PROBABILISTIC STRUCTURAL ANALYSIS

BY EXTREMUM METHODS

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Evaluation of structural system reliability is based on the determination of probabilistic response of structures. The study of probabilistic behavior of structural systems requires an underlying structural analysis model. The current system reliability analysis procedures utilize the structural analysis methods basically developed with a deterministic point of view. The applicability of these methods when dealing with a random situation needs examination. It may be that methods of structural analysis that are quite suitable for deterministic analysis are not as suitable for probabilistic analysis and vice versa. Development of structural system models especially suited for random state variables may be more efficient and likely to lead to important insights into random structural behavior. A recent National Science Foundation Workshop on Structural System Reliability held at the University of Colorado, Boulder [1] also emphasized the need for the development of structural analysis models from the probabilistic viewpoint.

The objective of this paper is to demonstrate discrete extremum methods of structural analysis as a tool for structural system reliability evaluation. Specifically, linear and multiobjective linear programming models for analysis of rigid plastic frames under proportional and multiparametric loadings, respectively, are considered. Kinematic and static approaches for analysis form a primal-dual pair in each of these models and have a polyhedral format. Duality relations link extreme points and hyperplanes of these polyhedra and lead naturally to dual methods for system reliability evaluation.

■ DIRECT METHOD

- **Formulation of the global limit state surface**

- **Computation of probability of random variables having an outcome in the safe set**

■ INDIRECT METHOD

- **Identification of all failure modes**

- **Computation of the probability of failure of individual modes**

- **Evaluation of system reliability from modal probabilities**

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It is known that the structural problems can be analyzed by idealizing these mathematically as either continuous or discrete models, leading respectively to the solution of differential or algebraic equations/inequalities. The underlying solution principles can be based on the solution of simultaneous equations or the use of extremum principles. In the sixties, Schmit and coworkers [2] briefly explored extremum methods for the deterministic structural analysis problems but found them to be not competitive with solution methods for simultaneous equations. However, in the use of simultaneous equations procedure for probabilistic structural analysis, one has to solve the structural analysis problem repeatedly for different realizations of random variables and this is compuationally costly. Use of extremum principles, on the other hand, elucidates the mathematical structure of the problem corresponding to various random realizations of state variables. This structure is extremely coherent with a definite pattern about the solutions of the problem. An understanding of such patterns leads one to gain important insights into response under random variables without analyzing the structure for all such combinations. This coupled with use of recent computational advances in algorithms [3] and vector processing of information on supercomputers are likely to make these methods extremely attractive for use in probabilistic analysis. For example, recent research [4] shows extremum methods to be ideally suited for structural analysis required in the system reliability assessment of structures with rigid plastic material behavior.

■ **STRUCTURAL ANALYSIS FOR RELIABILITY EVALUATION**

- **Discrete/Continuous models**

- **Classical methods, elastic/plastic**

- **Extremum methods**

■ **RESPONSE PATTERNS**

- **Polyhedral response regions**

- **Other response regions**

■ **ADVANTAGES OF EXTREMUM METHODS**

Structural analysis requires both the calculation of the distribution of forces throughout the structure and the displaced state of the system under the action of applied loads. One of the fundamental features of structural analysis is the possibility of using either forces or displacements as basic variables, with the respective approaches referred to as static and kinematic methods. The algebraic relationship of the static and kinematic approaches are the mathematical transpose of each other, a feature known as the static-kinematic duality. If the structure is statically determinate, the number of equations is same as the number of variables, and the forces and displacements can be found easily from the solution of system of algebraic equations. For statically indeterminate structures, additional conditions reflecting the material constitutive relations must be introduced.

As an example, for redundant frame structures, partial satisfaction of structural constraints generates a subspace in R^n containing the solutions of interest and the final solution can be reached by some optimality criterion. The optimal solution gives the result corresponding to the solution by traditional methods. The power of the extremum methods, however, is that all the suboptimal solutions may also be obtained from the model, and these suboptimal solutions correspond to various random realizations of the variables. This set of available solutions has a rich underlying mathematical structure and such patterns have recently been studied for rigid plastic frames under proportional and multiparametric loading [4].
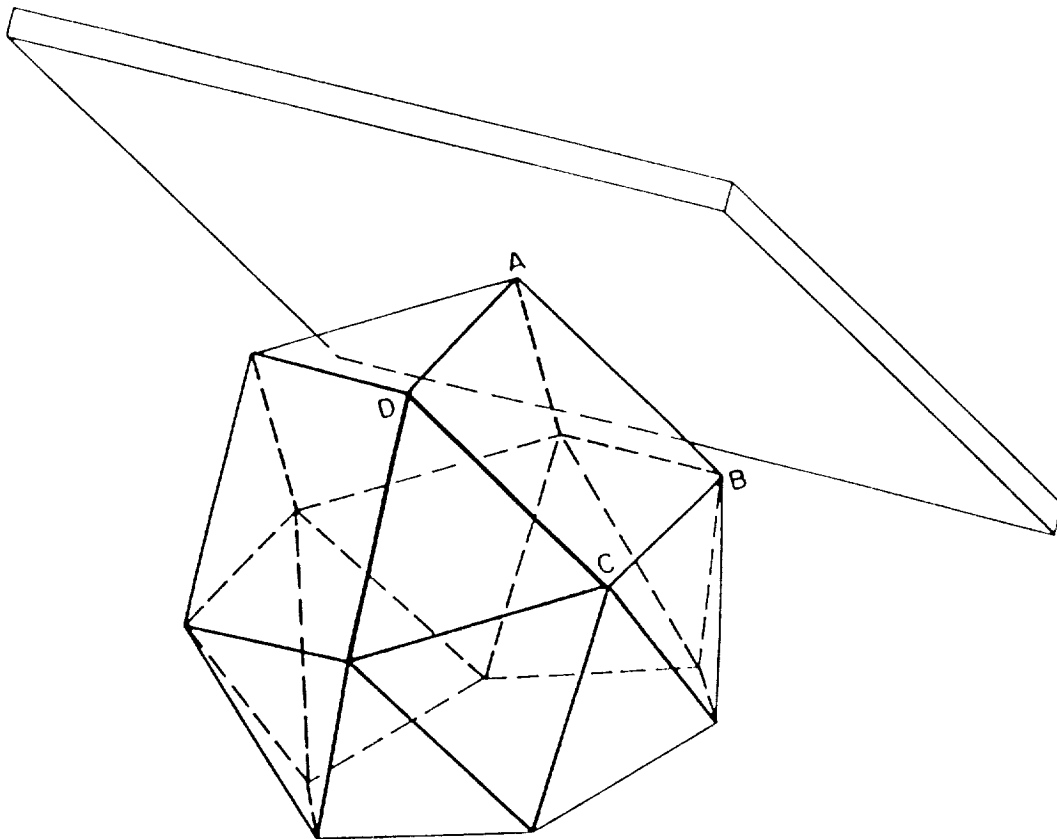
RIGID PLASTIC FRAMES WITH PROPORTIONAL LOADING

The problem of limit analysis of frames in which the plastic behavior is activated by a single stress resultant (such as flexure) may be formulated in terms of a Linear Programming (LP) model [5]. Plastic hinges are assumed to form at a set of discrete locations (j = 1, 2, ..., J) and the plastic moment capacity at the jth section is denoted by Mp_j. Models formulated from dual structural consideration of static and kinematic principles have been shown to be a primal-dual pair in the LP format. The variables are M_j = moment at section j, $M_j = M_j^+ - M_j^-$ where M_j^+ and M_j^- represent the positive and negative values of moments; θ_j = rotation at section j; t_k = a coefficient indicating the contribution of the kth elementary mode to collapse, $t_k = t_k^+ - t_k^-$ and λ^+, λ = collapse load factors for the kinematic and static LP's respectively. The parameters are θ_{kj} = hinge rotation of elementary mechanism k at joint j, e_k = external work associated with elementary mechanism k, M = number of equations of equilibrium and Mp_j = member capacity at section j.

KINEMATIC LP

$$\lambda = \text{Min} \left[\sum_{j=1}^{J} M_{p_j} (\theta_j^+ + \theta_j^-) \right]$$

$$\text{s.t.} \quad \theta_j^+ - \theta_j^- = \sum_{k=1}^{M} t_k \theta_{kj}$$

$$j = 1, \ldots, J$$

$$\sum_{k=1}^{M} t_k e_k = 1$$

$$\theta_j^+, \theta_j^- \geq 0$$

STATIC LP

$$\lambda = \text{Max} \ \lambda^-$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \theta_{kj} M_j = \lambda^- e_k$$

$$k = 1, \cdots, M$$

$$-M_{p_j}^- \leq M_j \leq + M_{p_j}^+$$

$$j = 1, \cdots, J$$

A study of the geometric structure of the primal and dual models shows the constraint regions for both to be polyhedral. The failure modes of the frame are given by the extreme points of the kinematic constraint region and the facets of the static constraint region. Duality transformations of LP actually map extreme points of one model to the hyperplanes of the other and vice versa. More generally, 1-dimensional subspaces of the model in R^n are linked to $(n-1)$ dimensional subspaces of the other model.

For primal, the random material properties specified by vector $\underline{Mp}$ occur only in the objective function and these determine the failure mode of the frame since the solution must surely belong to at least one of the extreme points. Therefore, unlike other procedures that require repeated solutions of structural model, one just needs to explore the polyhedral region for identifying the solution for a different (random) $\underline{Mp}$ vectors. Similarly, the vector $\underline{Mp}$ occurs only in the right hand side of static LP model and failure modes (facets) can be generated from the dual variables.

- **DUALITY**
  - **Static/Kinematic**
  - **Mathematical programming**

- **PHYSICAL INTERPRETATION**
  - **Limit states/Failure modes**
  - **Hyperplanes/Extreme points**
  - **Relationships**

Proportional loading indicates a system of concentrated loads, each of which is proportional to a parameter, λ. However, the actual loading on the structures may not satisfy the restriction of proportional loading. It is necessary, in such cases, to consider the independent variation of load factor parameters for the various loads acting on the frame. A static multiobjective linear programming (MOLP) model has recently been formulated. Q (q = 1, 2, ..., Q) denote independent load parameters and C_{kj} and D_{kq} are the constant coefficients. Unlike scalar optimization problems, the vector optimal solutions are not completely ordered and there is no unique 'optimal' solution. The notion of an optimal solution is replaced by the concept of weak noninferior solution.

The geometrical structure of the MOLP static model shows that it has two different associated polyhedra instead of just one, as in LP models. These polyhedral regions are defined by the feasible regions of the MOLP model in the objective (load) space and decision (basic variable) space, respectively. The two polyhedral feasible regions have frontiers made up of only polyhedral facets. It has been shown [6] that maximal facets of these polyhedra correspond to the failure modes of the structure and union of all maximal facets gives the global limit surface for the frame.

STATIC MOLP MODEL

$$\text{Max} \quad \bar{\lambda} = \text{Max} \{ \lambda_1, \ldots, \lambda_Q \}^T$$

$$\text{s.t.} \quad \sum_{j=1}^{J} C_{kj} M_j - \sum_{q=1}^{Q} D_{kq} \lambda_q = 0$$

$$k = 1, \cdots, M$$

$$-M_{p_j}^- \leq M_j = \leq M_{p_j}^+$$

$$j = 1, \cdots, J$$

These response patterns for frames define the limits of variation of random variables and once such patterns are generated for a given structure, different solutions corresponding to any random vector must belong to the defined regions. Therefore, by considering powerful algorithmic methods developed in mathematical and computer science literature for extremum problems, alternative structural responses can be predicted without reanalysis of the structure. Often, it is possible to further simplify the procedure in some cases, based on the decomposition of parametric space. These procedures decompose the parametric space into mutually exclusive (non-overlapping) and collectively exhaustive subdomains corresponding to various failure modes [7]. This enables one to replace the consideration of an infinite number of parameter combinations with a finite number of parametric regions, which are also polyhedral. Multiparametric procedures lead to partitioning of both the load and basic variable space. All these procedures do not in any way depend on the probabilistic information. The probability considerations can be subsequently introduced to evaluate structural system reliability. This facilitates investigation of different loading conditions and probabilistic assumptions since reliability evaluations can be obtained without any further structural analyses.
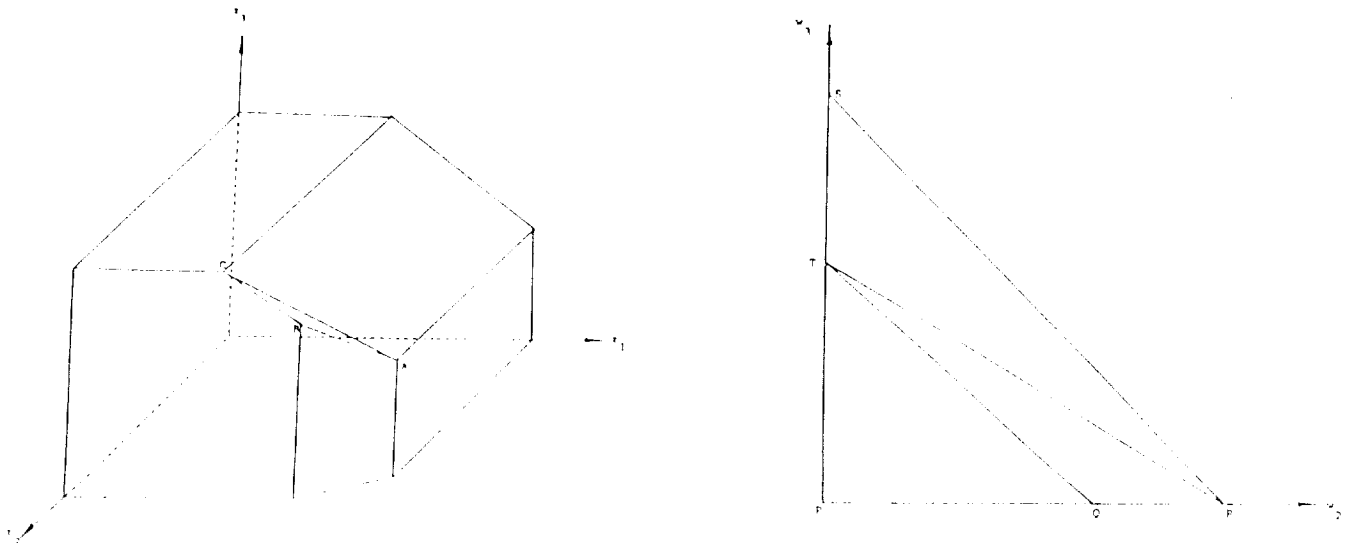
System reliability evaluation of frames for ultimate collapse is simplified by the use of structural responses generated by extremum procedures. For example, in the static case, a method has been proposed [8] which replaces the safe polyhedral response region of the frame with an analytically tractable region of equivalent volume, where the term volume is to be interpreted in a broad sense, since the volume may be of different dimensions and order. Reliability can be computed from the properties of the substituted region, which can be a parallelotope, hypersphere, hyperellipsoid, or any other suitable form. Use of hyperspherical equivalent region leads to the expression for structural system reliability in terms of the chi-square distribution.

System reliability evaluation of frames for ultimate collapse by the kinematic approach requires the enumeration of the failure modes, calculation of the probability of failure for each mode and then computation of the overall reliability by suitable aggregation. A simulation approach that exploits the special structure of the kinematic model has been proposed [9]. Load and resistance proportionalities are determined by each simulation, and the associated failure mode is identified as an extreme point of the LP model. The procedure gains its efficiency from the fact that every simulation derives an associated failure condition and its probability which are then combined into a system reliability.

■ SYSTEM RELIABILITY - STATIC APPROACH

- ### Replacement of response regions by an equivalent region

- ### Concept of equivalence

- ### Hypersphere, parallelotope, hyperellipsoid

■ SYSTEM RELIABILITY - KINEMATIC APPROACH

- ### Generation of failure modes

- ### Simulation procedure

Extremum methods of structural analysis offer significant promise for advances in the analysis of random structural systems and their reliability assessment. The complexity of the physical problem and the randomness of the variables makes the solution otherwise intractable. Fortunately, the mathematical nature of the problem lends itself to mathematical programming formulations and use of powerful algorithmic procedures. This has been illustrated by consideration of rigid plastic frames subject to collapse by flexural action. Linear and multiobjective linear programming models were discussed for structural systems analysis under proportional and multiparametric loading, respectively. Duality relations between the static and kinematic approaches for each of these models and their response patterns lead naturally to alternative methods for system reliability evaluation.

Author's ongoing research aims to demonstrate the use of extremum methods for the reliability analysis of different structural systems for varying material behavior, structural dynamics problems and stability analysis. It is planned to explore structural behavior patterns with the objective of gaining insights into random structural behavior, dual relationships of patterns from static and kinematic considerations, causes of redundancy and the feasibility of using insights for the development of simplified and efficient computational methods for structural reanalysis and system reliability evaluation.

- ## PROMISE OF EXTREMUM METHODS

- ## EXTENSIONS
  - **Other structures**
  - **Material behavior models**
  - **Structural dynamics problems**
  - **Stability analysis**

- ## REDUNDANCY

- ## STRUCTURAL REANALYSIS

- ## SIMPLIFIED METHODS FOR SYSTEM RELIABILITY

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