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Title: ANALYSIS OF THE LEADING EDGE EFFECTS
                   ON THE BOUNDARY LAYER TRANSITION

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1. Introduction.

In recent years, the active boundary-layer control by wall-heating has received a great deal of attention due to its potential applications to drag reduction in subsonic flow over an airfoil. To support an experimental investigation undertaken at Langley, a research program on the theoretical study of active flow control by surface heating/cooling has been initiated by the principal investigator.

The problem of boundary-layer control has been investigated by various researchers for many years, (see, e.g., Chapter 14, [1]). Recently Liepmann, Brown and Nosenchuck [2,3] introduced the wall-heating as an active control agent to suppress the Tollmien-Schlichting waves in the boundary-layer in a water tunnel. In their experiments, the boundary-layer distrubances are also excited by surface heating. They demostrated convincingly that, for a simple pluse-type of disturbances, the periodic surface-heating with properly adjusted phase and amplitude can effectively reduce the level of such disturbances. This new technique is very promising as a tool in the boundary-layer control. Later on Maestrello [4] has shown that, in a wind-tunnel, localized surface heating can be used to trigger instantaneous transition or to reduce the level of disturbances by coupling it to a feedback acoustic device. To provide a theoretical justification, Maestrello and Ting[5] performed an analysis of active control by surface heating.

So far the laboratory experiments have been conducted under ideal conditions, where, for instance, the flow distrubance is known to be simple periodic. The control problem becomes relatively easy. To realize such active control in a real flow situation with unpredictable disturbances, one needs an automatic

control device. Therefore a rational formulation of such a problem should be based on the optimal control theory [6], which has been the guiding principle in the course of our investigation. In contrast with other analytic studies [7,8], we seem to be the first ones who introduced the optimal control concept into the problem of active flow control by surface heating. A complete formulation of this problem would involve a coupled system of Navier-Stokes and energy equations subject to a boundary control by surface-heating elements. However, even without the control, the solution to such problem is already formidable. Clearly one has to start with a simplified model and then approaches the original problem step by step in improving the physical approximation.

Since the theory of optimal flow control is relatively new, the primary goal of this research project has been to explore the potential application of the optimal control theory, which is well developed for finite-dimensional systems, to a fluid dynamical system. Such system, as an example of the distributed-parameter system, is infinite-dimensional and, therefore, much more difficult to deal with. In this report, we shall present a general theory of the boundary Layer Control by surface heating, which is given in Section 2. In the next Section, we will describe some analytical results for a simplified model, i.e., the optimal control of temperature fluctuations in a shear flow. The result may provide a clue to the effectiveness of the active feedback control of a boundary layer flow by wall heating. In a practical situation, the feedback control may not be feasible from the instrumentational point of view. In this case the vibrational control introduced in systems science can provide a useful alternative. In Section 4 we briefly explain this principle

and applies it to the control an unstable wavepacket in a parallel shear flow. Application of such novel control technique to more complex fluid mechanical systems will be explored further.

2. General Theory of Boundary Layer Control by Heating

Consider the viscous flow over a semi-infinite plate lying on $y = 0, x \geq 0$, in the $x - y$ plane. Let the upstream $U_\infty(x)$ be perturbed by a disturbance $U(x, y, t)$ so that the upstream velocity components are

$$\tilde{u} = U_\infty + \epsilon U_0(x, y, t), \tilde{v} = 0$$

where ϵ is a small parameter measuring the magintude of the disturbance. Suppose u_0, u_0, p_0 and θ_0 are the Blasius velocity components, the pressure and the temperature, respecitvely. In the boundary-layer, we set $\tilde{u} = u_0 + \epsilon u, \tilde{v} = v_0 + \epsilon v, \tilde{p} = p_0 + \epsilon p$ and $\tilde{\theta} = \theta_0 + \epsilon \theta$. Then, by a linear stability analysis of the non-steady thermal boundary-layer equation, we get

$$(1) \qquad \frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} + v_0 \frac{\partial u}{\partial y} + \frac{\partial u_0}{\partial x} u + \frac{\partial u_0}{\partial y} v = \nabla \cdot (\mu \nabla u) + h.$$

$$(2) \qquad c(\frac{\partial \theta}{\partial t} + u_0 \frac{\partial \theta}{\partial x} + v_0 \frac{\partial \theta}{\partial y}) = k \frac{\partial^2 \theta}{\partial y2} + 2\mu(\frac{\partial u_0}{\partial y})(\frac{\partial u}{\partial y}).$$

$$(3) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

with the boundary conditions

$$(4) \qquad (u, v) = \begin{cases} (f, g), & t = 0 \\ (0, 0), & y = 0, x > 0 \\ (U_0, 0), & x \leq 0 \ or \ y = \infty \end{cases}$$

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(5) $\theta = \theta_0$, for $t = 0$, and at $x = 0$ or ∞ for $t > 0$,

$$k_0 \frac{\partial \theta}{\partial y} = -q(x,t) \text{ at } y = 0$$

In (1) the kinematic viscosiy μ is assumed to depend on the temperature and the source term h is given by

(6) $h(x,y,t) = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}(\frac{\partial U_0}{\partial t} + U_\infty \frac{\partial U_0}{\partial x} + U_0 \frac{\partial U_\infty}{\partial x})$,

as a result of the boundary-layer approximation (see Chapter. 15, [1]). In (2), c is the specific heat and the thermal diffusivity k is assumed to be constant. The heat flux q in (5) is the control.

As a first approximation, we follow a reasoning by Liepmann et al [2,3]. Since

(7) $-\frac{\partial \mu(\theta)}{\partial y} = \frac{\partial \mu}{\partial \theta}\frac{\partial \theta}{\partial y} = q(x,t)$,

which is proportional to the heat flux across the wall and $\frac{\partial \mu}{\partial y} \cong 0$ away from the wall, thus we have $\frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) = v_e \frac{\partial u}{\partial y} + \mu \frac{\partial^2 u}{\partial y^2}$, where v_e is an effective normal velocity given by $v_e = -(\frac{\partial \mu}{\partial y}) = q$ near $y = 0$, and $v_e \cong 0$, away from $y = 0$. Introduce the differential operator

(8) $Au = (\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial x} - \mu\Delta)u$,

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. By the above physical argument, the system (1) - (5) can be replaced by

(9) $Au + \beta v = h$, with $\beta = \frac{\partial u_0}{\partial y}$,

(10) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ for $x > 0, y > 0$ and $t > 0$,

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$$(u, v) = \begin{cases} (f, g) & \text{at } t = 0, \\ (0, v_e) & \text{at } y = 0, x > 0 \\ (U_0, 0) & \text{for } x \leq 0 \text{ or } y = \infty \end{cases}$$

where the energy equation (2) may be neglected.

The above systems form the state equations for the control problem where the effective velocity $v_e = q$ will be used as a boundary control.

Let $D = \{x > 0, y > 0\}$ be the region above the plate; Γ_0 and Γ_1 be two disjoint parts of the wall $\Gamma = \{x > 0, y = 0\}$. For a feedback control, we measure the shear-stress distribution τ over Γ_1 to yield the observation equation

$$(11) \qquad \tau(x, t) = \rho\mu \frac{\partial u}{\partial y} \text{ on } \Gamma_1.$$

For the optimality criterion, we introduce an objective functional or performance index $J(v_e)$. For example, one may choose

$$(12) \quad J(v_e) = \int_0^T \int_{\Gamma_1} \tau^2(x, t)\rho(x, t) dx dt + \int_0^T \int_{\Gamma_0} v_e(x, t)\gamma(x, t) dx dt,$$

where the second integral represents the control cost function, while ρ and γ are appropriate weight functions.

Let Q be the admissible set of controls v_e of functions of t and x, for $0 \leq t \leq T$ and x in Γ_0 (the porous region). Then the optimal control problem can be stated as follows:

Find an optimal surface-heating rule $v_e^*(\tau)$ from the admissible class Q such that v_e^* minimizes the observed wall-shear and the control cost, i.e.

$$(13) \qquad J(v_e^*) = \min_{v_e^* \text{ in } Q} J(v_e).$$

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This can be formulated as a problem in variational calculus. If such a solution $q\star$ can be constructed, it will yield a feedback control law

(14) $$v_e = v_e^\star(\tau).$$

This control law will regulate the injection velocity distribution automatically based on the wall-shear input.

To be specific, let us consider two special types of problems:

1). Feedback Control of Normal - Mode Instabilities.

For a parallel shear flow subject only to an initial perturbation, we set $v_0 = 0$ and $h = 0$ in Equation (8). By introducing a stream function φ, the system (9) - (10) yields

(15) $$[\frac{\partial}{\partial t} + u_0(y)\frac{\partial}{\partial x}]\Delta\varphi = u_0''(y)\frac{\partial}{\partial x}\varphi + R^{-1}\Delta^2\varphi,$$

$$\varphi(x, y, 0) = \varphi_0(x, y),$$

(16) $$\begin{cases} \frac{\partial}{\partial x}\varphi \mid_{y=0} = v_e(x, t), \frac{\partial}{\partial y}\varphi \mid_{y=0} = 0, \\ \nabla\varphi = 0 \text{ as } y \to \infty, \end{cases}$$

where R is the Reynolds number.

Given the observed wall-shear τ, the feedback control law is assumed to be of the form:

(17) $$v_e(x, t) = \int g(x - \xi)\tau(\xi, t)d\xi = \mu \int g(x - \xi)\varphi_{yy}(\xi, 0, t)d\xi,$$

where g is an optimal transfer function as yet to be determined. For a normal mode analysis, let

(18) $$\hat{\varphi}(y, t, k) = \int \varphi(x, y, t)e^{ikx}dx = \theta(y, k)e^{iwt}$$

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Then the system (15) - (16) reduces to an eigenvalue problem for the Orr-Sommerfeld equation with a modified boundary condition:

$$(19) \qquad L\theta = \lambda M\theta, y > 0$$

$$ik\theta = \mu\hat{g}\theta'' \text{ and } \theta' = 0 \text{ at } y = 0,$$

$$\theta = \theta' = 0 \text{ at } y = \infty.$$

Here we have put

$$L = -(D^2 k^2)^2, D = \frac{d}{dx},$$

$$M = (U - c)(D^2 - k^2) - U; \ U = U_0/c; \ c = w/k,$$

$$\lambda = (-iw),$$

and

$$\hat{g}(k) = \int g(x)e^{ikx}dx.$$

Since the real part $Re \ \{\lambda\}$ of the eigenvalue λ yields the growth rate (if positive) for the normal mode with wave number k. Therefore we choose the performance index $J(v_e) = Re \ \{\lambda(v)\}$. The optimal control problem is to determine the transfer function $\hat{g}$ (or g) so as to minimize the growth rate $J(v_e) = \tilde{J}(g)$. The optimal transfer functions g^* will give the optimal control law via Eq. (17).

2). Optimal Control of Externally Excited Instabilities.

In constrast with the previous problem, the external perturbation h in Eq. (8) is non-zero. By method of superposition, one assumes that the normal mode (18) and

$$(20) \qquad \hat{h}(y, t, k) = \int h(x, y, t)e^{ikx}dx = \tilde{h}(y, k)e^{-iwt}.$$

In view of (20), a Fourier transform of the system (15)-(16) yields a nonhomogeneous Orr-Sommerfeld equation with a modified boundary condition:

(21)
$$(L - \lambda M)\theta = \tilde{h}(y, k), y > 0,$$

$$ik\theta = \mu \hat{g} \theta'' \text{ and } \theta' = 0 \text{ at } y = 0,$$

$$\theta = \theta' = 0 \text{ at } y = \infty,$$

which is a nonhomogeneous boundary-value problem. For a feedback control of the form (17), we choose the mean kinetic energy of the pertubation as the performance index:

$$J(v_e) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int \int \{\varphi_x^2(x, y, t) + \varphi_y^2(x, y, t)\} dt \, dx \, dy,$$

which in terms of θ, reduces to

(22)
$$J(\hat{v}_e) = \int_0^\infty \{| \theta'(y, k) |^2 + k^2 | \theta(y, k) |^2\} dy.$$

Since $J(\hat{v}_e) = \tilde{J}(\hat{g})$, the control problem at hand is to determine the transfer function g^* which minimizes the mean kinetic energy J. This problem may be solved by the method of Green's function. Let $G(y, \eta, k)$ be the Green's function for the boundary-value problem (21) so that

(23)
$$\theta(y, k) = \int_0^\infty G(y, \eta, k) \tilde{h}(\eta, k) d\eta.$$

Note that, through a boundary condition in (21), G depends smoothly on $\tilde{g}$ (or g). A substitution of (23) into (22) shows that $\tilde{J}(\hat{g})$ is a functional of $\hat{g}$. Therefore the optimal $\hat{g}^*$ (or g^*) can be determined by the variational equation

(24)
$$\delta \tilde{J}(\hat{g}) = 0.$$

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Thereby the optimal control law can be determined. For this problem the major task is to carry out the variational analysis. Then numerical computation may be done to show the effectiveness of the optimal control and its dependence on various physical parameters.

3. Solution of a Simplified Control Problem

As mentioned before, due to the complexity of the surface heating control problem, an analytical solution is unattainable. Even the numerical solution is difficult. In this section we shall present a simplified model problem which can be analyzed rather completely.

Recall that the analysis of heated boundary layers is based on a coupled system of the momentum equation (1) and the energy equation (2). One notes that the coupling between these equation is due to the dependence of the viscosity on the temperature. It seems safe to say that, in the absence of mechanical disturbance, the flow instability can be achieved by reducing the temperature fluctuation, which is the only source of external excitations. Furthermore we regard the dynamical heat production term in the RHS of Eq. (2) as an external perturbation denoted by, say, $cf(x, y, t)$, and assume that the flow is parallel to the plate with $u_0 = U(y)$ and $v_0 = 0$. Then we are led to considering the following optimal heat regulating problem:

$$(25) \qquad \left(\frac{\partial}{\partial t} + U\, \frac{\partial}{\partial x}\right)\theta = \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta + f(x, y, t),$$

$$(26) \qquad k_o\frac{\partial\theta}{\partial y}(x, o, t) = -q(x, t),$$

for $t > 0, -\infty < x < \infty$, and $y > 0$, where $\nu = k/c$ is the thermal diffusivity and f is an equivalent source of thermal disturbance. Of particular physical

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interest is the case where the perturbation is persistent in time and localized in space. Thus the perturbing source function f is assumed to satisfy the mean-square integrability condition

$$(27) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} |f(x,y,t)|^2 dt \ dx \ dy < \infty.$$

As a consequence, the heat flux q, the active control, is expected to have a similar property

$$(28) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{-\infty}^{\infty} |q(x,t)|^2 dt \ dx < \infty.$$

As an optimality criterion, the objective function J given below will be minimized,

$$(29) \qquad J(q) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{-\infty}^{\infty} \{|\theta(x,0,t)|^2 + N|q(x,t)|^2 dt \ dx$$

where the first term in the integrand yields the mean-square temperature fluctuation along the wall, while the second one is a measure of the mean-square control cost, where $N > 0$ is a cost parameter. For $N = 0$, the minimization of J gives the optimal control $\tilde{q}$ which is the best possible for the system to realize. On the other hand, for $N > 0$, the optimal solution $\tilde{q}$ will be to minimize the wall temperature fluctuation without excessive use of the control action. The choice of the number N depends subjectively on the relative importance assigned to two competing factors in the optimality criterion. Now the mathematical problem for the control of thermal disturbances can be formulated as follows: For a given heat source perturbation $f(x,y,t)$ satisfying the condition (27), find the optimal control $\tilde{q}(x,t)$, with the property (28), which minimizes the objective function (29) . Here the

"state" θ is the solution of the system (25) and (26) subject to the zero initial condition. Note that, since the transient part of the solution to the system will be wiped out by the time average in (29), without loss of generality the initial condition may be taken to be zero.

Since the governing equation is linear and the functional J is quadratic, the problem can be solved by the principle of superposition. Therefore it is sufficient to consider a time-harmonic perturbation

$$(30) \qquad f(x, y, t) = g(x, y)e^{-i\omega t}.$$

Then it is possible to seek a time-harmonic solution

$$(31) \qquad \theta(x, y, t) = \phi(x, y)e^{-i\omega t},$$

and

$$(32) \qquad q(x, t) = r(x)e^{-i\omega t}$$

A substitution of (30)–(32) into (25), (26), and (29) yields

$$(33) \qquad L\phi = \nu \nabla^2 \phi - U(y)\frac{\partial \phi}{\partial x} + i\omega\phi = -g(x, y)$$

$$(34) \qquad k_o \frac{\partial \phi}{\partial y}(x, o) = -r(x),$$

$$(35) \qquad J(r) = \int_{-\infty}^{\infty} \{|\phi(x, o)|^2 + N|r(x)|^2\}dx,$$

where ϕ vanishes as $|x| \to \infty$ or as $y \to \infty$. It is well known that the necessary condition for J being minimal is

$$(36) \qquad \delta J(r) = 0,$$

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where δJ means the variation of J with respect to r. Note that the quantities ϕ, r are complex. Denote their complex conjugates by ϕ^* and r^* respectively. By introducing an adjoint state ψ to ϕ, one can derive the following optimality system [6]

$$(37) \qquad L\phi = \nu\nabla^2\phi - U(y)\,\frac{\partial\phi}{\partial x} + i\omega\phi = -g(x,y),$$

$$(38) \qquad L^*\psi = \nu\nabla^2\psi + U(y)\,\frac{\partial\psi}{\partial x} - i\omega\psi = 0,$$

$$(39) \qquad k_o\frac{\partial\phi}{\partial y}(x,o) - \frac{1}{N}\psi(x,o) = 0,$$

$$(40) \qquad \phi(x,o) + k_o\frac{\partial\psi}{\partial y}(x,o) = 0.$$

It is seen that, in order to find the optimal control $\tilde{r}$, one must solve an extended system of coupled equations (37)–(40). Then the optimal solution is given by

$$(41) \qquad r = \tilde{r}(x) = -\frac{1}{N}\psi(x,o)$$

and

$$(42) \qquad \tilde{q}(x,t) = \tilde{r}(x)\,e^{-i\omega t}$$

gives the optimal control for the time-harmonic disturbance (30). In the multiple frequency case, the optimal control $\tilde{q}$ is a sum of the single-frequency solutions by superposition.

Clearly, by a Fourier transform in x, the optimality system (37)–(40) is reducible to a one-dimensional problem:

$$(43) \qquad \nu \hat{\phi}'' - [\nu \lambda^2 + i(\lambda U + \omega)]\hat{\phi} = -\hat{g}(\lambda, y),$$

$$(44) \qquad \nu \hat{\psi}'' - [\nu \lambda^2 - i(\lambda U + \omega)]\hat{\psi} = 0,$$

$$(45) \qquad k_o \hat{\phi}'(\lambda, o) - \frac{1}{N}\hat{\psi}(\lambda, o) = 0,$$

$$(46) \qquad k_o \hat{\psi}'(\lambda, o) + \hat{\phi}(\lambda, o) = 0,$$

where the Fourier transform $\hat{f}(\lambda)$ of a function $f(x)$ is defined as

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx.$$

Let $\hat{\phi}_p$ be a particular solution and $\hat{\phi}_1, \hat{\phi}_2$, two linearly independent complementary solutions of the equation (43) such that $|\hat{\phi}_2| \to \infty$ as $y \to \infty$. Then the complex conjugates $\hat{\phi}_1^\star$ and $\hat{\phi}_2^\star$ are linearly independent solutions of (44). Therefore the bounded solutions of (43) and (44) are given by

$$(47) \qquad \hat{\phi}(\lambda, y) \;=\; \hat{\phi}_p(\lambda, y) - a\hat{\phi}_1(\lambda, y),$$

$$(48) \qquad \hat{\psi}(\lambda, y) \;=\; b\hat{\phi}_1^\star(\lambda, y),$$

where, in view of the boundary conditions (45) and (46), the constants a and b are found to be

$$a(\lambda) \;=\; [k_o^2(\hat{\phi}_1^\star)'\hat{\phi}_p' + \frac{1}{N}\hat{\phi}_1^\star\hat{\phi}_p](\lambda, o)/d(\lambda),$$

$$b(\lambda) \;=\; k_o(\hat{\phi}_1\hat{\phi}_p' - \hat{\phi}_1'\hat{\phi}_p)(\lambda, o)/d(\lambda),$$

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where

$$d(\lambda) = (k_0^2|\hat{\phi}_1'|^2 + \frac{1}{N}|\hat{\phi}_1|)(\lambda, o).$$

In view of the equations (43) and (48), by an inverse Fourier transform, the optimal control law is obtained,

$$(49) \qquad \tilde{r}(x) = -\frac{1}{2\pi N} \int_{-\infty}^{\infty} b(\lambda)\hat{\phi}_1(\lambda, o)e^{-i\lambda x}d\lambda.$$

Under the optimal control, the state of thermal fluctuation is given by the inverse transform of (47):

$$(50) \qquad \phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x}[\hat{\phi}_p(\lambda, y) - a(\lambda)\hat{\phi}_1(\lambda, y)]d\lambda.$$

On the other hand, when there is no control ($r \equiv 0$), the temperature fluctuation $\phi_o(x, y)$ is given by

$$(51) \qquad \phi_o(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x}[\hat{\phi}_p(\lambda, y) - a_o(\lambda)\hat{\phi}_1(\lambda, y)]d\lambda,$$

where

$$a_o(\lambda) = \hat{\phi}_p'(\lambda, o)/\hat{\phi}_1'(\lambda, o).$$

To assess the effectiveness of the control, we may introduce either the uniform reduction ratio

$$(52) \qquad \varepsilon_o = \max \varepsilon(x) = \max \cdot |\frac{\phi(x, o)}{\phi_o(x, o)}|,$$

or the (root) mean reduction ratio

$$(53) \qquad \varepsilon_m = \left\{ \int_{-\infty}^{\infty} |\phi(x, o)|^2 dx / \int_{-\infty}^{\infty} |\phi_o(x, o)|^2 dx \right\}^{1/2}$$

$$= \left\{ \int_{-\infty}^{\infty} |\hat{\phi}(\lambda, o)|^2 d\lambda / \int_{-\infty}^{\infty} |\hat{\phi}_o(\lambda, o)|^2 d\lambda \right\}^{1/2},$$

by Parseval's equality in Fourier transform. To give a qualitative physical interpretation of our results, it is instructive to go through some examples in detail.

Two examples corresponding to a time-harmonic perturbation $f(x, y, t)$ of the form (30) will be considered. In particular the spatial distribution function g is assumed to be

$$(54) \qquad g(x, y) = A \ \exp\{i\kappa x - \frac{1}{2}\alpha x^2 - \beta y\}$$

which represents a decaying surface-wave disturbance with amplitude A, the wave number κ, and the exponential decay parameters α and β. Then the Fourier transform of g in x is given by

$$(55) \qquad \hat{g}(\lambda, y) = \hat{g}_o(\lambda) e^{-\beta y},$$

where

$$(56) \qquad \hat{g}_o(\lambda) = A \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{1}{2\alpha}(\lambda + \kappa)^2}.$$

As the first example, let the velocity $U = U_o$ be a constant. Then the transformed optimality system (43)–(46) with g given by (54) can be solved analytically. The transformed optimal solutions (47) and (48) become

$$(57) \qquad \hat{\phi}(\lambda, y) = h(\lambda) \left[e^{-\beta y} - \left(\frac{1 + N k_0^2 \beta \xi^*}{1 + N k_0^2 |\xi|^2} \right) e^{-\xi^* y} \right],$$

$$(58) \qquad \hat{\psi}(\lambda, y) = h(\lambda) \left[\frac{N k_o (\xi - \beta)}{1 + N k_o^2 |\xi|^2} \right] e^{-\xi^* y},$$

where

$$(59) \qquad h(\lambda) = \frac{\hat{g}_o(\lambda)}{\xi^2(\lambda) - \beta^2},$$

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and $\xi^*(\lambda)$ is the complex conjugate of $\xi(\lambda)$ defined by

$$(60) \qquad \xi(\lambda) = \{\lambda^2 + i(\lambda U_o + \omega/\nu)\}^{1/2}, Re\{\xi(\lambda)\} > 0.$$

By the inverse Fourier transform (49), the optimal control law is found to be

$$(61) \qquad \tilde{r}(x) = -\frac{k_o}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{g}_o(\lambda)e^{-i\lambda x}}{(1 + Nk_o^2|\xi(\lambda)|^2)[\xi(\lambda) + \beta]}.$$

Along the wall, from (50) and (51), the controlled and the uncontrolled temperature fluctuations are given by

$$(62) \qquad \phi(x,o) = \frac{Nk_o^2}{2\pi} \int_{-\infty}^{\infty} \frac{\xi^*(\lambda)\hat{g}_o(\lambda)}{[\xi(\lambda) + \beta](1 + Nk_o^2|\xi(\lambda)|^2)} e^{-i\lambda x} dx,$$

$$(63) \qquad \phi_o(x,o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{g}_o(\lambda)}{\xi(\lambda)[\xi(\lambda) + \beta]} e^{-i\lambda x} d\lambda.$$

One observes that, if N is large, then $\tilde{r}(x) \sim o$ and $\phi(x,o) \sim \phi_o(x,o)$. That is, when the control is costly, no control action is taken so that the state is unchanged. While, if the control cost parameter N is small, we get

$$(64) \qquad \tilde{r}(x) \sim \int_{-\infty}^{\infty} \frac{\hat{g}_o(\lambda)e^{-i\lambda x}}{\xi(\lambda) + \beta} d\lambda,$$

and $\phi(x,o) \sim 0$. Thus the unrestricted thermal control almost completely eliminates the temperature fluctuation along the wall. In fact, in this case, the control action is to cancel out the heat flux across the wall at any cost.

To simplify the results, let the decay factor α be small. In view of (56), as $\alpha \to 0, \hat{g}$ approaches a δ-function and the otpimal control (55) reduces to

$$(65) \qquad \tilde{r}(x) \sim -\frac{Ak_o e^{i\kappa x}}{[\xi(-\kappa) + \beta](1 + Nk_o^2|\xi(-\kappa)|^2)}.$$

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The corresponding residual temperature fluctuation (51) becomes

$$(66) \qquad \phi(x,o) \sim \frac{ANk_0^2\xi^*(-\kappa)e^{i\kappa x}}{[\xi(-\kappa)+\beta](1+Nk_o^2|\xi(-\kappa)|^2)}.$$

Recall, by noting (54), that

$$(67) \qquad \xi(k) = \{k^2 + i(kU_o + \omega/\nu\}^{1/2}, Re\{\xi\} > 0,$$

which gives

$$|\xi(k)|^2 = \{k^4 + (kU_o + \omega)^2/\nu^2\}^{1/2}.$$

As it turns out, in this case, the uniform reduction ratio and the mean reduction ratio, defined by (52) and (53) respectively, coincide. In fact we have

$$(68) \qquad \varepsilon = \frac{Nk_o^2|\xi(-\kappa)|^2}{1+Nk_o^2|\xi(-\kappa)|^2} < 1,$$

where $\varepsilon = \varepsilon_o = \varepsilon_m$.

From (54) one sees that, as $\alpha \to 0$, the excited disturbance is a surface plane wave. The otpimal response $\tilde{r}$ in (65) is also of the same wave form with a phase-shift $\arg.\{\xi(-\kappa)+\beta\}$. Under the control, the surface temperature fluctuation is reduced to (66), which has the reduction factor ε given by (68). It shows clearly that ε increases from 0 to 1 as $\eta = Nk_o^2|\xi(-\kappa)|^2$ increases from 0 to ∞. For a fixed control cost parameter N, η is an increasing function of κ and ω, but a decreasing function of ν. Therefore the reduction ratio ε and, hence, the effectiveness of the thermal control decreases as the wave number κ or the frequency ω of the distrubance increase, while they increase as the thermal diffusivity ν of the fluid increases. These results are physically

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plausible and may provide some insights into a certain flow control problem by surface heating or cooling.

As the second example, assume the shear velocity U varying slowly from zero at the wall to the free stream velocity U_o at the height $y = \frac{1}{\delta}$, where δ is small. Then, by applying the multiple-scale or the WKB Method [9], one can construct asymptotic solutions to the optimality system (43)–(46), by using δ as the small parameter. Without giving the derivation, it can be shown that the first-order asymptotic approximation yields the complementary and particular solutions to equation (43) as follows:

(69) $$\hat{\phi}_{1,2}(\lambda, y) \sim a(\lambda, y) e^{\mp \Omega(\lambda, y)}$$

and

(70) $$\begin{aligned} \hat{\phi}_p(\lambda, y) \quad \sim \quad & [2\eta(\lambda, o)]^{-1} \{ \hat{\phi}_1(\lambda, y) \int_0^y \hat{\phi}_2(\lambda, s) \hat{g}(\lambda, s) ds \\ & + \hat{\phi}_2(\lambda, y) \int_y^\infty \hat{\phi}_1(\lambda, s) \hat{g}(\lambda, s) ds \} \end{aligned}$$

where

(71) $$\begin{aligned} \eta(\lambda, y) \quad &= \quad \{\lambda^2 + i \, \nu^{-1} [\lambda U(y) + \omega]\}^{1/2}, Re\eta > 0, \\ a\,(\lambda, y) \quad &= \quad \left[\frac{\eta(\lambda, o)}{\eta(\lambda, y)} \right]^{1/2}, \end{aligned}$$

(72)

and

$$\Omega(\lambda, y) = \int_0^y \eta(\lambda, s) ds.$$

For a small decay factor, $\alpha \approx 0$, by some tedious by straight-forward algebraic manipulations, one can obtain the otpimal control $\tilde{r}$ and the residual

wall temperature fluctuation $\phi(x, o)$ in a simple form:

$$(73) \qquad \tilde{r}(x) \sim -A \frac{k_o \gamma(-\kappa) e^{i\kappa x}}{(1 + N k_0^2 |\eta(-\kappa, o)|^2)},$$

$$(74) \qquad \phi(x, o) \sim \gamma(-\kappa) \left[\frac{k_o^2 N \eta^\star(-\kappa, o)}{1 + k_o^2 N |\eta(-\kappa, o)|^2} \right] e^{i\kappa x},$$

where

$$(75) \qquad \gamma(\lambda) = \int_0^\infty a(\lambda, \tau) e^{-\Omega(\lambda, \tau) - \beta \tau} d\tau$$

From (66) it follows that the reduction ratio

$$(76) \qquad \varepsilon = \frac{N k_o^2 |\eta(-\kappa, o)|^2}{1 + N k_o^2 |\eta(-\kappa, o)|^2}.$$

In contrast with the results (65)–(68) for a uniform shear profile, the above results (72)–(75) show that, for a slowly varying shear flow, the velocity variation introduces a shape factor γ as defined by (69) which modifies the control and the corresponding temperature fluctuation. In fact it is easy to verify that if $u \equiv U_o$, the results (72)–(75) reduce to the previous ones (65)–(68) with a uniform shear velocity, as they should. Therefore the previous physical interpretation of those results is still valid, at least qualitatively.

To issustrate the results graphically, some numerical calculations have been performed when the shear velocity profile is uniform. The numerical results are displayed in Figs. 2–8. Figure 2 shows the periodic variation of the thermal control input along the wall as the nondimensionalized wave number κ varies from 0 to 2, where $r_1 = Re\{\tilde{r}\}$, and $A = k_o = \beta = N = 1$. It is interesting to note that, in response to the periodic distrubance, the

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controlling heat input rises steeply, as κ increases, and then falls off sharply. This trend is clearly seen in Fig. 3. In Fig. 4, two sets of curves correspond to the controlled wall-temperature distribution ($\zeta_1 = Re\zeta$) and the uncontrolled one ($\zeta_1 = Re\zeta_o$), for different values of κ. The associated optimal control is given in Fig. 2. The dependence of the amplitudes of the controlled and the uncontrolled wall-temperature fluctuations on the wave number κ is shown in Fig. 5. The above two figures show clearly the effect of the thermal control on the wall temperature fluctuation at a moderate control cost factor $a = Nk_o^2 = 1$. With $\kappa = 1$ and the rest of the parameters fixed as before, the set of curves in Fig. 6 displays the residual wall-temperature distributions under the optimal control for several values of a, while the corresponding amplitudes are plotted in Fig. 7 against the effective control cost parameter a for several values of κ. Finally in Fig. 8, the reduction ratio ε, which measures the effectiveness of the control, is sketched as function of the wave number κ, for several values of a. Obviously the optimal control is most effective in reducing the thermal disturbances with small wave numbers and a low control cost parameter. The control loses its effectiveness for short-wave disturbances or when the control cost parameter gets too high. These findings seem to be consistent with one's physical intuition. Also it is interesting to note that, for small wave numbers, the reduction ratio dips into a minimum value before it takes off and increases monotonically to one.

4. Vibrational Control of Unstable Wavepackets

The principle of vibrational control of dynamical systems was proposed by Meerkov [10] for control problems where feedback and feedforward principles

vibrational control $\varphi(wt)$ with a high frequency $\omega \gg 1$. To this end, let us scale the time by setting $s = \omega t$ and $\varepsilon = 1/\omega$. Then Eq. (77) can be written as

$$(79) \qquad \frac{dx}{ds} = \varepsilon F(x, t, \varphi(s)),$$

where $t = \varepsilon s$ becomes a slow variable. Note that Eq. (79) is in the canonical form, in the language of nonlinear oscillation theory. Thus, by the averaging principle of Bogoliubov and Mitropolski [13], the average state $y(t)$ satisfies the equation:

$$(80) \qquad \frac{dy}{dt} = \hat{F}(y, t),$$

where

$$\hat{F}(y, t) = \int_0^1 F(y, t, \varphi(s)) ds$$

is the average of F is s over a period with y, t held fixed. By assumption, $y = x_0$ is also an equilibrium point of Eq. (79). If it is possible to find a mean-zero periodic excitation $\varphi(\omega t)$ at a high frequency ω in Eq. (78) so that the average equation (79) is stable at x_0 in the mean, then the system (77) with $\mu = 0$ is said to be vibrationally controllable near $x = x_0$. There are two ways to excite the system: additive excitation and multiplicative (or parametric) excitation. In the following application, we shall be concerned with the case of additive vibrational control only. In this case $F(x, ; \mu) = F_0(x, t) + \mu$ so that Eq. (77) reads

$$(81) \qquad \frac{dx}{dt} = F_0(x, t) + \mu.$$

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For vibrational control, we set $\mu = \frac{d}{dt}\varphi(\omega t) = \dot{\varphi}(\omega t)$ in the above equation to get the controlled system:

$$(82) \qquad \frac{dx}{dt} = F_0(x, t) + \dot{\varphi}(\omega t).$$

Now let

$$x(t) = \varphi(\omega t) + \zeta(t),$$

which is then substituted into Eq. (82) to get

$$(83) \qquad \frac{d\zeta}{dt} = F_0(\varphi(\omega t) + \zeta, t).$$

If we set $F_0(\varphi(\omega t) + \zeta, t) = F(\zeta, t; \varphi(\omega t))$, then Eq. (83) becomes a special case of the system (78) and the vibrational control method described above in applicable to this problem. In fact, since $\varphi(\omega t)$ has a zero mean, the controlability of Eq. (82) near x_0 implies the same for Eq. (81). Now we are going to apply this technique to control the instability of wavepackets in a shear flow as announced.

Consider a parallel shear flow in the x-direction. For a two-dimensional problem, let $u(x, t)$ be the velocity perturbation about the mean flow U at the Reynolds number R near the critical value R_c. Let us write

$$R = R_c \pm \delta^2, \delta = |R - R_c|^{1/2}$$

and

$$u(x, t) = 2Re\left\{A(\xi, \tau)e^{i(\alpha_c x - \bar{\omega}_c t)}\right\}$$

where $\xi = \delta x$ and $\tau = \delta^2 t$, that is, we are seeking a solution in the form of slowly varying wavepacket. It was shown that the variation of the amplitude

A of the most unstable mode satisfies the equation [12].

$$(84) \qquad \frac{\partial A}{\partial \xi} + c_g \frac{\partial A}{\partial \xi^2} + kA - \frac{1}{2}\delta^2 \ell |A|^2 A,$$

where c_g is the group velocity; ℓ is the Landau constant, and the complex constants c and a appear in the following expansion:

$$\lambda = -i\tilde{\omega} = i\tilde{\omega}_c - ic_g(\alpha - \alpha_c) \pm k\delta^{1/2} - a(\alpha - \alpha_c)^2 + \dots.$$

To simplify Eq. (84), we define

$$\eta = (\xi - c_g\tau) \quad \text{and} \quad B = A/\delta,$$

so that it yields:

$$(85) \qquad \frac{\partial B}{\partial \tau} - a \frac{\partial^2 B}{\partial \eta^2} = k\,B - \frac{1}{2}\ell |B|^2 B.$$

It is known that a and k have a positive real part and that, if $R > R_c$ and $\ell_r = Re\,\{\ell\} < 0$, the amplitude may tend to infinity in finite time. Such a burst is reminiscent of sudden transition to turbulence, (see p 452, [12]). In order to control the instability, we introduce an additive vibriational control $\psi(\eta, \omega\tau)$ and consider the controlled equation:

$$(86) \qquad \frac{\partial B}{\partial \tau} - a \frac{\partial^2 B}{\partial \eta^2} = kB - \frac{1}{2}\ell |B|^2 B + \psi(\eta, \omega t).$$

Let us consider a plane wave disturbance of the form

$$B(\eta, \tau) = b(\tau)e^{i\beta\eta}, Im\,\beta = 0,$$

and the corresponding control

$$\psi(\eta, \omega t) = \dot{\varphi}(\omega t)e^{i\beta\eta}.$$

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Then Eq. (86) yields

$$(87) \qquad \frac{db}{d\tau} = (k - a\beta^2)b - \frac{1}{2}\ell|b|^2 b + \dot\varphi(\omega\tau),$$

which is of the form (82). If there is no control ($\varphi = 0$), we can derive from (87) the following equation for $R = |b|^2$:

$$(88) \qquad \frac{dR}{d\tau} = 2\sigma R - \ell_r R^2,$$

where $\sigma = Re\,\{k - a\beta^2\}$ and $\ell_r = Re\,\{\ell\}$. The logistic equation (88) has the solution

$$(89) \qquad R(\tau) = c\,R_1 e^{2\sigma\tau}/(1 + c\,e^{2\sigma\tau}),$$

where $c = \frac{R_0}{R_1 - R_0}$, $R_0 = R(0)$ and $R_1 = \frac{2\sigma}{\ell_r}$. From (89) it is clear that, if $Re\,\sigma > 0$ and $R_1 > R_0$, the equilibrium $R = 0$ is unstable. On the other hand, if $Re\,\sigma < 0$ and $R_1 > R_0$, $R = 0$ is asymptotically stable. However, for any σ, a burst may occur if $R_1 < R_0$. Now returning to the controlled equation (87), we let

$$(90) \qquad b(\tau) = \varphi(\omega\tau) + b^\varepsilon(s),$$

where $s = \tau/\varepsilon, \varepsilon = \frac{1}{\omega}$. Then b^ε satisfies

$$\frac{db^\varepsilon}{ds} = \varepsilon\{\sigma[\varphi(s) + b^\varepsilon] - \frac{1}{2}\ell|\varphi + b^\varepsilon|^2(\varphi + b^\varepsilon)\},$$

which yields the following average equation:

$$(91) \qquad \frac{d\hat{b}}{d\tau} = \sigma\hat{b} - \ell\hat\varphi^2(\hat{b} + \frac{1}{2}\hat{b}^*) - \frac{1}{2}\ell|\hat{b}|^2\hat{b},$$

where $\hat\varphi^2 = \int_0^1 \varphi^2(s)ds$ and $\hat\varphi^3 = 0$. In contrast with Eq. (87) with $\dot\varphi = 0$, one cannot solve for $\hat{R} = |\hat{b}|^2$ in a closed form. To see the effect of the

vibrational control, let us consider the linear stability near $\hat{b} = 0$. Writing $\hat{b}_r = Re\ \hat{b}, \hat{b}_i = Im\ \hat{b}$, etc., the linearized equation of (91) can be written as:

$$(92) \qquad \frac{d\hat{b}_r}{d\tau} = \gamma_1 \hat{b}_r - \bar{\gamma}_2 \hat{b}_i,$$

$$\frac{d\hat{b}_i}{d\tau} = \gamma_2 \hat{b}_r + \bar{\gamma}_1 \hat{b}_i,$$

where

$$\gamma_1 = (\sigma_r - \frac{3}{2}\ell_r\hat{\varphi}^2), \bar{\gamma}_1(\sigma_r - \frac{1}{2}\ell_r\hat{\varphi}^2),$$

and

$$\gamma_2 = (\sigma_i - \frac{3}{2}\ell_i\hat{\varphi}^2), \bar{\gamma}_2 = (\sigma_i - \frac{1}{2}\ell_i\hat{\varphi}^2).$$

Let us substitute for $(\hat{b}_r, \hat{b}_i)$ by $(a_r, a_i)e^{\lambda\tau}$ in Eq. (92) to yield an eigenvalue problem, for which the characteristic equation reads

$$\begin{vmatrix} (\lambda - \gamma_1) & \bar{\gamma}_2 \\ -\gamma_2 & (\lambda - \bar{\gamma}_1) \end{vmatrix} = 0.$$

The roots of the equation are

$$\lambda_{1,2} = (\sigma_r - \ell_r\hat{\varphi}^2) \pm \{\frac{1}{4}|\ell\hat{\varphi}^2|^2 - (\sigma_i - \ell_i\hat{\varphi}^2)^2\}^{1/2}.$$

By the method of finding a maximum in calculus, it is easy to show that

$$Re\ \lambda \leq (\sigma_r - \ell_r\hat{\varphi}^2) + \{\frac{1}{4}|\ell_r\hat{\varphi}^2|^2 + \frac{1}{3}\sigma_i^2\}^{1/2}.$$

Thus, if $\sigma_i = 0$, the equilibrium $\hat{b} = 0$ is stable provided that $\sigma_r < \frac{1}{2}\ell_r\hat{\varphi}^2$. In contrast with the stability condition $\sigma_r < 0$ for the uncontrolled case, the vibrational control can stabilize the system if we make $\hat{\varphi}^2$ large enough. For any $\sigma_i(\neq 0)$, this is also true. Of course we pay a price for stability by allowing

a small zero-mean oscillation about the equilibrium. Further applications of vibrational control to fluid mechanical systems will be discussed in our future work.

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