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Evaluation of Alternatives for Best-Fit Paraboloid for Deformed Antenna Surfaces

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Abstract

Paraboloid antenna surfaces suffer performance degradation due to structural deformation. A first step in the prediction of the performance degradation is to find the best-fit paraboloid to the deformed surface. This paper examines the question of whether rigid body translations perpendicular to the axis of the paraboloid should be included in the search for the best-fit paraboloid. It is shown that if these translations are included the problem is ill-conditioned, and small structural deformation can result in large translations of the best-fit paraboloid with respect to the original surface. The magnitude of these translations then requires nonlinear analysis for finding the best-fit paraboloid. On the other hand, if these translations are excluded, or if they are limited in magnitude, the errors with respect to the restricted "not-so-best-fit" paraboloid can be much greater than the errors with respect to the true best-fit paraboloid.

Introduction

Paraboloid antenna surfaces suffer performance degradation due to structural deformation. A first step in the prediction of the performance degradation is to find the best-fit paraboloid to the deformed surface. The process of finding this best-fit paraboloid has received some attention in the past, (e.g. Refs. 1,2) but there is no clear agreement on a procedure that should be followed. In particular, questions that arise are whether it is acceptable to change the focus length in choosing the best-fit paraboloid and which rigid body motions should be included in moving from the original paraboloid to the best-fit one. The present paper attempts to shed some light on this second question.

The choice of rigid body modes to be considered is associated with ill-conditioning of the numerical process of obtaining the best-fit paraboloid. If z denotes the paraboloid axis symmetry, then the ill-conditioning is associated with translations in the x and y directions. Because antenna surfaces are typically shallow paraboloids, finding x and y translations required to move the original paraboloid to the best-fit one leads to an ill-conditioned set of equations. It is possible to eliminate these translations by, for example, setting them to be equal to the corresponding translations at the apex. However, it is not clear how much is lost in terms of the root mean square (rms) surface error. This paper shows that not including these translations can indeed result in substantial increase in rms errors, but that to include them one must resort to complicated and costly nonlinear calculations. This is demonstrated first by the simpler case of a best-fit parabola.

Best-Fit Parabola

The undeformed shape of the parabola is given as

$$y_1 = ax_1^2$$

The distortions in the x_1 and y_1 directions are given by ξ and η (see Figure 1) so that a point A moves to position A'. The best-fit parabola is given as

$$y = cx^2 \tag{2a}$$

$$c = a + b \tag{2b}$$

The two coordinate systems shown in Figure 1 have unit vectors $\bar{i}_1$ and $\bar{j}_1$ associated with the undeformed parabola, and $\bar{i}$ and $\bar{j}$ associated with the best-fit parabola.

The radius vector $\overline{R}'_A$ from the apex of the original parabola to A' is given as

$$\overline{R}'_A = (x_A + \xi)\bar{i}_1 + (ax_A^2 + \eta)\bar{j}_1 \tag{3}$$

The closest point to A' on the best-fit parabola is denoted B (Figure 1) and has the coordinates $[x_B, (a+b)x^2\beta]$ in the (x, y) coordinate system. The radius vector from the origin of the original parabola to β is given as

$$\overline{R}_B = \beta_1\bar{i}_1 + \beta_2\bar{j}_1 + x_B\bar{i} + (a+b)x_B^2\bar{j} \tag{4}$$

where β_1 and β_2 are the coordinates of the origin of the best-fit parabola. Denoting the angle between the x, and x axes as β_3 we have

$$\bar{i}_1 = \bar{i}\cos\beta_3 - \bar{j}\sin\beta_3 \tag{5a}$$

$$\bar{i}_2 = \overrightarrow{i}\sin\beta_3 + \bar{j}\cos\beta_3 \tag{5b}$$

Using Eqs. (5) we can obtain from Eqs. (3) and (4) the error $\bar{\nu}$ of A' with respect to the best-fit parabola

$$\bar{\nu} = \overline{R}_B - \overline{R}'_A = (x_B - \bar{x}_A)\bar{i} + [(a+b)x_B^2 - \bar{y}_A]\bar{j} \tag{6}$$

where $(\overline{x}_A, \overline{y}_A)$, the coordinates of A' in the best-fit coordinate system are given as

$$\overline{x}_A = (x_A + \xi)\cos\beta_3 + (ax_A^2 + \eta)\sin\beta_3 - \beta_1\cos\beta_3 - \beta_2\sin\beta_3 \tag{7a}$$

$$\overline{y}_A = -(x_A + \xi)\sin\beta_3 + (ax_A^2 + \eta)\cos\beta_3 + \beta_1\sin\beta_3 - \beta_2\cos\beta_3 \tag{7b}$$

The point β on the best-fit parabola is found by minimizing $\|\overline{R}_B - R'_A\|$ with respect to x_β. In the present work this is done with a Newton-Raphson iteration using x_A as an initial guess (x_B is the solution to a cubic equation).

The parameters $b, \beta_1, \beta_2, \beta_3$ are found by minimizing the root-mean-square (rms) distortion

$$\nu_{rms}^2 = \frac{1}{2h}\int_{-h}^{h} \nu \cdot \nu dx \cong \sum_{i=1}^{n} c_i \nu^2(x_i) \tag{8}$$

where $\pm$h are the limits of the parabola, c_i are quadrature weight and x_i are points where the deformed parabola coordinates are given. In this work the minimization was performed by a conjugate-gradient method using finite-difference derivatives.

Instead of performing this nonlinear analysis it is standard practice to linearize the problem. First we set $\cos\beta_3 = 1$, $\sin\beta_3 = \beta_3$ and neglect higher order terms in Eqs. (7) (assuming $\xi, \eta, \beta_1, \beta_2, \beta_3$ are small) to get

$$\overline{x}_A = x_A + \xi + ax_A^2\beta_3 - \beta_1 \tag{9a}$$

$$\overline{y}_A = -x_A\beta_3 + ax_A^2 + \eta - \beta_2 \tag{9b}$$

Next we use a linear approximation to the minimum distance as follows: We set $x_B = x_A$ in Eq. (6) and take only the component of $\bar{\nu}$ normal to the best-fit parabola at $x_B = x_A$. This normal $\overline{n}$ is given as

$$\overline{n} = \frac{\bar{j} - 2(a+b)x_A\bar{i}}{\sqrt{1 + 4(a+b)^2 x_{A^2}}} \tag{10}$$

Neglecting higher order terms the normal component of ν can be written as

$$\nu_n = \overline{\nu} \cdot \overline{n} = -\nu_\circ + \ell^T \alpha \tag{11}$$

where

$$\nu_\circ = (\eta - 2ax_A\xi)/\sqrt{1 + 4a^2 x_A^2} \tag{12}$$

and

$$\ell^T = [x_A^2, -2ax_A, 1, xa + 2a^2 x_A^3]/\sqrt{1 + 4a^2 x_A^2} \tag{13}$$

The rms error is now defined as

$$\nu_{rms}^2 = \frac{1}{2h}\int_{-h}^{h} \nu_n^2 dx \tag{14}$$

To minimize it we differentiate Eq. (14) with respect to α to obtain

$$\frac{1}{2h}\int_{-h}^{h} \nu_n \frac{\partial \nu_n}{\partial \alpha_j} dx = 0 \qquad j = 1, ..., 4 \tag{15}$$

Using Eq. (11), Eq. (15) becomes

$$A\alpha = f \tag{16}$$

where

$$A = \int_{-h}^{h} \ell\ell^T dx = \sum_{i=1}^{n} c_i \ell_i(x_i) \ell_i^T(xi) \tag{17a}$$

$$f = \int_{-h}^{h} \nu_o \ell dx = \sum_{i=1}^{n} c_i \nu_o(x_i) \ell(x_i) \tag{17b}$$

The matrix A is almost singular so that small deformations ξ, η can result in large values of β_1 (the x-translation) and β_3 (the rotation). We can minimize ν_{rms}^2 with an additional limitation on the size of α of the form

$$\alpha^T \alpha \leq \xi \tag{18}$$

and this leads to a system of equations

$$(A + \lambda I)\alpha = f \tag{19}$$

where λ is a Lagrange multiplier (chosen to satisfy Eq. (18)) and I the identity matrix.

Best-Fit Paraboloid

The derivations for the paraboloid parallel the derivations for the parabola given in the previous section. The undeformed shape of the paraboloid is given as

$$z_1 = a\rho_1^2 \tag{20}$$

in a coordinate system shown in figure 2. The distortions in the ρ_1, θ_1 and z_1 directions are given by ξ, η and ζ, respectively, so that point A in Figure 2 moves to position A$'$.

The best-fit paraboloid is given as

$$z = c\rho^2 \tag{21a}$$

$$c = a + b \tag{21b}$$

The radius vector $\bar{R}'_A$ from the apex of the original paraboloid to A' is given as

$$\begin{aligned}\bar{R}'_A &= [(\rho_A + \xi)cos\ \theta_A - \eta\ sin\ \theta_A]\bar{i}_1 \\ &+ [(\rho_A + \xi)sin\ \theta_A + \eta\ cos\ \theta_A]\ \bar{j}_1 + (a\rho_A^2 + \zeta)\bar{k}_1 \\ &= x'_A\bar{i}_1 + y'_A\bar{j}_1 + z'_A\bar{k}_1\end{aligned} \tag{22}$$

The closest point to A' on the best-fit paraboloid is denoted B (Figure 2) and has the cylindrical coordinates $[\rho_B, \theta_B, (a+b)\rho_B^2]$ in the (x, y, z) cordinate system. The radius vector from the origin of the original paraboloid to B is given as

$$\bar{R}_B = \beta_1\bar{i}_1 + \beta_2\bar{j}_1 + \beta_3\bar{k}_1 + \rho_B\ cos\ \theta_B\bar{i} + \rho_B\ sin\ \theta_B\bar{j} + (a+b)\ \rho_B^2\bar{k} \tag{23}$$

where now β_1, β_2 and β_3 are the coordinates of the apex of the best-fit paraboloid in the $(x_1,\ y_1,\ z_1)$ system. The relationship between the unit vectors in the original and best-fit systems is given as

$$\left\{\begin{matrix}\bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1\end{matrix}\right\} = T\left\{\begin{matrix}\bar{i} \\ \bar{j} \\ \bar{k}\end{matrix}\right\} \tag{24}$$

where

$$T = \begin{bmatrix}1 & 0 & 0 \\ 0 & cos\beta_4 & -sin\beta_4 \\ 0 & sin\beta_4 & cos\beta_4\end{bmatrix}\begin{bmatrix}cos\beta_5 & 0 & sin\beta_5 \\ 0 & 1 & 0 \\ -sin\beta_5 & 0 & cos\beta_5\end{bmatrix}\begin{bmatrix}cos\beta_6 & -sin\beta_6 & 0 \\ sin\beta_6 & cos\beta_6 & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{25}$$

and β_4, β_5 and β_6 are rotations around the axes x_1, y_1, z_1, respectively.

Using Eq. (25) we can obtain from Eq. (22), (23) and (24) the error $\bar{\nu}$ of A' with respect to the best-fit paraboloid.

$$\bar{\nu} = \bar{R}_B - \bar{R}'_A = (\rho_B \cos\theta_B - \bar{x}_A)\bar{i} + (\rho_B \sin\theta_B - \bar{y}_A)\bar{j} + (c\rho_B^2 - \bar{z}_A)\bar{k} \quad (26)$$

where $(\bar{x}_A, \bar{y}_A, \bar{z}_A)$, the coordinates of A' in the best-fit coordinate system, are given as

$$\begin{Bmatrix} \bar{x}_A \\ \bar{y}_A \\ \bar{z}_A \end{Bmatrix} = T^t \begin{Bmatrix} x'_A - \beta_1 \\ y'_A - \beta_2 \\ z'_A - \beta_3 \end{Bmatrix} \quad (27)$$

The point B on the best-fit paraboloid is found by minimizing $\| \bar{\nu} \|^2$ with respect to ρ_B and θ_B. Doing so we obtain

$$\theta_B = \theta_A + \phi \quad (28)$$

where

$$tan\phi = \frac{\bar{y}_A \cos\theta_A - \bar{x}_A \sin\theta_A}{\bar{x}_A \cos\theta_A + \bar{y}_A \sin\theta_A} \quad (29)$$

and ρ_B is the solution of the cubic equation

$$2c^2\rho_B^3 + \rho_B(1 - 2c\bar{z}_A) - (\bar{x}_A \cos\theta_B + \bar{y}_A \sin\theta_B) = 0 \quad (30)$$

which is closest to ρ_A. The parameters $b, \beta_1, ..., \beta_6$ are found by minimizing the root-mean-square (rms) distortion

$$\nu_{\mathrm{rms}}^2 = \frac{1}{\pi \mathrm{h}^2} \int_0^{\mathrm{h}} \int_0^{2\pi} \bar{\nu}.\bar{\nu}\rho \mathrm{d}\theta \cong \sum_{\mathrm{i}=1}^{\eta} \mathrm{c_i}\nu^2(\rho_\mathrm{i}, \theta_\mathrm{i}) \quad (31)$$

where h is the limit of ρ for the paraboloid. As in the case of the parabola the minimization was performed by a conjugate-gradient method using finite-difference derivatives. As in the case of the parabola we consider also a linear analysis setting $cos\beta_i = 1,\ sin\beta_i = \beta_i$ for $i = 4, 5, 6$. The linear approximation to the minimum distance is obtained by a similar procedure to the two-dimensional case: We set $\rho_B = \rho_A$ and $\theta_B = \theta_A$ in Eq. (26) and take only the component of $\bar{\nu}$ normal to the best-fit paraboloid at $\rho_B = \rho_A$ and $\theta_B = \theta_A$. The normal $\bar{n}$ is given by

$$\bar{n} = \frac{\bar{k} - 2c\rho_A cos\theta_A \bar{i} - 2c\rho_A sin\theta_A \bar{j}}{\sqrt{1 + 4c^2\rho_A^2}} \tag{32}$$

Neglecting higher-order terms the normal component of $\bar{\nu}$ can be written as

$$\nu_n = \bar{\nu}.\bar{n} = \ -\nu_0 + \ell^t\alpha \tag{33}$$

where

$$\nu_0 = (\zeta - 2a\rho_A\xi)/\sqrt{1 + 4a^2\rho_A^2} \tag{34}$$

and

$$\ell^T = [\rho_A^2, -2a\rho_A cos\theta_A, -2a\rho_A sin\theta_A, 1, \rho_A sin\theta_A(1+2a^2\rho_A^2), -\rho_A cos\theta_A(1+2a^2\rho_A^2)]$$

$$/\sqrt{1 + 4a^2\rho_A^2} \tag{35}$$

and

$$\alpha^t = [b, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5] \tag{36}$$

Note that, as expected because of axial symmetry, η and β_6 do not influence the error. The rms error is again defined as

$$\nu_{\mathrm{rms}}^2 = \frac{1}{\pi \mathrm{h}^2} \int_0^{\mathrm{h}} \int_0^{2\pi} \nu_{\mathrm{n}}^2 \rho \mathrm{d}\rho \mathrm{d}\theta \tag{37}$$

and the minimization again leads to a system of linear equations.

$$A\alpha = f \tag{38}$$

where

$$A = \int_0^h \int_0^{2\pi} \ell \ell^t \rho d\rho d\theta \ = \sum_{L=1}^{n} c_i \ell_i(\rho_i, \theta_i) \ell_i^t(\rho_i, \theta_i) \tag{39a}$$

$$f = \int_0^h \int_0^{2\pi} \nu_0 \ell \rho d\rho d\theta = \sum_{L=1}^{n} c_i \nu_0(\rho_i, \theta_i) \ell(\rho_i, \theta_i) \tag{39b}$$

Results for Best-Fit Parabola

To illustrate the problems associated with finding the vector α which defines the best-fit parabola consider a distortion of the form

$$\xi = \xi_C (1 - \cos \frac{2\pi x}{h}) \tag{40}$$

with $\eta = 0$. A parabola with a/h = 0.2 corresponding to focal length over diameter ratio of 0.625 was used. The best-fit parabola was calculated for a very small disturbance $\xi_C/h = 0.005$. The best-fit parabola corresponding to this distortion was calculated three different ways:

(i) By using conjugate gradient optimization procedure to minimize ν_{rms}^2 based on the nonlinear expressions in Eqs. (6) and (8). The resulting surface error is denoted ν_{NL}.

(ii) By solving the linearized problem Eq. (16). The corresponding linear approximation to the surface distortion is denoted ν_L.

(iii) By solving the size-limited problem, Eq. (19) for various values of λ.

The results are summarized in Table 1. The first line shows the results obtained with conjugate gradient minimization of the nonlinear expression for the error. It is seen that the rms value of the error can be reduced by a factor of three. However, there is great amplification of the disturbance with the normalized translation β_1/h being equal to 0.1571. The linear analysis based on Eq. (16) yielded similar values for the components of α. However, because of the large values of these components the prediction of that linear analysis was erroneous. The predicted rms value of 4.9×10^{-4} compares with a nonlinear value of 6.37×10^{-3}. Thus while the linear analysis predicted a reduction of the initial rms by a factor of 3 the nonlinear analysis predicted that the best-fit linear parabola actually *increased* the error by a factor of three and a half.

The next three lines in Table 1 include size-limited solutions obtained from Eq. (19) with various values of λ. It is seen that as λ is increased the size of α decreases so that the linear and nonlinear predictions become close. However, this is accompanied with substantial increase in the best-fit rms error. The last line in the table shows a 3-variable solution obtained by setting β_1 to the apex x-translation (zero here). This solution is close to the large-λ solution from Eq. (19).

Table 1 shows that we have a dilemma in the construction of a best-fit parabola. Linear analysis requires that we eliminate one of the variables or restrict the size of the solution. These limitations, however, substantially increase the error rms of the now 'not-so-best-fit' parabola. The alternative nonlinear solution is complex and costly.

This type of difficulty is not encountered when the distortion does not require β_1 and β_3 for its correction. As an example consider a distortion of the form

$$\xi = \xi_s \sin\frac{2\pi x}{h} \tag{41}$$

The results of the nonlinear and linear solutions are shown in Table 2 for a substantial value of $\xi_s/h = 0.04$. It is seen that there is hardly any difference between the linear and nonlinear solutions.

Results for Best-Fit Paraboloid.

Similar results were obtained for the best-fit paraboloid for $a/h = 0.2$. For example, a distortion of the form

$$\zeta = \zeta_s \mathrm{sin}(\pi\rho/\mathrm{h})\mathrm{sin}\theta \tag{42}$$

was considered, and the results for $\zeta_s/h = 0.001$ are summarized in Table 3. The first column shows the results of the nonlinear analysis coupled with the conjugate gradient minimization. The rms error is reduced by about a factor of 7, however there is again amplification of the distortion due to the ill-conditioning of the problem with $\beta_2/h = 0.09$. The linear analysis shown in the second column produces very similar solution, predicting even better reduction in rms (about a factor of 10). However, when the nonlinear solution is analyzed using the nonlinear analysis we find that the error actually *increased* by a factor of 3.

The next three columns in Table 3 show the size-limited solutions based on Eq. (19). It is seen that, as we put more and more stringent limits on the magnitude of the solution, the agreement between the linear and nonlinear solution improves. However, much of the reduction in the error is lost, so that we have a 'not-so-best-fit-paraboloid'. Similar results are obtained by setting β_1 and β_2 equal to x and y translation of the apex (zero for the example) and solving a reduced 4-variable problem.

While this dilemma of how to calculate the best-fit-paraboid is difficult, there is a bright side to it. The linear analysis gives a reasonable idea of the magnitude of error reduction possible with the best-fit paraboloid.

Concluding Remarks

An investigation of alternatives for calculating the best-fit paraboloid to a deformed paraboloid surface was investigated. In particular we focused on the ill-conditioning associated with the translations perpendicular to the axis of the paraboloid. It was shown that this ill-conditioning results in disturbance amplification so that small deformation can result in large translations and rotations for the best-fit paraboloid. It was also found that eliminating the two translations or restricting their magnitude may result in large increases in rms errors.

The amplification of translations and rotations for the best-fit paraboloid results in grossly inaccurate prediction by linear analysis of the rms error. However, the linear analysis may be less inaccurate in predicting the achievable reduction in rms error.

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Table 1: Best-fit parabola with various fitting schemes, $\xi_c/h = 0.005$, initial error $\nu_{orms}/h = 1.75 \times 10^{-3}$

Fitting scheme	b/h	β_1/h	β_2/h	β_3/h	ν_L/h	ν_{NL}/h
					rms values	
4-variable						
nonlinear	.00104	.1571	.00426	.05854		4.90×10^{-4}
linear	0.	.1574	0.	.05844	4.90×10^{-4}	6.37×10^{-3}
$\lambda = .00001$	0.	.1238	0.	.05844	4.90×10^{-4}	3.90×10^{-3}
$\lambda = .0001$	0.	.0426	0.	.0146	8.91×10^{-4}	9.74×10^{-4}
$\lambda = .0002$	0.	.0248	0.	.00785	9.89×10^{-4}	9.95×10^{-4}
$\lambda = .0005$	0.	.0113	0.	.00267	1.07×10^{-3}	1.07×10^{-3}
3-variable	0.	0.	0.	-.00163	1.13×10^{-3}	1.13×10^{-3}

Table 2: Best-fit parabola with various fitting schemes, $\xi_s/h = 0.04$, $\nu_{orms}/h = 8.69 \times 10^{3}$

Fitting scheme	b/h	β_1/h	β_2/h	β_3/h	ν_L/h	ν_{NL}/h
					rms values	
nonlinear	.0142	0.	-.00228	0.		6.02×10^{-3}
linear	.0141	0.	-.00225	0.	5.69×10^{-3}	6.02×10^{-3}

Table 3: Best-fit parabola with various fitting schemes, $\zeta_s/h = 0.001$, initial error $\nu_{orms}/h = 4.868 \times 10^{-4}$

Fitting scheme	6-variable					4-variable
	nonlinear	linear				
		$\lambda = 0$	$\lambda = 5 \times 10^{-6}$	$\lambda = 2 \times 10^{-5}$	$\lambda = 5 \times 10^{-5}$	
b/h	2.33×10^{-4}	0.	0.	0.	0.	0.
β_1/h	0.	0.	0.	0.	0.	0.
β_2/h	-0.09055	-0.09055	-0.06281	-0.03280	-0.01686	0.
β_3/h	1.41×10^{-3}	0.	0.	0.	0.	0.
β_4/h	-0.03366	-0.03366	-0.02313	-0.01174	0.00569	7.12×10^{-4}
β_5/h	0.	0.	0.	0.	0.	0.
ν_L/h		5.19×10^{-5}	1.12×10^{-4}	2.14×10^{-4}	2.70×10^{-4}	3.29×10^{-4}
ν_{NL}/h	6.91×10^{-5}	1.47×10^{-3}	7.05×10^{-4}	2.79×10^{-4}	2.73×10^{-4}	3.29×10^{-4}

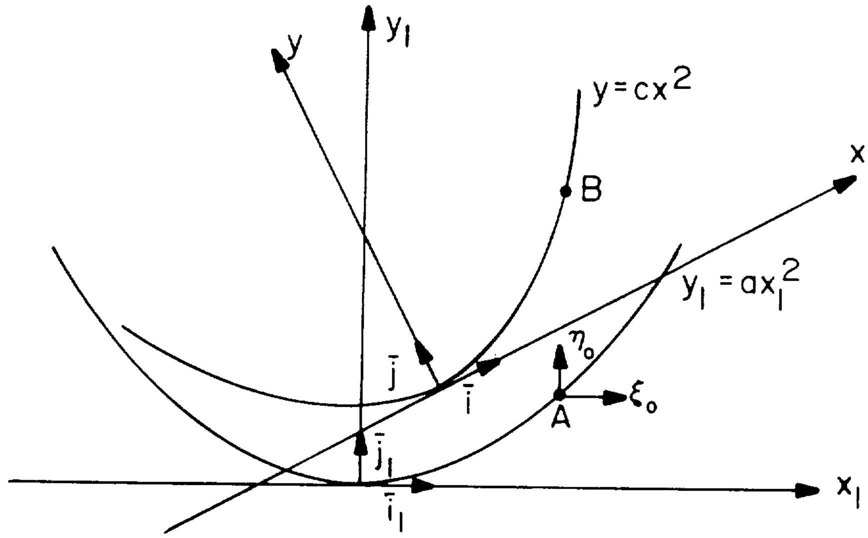

Figure 1: Parabola geometry

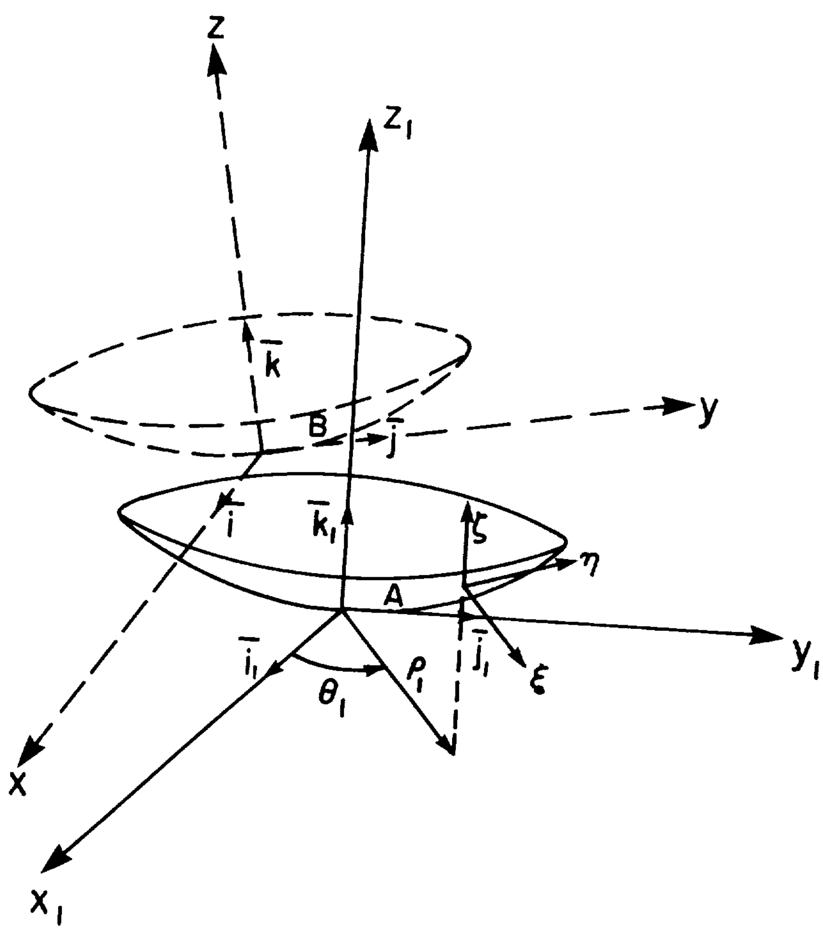

Figure 2: Paraboloid Geometry

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