Measurement System and Timing Points

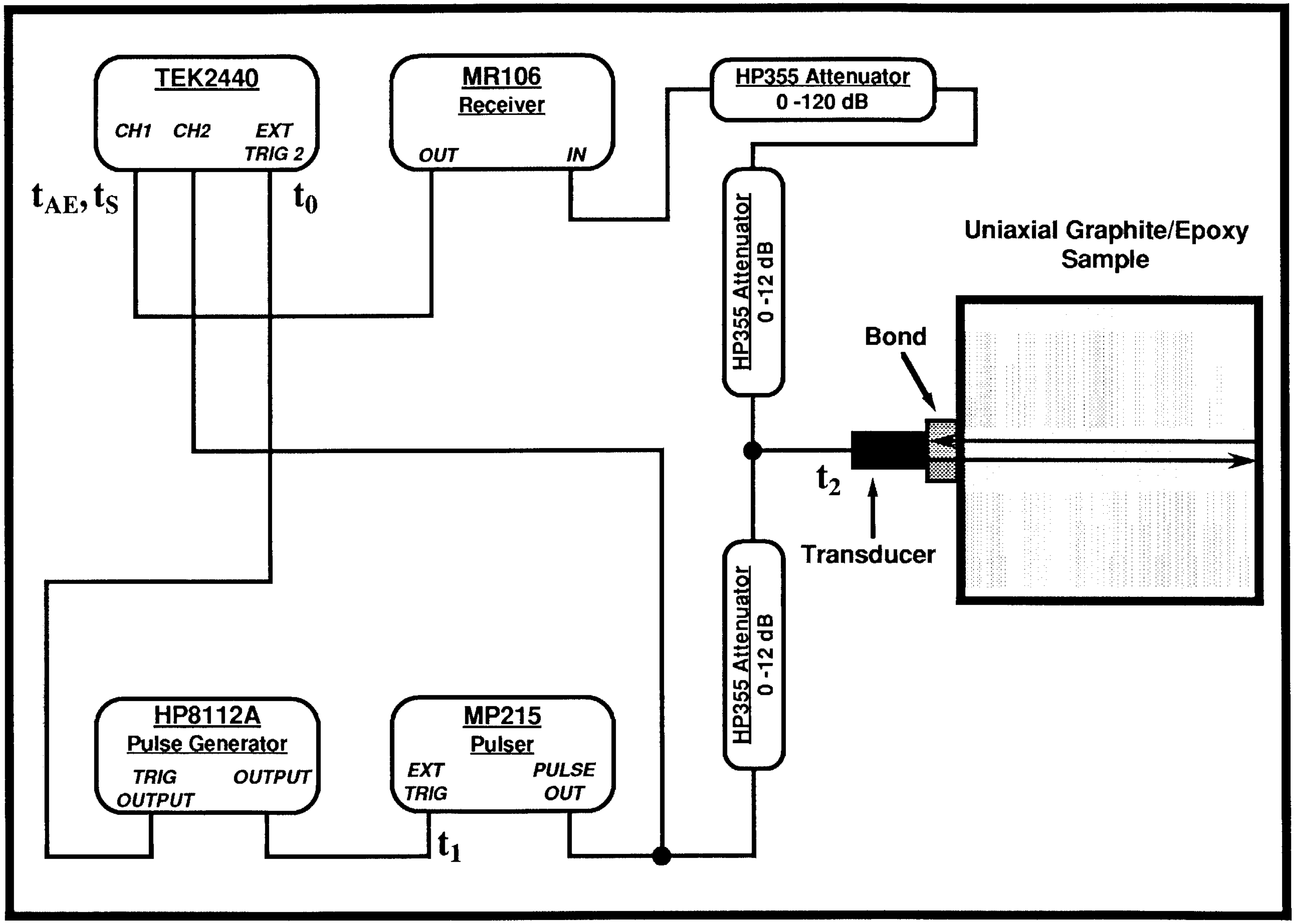

Figure 3: Reflection mode measurement system and timing points for time-of-flight determination.

The timing diagram for the measurement system is illustrated in Figure 4.

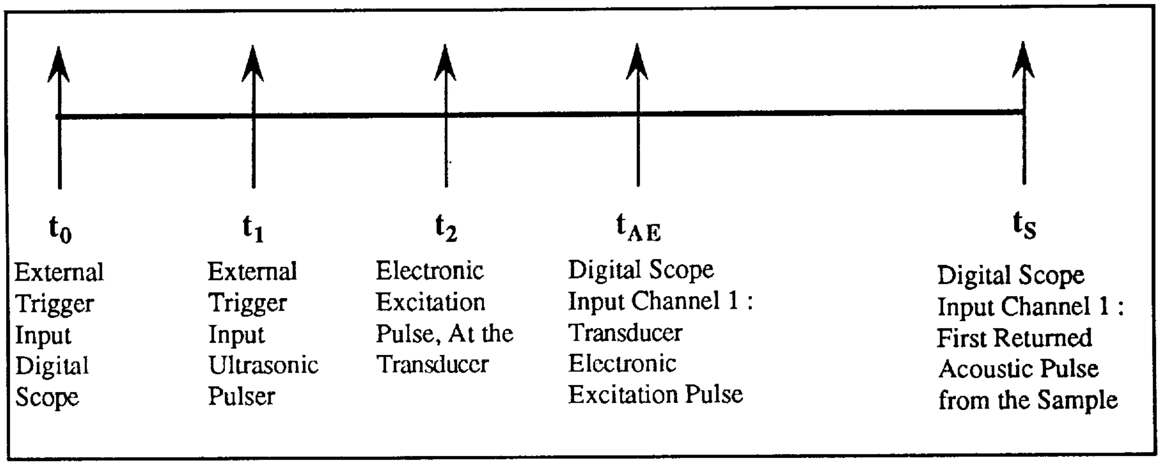

Figure 4: Timing diagram (not to scale) for the measurement of the time-of-flights.

Only the first returned RF pulse was used, in order to minimize the bonding effects of the transducer to the sample. The difference in time between when the transducer starts to respond to the electronic pulse and the returned ultrasonic RF pulse determines the time-of-flight for a given measurement.

$$Time\text{–}of\text{–}Flight \equiv \mathbf{t_S} - \mathbf{t_{AE}} \tag{2}$$

Because the time-of-flight is obtained by taking a difference in time, the measurement is independent of the electronic (receiver) and coaxial cable propagation delays.

For each of the five time-of-flights required for the determination of the five elastic stiffness coefficients the following measurement criteria were carried out. Three spatial sites were insonified for a given measurement. All signal amplitudes, routed to the input of the digital scope, were adjusted to make maximum use of the digitization range of the scope while maintaining linearity in the measurement system. A survey trace with the timebase of the digital scope set to 1 or 2 μsec/div was used to obtain an overall view of the distribution of returned ultrasonic pulses. The timebase on the digital scope was set to 10 nsec/div and the pulse corresponding to the transducer excitation pulse was captured for determination of the timing point $\mathbf{t_{AE}}$. The first returned RF pulse was captured using 500 nsec/div, 200 nsec/div and 100 nsec/div timebases for the determination of the timing point t_S. The data are currently being analyzed and will be reported in the March 1991 Progress Report.

III. Classical Bulk Wave Propagation

In this Section we obtain fundamental results from the classical wave equation which are useful in increasing physical intuition for wave propagation in linear elastic media. The concept of phase and group velocity are discussed for monochromatic plane waves and the necessary conditions imposed upon them to satisfy the classical wave equation. In what follows we will assume that the media is lossless and homogeneous and the strength of the acoustical disturbance is small so that linear theory is applicable.

Homogeneous Classical Wave Equation:

We will begin by investigating what general information can be obtained from one form of the homogeneous classical wave equation for a three-dimensional medium. The classical wave equation (neglecting body forces) can be written as

$$\nabla_r^2 \mathbf{u}(\mathbf{r}, t) = \frac{1}{[c(k_x, k_y, k_z)]^2} \frac{\partial^2 \mathbf{u}(\mathbf{r}, t)}{\partial t^2}, \tag{3}$$

where $\mathbf{u}(\mathbf{r}, t)$ is a vector that represents the particle displacement. The above equation states that the solutions for the differential equation must have its second-order spatial derivatives equal to the second-order time derivatives *times* a proportionality term independent of space and time coordinates. In general the proportionality term can be a function of k_x, k_y, k_z, the vector components of the propagation wave number. Solutions of the form $f(\mathbf{k}\cdot\mathbf{r} \pm \omega t)$ will satisfy the equation along with the appropriate initial and boundary conditions.

Plane Harmonic Wave Solution:

Angular Frequency Function: "Dispersion Relation"

One class of solutions which satisfy the classical wave equation is the monochromatic plane harmonic wave which has the following form

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{U}_0 \, e^{i[\mathbf{k}\cdot\mathbf{r} - \omega(k_x, k_y, k_z)t]} \tag{4}$$

where $\mathbf{U}_0$ is assumed to be a constant vector. First, we will substitute Equation (4) into Equation (3) and see what general information we can extract from the wave equation. Since

$$\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} \Rightarrow -i\,\omega(k_x, k_y, k_z)\,\mathbf{u}(\mathbf{r}, t), \qquad \frac{\partial^2 \mathbf{u}(\mathbf{r},t)}{\partial t^2} \Rightarrow -[\omega(k_x, k_y, k_z)]^2\,\mathbf{u}(\mathbf{r}, t) \tag{5}$$

and

$$\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial x_j} \Rightarrow i\,k_j\,\mathbf{u}(\mathbf{r}, t), \qquad \frac{\partial^2 \mathbf{u}(\mathbf{r}, t)}{\partial x_j^2} \Rightarrow -k_j^2\,\mathbf{u}(\mathbf{r}, t)$$

$$\nabla_r^2 \mathbf{u}(\mathbf{r}, t) \Rightarrow -(k_x^2 + k_y^2 + k_z^2)\,\mathbf{u}(\mathbf{r}, t) = -|\mathbf{k}|^2\,\mathbf{u}(\mathbf{r}, t), \tag{6}$$

it follows that

$$|\mathbf{k}|^2 \ \mathbf{u}(\mathbf{r}, t) \quad = \quad \left[\frac{\omega(k_x, k_y, k_z)}{c(k_x, k_y, k_z)} \right]^2 \ \mathbf{u}(\mathbf{r}, t) \quad . \tag{7}$$

Thus, for the plane harmonic wave to be a solution of the wave equation the differential equation demands that

$$|\mathbf{k}|^2 \quad = \quad \left[\frac{\omega(k_x, k_y, k_z)}{c(k_x, k_y, k_z)} \right]^2 \tag{8}$$

or

$$[\omega(k_x, k_y, k_z)]^2 \quad = \quad |\mathbf{k}|^2 \ [c(k_x, k_y, k_z)]^2 \tag{9}$$

or

$$\omega(k_x, k_y, k_z) \quad = \quad \pm \ |\mathbf{k}| \sqrt{[\ c(k_x, k_y, k_z)\ ]^2} \quad . \tag{10}$$

The above equation is a "dispersion relation" which states that the angular frequency is a function of the components of the wave number vector, **k**. The proportionality term has units of [m/sec] and thus is a velocity-like term. We will soon see that the proportionality term of the wave equation (which defines the functional form of the angular frequency) is the point at which the physical properties of the medium (density, elastic coefficients, etc.) are inserted into the wave equation. This proportionality term plays a major role in the determination of the allowed propagation modes, the resultant localized particle motion and the elastic wave propagation velocity for each mode.

The form of this equation, ω as a function of the wave number vector components, is counter intuitive to the way one normally thinks about initiating the wave phenomena. In the laboratory we use a transducer, coupled to the medium being investigated, to generate pressure waves to start the wave propagating. We assume a simple model for the generation of the pressure waves. A pulse of acoustic energy is radiated by an ultrasonic transducer. Plane waves are generated and initially propagate along the axis of the transducer. The axis of the transducer defines the initial direction of the **k** vector. The wave-packet is limited in the lateral dimensions by the size of the transducer and in the third dimension by the pulse length. We control the frequency components of the wave-packet by whether we choose to perform a CW, tone-burst, or wide-band measurement. Therefore, ω is an independent parameter in the measurement system, which we control. In the following mathematical analysis of the wave equation, instead of letting ω and the direction of **k** be the independent parameters, we will consider **k** to be the independent parameter. Since the angular frequency function defines a relationship between these two parameters we will have the ability to invert the final results to convey the information in a more conventional form.

Phase Velocity:

Starting with the monochromatic plane harmonic solution to the wave equation we can define a set of planes in space, called *surfaces of constant phase*, by requiring the argument of the exponential function to be a constant.

$$\mathbf{k} \cdot \mathbf{r} \; - \; \omega(k_x, k_y, k_z)\, t \; = \; constant$$

$$\frac{d}{dt}\{\, \mathbf{k} \cdot \mathbf{r} \; - \; \omega(k_x, k_y, k_z)\, t \,\} \; = \; 0$$

Let $\xi \equiv \hat{\mathbf{k}} \cdot \mathbf{r}$ the component of the position vector along the $\mathbf{k}$ vector.

$$\frac{d}{dt}\{\, |\mathbf{k}|\, \xi \; - \; \omega(k_x, k_y, k_z)\, t \,\} \; = \; 0$$

$$|\mathbf{k}|\, \frac{d\xi}{dt} \; = \; \omega(k_x, k_y, k_z)$$

$$V_{Phase}(k_x, k_y, k_z) \; \equiv \; \frac{d\xi}{dt} \; = \; \frac{\omega(k_x, k_y, k_z)}{|\mathbf{k}|} \tag{11}$$

The direction cosines of the planes of constant phase are proportional to the components of the wave number vector $\mathbf{k}$. Thus, $\mathbf{k}$ is normal to the surfaces of constant phase and these surfaces move in the direction of $\mathbf{k}$ at a rate equal to the *phase velocity*. The phase velocity can be written in a general vector form as

$$\mathbf{V}_{\mathbf{Phase}}(k_x, k_y, k_z) \; \equiv \; \frac{\omega(k_x, k_y, k_z)}{|\mathbf{k}|}\, [\, (\hat{\mathbf{k}} \cdot \hat{\mathbf{x}})\; \hat{\mathbf{x}} + (\hat{\mathbf{k}} \cdot \hat{\mathbf{y}})\; \hat{\mathbf{y}} + (\hat{\mathbf{k}} \cdot \hat{\mathbf{z}})\; \hat{\mathbf{z}}\,] \tag{12}$$

From the above equation we see that the magnitude of the phase velocity is equal to the angular frequency $\omega(k_x, k_y, k_z)$ divided by the magnitude of the wave vector $\mathbf{k}$ and in general is a function of the wave number vector components.

Velocity of Modulation on a Wave - "Group Velocity":

To find the group velocity we start as we did for the surfaces of constant phase by looking at the argument of the exponential function. But this time we perform a gradient operation with respect to the wave number vector components before the time derivative.

$$\nabla_{\mathbf{k}} \left\{ \frac{d}{dt} \{ \mathbf{k} \cdot \mathbf{r} \; - \; \omega(k_x, k_y, k_z)\, t \} \right\} \; = \; 0$$

$$\nabla_{\mathbf{k}} \left\{ \frac{d}{dt} \{ \mathbf{k} \cdot \mathbf{r} \} \right\} \; = \; \nabla_{\mathbf{k}} \{ \omega(k_x, k_y, k_z) \}$$

$$\frac{d}{dt} \{ \nabla_{\mathbf{k}} \{ \mathbf{k} \cdot \mathbf{r} \} \} \; = \; \nabla_{\mathbf{k}} \{ \omega(k_x, k_y, k_z) \}$$

$$\frac{d\mathbf{r}}{dt} \; = \; \nabla_{\mathbf{k}} \{ \omega(k_x, k_y, k_z) \} \tag{13}$$

The quantity $\nabla_{\mathbf{k}}\, \omega(k_x, k_y, k_z)$ is defined as the *velocity of modulation on a wave* or *group velocity* . This quantity can be obtained from the angular frequency function as follows,

$$\mathbf{V}_{\mathbf{Group}}(k_x, k_y, k_z) \equiv \nabla_{\mathbf{k}}\, \omega(k_x, k_y, k_z) \; = \; \frac{\partial \omega(k_x, k_y, k_z)}{\partial k_x} \; \hat{\mathbf{x}} + \frac{\partial \omega(k_x, k_y, k_z)}{\partial k_y} \; \hat{\mathbf{y}} + \frac{\partial \omega(k_x, k_y, k_z)}{\partial k_z} \; \hat{\mathbf{z}} \; . \tag{14}$$

The angular frequency function will be constrained in some manner to satisfy the classical wave equation. The physical properties of the medium play a major role in determining the actual functional form of the angular frequency function.

Case 1:

For an isotropic medium the angular frequency relation is $\omega(k_x, k_y, k_z) \; = \; \omega(|\mathbf{k}|) \; =$ *constant* times $|\mathbf{k}|$ and is independent of direction of propagation.

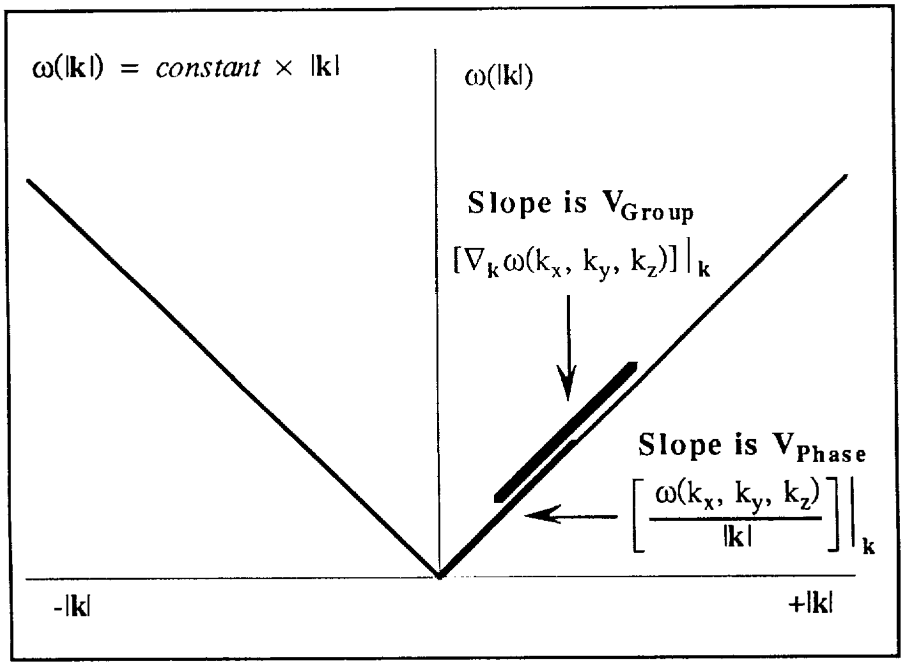

Figure 5: For a monochromatic plane harmonic wave propagating in a lossless isotropic media the phase and group velocity are collinear and equal in magnitude.

For an isotropic medium the curve $\omega(|\mathbf{k}|)$ is a straight line having a slope equal to the phase velocity. The derivative of the angular frequency curve with respect to the wave number is the slope, therefore, the magnitude of the phase and group velocities are equal. Since the angular frequency function is a constant times $|\mathbf{k}|$, the gradient of ω yields a vector direction along the direction of the wave number vector. Thus the phase and group velocity are collinear for an isotropic medium. In general for an isotropic medium the angular frequency relation can be written as

$$\omega(|\mathbf{k}|) = V_{Phase}\sqrt{k_x^2 + k_y^2 + k_z^2} \quad . \tag{15}$$

We see that ω scales directly with $|\mathbf{k}|$, therefore, the phase and group velocity surfaces are spheres of constant radii V_{Phase} and V_{Group}, respectively. If we consider the functional form when $k_z = 0$ we see that the angular frequency has the form of a right circular cone with

$$\omega(|\mathbf{k}|) = V_{Phase}\sqrt{k_x^2 + k_y^2} \tag{16}$$

used as the *generator* for the surface of revolution. The magnitude of these velocities correspond to the slope of the $\omega(|\mathbf{k}|)$ curve as depicted in Figure 5. Inverting Equation (15) yields

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega(|\mathbf{k}|)}{V_{Phase}}\right)^2 \quad . \tag{17}$$

If we were to think 3-dimensionally we could construct **k** space which would correspond to concentric spheres scaled by $\omega(|\mathbf{k}|)$. The 3-dimensional surfaces in **k** space can be thought of as equipotential surfaces such that the normal at a point on a surface points in the direction of the energy flow. Slowness or inverse velocity space is defined as $|\mathbf{k}|/\omega(k_x,k_y,k_z)$. Since ω is a homogeneous function of degree one in $|\mathbf{k}|$ for an isotropic medium, the slowness is a sphere of constant radius $1/V_{Phase}$. The slowness is a function independent of the angular frequency and the magnitude and direction of the wave number vector for an isotropic medium.

Case 2:

For this case we will consider an angular frequency relation which is a function of the wave vector components.

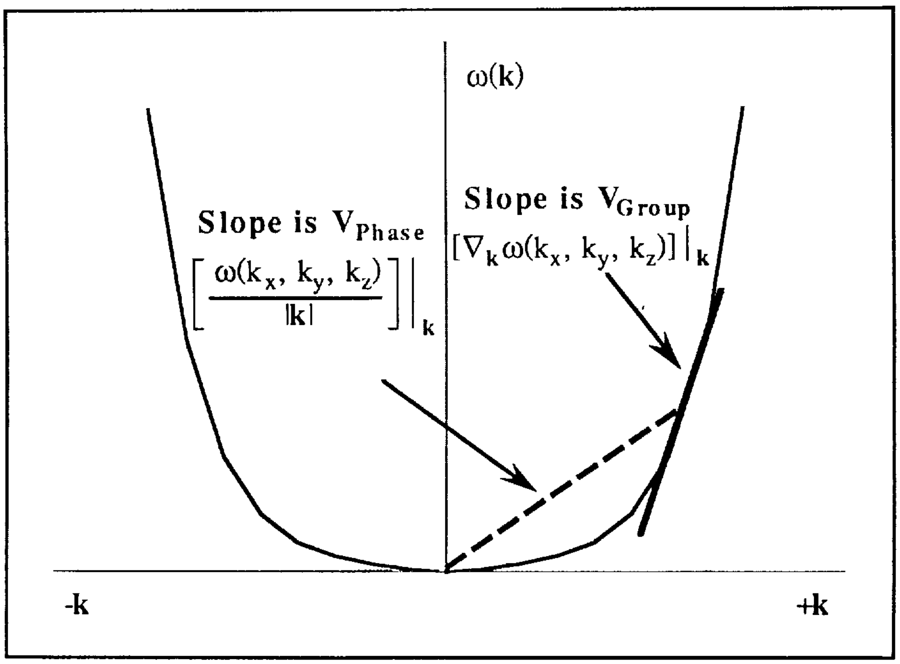

Figure 6: For a monochromatic plane harmonic wave propagating in a lossless medium the magnitude of the phase and the group velocities need not be equal for angular frequency functions for which the proportionality term in the wave equation is a function of the wave number components.

Since the phase velocity is the value of ω (evaluated at **k**) divided by the magnitude of **k** , this corresponds to the slope of the straight line drawn from the origin to $\omega(\mathbf{k})$. The group velocity is defined as the derivative with respect to **k** evaluated at **k**. From Figure 6 we see that the phase and group velocity are not equivalent for a medium in which the angular frequency relation is not solely a function of the magnitude of **k**.

Since the surfaces of constant phase travel at a speed $\omega(k_x, k_y, k_z)/|\mathbf{k}|$ and the group velocity travels at a speed of $|\nabla_{\mathbf{k}}\omega(k_x, k_y, k_z)|$, we are able to determine some information about the functional form of the angular frequency function by performing a little math and applying dimensional analysis.

$$
\begin{aligned}
V_{Phase}(k_x, k_y, k_z) &= \frac{\omega(k_x, k_y, k_z)}{|\mathbf{k}|} \\
\nabla_{\mathbf{k}}\, \omega(k_x, k_y, k_z) &= \nabla_{\mathbf{k}} \{\, |\mathbf{k}|\ V_{Phase}(k_x, k_y, k_z) \,\} \\
\nabla_{\mathbf{k}}\, \omega(k_x, k_y, k_z) &= V_{Phase}(k_x, k_y, k_z)\ \nabla_{\mathbf{k}}\{\, |\mathbf{k}|\} \;+\; |\mathbf{k}|\ \nabla_{\mathbf{k}}\{V_{Phase}(k_x, k_y, k_z)\} \\
\nabla_{\mathbf{k}}\, \omega(k_x, k_y, k_z) &= V_{Phase}(k_x, k_y, k_z)\ \hat{\mathbf{k}} \;+\; |\mathbf{k}|\ \nabla_{\mathbf{k}}\{V_{Phase}(k_x, k_y, k_z)\} \\
\left[\frac{\text{meters}}{\text{second}}\right] &= \left[\frac{\text{meters}}{\text{second}}\right] \;+\; \left[\frac{1}{\text{meters}}\right]\left[\frac{\text{meters}^2}{\text{second}}\right]
\end{aligned}
\tag{18}
$$

We see that in general the group velocity is the superposition of two vector quantities. One with a magnitude of the phase velocity along the direction of $\mathbf{k}$, and one along a direction defined by the gradient operation on the phase velocity. For an isotropic medium the gradient of the phase velocity is zero and we obtain the expected result that the phase and group are equivalent in magnitude and direction.

Physical Interpretation of the Group Velocity:

Why the gradient of the angular frequency function is called the group velocity can be seen by considering the propagation of two monochromatic plane harmonic waves.

$$
\begin{aligned}
\mathbf{u}(\mathbf{r}, t) &= \mathbf{u}_1(\mathbf{r}, t) + \mathbf{u}_2(\mathbf{r}, t) \\
&= \mathbf{U}_{01}\, e^{i\,[\mathbf{k}_1 \cdot \mathbf{r} \,-\, \omega_1(k_{x1}, k_{y1}, k_{z1})\, t\,]} + \mathbf{U}_{02}\, e^{i\,[\mathbf{k}_2 \cdot \mathbf{r} \,-\, \omega_2(k_{x2}, k_{y2}, k_{z2})\, t\,]}
\end{aligned}
\tag{19}
$$

Inserting Equation (19) into the wave equation yields

$$
\begin{aligned}
|\mathbf{k}_1|^2\, \mathbf{u}_1(\mathbf{r}, t) + |\mathbf{k}_2|^2\, \mathbf{u}_2(\mathbf{r}, t) &= \left[\frac{\omega_1(k_{x1}, k_{y1}, k_{z1})}{c(k_{x1}, k_{y1}, k_{z1})}\right]^2 \mathbf{u}_1(\mathbf{r}, t) \\
&\quad + \left[\frac{\omega_2(k_{x2}, k_{y2}, k_{z2})}{c(k_{x2}, k_{y2}, k_{z2})}\right]^2 \mathbf{u}_2(\mathbf{r}, t) \quad ,
\end{aligned}
\tag{20}
$$

the linear superposition of the two waves. In general, the resultant acoustic disturbance will depend on the initial wave number directions and the angular frequency of each of the monochromatic waves.

Group Velocity for an Isotropic Medium: Modulated Harmonic Wave

Consider the case for two monochromatic plane waves, having slightly different frequencies but collinear wave number vectors, propagating in an isotropic medium. The angular frequency relation for each wave is given by

$$\omega_1(k_{x1}, k_{y1}, k_{z1}) = \omega(|\mathbf{k}_1|) = |\mathbf{k}_1| \; constant = |\mathbf{k}_1| \; V_{phase}$$
$$\omega_2(k_{x2}, k_{y2}, k_{z2}) = \omega(|\mathbf{k}_2|) = |\mathbf{k}_2| \; constant = |\mathbf{k}_2| \; V_{phase} \, . \tag{21}$$

Each angular frequency is directly proportional to the magnitude of the wave number vector and independent of its direction. The gradient of the angular frequency relation for each wave, considered individually, is equal to the constant phase velocity. Thus, when each wave is considered alone the group velocities and phase velocities have equal magnitude and direction, and the group and phase velocity for one wave is the same as for the other wave.

Consider each wave launched such that at time $t = 0$ seconds they each have coincident nodes (see Figure 7a). The surfaces of constant phase for each wave travel at a speed V_{Phase} which implies that the ratio $\omega(|\mathbf{k}|)/|\mathbf{k}|$ is constant. That is, for each individual wave the speed at which its surfaces of constant phase travel is independent of the magnitude and direction of the wave number vector and therefore of frequency. But since the two waves have different wavelengths, even though they each start out at a node, their nodes do not line up as time progresses because the distance between corresponding nodes is different for the two waves ($\lambda_1 \neq \lambda_2$).

$$\frac{\lambda_1}{\Delta t_1} = \frac{\lambda_2}{\Delta t_2} = V_{Phase} \, , \quad (\text{ a constant })$$
$$\Delta t_1 = \frac{2\,\pi}{\omega_1} \, , \qquad \Delta t_2 = \frac{2\,\pi}{\omega_2} \tag{22}$$

If we let the two monochromatic plane waves have equal amplitudes, then

$$\mathbf{U}_{00} \equiv \mathbf{U}_{01} = \mathbf{U}_{02}$$
$$\mathbf{k_1} \equiv (k_0 - \Delta k) \; \hat{\mathbf{k}}$$
$$\mathbf{k_2} \equiv (k_0 + \Delta k) \; \hat{\mathbf{k}}$$
$$\text{where} \quad \Delta k << k_0$$
$$\omega_1(k_{x1}, k_{y1}, k_{z1}) \equiv \omega(|\mathbf{k}_0|) - \Delta\omega$$
$$\omega_2(k_{x2}, k_{y2}, k_{z2}) \equiv \omega(|\mathbf{k}_0|) + \Delta\omega \, . \tag{23}$$

Substituting these expressions into the plane harmonic solution yields

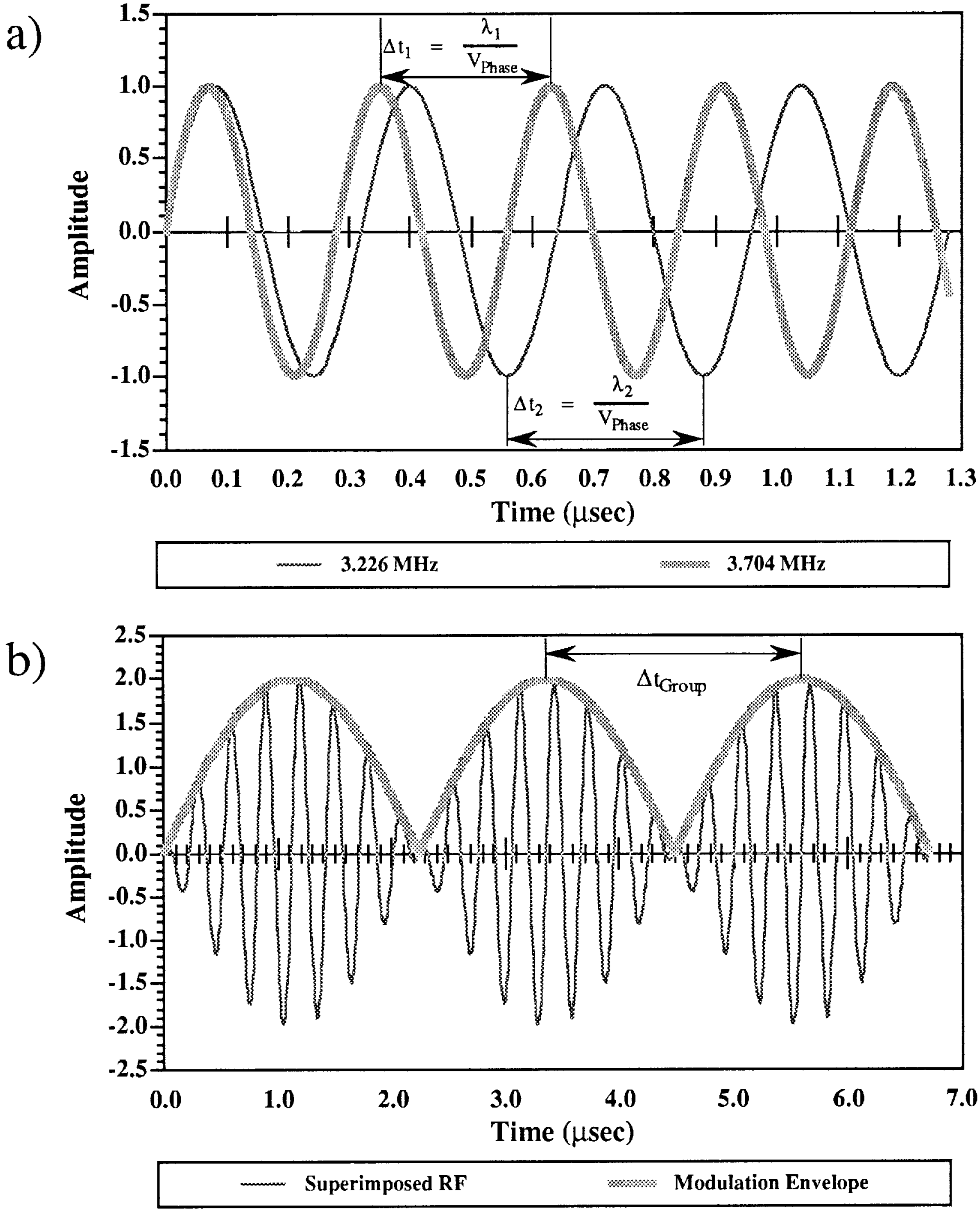

Figure 7: (a) Two monochromatic RF plane waves having equal amplitudes and slightly different frequencies propagating with collinear **k** vectors. (b) Superimposed RF signal plus the modulation envelope.

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{U}_{00}\, e^{i\,[(k_0 - \Delta k)\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; [\omega(|\mathbf{k}_0|) - \Delta\omega]\, t\,]} + \mathbf{U}_{00}\, e^{i\,[(k_0 + \Delta k)\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; (\omega(|\mathbf{k}_0|) + \Delta\omega)\, t\,]} \tag{24}$$

or after some algebra

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{U}_{00}\, e^{i\,[k_0\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; \omega(|\mathbf{k}_0|)\, t\,]} \left\{ e^{i\,[\Delta k\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; \Delta\omega\, t\,]} + e^{-i\,[\Delta k\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; \Delta\omega\, t\,]} \right\}$$

$$\mathbf{u}(\mathbf{r}, t) = 2\,\mathbf{U}_{00}\, e^{i\,[k_0\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; \omega(|\mathbf{k}_0|)\, t\,]} \cos\left(\Delta k\,(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; \Delta\omega\, t\right) \tag{25}$$

or

$$\mathbf{u}(\mathbf{r}, t) = 2\,\mathbf{U}_{00}\; e^{i\,[\,|k_0|(\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; \omega(|\mathbf{k}_0|)\, t\,]}\; \cos\left(\Delta k\,((\hat{\mathbf{k}} \cdot \mathbf{r}) \; - \; V_{Group}\, t\,)\right) . \tag{26}$$

The form of the plane harmonic solution looks like a carrier wave oscillating at an angular frequency of $\omega(|\mathbf{k}_0|)$ $[\omega_1(|\mathbf{k}_1|) < \omega_0(|\mathbf{k}_0|) < \omega_2(|\mathbf{k}_2|)]$ multiplied by a cosine term (see Figure 7b). The resultant wave is an amplitude modulated wave whose carrier frequency is given by the exponential term and whose amplitude (spatial/temporal distribution) is determined by the cosine term. The argument of the exponential term yields the carrier phase velocity and the argument of the cosine term the modulation envelope or group velocity. The modulation envelope changes slowly in space and time compared with the carrier term. The density of the wave energy is concentrated in the area where the modulation term is large and thus propagates at the group velocity (for a lossless media). Strictly speaking the concept of phase velocity only applies when the form of the wave remains constant throughout its length. This condition is necessary to be able to measure the wavelength by taking the distance between any two successive corresponding points on the wave.

What the above relations tell us is that for a monochromatic plane harmonic wave to be a solution of the classical wave equation the following relations in Table 2 must be satisfied.

What the Classical Wave Equation Tells Us About a Monochromatic Plane Harmonic Wave Solution	
For a plane harmonic wave to be a solution of the classical wave equation the angular frequency, ω, must be equal to the magnitude of the wave number vector times a term which may be a function of the wave number vector components.	$\omega(k_x, k_y, k_z) = \pm \|\mathbf{k}\| \sqrt{[\, c(k_x, k_y, k_z)\,]^2}$
The ratio of the angular frequency, ω, with respect to the magnitude of the wave number is the speed at which surfaces of constant phase travel in the direction of the **k** vector for a monochromatic plane wave having the frequency ω.	$V_{Phase}(k_x, k_y, k_z) = \dfrac{\omega(k_x, k_y, k_z)}{\|\mathbf{k}\|}$
The gradient of the angular frequency curve, ω, with respect to the wave vector components is defined to be the modulation on a wave or group velocity. The energy of the wave travels in the direction and with the speed of the group velocity for a lossless medium.	$V_{Group}(k_x,k_y,k_z) \equiv \nabla_k\, \omega(k_x, k_y, k_z)$

Table 2

IV. Classical Bulk Wave Propagation - Linear Elastic Medium

In this Section we assume a linear elastic model for the material in which the acoustical waves propagate. As stated in Section III the physical parameters that describe the structure of the material are inserted in the proportionality term of the classical wave equation. Starting with the acoustic field equations (acoustical analogue of Maxwell's equations for electromagnetics) we obtain the linear elastic wave equation. By investigating the functional form of the phase and group velocity we see that these velocities will be dependent on the density, the elastic stiffness coefficients that describe the medium, and the direction of the wave number vector. We also see that they are independent of the magnitude of the wave number vector and the frequency of the monochromatic plane wave. The independence of the phase and group velocity on the frequency and magnitude of the wave number vector allow the generation of three-dimensional surfaces that completely describe the wave propagation parameters for a linear elastic medium (phase velocity, energy velocity, group velocity and slowness). The following analysis assumes that the medium is lossless and homogeneous and that there exist a constitutive relation that relates the applied stresses to the resulting homogeneous strains via the elastic stiffness constants.

The linear elastic wave equation is obtained using the 81 component notation so that directional information contained in the equation may be more easily seen. The wave equation written in the 36 component notation (Voigt) is also given for computational purposes.

Linear Elastic Acoustic Field Equations:

We can write the acoustic field equations in a form similar to the electromagnetic field equations (Maxwell's equations) as shown in Table 3.

Acoustic Field Equations	
Equation of Motion	$\nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{v}}{\partial t} - \mathbf{F_B}$
Strain Displacement Relation	$\nabla_s \mathbf{v} = \frac{\partial \mathbf{S}}{\partial t}$

Table 3: $\mathbf{T}$ denotes the stress, $\mathbf{v}$ the particle velocity, $\mathbf{F_B}$ external body forces, and $\mathbf{S}$ the strain

Constitutive Relations:

For a linear elastic medium the stress and strain are related by the constitutive relations

$$\mathbf{S} = \mathbf{s} : \mathbf{T} \quad \text{and} \quad \mathbf{T} = \mathbf{c} : \mathbf{S} \tag{27}$$

where $\mathbf{c}$ corresponds to the elastic stiffness constants and $\mathbf{s}$ the elastic compliance constants. These elastic constants contain the information about the physical structure of the medium.

Homogeneous Linear Elastic Wave Equation:

Starting with the acoustic field equations we can obtain a wave equation as follows. The equation of motion (neglecting body forces) can be written as

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} . \tag{28}$$

Rewriting this equation in matrix form yields

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x_i} . \tag{29}$$

Since the stress is related to the strain by the linear elastic constitutive relation

$$T_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} S_{kl} , \tag{30}$$

the physical structure of the medium can be inserted into the wave equation to produce

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} \frac{\partial S_{kl}}{\partial x_i} . \tag{31}$$

We see that each component of the particle displacement's acceleration is dependent on the spatial derivatives of the strain (weighted by the elastic stiffness coefficients) with respect to all three spatial directions. In general for a medium not exhibiting high degrees of symmetry each component of the particle displacement or velocity will be a complicated set of coupled equations. By making use of the strain definition

$$S_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \tag{32}$$

Equation (31) can be expanded as

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} \frac{\partial S_{kl}}{\partial x_i} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{1}{2} c_{ijkl} \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) . \tag{33}$$

Since the strain is symmetric under the permutation of the indices k and l we obtain

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} \frac{\partial^2 u_k}{\partial x_i \partial x_l} . \tag{34}$$

Equation (34) has the form of a wave equation. In general, Equation (34) describes a set of three coupled equations for each allowed mode of propagation in the medium. The physical properties and structure of the medium are contained in the density and elastic stiffness coefficients.

Monochromatic Plane Wave Solutions:

One class of solutions for the linear elastic wave equation is the monochromatic plane wave having the form

$$u_j \propto e^{i[\mathbf{k}\cdot\mathbf{r} - \omega(k_x, k_y, k_z)t]} \quad . \tag{35}$$

As we have seen in Section IIII of this Progress Report a relationship between $\omega(k_x, k_y, k_z)$ and **k** is required for the solution to satisfy the wave equation. By noting that

$$\frac{\partial u_j}{\partial t} \Rightarrow -i\,\omega(k_x,k_y,k_z)\,u_j \quad \text{and} \quad \frac{\partial u_k}{\partial x_i} \Rightarrow i\,k_i\,u_k \tag{36}$$

we can substitute the plane wave solution into the wave equation to obtain an eigenvalue equation.

$$\rho\,[\omega(k_x,k_y,k_z)]^2\,u_j = \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}\,k_i\,k_l\,u_k \tag{37}$$

By rearranging and summing over the index j we have

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} [\,\rho\,[\omega(k_x,k_y,k_z)]^2\,\delta_{jk} - c_{ijkl}\,k_i\,k_l\,]\,u_k = 0 \quad . \tag{38}$$

This equation defines a system of homogeneous equations that have non-trivial solutions if and only if the determinant is equal to zero.

$$\left|\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} [\,\rho\,[\omega(k_x,k_y,k_z)]^2\,\delta_{jk} - c_{ijkl}\,k_i\,k_l\,]\right| = 0 \tag{39}$$

The determinant or characteristic equation is a cubic equation in ω^2. For a given wave number vector **k** the equation has three roots $\omega_1^2, \omega_2^2, \omega_3^2$, which in general are different for anisotropic media. Each root gives the angular frequency as a function of the wave number vector components. Substituting a particular root for ω^2 into Equation (38) yields the corresponding orthonormal components for the displacement vector for that mode of propagation (since the equations are homogeneous only the normalized vector components or direction cosines of the displacement will be obtained).

By inspection of Equation (39) we see that the angular frequency $\omega(k_x, k_y, k_z)$ is a homogeneous function of degree one of the components of wave vector **k**. If instead of solving for $\omega(k_x, k_y, k_z)$ we solve for the ratio $\omega(k_x, k_y, k_z)/|\mathbf{k}|$, we obtain the speed at which the surfaces of constant phase travel *along the **k** direction.* Since the elastic stiffness coefficients in the equation do not depend on the magnitude of **k**, the velocity of propagation of the surfaces of constant phase along the **k** direction is a homogeneous function of degree zero of k_i. Therefore, the phase velocity is a function of its direction (which is determined by **k**) but not the magnitude of

k or its frequency. Since, in general, there are three possible modes of propagation for a given **k** direction, there will be three angular frequency relations, thus three different phase velocities.

When we assume a given **k** vector the determinant of the eigenvalue equation determines the principal values $\rho\,\omega^2$ of a tensor of rank two, $c_{ijkl}\, k_i\, k_l$, which is symmetrical with respect to the indices *j,k*. Equation (38) yields the principal axes of this tensor (the particle displacement directions) which are *always* mutually orthogonal.[1]

Modulation Envelope or Group Velocity for a Lossless Linear Elastic Medium:

The group velocity (modulation envelope velocity) is defined to be the gradient of the angular frequency relation with respect to the wave vector components. As we have seen in Section III for a single monochromatic plane wave propagating in an isotropic medium the phase velocity and group velocity were equivalent both in magnitude and direction of propagation. A monochromatic plane wave propagating in an isotropic medium is a very special case of bulk wave propagation in a general linear elastic medium. In order for the group velocity to have its own distinct meaning in an isotropic media we had to consider the superposition of at least two monochromatic plane waves propagating in the medium (hence the name group). From the analysis in Section III we saw that the energy in the superimposed monochromatic waves travelled at a speed equal to the group velocity *not* the phase velocity for each of the individual monochromatic plane waves. Therefore, the magnitude of the group velocity was different from the magnitude of the phase velocity. A very important piece of information was lost in this analysis because of the isotropic nature of the medium.. When we carry out the analysis in a medium that has a physical structure that influences the direction of the resultant wave propagation we see that even for a single monochromatic plane wave the phase and group velocities are distinct entities.

In an isotropic medium we have one purely longitudinal wave and two degenerate transverse waves. For the longitudinal mode the particle displacement is collinear with the **k** vector and the particle displacement for the transverse modes are orthogonal to the **k** vector, always. For each mode of propagation in an anisotropic medium there corresponds a wave in which the displacement vector can have components both parallel and perpendicular to the wave number direction. This implies that the response of the medium, to the application of external stresses, is determined by the physical parameters (c_{ijkl}) that describe the structure of the material. The resultant particle displacement for a given propagation mode and **k** vector may not be along the **k** vector direction.

Energy Velocity:

For a general linear elastic medium the direction of the energy velocity is the direction in which the energy contained in the wave propagates. In anisotropic media the direction of the energy flow need not be collinear with the initial **k** vector direction. This is a direct consequence of the medium having a physical structure in which there are preferred directions. In other words the response of an anisotropic medium to the application of a stress stimulus depends on the physical structure (symmetries or lack of symmetry) of the medium. Consider a stress stimulus that varies sinusoidally with time applied to a uniaxial composite medium as illustrated in Figure 8.

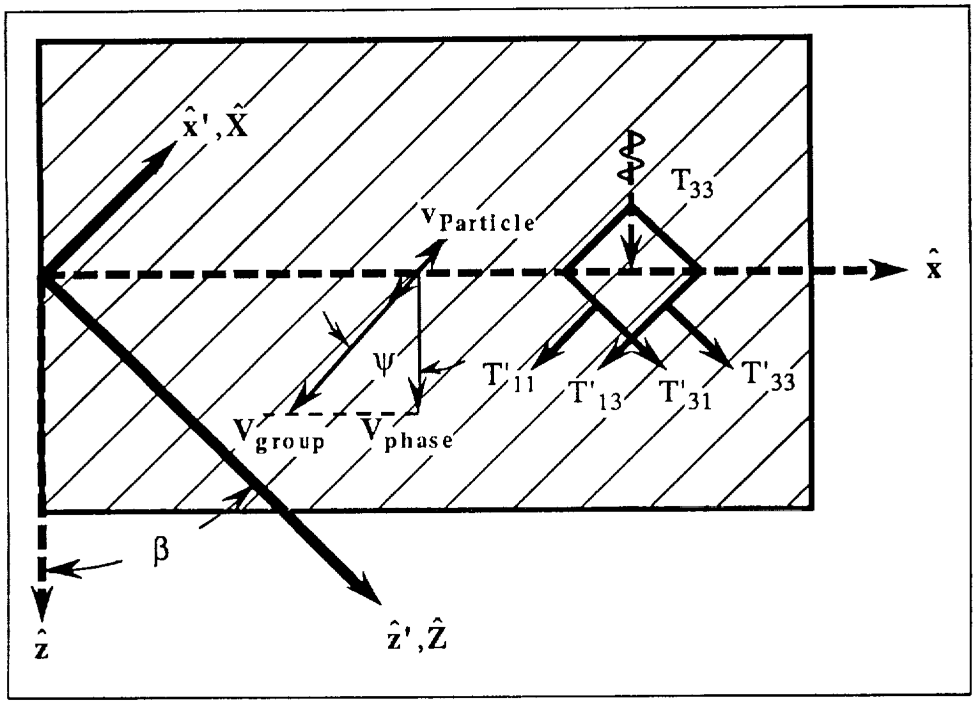

Figure 8: Illustration of the various velocity directions resulting from the applied stress T_{33}.

Because the response of the medium depends on the symmetries of the medium in general the resultant displacement will not be along **k**. Therefore, the longitudinal mode has a polarization (particle velocity) that is not parallel to the **k** vector direction. The acoustic Poynting vector (the energy flow vector), for a given **k** vector direction and a given mode of propagation, is proportional to the particle velocity *dotted* with the stress stimulus.

$$V_{\text{Energy}} \propto \mathbf{P} \propto -\mathbf{v} \cdot \mathbf{T} \tag{40}$$

If all the stresses are zero except T_{33}, as illustrated in Figure 8, and we rotate the system to align it with the principal axes, the stress can be written in terms of the rotated coordinate system as

$$[T'_{ij}] = \begin{bmatrix} [\sin(\beta)]^2 T_{33} & 0 & -\cos(\beta)\sin(\beta) T_{33} \\ 0 & 0 & 0 \\ -\cos(\beta)\sin(\beta) T_{33} & 0 & [\cos(\beta)]^2 T_{33} \end{bmatrix} . \tag{41}$$

The equations of motion for a given mode of propagation are

$$\frac{\partial v'_1}{\partial t} = \frac{1}{\rho}[\nabla' \cdot \mathbf{T}']_1 = \frac{1}{\rho}\left\{ \frac{\partial T'_{11}}{\partial x'_1} + \frac{\partial T'_{31}}{\partial x'_3} \right\}$$

$$\frac{\partial v'_3}{\partial t} = \frac{1}{\rho}[\nabla' \cdot \mathbf{T}']_3 = \frac{1}{\rho}\left\{ \frac{\partial T'_{13}}{\partial x'_1} + \frac{\partial T'_{33}}{\partial x'_3} \right\} . \qquad (42)$$

The acoustic Poynting vector in the rotated system is

$$P_1' \propto -\{ v_1' T'_{11} + v_3' T'_{31} \}$$

$$P_2' = 0$$

$$P_3' \propto -\{ v_1' T'_{13} + v_3' T'_{33} \} \qquad (43)$$

or after inserting the constitutive relation

$$P'_1 \propto -\{ v'_1 [c'_{11}S'_1 + c'_{13}S'_3 + c'_{15}S'_5] + v'_3 [c'_{15}S'_1 + c'_{35}S'_3 + c'_{55}S'_5] \}$$

$$P'_2 = 0$$

$$P'_3 \propto -\{ v'_1 [c'_{15}S'_1 + c'_{35}S'_3 + c'_{55}S'_5] + v'_3 [c'_{13}S'_1 + c'_{33}S'_3 + c'_{35}S'_5] \} . \qquad (44)$$

We see that the direction of the energy flow is determined by the resultant particle velocity, elastic stiffness coefficients and the resultant strains. For lossless linear elastic media it can be shown that the group and energy velocity are equivalent.[2]

Wave Equation in Reduced Subscript Notation:

In general there can be at most 81 independent elastic coefficients. Because we have required the stress and strain to be symmetric tensors, this number is reduced to 36. Materials for which an elastic strain energy density function can be written have at most 21 independent elastic coefficients. If the material exhibits other symmetries then a further reduction in the number of independent coefficients can be achieved. The linear elastic wave equation can be written in reduced subscript notation (Voigt notation) as

$$\sum_{j=1}^{3}\sum_{K=1}^{6}\sum_{L=1}^{6} [\, |k|\, l_{iK}\; c_{KL}\; |k|\, l_{Lj} \,]\, v_j = \rho\, [\omega(k_x,k_y,k_z)]^2\, v_i \qquad (45)$$

where l_{iK} and l_{Lj} are functions of the direction cosines of the wave number vector.

Uniaxial Symmetry: Fiber Axis Along the x Axis

The physical structure of a uniaxial graphite/epoxy composite can be approximated by a material exhibiting hexagonal symmetry. The density and the five independent elastic stiffness constants determine the physical structure and the wave propagation parameters for the material. The angular frequency relations for wave propagation in a *xy* meridian plane are:

Quasi-Longitudinal Mode:

$$[\omega(k_x, k_y, k_z)]_{qL} = \left\{ \frac{\begin{array}{l} c_{11}\, k_x^2 + c_{22}\, k_y^2 + c_{55} \\ \quad + \{ [\, (c_{11} - c_{55})\, k_x^2 + (c_{22} - c_{55})\, k_y^2 \,]^2 \\ \qquad + 4\, [(c_{12} + c_{55})^2 - (c_{11} - c_{55})(c_{22} - c_{55})]\, k_x^2\, k_y^2 \,\}^{1/2} \end{array}}{2\, \rho} \right\}^{1/2} \tag{46}$$

Quasi-Shear Mode:

$$[\omega(k_x, k_y, k_z)]_{qS} = \left\{ \frac{\begin{array}{l} c_{11}\, k_x^2 + c_{22}\, k_y^2 + c_{55} \\ \quad - \{ [\, (c_{11} - c_{55})\, k_x^2 + (c_{22} - c_{55})\, k_y^2 \,]^2 \\ \qquad + 4\, [(c_{12} + c_{55})^2 - (c_{11} - c_{55})(c_{22} - c_{55})]\, k_x^2\, k_y^2 \,\}^{1/2} \end{array}}{2\, \rho} \right\}^{1/2} \tag{47}$$

The particle displacement/velocity is contained in the *xy* meridian plane for the quasi-longitudinal and quasi-shear mode.

Pure-Shear Mode:

$$[\omega(k_x, k_y, k_z)]_{pS} = \left\{ \frac{\frac{1}{2}(c_{22} - c_{23})\, k_y^2 \;+\; c_{55}\, k_x^2}{\rho} \right\}^{1/2} \tag{48}$$

For the pure-shear mode the particle displacement/velocity is always perpendicular to the *xy* plane for all **k** vectors contained within the plane. From the above equations we explicitly see that the angular frequency relations are homogeneous equations of degree one with respect to the wave number vector components.

Case 3:

Consider a monochromatic plane harmonic solution to the wave equation for a material exhibiting uniaxial symmetry with the fiber axis aligned along the x axis. The angular frequency function for the pure-shear mode propagating in the *xy* meridian plane is

$$[\omega(k_x, k_y, k_z)]_{pS} = \left\{ \frac{\frac{1}{2}(c_{22} - c_{23})\, k_y^2 \;+\; c_{55}\, k_x^2}{\rho} \right\}^{1/2} . \tag{49}$$

The phase velocity can be obtained by taking the ratio of the angular frequency function and the magnitude of the wave number vector.

$$[V_{Phase}(k_x, k_y, k_z)]_{pS} = \left[\frac{\omega(k_x, k_y, k_z)}{|\mathbf{k}|} \right]_{pS} = \left\{ \frac{\frac{1}{2}(c_{22} - c_{23}) \left(\frac{k_y}{|\mathbf{k}|}\right)^2 + c_{55} \left(\frac{k_x}{|\mathbf{k}|}\right)^2}{\rho} \right\}^{1/2}$$

$$[V_{Phase}(l_x, l_y, l_z)]_{pS} = \left[\frac{\omega(k_x, k_y, k_z)}{|\mathbf{k}|} \right]_{pS} = \left\{ \frac{\frac{1}{2}(c_{22} - c_{23})\, l_y^2 + c_{55}\, l_x^2}{\rho} \right\}^{1/2} \tag{50}$$

Where $\mathbf{l}$ is the wave number unit vector and l_x, l_y, l_z are the direction cosines of the wave number vector.

$$l_x \equiv \frac{\mathbf{k}\cdot\hat{\mathbf{x}}}{|\mathbf{k}|} = \frac{k_x}{|\mathbf{k}|}, \quad l_y \equiv \frac{\mathbf{k}\cdot\hat{\mathbf{y}}}{|\mathbf{k}|} = \frac{k_y}{|\mathbf{k}|}, \quad l_z \equiv \frac{\mathbf{k}\cdot\hat{\mathbf{z}}}{|\mathbf{k}|} = \frac{k_z}{|\mathbf{k}|} \tag{51}$$

The above equation explicitly shows that the phase velocity is a function of the density, linear elastic stiffness constants and the direction cosines of the wave number vector It is independent of the magnitude of the wave number vector and the frequency of the monochromatic wave.

The group velocity is obtained by taking the gradient of the angular frequency relation with respect to the wave number vector components.

$$[\mathbf{V}_{\mathbf{Group}}(k_x, k_y, k_z)]_{pS} = \nabla_{\mathbf{k}}[\omega(k_x, k_y, k_z)]_{pS} = \nabla_{\mathbf{k}} \left\{ \left\{ \frac{\frac{1}{2}(c_{22} - c_{23})\, k_y^2 + c_{55}\, k_x^2}{\rho} \right\}^{1/2} \right\} \tag{52}$$

$$[V_{Group_x}(k_x, k_y, k_z)]_{pS} = \left[\frac{\partial \omega}{\partial k_x} \right]_{pS} = \frac{k_x\, c_{55}}{\rho\, [\omega(k_x, k_y, k_z)]_{pS}}$$

$$[V_{Group_y}(k_x, k_y, k_z)]_{pS} = \left[\frac{\partial \omega}{\partial k_y} \right]_{pS} = \frac{k_y \left(\frac{c_{22} - c_{23}}{2} \right)}{\rho\, [\omega(k_x, k_y, k_z)]_{pS}} \tag{53}$$

If we divide both the numerator and denominator by the magnitude of the wave number vector

$$[V_{Group_x}(k_x, k_y, k_z)]_{pS} = \frac{\frac{k_x}{|\mathbf{k}|}\, c_{55}}{\rho\, \frac{[\omega(k_x, k_y, k_z)]_{pS}}{|\mathbf{k}|}} = \frac{\frac{k_x}{|\mathbf{k}|}\, c_{55}}{\rho\, [V_{Phase}(k_x, k_y, k_z)]_{pS}}$$

$$[V_{Group_y}(k_x, k_y, k_z)]_{pS} = \frac{\frac{k_y}{|\mathbf{k}|} \left(\frac{c_{22} - c_{23}}{2} \right)}{\rho\, \frac{[\omega(k_x, k_y, k_z)]_{pS}}{|\mathbf{k}|}} = \frac{\frac{k_y}{|\mathbf{k}|} \left(\frac{c_{22} - c_{23}}{2} \right)}{\rho\, [V_{Phase}(k_x, k_y, k_z)]_{pS}} \tag{54}$$

then we can explicitly show that the group velocity is a function of only the direction cosines of the wave number vector.

$$[V_{Group_x}(l_x, l_y, l_z)]_{pS} = \frac{l_x\, c_{55}}{\rho\, [V_{Phase}(l_x, l_y, l_z)]_{pS}}$$

$$[V_{Group_y}(l_x, l_y, l_z)]_{pS} = \frac{l_y \left(\frac{c_{22} - c_{23}}{2} \right)}{\rho\, [V_{Phase}(l_x, l_y, l_z)]_{pS}} \tag{55}$$

For a given **k** direction in the *xy* meridian plane the magnitude of the group velocity is given by

$$\left| [\mathbf{V}_{Group}(l_x,l_y,l_z)]_{pS} \right| = \frac{1}{[V_{Phase}(l_x,l_y,l_z)]_{pS}} \left\{ \frac{c_{55}^2\, l_x^2 + \left(\frac{c_{22} - c_{23}}{2} \right)^2 l_y^2}{\rho^2} \right\}^{1/2} \tag{56}$$

or

$$\left| [\mathbf{V}_{Group}(l_x,l_y,l_z)]_{pS} \right| = \frac{\left\{ \frac{c_{55}^2\, l_x^2 + \left(\frac{c_{22} - c_{23}}{2} \right)^2 l_y^2}{\rho^2} \right\}^{1/2}}{\left\{ \frac{c_{55}\, l_x^2 + \left(\frac{c_{22} - c_{23}}{2} \right) l_y^2}{\rho} \right\}^{1/2}} . \tag{57}$$

We explicitly see that the group velocity is also a function of only the direction cosines of the wave number vector not its magnitude.

Functional Dependence of the Phase Velocity on Wave Number Vector:

The phase velocity for an isotropic linear elastic medium can be written in general terms as

$$V_{Phase} = \sqrt{\frac{\text{elastic stiffness coefficient}}{\text{density}}} = \textit{constant} . \tag{58}$$

Using the fact that the elastic stiffness coefficients have dimensions of [Newtons/meters^2] we can check the dimensions of Equation (58). Dimensionally we have

$$\left[\frac{\text{meters}}{\text{second}} \right] = \left[\frac{\text{Newtons/meter}^2}{\text{kg/meter}^3} \right]^{1/2} = \left[\frac{\frac{\text{kg meters}}{\text{seconds}^2\ \text{meters}^2}}{\frac{\text{kg}}{\text{meters}^3}} \right]^{1/2} = \left[\frac{\text{meters}}{\text{second}} \right] . \tag{59}$$

For the general anisotropic medium the numerator will be a linear combination of the elastic stiffness coefficients. In order to include the wave number vector information in the equation and to maintain the proper dimensions only dimensionless quantities such as the direction cosines of **k** can be involved in the numerator term. As we have seen above this implies that for a linear elastic medium the phase velocity is a function of the density, elastic coefficients, and the direction of the wave number vector and not the magnitude of the wave number or the frequency. Thus, for the general anisotropic linear elastic medium we should rewrite the numerator as a linear combination of the elastic stiffness coefficients weighted by the direction cosines of the wave number vector.

$$V_{\text{Phase}}\left(\frac{k_x}{|\mathbf{k}|}, \frac{k_y}{|\mathbf{k}|}, \frac{k_z}{|\mathbf{k}|}\right) = \left\{ \frac{\begin{array}{c}\text{wave number } \textit{direction cosine} \text{ weighted} \\ \text{linear combination of the elastic stiffness constants}\end{array}}{\text{density}} \right\}^{1/2} \tag{60}$$

This dependence is explicitly seen from the eigenvalue form of the linear elastic wave equation. The exact linear combination and weighting of the elastic stiffness constants depends on the particular linear elastic system (cubic, hexagonal, etc.) that describes the physical properties of the medium.

In Table 4 we summarize the results for the homogeneous classical wave equation for a linear elastic medium.

Monochromatic Plane Harmonic Waves in Linear Elastic Media	
The angular frequency, ω, is equal to the magnitude of the wave number vector times a term which is a function of the density, elastic stiffness coefficients and wave number vector directions.	$\omega(k_x,k_y\ k_z) = \pm\lvert\mathbf{k}\rvert\sqrt{\left[f\left(\rho,c_{IJ},\frac{k_x}{\lvert\mathbf{k}\rvert},\frac{k_y}{\lvert\mathbf{k}\rvert},\frac{k_z}{\lvert\mathbf{k}\rvert}\right)\right]^2}$
The phase velocity can only be a function of the density, elastic stiffness coefficients, and the direction cosines of the wave number vector. The phase velocity is independent of frequency.	$V_{Phase}\left(\frac{k_x}{\lvert\mathbf{k}\rvert},\frac{k_y}{\lvert\mathbf{k}\rvert},\frac{k_z}{\lvert\mathbf{k}\rvert}\right) = \frac{\omega(k_x,k_y,k_z)}{\lvert\mathbf{k}\rvert}$
The group velocity can only be a function of the density, elastic stiffness coefficients, and the direction cosines of the wave number vector. The group velocity is independent of frequency. The energy of the wave travels in the direction and with the speed of the group velocity for a lossless medium.	$V_{Group}\left(\frac{k_x}{\lvert\mathbf{k}\rvert},\frac{k_y}{\lvert\mathbf{k}\rvert},\frac{k_z}{\lvert\mathbf{k}\rvert}\right) = \nabla_{\mathbf{k}}\omega(k_x,k_y,k_z)$

Table 4

V. Group Velocity for Anisotropic Media: General Physical Interpretation

In an isotropic medium for the concept of group velocity to have any meaning we had to consider at least two monochromatic waves propagating in the medium in order for the phase and group velocity not to be equivalent. We have to remember that the isotropic medium is a very special case of the general anisotropic medium. Since the angular frequency, for an isotropic medium, is a function of the magnitude of the wave number vector and not its direction, the phase velocity is a function independent of ω and the direction of $\mathbf{k}$. The phase velocity is a function of the density and the elastic stiffness constants only. This is a very special case.

In general, for an anisotropic linear elastic medium, the phase velocity is a function of the direction of **k** as well. For this case the phase and group velocity are not equivalent even for a monochromatic plane wave. The reason for this is that the particle displacement must be considered. Let us consider a longitudinal linear elastic wave propagating in an isotropic medium. As we push on the material the resultant particle displacement vector is along the direction of the push (**k**). Therefore, there are no components of the displacement vector orthogonal to the wave number vector (for a given mode of propagation the 3 vector components of the displacement are decoupled in the eigenvalue equations). In an anisotropic medium as we push on the material there is no guarantee that the resultant displacement vector will be along the direction of **k** (in general for a given mode of propagation the vector components for the displacement are described by 3 coupled eigenvalue equations). In fact, due to the anisotropic nature of the material, in general the resultant displacement vector will have components both parallel and perpendicular to **k**. Keeping this in mind, we see that the physical picture we obtained from the case of two slightly different monochromatic plane harmonic waves propagating in the same direction in an isotropic medium is not the complete picture for what the group velocity means. Since for a lossless linear elastic medium the group and energy velocity are equivalent we should view the group velocity as the direction and speed the energy in the elastic wave travels. Now even for a monochromatic plane harmonic wave the group velocity has its own distinct meaning. The launching of a single monochromatic wave in a given **k** direction implies that the surfaces of constant phase are planes perpendicular to the direction of **k**. But because the resultant displacement vector may have components orthogonal to **k**, the resultant displacement vector may not be collinear with **k**. This implies that the elastic wave does not propagate in a direction perpendicular to the surfaces of constant phase (defined by the monochromatic components of the wave). The energy contained in the wave (for a lossless medium) propagates in the direction of the group velocity. As we saw above, this is a direct consequence of the proportionality term in the classical wave equation having the ability to be a function of the direction cosines of the **k** vector.

Graphical Interpretation of the Relationship Between Phase and Group Velocity Surfaces:

In the March 1990 Progress Report 3-dimensional surfaces were presented for the group velocities for wave propagation in uniaxial graphite/epoxy composites. As was discussed the interpretation of these surfaces was not as straight-forward as the phase velocity and slowness surfaces presented in the September 1989 Progress Report. The physical interpretation of the phase velocity surface can be understood by placing an observer at the origin of the 3-dimensional surface and allowing the observer to look in any direction. Where his/her line-of-sight intercepts the surface defines a vector whose direction indicates the direction of **k**, and magnitude the phase velocity for the given mode of wave propagation. The corresponding group velocity surface can not be interpreted in this manner. As we have seen in Section IV of this Progress Report the resultant direction of the energy is in general not in the direction of the wave number vector **k** but is in the direction of the group velocity. This implies that for ultrasonic measurements made in transmission mode the receiving transducer may have to be offset in order to intercept the ultrasonic wave. The resultant group velocity direction for a given propagation mode can easily be determined by graphical means by making use of the slowness surfaces as described in previous Progress Reports. The direction *and* magnitude of the group velocity can be determined by making use of a relationship between the phase and energy/group velocity surfaces.

The *wave vector surface* is the plot of the wave number vector as a function of its direction for a given $\omega(k_x,k_y,k_z)$.

$$|\mathbf{k}| = \frac{\omega(k_x,k_y,k_z)}{|\mathbf{V}_{Phase}(l_x,l_y,l_z)|} \tag{61}$$

It can be shown that for a lossless linear elastic medium the wave number vector $\mathbf{k}$ is always normal to the *ray surface* (energy flow direction, $\mathbf{V}_{Energy}$).[2] This statement implies that

$$\frac{\mathbf{k}\cdot\mathbf{V}_{Energy}}{\omega(k_x, k_y, k_z)} = 1$$

$$\frac{|\mathbf{k}|\,(\hat{\mathbf{k}}\cdot\mathbf{V}_{Energy})}{\omega(k_x, k_y, k_z)} = 1$$

$$(\hat{\mathbf{k}}\cdot\mathbf{V}_{Energy}) = \frac{\omega(k_x, k_y, k_z)}{|\mathbf{k}|} = V_{Phase}\left(\frac{k_x}{|\mathbf{k}|}, \frac{k_y}{|\mathbf{k}|}, \frac{k_z}{|\mathbf{k}|}\right), \tag{62}$$

therefore,

$$|\mathbf{V}_{Phase}| = |\mathbf{V}_{Energy}|\,(\hat{\mathbf{k}}\cdot\mathbf{V}_{Energy}) = |\mathbf{V}_{Energy}|\cos(\psi)\,. \tag{63}$$

The angle ψ is the angle between the energy velocity and the wave number vector direction. Since Equation (62) must apply at every point on the *normal surface* (phase velocity surface), the *ray surface* (energy velocity surface) is the envelope of the planes normal to $\mathbf{V}_{Phase}$ (normal to $\mathbf{k}$). Since the phase fronts of a plane wave are normal to $\mathbf{k}$ each portion of the *ray surface* corresponds to the phase front for a plane wave with energy traveling in that direction (see Figure 9).

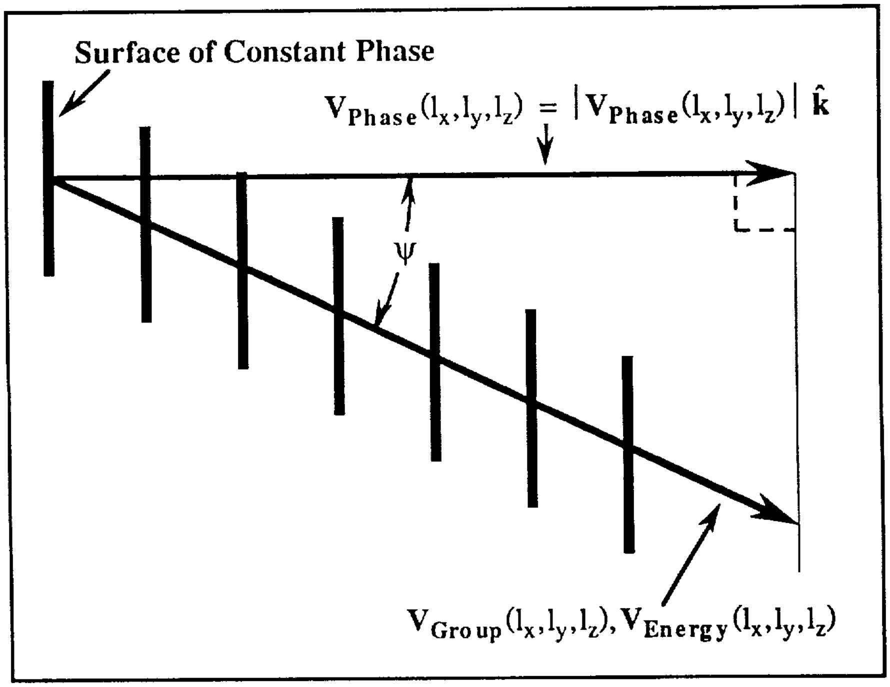

Figure 9: The surfaces of constant phase are perpendicular to **k**, the speed and direction for which a point on a given surface of constant phase travels is determined by the group/energy velocity.

If we superimpose the phase and the group velocity surfaces, for a given mode of wave propagation, we can graphically obtain both the direction and magnitude of the resulting group velocity for any given **k** direction. As described above we let the observer be placed at the origin. Where his/her line-of-sight intercepts the phase velocity surface defines a vector whose direction indicates the **k** direction and magnitude the phase velocity for the given mode of wave propagation. Next we find the plane normal to the phase velocity vector. This plane will be tangent to group velocity surface at a point. If the observer looks in the direction where the plane is tangent to group velocity surface then where his/her line-of-sight intercepts the group velocity surface defines the direction and magnitude of the resulting group velocity for the given **k** direction. In Figure 10 we demonstrate this graphical technique for a quasi-longitudinal mode propagating in a meridian plane of a uniaxial graphite/epoxy composite material. Since the group and energy velocity are equivalent for a lossless linear elastic material, this technique allows the determination of the placement of the receiving transducer by a simple graphical technique. In order to more clearly visualize the connection between the phase velocity and the resulting energy/group velocity directions and magnitudes, we are currently making a video tape for delivery to NASA Langley Research Center illustrating this relationship.

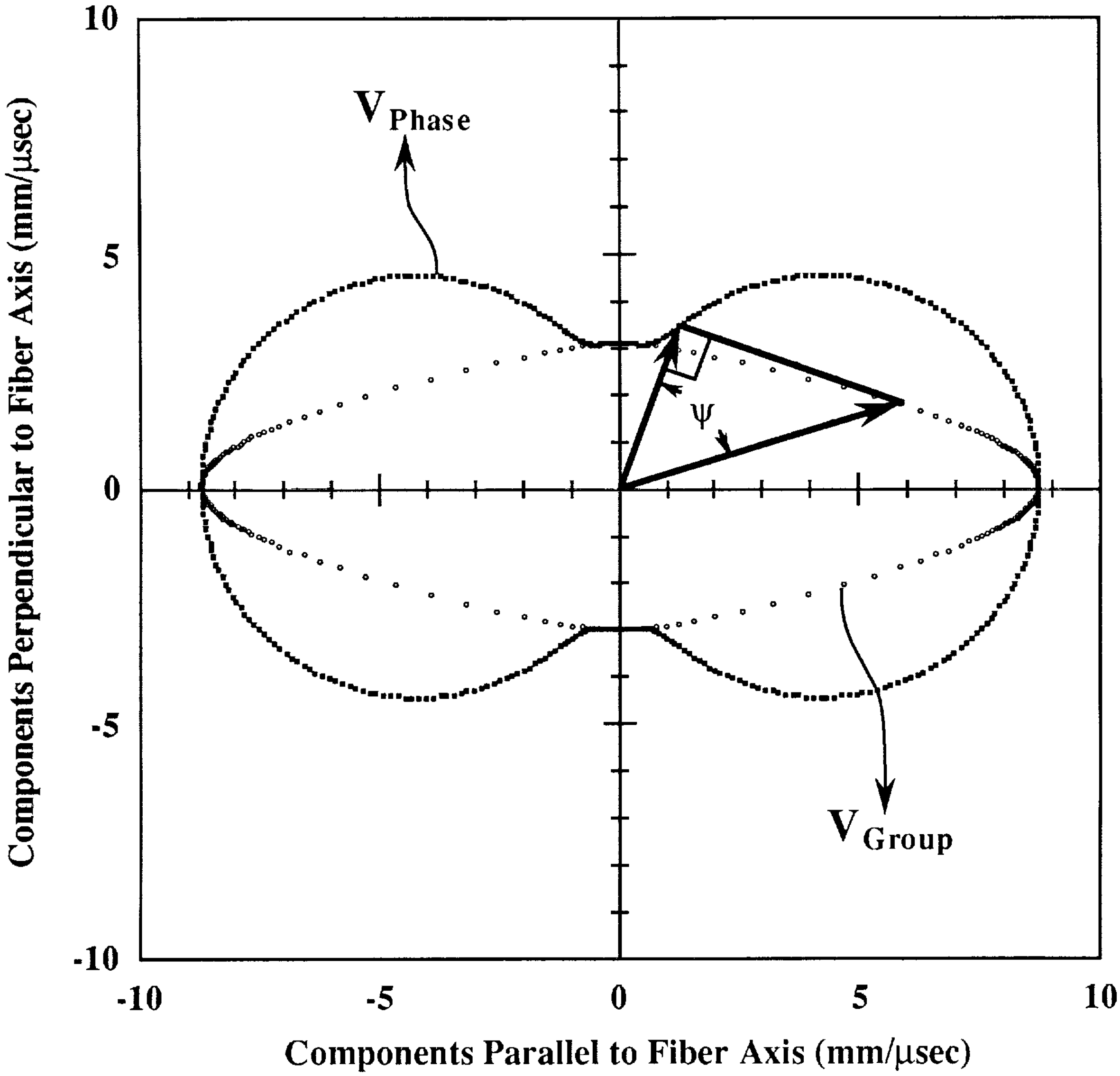

Figure 10: Graphical determination of the group velocity from the phase velocity for a quasi-longitudinal mode propagating in a meridian plane of a uniaxial graphite/epoxy composite.

Summary:

The velocities of wave propagation in an lossless anisotropic linear elastic material should be viewed as follows. For a monochromatic plane wave the direction of **k** is normal to the surfaces of constant phase always. If we pick a point on a particular surface of constant phase and follow it in time we see that it follows a path determined by the group velocity *not* by the phase velocity or **k** vector direction. This is a consequence of the fact that the proportionality term in the wave equation can be a function of the vector components of **k**. That is, each successive snapshot in time displays the surface of constant phase always perpendicular to **k** but the direction in which any given point on the surface travels is in the direction of the group velocity. The phase velocity for a monochromatic plane wave should be thought of as the *component of the elastic wave velocity along the* ***k*** *direction.*

The preceding description focussed on the surfaces of constant phase. We can also describe the interaction of the monochromatic plane wave in terms of the resultant particle velocity and homogeneous strains produced by the response of the material to the harmonic stresses applied to the material. This description lead to the energy velocity in terms of the acoustic Poynting vector (energy flow vector). As we saw this was a direct consequence of the fact that the response of an anisotropic material (particle displacement/velocity) to a push along a given direction may have components both parallel and perpendicular to the push direction. As discussed above for a lossless linear elastic material the group and energy velocity are equivalent. Therefore, both descriptions lead to the same conclusion; *the energy in a linear elastic wave travels in the direction of the group velocity.*

The emphasis in the latter Sections of this Progress Report has been on obtaining a more physical understanding of anisotropic nature of graphite/epoxy materials. The allowance of the proportionality term in the classical wave equation to be a function of the wave number components provided the starting point towards the understanding of why the phase and energy/group velocities are distinct quantities in an anisotropic material for even a monochromatic plane wave. Since the energy/group velocity is the measurable quantity in a nondestructive ultrasonic measurement system, a better understanding of the anisotropic nature of this velocity should prove valuable towards the design of advanced ultrasonic measurement systems.

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