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SUMMARY

The aim of this research is to recast the indicial response approach in a form appropriate to the study of vortex-induced oscillations phenomena. We demonstrate that an appropriate form for the indicial response of the velocity field may be derived directly from the Navier-Stokes equations. For convenience, the study is divided into three parts. In Part I, we demonstrate, on the basis of the Navier-Stokes equations, how a form of the velocity response to an arbitrary motion may be determined. To establish its connection with our previous work, the new approach is applied first to the simple situation wherein the indicial response has a time-invariant equilibrium state. Results for the aerodynamic response to an arbitrary motion are shown to confirm the form that we have obtained previously.

INTRODUCTION

The motivation behind this research originates with a recent review article by Tobak et al., which is an exposition of their approach to the modeling of the aerodynamic contribution to the inertial equations of motion of a maneuvering aircraft (ref. 1). The approach features the use of nonlinear indicial responses and generalized superposition integrals. Recognizing its natural connection with ideas from bifurcation theory, the authors of reference 1 extended their modeling approach to accommodate the potential occurrence of bifurcations involving time-dependent equilibrium states. Of particular interest is the first of these, called a "Hopf bifurcation," wherein a formerly stable time-invariant equilibrium state is replaced by a time-varying periodic equilibrium state. In the implementation, however, Tobak et al. discovered that an important additional issue arises along with a Hopf bifurcation: the necessity of specifying phase as well as amplitude and frequency to completely determine the periodic equilibrium state. Although the amplitude and frequency of the periodic state are determined entirely by flow conditions that are independent of an origin in time, specifying the phase requires additional information involving an initial condition. Through specification of the phase, the equilibrium state is required to acknowledge an origin in time.

In aerodynamic applications, the physical origin of a large-scale periodic equilibrium state often is the onset of vortex-shedding. Of the many examples, Tobak et al. (ref. 1) cited the occurrence of stall on an airfoil when the angle of attack exceeds a critical value, and the wake of the flow past a cylinder when the Reynolds number exceeds 50. The latter example, vortex-shedding from a cylinder or, more generally, a bluff body, is itself a subject that has attracted the attention of researchers for many decades. When the body is able to interact with the vortex-shedding through, for example, elastic mountings, the consequences are varied and potentially dramatic. This subject, which has come to be known as "vortex-induced oscillations," has also inspired a large literature, reflecting the wide range of practical circumstances in which structures may be endangered by undergoing such oscillations. Excellent surveys of the field have been published by Sarpkaya (ref. 2) and more recently by Bearman (ref. 3). Both authors noted that in spite of the long and intense interest, an adequate rationally based mathematical model of the phenomenon still does not exist.

In a first attempt to fulfill the need for a suitable mathematical model of the aerodynamic contribution to the equation governing the motion of an elastically mounted cylinder immersed in a uniform oncoming stream, the authors of reference 1 proposed a simple model based on one of the alternative approaches studied therein. However, the model does not incorporate the memory effects that the specification of phase would have required. A careful numerical study of the resulting equation of motion, carried out in reference 4 and by us, suggests that the model proposed is unable to predict the occurrence of the self-sustained oscillations that experiments have revealed.

To that end, this study has two purposes: (1) to show how the deficiency of the model proposed in reference 1 may be remedied to include the effects of memory; and (2) to apply the results of the analysis to the particular case of flow past a cylinder that is in periodically forced transverse motion. As it turns out, revising the reference 1 model to include memory effects must be done at the level of the velocity field itself. We demonstrate that a form for the indicial response of the velocity field may be derived directly from the equations governing the fluid motion. These are taken to be the Navier-Stokes equations for time-dependent incompressible flow.

For convenience, the study will be divided into three parts. In Part I, we demonstrate on the basis of the Navier-Stokes equations how an indicial response for the velocity field may be formulated, and then used within a generalized superposition integral to determine the form of the velocity response to an arbitrary motion. To establish its connection with our previous work, the new approach is applied first to the simple situation wherein the indicial response has a time-invariant equilibrium state. Results for the aerodynamic response to an arbitrary motion are shown to confirm the form that we have obtained previously by a variety of approaches (ref. 1). In Part II, the analysis is then directed to the new situation where the equilibrium state of the indicial response is periodic in time. A method is derived for specifying its phase, and, by allowing a correct incorporation of memory effects, this proves sufficient to remedy the deficiency of our previous model.

Part III is the object of the second purpose, namely, to apply the results of the analysis to the particular case of flow past a cylinder that is in periodically forced transverse motion. We show that our approach captures the distinctive features of vortex-induced oscillations that have been revealed by the results of careful experiments on this motion (reported in refs. 2, 3). Finally, we show how the indicial response approach may be reconciled with a currently popular approach based on the use of amplitude equations.

GENERALIZED SUPERPOSITION INTEGRAL BASED ON INDICIAL RESPONSE APPROACH

We derive the indicial response approach with emphasis on the physical postulates involved.

For generality, consider an aircraft that has started from rest in the distant past with fixed axial velocity U_0 and zero vertical velocity (cf. fig. 1). Its motion is referred to an X, Y coordinate system that is fixed in space. It passes through the origin at the arbitrarily chosen initial instant $\xi = 0$, maintaining the constant axial velocity U_0 and simultaneously translating vertically, with the vertical velocity v_c at the center of gravity being an arbitrary function of time ξ. The angle of attack α is defined as the angle between the resultant velocity vector and the aircraft's longitudinal axis:

$$\alpha = tan^{-1}\left(\frac{v_c(\xi)}{U_0}\right) \tag{1}$$

Let us note that we specify a constant axial velocity U_0 to be in accord with normal operating conditions in wind- or water-tunnel experiments, wherein the uniform oncoming flow, normally held at constant velocity, would supply the corresponding value of U_0. To form the indicial response, we need to consider two motions. In the first one, the aircraft undergoes a variation of angle of attack $\alpha(\xi)$ from time zero to a time $\xi = \tau$ (cf. fig. 2). Subsequent to time τ, the angle of attack is held constant at $\alpha(\tau)$. The first motion history therefore is designated $\alpha_\tau(\xi)$:

$$\alpha_\tau(\xi) = \begin{cases} \alpha(\xi) & : \quad if\, 0 < \xi < \tau \\ \alpha(\tau) & : \quad if\, \xi \geq \tau \end{cases} \tag{2}$$

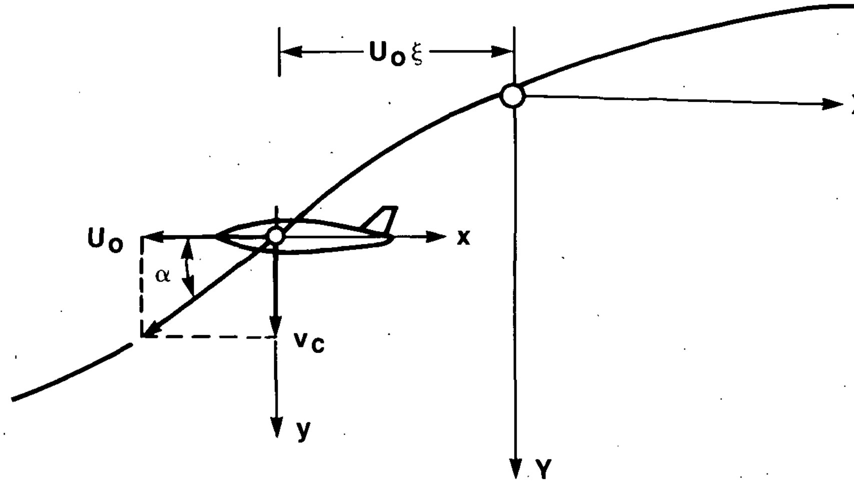

Figure 1. Maneuver referred to space-fixed (X, Y) and moving (x, y) coordinates, the latter attached to the airplane.

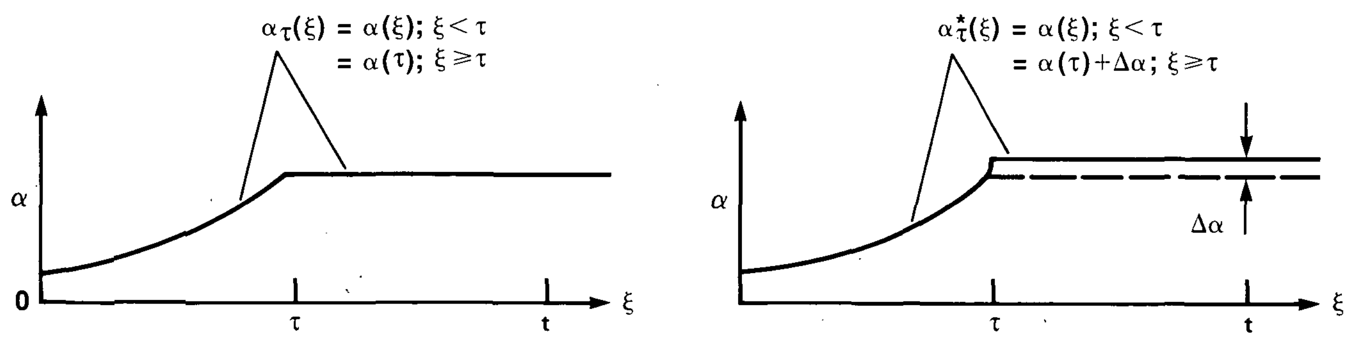

Figure 2. Formation of indicial response.

The angle-of-attack history $\alpha_\tau(\xi)$ is said to belong to a family of motions, all of which are generated by the "natural" motion history $\alpha(\xi)$, and all of which have the characteristic of remaining constant beyond the instant designated by the subscript, here τ. Accordingly, the angle-of-attack history $\alpha_{\tau+\Delta\tau}(\xi)$ belongs to the same family, being the extension of the history $\alpha(\xi)$ to the instant $\tau+\Delta\tau$, beyond which α remains constant at $\alpha(\tau+\Delta\tau)$. The second motion needed to form the indicial response, however, does not belong to this family. In the second motion, the aircraft undergoes the same angle-of-attack history $\alpha(\xi)$ up to time τ. Subsequent to τ, the angle of attack again is held constant, but is given an incremental step change $\Delta\alpha$ over its previous value of $\alpha(\tau)$. The second motion, designated $\alpha^\star_\tau(\xi)$, is represented as

$$\alpha^\star_\tau(\xi) = \alpha_\tau(\xi) + \Delta\alpha \cdot H_\tau(\xi) \tag{3}$$

where $H_\tau(\xi)$ is the Heaviside function:

$$H_\tau(\xi) = \left\{ \begin{array}{lll} 0 & : & if\, 0 < \xi < \tau \\ 1 & : & if\, \xi \geq \tau \end{array} \right. \tag{4}$$

Let us define by $\vec{u}(\vec{x}, t, \tau)$ ($\vec{x}$: spatial coordinates, t: time of observation subsequent to τ) the velocity field in the neighborhood of the aircraft.

The indicial response approach is established by making use of the following postulates.

POSTULATE 1: Corresponding to a motion history belonging to the family of motion histories $\alpha_\tau(\xi)$ with $\tau \in [0, t]$, there exists a velocity field $\vec{u}[\vec{x}, \alpha_\tau(\xi); t, \tau]$ well defined and supposed known by some means.

POSTULATE 2: To each member of the family of motion histories $\alpha^\star_\tau(\xi)$ with $\tau \in [0, t]$, there exists a velocity response $\vec{u}[\vec{x}, \alpha^\star_\tau(\xi); t, \tau]$. For any observation time $t \geq \tau + \Delta\tau$, this velocity response reproduces the velocity response corresponding to the motion history $\alpha_{\tau+\Delta\tau}(\xi)$ within a negligible error of $\vartheta(\Delta\alpha)^2$:

$$\vec{u}[\vec{x}, \alpha^\star_\tau(\xi)] \equiv \vec{u}[\vec{x}, \alpha_\tau(\xi) + \Delta\alpha \cdot H_\tau(\xi)] = \vec{u}[\vec{x}, \alpha_{\tau+\Delta\tau}(\xi)] + \vartheta(\Delta\alpha)^2 \tag{5}$$

The increments $\Delta\alpha$ and $\Delta\tau$ are connected by the rate-of-change of the angle of attack $\dot{\alpha}(\tau) = \frac{\Delta\alpha}{\Delta\tau}$, which is presumed to be defined for each value of $\tau \in [0, t]$.

POSTULATE 3: For every value of $\tau \in [0, t]$, there exists an indicial response $\vec{u}_\alpha[\vec{x}, \alpha(\xi); t, \tau]$ defined as

$$\vec{u}_\alpha[\vec{x}, \alpha(\xi); t, \tau] = lim_{\Delta\alpha \to 0} \frac{1}{\Delta\alpha} \{\vec{u}[\vec{x}, \alpha_\tau(\xi) + \Delta\alpha . H_\tau(\xi); t, \tau] - \vec{u}[\vec{x}, \alpha_\tau(\xi); t, \tau]\} \tag{6}$$

Given a motion history $\alpha_t(\xi : \xi \in [-\infty, t])$, it is possible to choose an initial time τ_0 and partition the time interval $[\tau_0, t]$ into $[\tau_0, \tau_1, \ldots, \tau_n]$ such that

$$\tau_n = t \ , \ n\Delta\tau = t \ , \ \tau_i - \tau_{i-1} = \Delta\tau \qquad (i = 1, \ldots, n) \tag{7}$$

One has the following relations:

$$\left. \begin{array}{rcl} \vec{u}_1[\alpha_{\tau_0}(\xi) + \Delta\alpha \cdot H_{\tau_0}(\xi)] - \vec{u}_0[\alpha_{\tau_0}(\xi)] & = & \vec{u}_\alpha[\alpha_{\tau_0}(\xi)]\Delta\alpha + \vartheta(\Delta\alpha)^2 \\ \vec{u}_2[\alpha_{\tau_1}(\xi) + \Delta\alpha \cdot H_{\tau_1}(\xi)] - \vec{u}_1[\alpha_{\tau_1}(\xi)] & = & \vec{u}_\alpha[\alpha_{\tau_1}(\xi)]\Delta\alpha + \vartheta(\Delta\alpha)^2 \\ & \vdots & \\ \vec{u}_n[\alpha_{\tau_{n-1}}(\xi)] + \Delta\alpha \cdot H_{\tau_{n-1}}(\xi)] - \vec{u}_{n-1}[\alpha_{\tau_{n-1}}(\xi)] & = & \vec{u}_\alpha[\alpha_{\tau_{n-1}}(\xi)]\Delta\alpha + \vartheta(\Delta\alpha)^2 \end{array} \right\} \tag{8}$$

Note in (8) that the additional dependencies of $\vec{u}_i$ on $\vec{x}, t, \tau_i (i = 1, \ldots, n)$ have been omitted for brevity.

By summing up the relations (8) and using Postulate 2, one gets

$$\vec{u}_n[\alpha_{\tau_n}(\xi)] - \vec{u}_0[\alpha_{\tau_0}(\xi)] = \sum_{i=0}^{n-1} \vec{u}_\alpha[\alpha_{\tau_i}(\xi)]\Delta\alpha + \vartheta(\Delta\alpha)^2 \tag{9}$$

or, in the limit as $\Delta\tau \to 0$,

$$\vec{u}[\vec{x}, \alpha_t(\xi); t, t] = \vec{u}[\vec{x}, \alpha_{\tau_0}(\xi); t, \tau_0] + \int_{\tau_0}^{t} d\tau\, \vec{u}_\alpha[\vec{x}, \alpha_\tau(\xi); t, \tau]\frac{d\alpha}{d\tau} + \vartheta(\Delta\alpha)^2 \tag{10}$$

The relation (10) constitutes our main result that will be used in subsequent sections. Notice that it relies on three physical postulates which have to be satisfied.

CONSTRUCTION OF VELOCITY RESPONSE FOR MOTION HISTORY: $\alpha_\tau(\xi)$

In order to derive the aerodynamic response on the basis of relation (10), we need to establish forms for the velocity responses to the motions $\alpha_\tau(\xi)$ and $\alpha_\tau^*(\xi)$ over the time interval $t - \tau > 0$. In this and the following section, we shall demonstrate that the forms can be derived analytically from direct consideration of the Navier-Stokes equations.

For simplicity, we neglect compressibility effects and assume that the fluid motion is governed by the Navier-Stokes equations for an incompressible fluid. In a coordinate system attached to the body, the Navier-Stokes equations have an additional term $\dot{\vec{v}}_c$ (where $\vec{v}_c$ is the vertical velocity at the mass center and the dot denotes a derivative with respect to time) to account for the acceleration of the coordinate system relative to inertial space:

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}.\vec{\nabla})\vec{u} + \vec{\nabla}p - \nu\nabla^2\vec{u} = -\dot{\vec{v}}_c \tag{11}$$

Here, $\vec{u}$ is the velocity field, p is the pressure (normalized with respect to constant density ρ), and ν is the kinematic viscosity. We wish to derive a form for the velocity response to the motion $\alpha_\tau(\xi)$ over the time interval $t - \tau > 0$. Since $\alpha_\tau(\xi) = \alpha(\tau) =$ constant for $t - \tau > 0$, boundary conditions determining the velocity response over the interval $t - \tau > 0$ are perfectly steady. Consequently, we expect the velocity response to attain an equilibrium state as the interval becomes large, that is, as $t - \tau \to \infty$.

The response of the velocity field can be decomposed into an equilibrium state $\vec{u}_{equil}$ and a transient component $\vec{u}_{trans}$ that decays as time increases. The principal condition that we impose in Part I is that the equilibrium state $\vec{u}_{equil}$ be time-invariant. Thus, as time is referred to τ (the instant after which the angle-of-attack is to be kept constant), we have

$$\left.\begin{aligned} \vec{u}(\vec{x}, t-\tau>0) &= \vec{u}_{equil}(\vec{x}) + \vec{u}_{trans}(\vec{x}, t-\tau>0) \\ p(\vec{x}, t-\tau>0) &= p_{equil}(\vec{x}) + p_{trans}(\vec{x}, t-\tau>0) \end{aligned}\right\} \tag{12}$$

To simplify the notation, we define

$$t_+ = t - \tau > 0 \tag{13}$$

Let us assume that the equilibrium state $(\vec{u}_{equil}(\vec{x}), p_{equil}(\vec{x}))$ is available, for example, from a solution of the time-independent $(\frac{\partial}{\partial t}\vec{u} = \dot{\vec{v}}_c = 0)$ version of equation (11).

Substituting (12) into (11) and noting that $\dot{\vec{v}}_c = 0$ at $t_+ = 0(+)$, we get the following for the equations governing $\vec{u}_{trans}$:

$$\left.\begin{aligned} \frac{\partial}{\partial t_+}\vec{u}_{trans} + (\vec{u}_{equil}.\vec{\nabla})\vec{u}_{trans} + (\vec{u}_{trans}.\vec{\nabla})\vec{u}_{equil} + \vec{\nabla}p_{trans} - \nu\nabla^2\vec{u}_{trans} + (\vec{u}_{trans}.\vec{\nabla})\vec{u}_{trans} &= 0 \\ \vec{\nabla}.\vec{u}_{trans} &= 0 \end{aligned}\right\} \tag{14}$$

with $\vec{u}_{trans} = 0$ on appropriate boundaries $\partial\Omega$.

To find a form for $\vec{u}_{trans}$, we shall first consider a small disturbance $\vec{u}'$ which satisfies the linearized version of equations (14)

$$\left.\begin{aligned} \frac{\partial}{\partial t_+}\vec{u}' + (\vec{u}_{equil}.\vec{\nabla})\vec{u}' + (\vec{u}'.\vec{\nabla})\vec{u}_{equil} + \vec{\nabla}p' - \nu\nabla^2\vec{u}' &= 0 \\ \vec{\nabla}.\vec{u}' &= 0 \end{aligned}\right\} \tag{15}$$

with $\vec{u}' = 0$ on $\partial\Omega$. Equations (15) constitute a linear eigenvalue problem. Since $\vec{u}_{equil}$ is independent of time, the eigensolutions $\vec{\gamma}_n$ are also independent of time. It is possible to choose them such that (from ref. 5)

$$\vec{u}' = \vec{u}'_n = e^{\lambda_n t_+}\vec{\gamma}_n(\vec{x}) \quad , \quad \vec{\nabla}.\vec{\gamma}_n = 0 \quad , \quad \vec{\gamma}_n|_{\partial\Omega} = 0 \tag{16}$$

From equations (15), the corresponding expression for p' is

$$p' = e^{\lambda_n t_+}p'_n(\vec{x}) \tag{17}$$

The eigensolutions $\vec{\gamma}_n(\vec{x})\,(n = 1,\ldots)$ are associated with the eigenvalues λ_n, the latter all having negative real parts in the physical situation under consideration, wherein the equilibrium state $\vec{u}_{equil}(\vec{x})$ is supposed to be stable to small perturbations. There is an equation adjoint to (15) having a set of eigensolutions $\vec{\gamma}_n^*$ such that (from ref. 5)

$$\left.\begin{aligned} \langle\vec{\gamma}_j^*,\vec{\gamma}_i\rangle &\equiv \int_V dV\, \overline{\vec{\gamma}_j^*}(\vec{x}).\vec{\gamma}_i(\vec{x}) = \delta_{ij} \qquad \forall\, i,j \\ \langle\overline{\vec{\gamma}_j^*},\vec{\gamma}_i\rangle &\equiv \int_V dV\, \vec{\gamma}_j^*(\vec{x}).\vec{\gamma}_i(\vec{x}) = 0 \qquad \forall\, i,j \end{aligned}\right\} \tag{18}$$

where the brackets $\langle\quad\rangle$ denote the scalar product over space. The eigensolutions $\vec{\gamma}_n(\vec{x})$ span a complete functional space, and one can use this fact to construct a suitable solution form for $\vec{u}_{trans}(\vec{x},t_+)$. Returning now to the full nonlinear equations (14) governing $\vec{u}_{trans}$, let us assign $\vec{u}_{trans}$ the form resulting from its projection onto the functional space of the $\vec{\gamma}_n$:

$$\vec{u}_{trans}(\vec{x},t_+) = \sum_n \left(d_n(t_+)\vec{\gamma}_n(\vec{x}) + \overline{d}_n(t_+)\overline{\vec{\gamma}}_n(\vec{x})\right) \tag{19}$$

where the barred terms denote complex conjugates (abbreviated c.c.). It will be convenient to break the pressure term p_{trans} in (12) into two parts: one following the form of equation (19) and correponding to the linear part of the disturbance equations, and the other (denoted $\tilde{p}$) corresponding to the nonlinear part of the disturbance equations:

$$p_{trans}(\vec{x},t_+) = \sum_n \left(d_n(t_+)p'_n(\vec{x}) + \overline{d}_n(t_+)\overline{p}'_n(\vec{x})\right) + \tilde{p}(\vec{x},t_+) \tag{20}$$

After inserting equations (19) and (20) into (14) and eliminating the set of terms that identically satisfies the linear form (15), we find that the coefficients $d_n(t_+)$ satisfy the following equation:

$$\begin{aligned} \sum_n \left(\dot{d}_n - \lambda_n d_n\right)\vec{\gamma}_n + \sum_n (\dot{\overline{d}}_n - \overline{\lambda}_n\overline{d}_n)\overline{\vec{\gamma}}_n + \vec{\nabla}\tilde{p} = -\sum_{n,m} \Big(& d_n d_m(\vec{\gamma}_n.\vec{\nabla})\vec{\gamma}_m + d_n\overline{d}_m(\vec{\gamma}_n.\vec{\nabla})\overline{\vec{\gamma}}_m \\ & + \overline{d}_n d_m(\overline{\vec{\gamma}}_n.\vec{\nabla})\vec{\gamma}_m + \overline{d}_n\overline{d}_m(\overline{\vec{\gamma}}_n.\vec{\nabla})\overline{\vec{\gamma}}_m\Big) \end{aligned} \tag{21}$$

Multiplying equation (21) by the vector $\overline{\vec{\gamma}_j^\star}(\vec{x})$, integrating over space, and using the properties of adjoint vectors defined by (18), we obtain

$$\begin{aligned}\dot{d}_j(t_+) - \lambda_j d_j &= -\sum_{n,m}\Big(d_n d_m\langle\vec{\gamma}_j^\star, (\vec{\gamma}_n.\vec{\nabla})\vec{\gamma}_m\rangle + d_n\overline{d}_m\langle\vec{\gamma}_j^\star, (\vec{\gamma}_n.\vec{\nabla})\overline{\vec{\gamma}}_m\rangle \\ &+\overline{d}_n d_m\langle\vec{\gamma}_j^\star, (\overline{\vec{\gamma}}_n.\vec{\nabla})\vec{\gamma}_m\rangle + \overline{d}_n\overline{d}_m\langle\vec{\gamma}_j^\star, (\overline{\vec{\gamma}}_n.\vec{\nabla})\overline{\vec{\gamma}}_m\rangle\Big)\; j = 1, \ldots \qquad (22)\end{aligned}$$

Notice that the pressure gradient term $\vec{\nabla}\tilde{p}$ in equation (21) makes no appearance in equation (22). The properties of $\vec{\gamma}_n$ and $\vec{\gamma}_n^\star$ (solenoidal and vanishing on $\partial\Omega$) ensure that the scalar products they form with gradient terms (such as $\vec{\nabla}\tilde{p}$) will be identically zero. By defining appropriate coefficients A_{jnm}, B_{jnm}, C_{jnm} and D_{jnm}, we rewrite (22) as:

$$\begin{aligned}\dot{d}_j(t_+) - \lambda_j(\alpha(\tau))d_j(t_+) &= \sum_{n,m}\Big(A_{jnm}(\alpha(\tau))d_n(t_+)d_m(t_+) + B_{jnm}(\alpha(\tau))d_n(t_+)\overline{d}_m(t_+) \\ &+C_{jnm}(\alpha(\tau))\overline{d}_n(t_+)d_m(t_+) \\ &+D_{jnm}(\alpha(\tau))\overline{d}_n(t_+)\overline{d}_m(t_+)\Big)\; j = 1, \ldots \qquad (23)\end{aligned}$$

Equations (23) are transformed into integral equations:

$$\begin{aligned}d_j(t_+) &= e^{\lambda_j(\alpha(\tau))t_+}d_j(t_+ = 0) + \sum_{n,m}A_{jnm}(\alpha(\tau))\int_0^{t_+} ds\; e^{\lambda_j(\alpha(\tau))(t_+-s)}d_n(s)d_m(s) \\ &+\sum_{n,m}B_{jnm}(\alpha(\tau))\int_0^{t_+} ds\; e^{\lambda_j(\alpha(\tau))(t_+-s)}d_n(s)\overline{d}_m(s) \\ &+\sum_{n,m}C_{jnm}(\alpha(\tau))\int_0^{t_+} ds\; e^{\lambda_j(\alpha(\tau))(t_+-s)}\overline{d}_n(s)d_m(s) \\ &+\sum_{n,m}D_{jnm}(\alpha(\tau))\int_0^{t_+} ds\; e^{\lambda_j(\alpha(\tau))(t_+-s)}\overline{d}_n(s)\overline{d}_m(s) \qquad j = 1, \ldots \qquad (24)\end{aligned}$$

Solutions of the system of equation (24) can be obtained by using Picard's method of successive approximations:

$$\begin{aligned}d_j^{(0)}(t_+) &= e^{\lambda_j(\alpha(\tau))t_+}d_j(t_+ = 0) \\ d_j^{(1)}(t_+) &= d_j^{(0)}(t_+) + \sum_{n,m}A_{jnm}(\alpha(\tau))\int_0^{t_+} dse^{\lambda_j(t_+-s)}d_n^{(0)}(s)d_m^{(0)}(s) \\ &+\; \ldots \\ &\vdots \\ d_j^{(n+1)}(t_+) &= d_j^{(n)}(t_+) + \sum_{n,m}A_{jnm}(\alpha(\tau))\int_0^{t_+} dse^{\lambda_j(t_+-s)}d_n^{(n)}(s)d_m^{(n)}(s) \\ &+\; \ldots \\ &\vdots\end{aligned} \qquad (25)$$

where for brevity we have omitted writing the nonlinear terms in $\overline{d}_n d_m$, $d_n\overline{d}_m$, $\overline{d}_n\overline{d}_m$.

Solutions $d_j(t_+)$ of equations (25) have the form of functional expansion series:

$$\begin{aligned} d_j(t_+) &= d_j^{(0)}(t_+) + \sum_{n,m} A_{jnm}(\alpha(\tau)) \int_0^{t_+} ds_0\, e^{\lambda_j(t_+-s_0)} d_n^{(0)}(s_0) d_m^{(0)}(s_0) \\ &+ \sum_{n,m} A_{jnm}(\alpha(\tau)) \int_0^{t_+} ds_0 \int_0^{s_0} ds_1\, e^{\lambda_j(t_+-s_0)} \left\{ d_n^{(0)}(s_0).e^{\lambda_m(s_0-s_1)} \sum_{p,q} A_{mpq}(\alpha(\tau)) \right. \\ &\left. \times d_p^{(0)}(s_1) d_q^{(0)}(s_1) + e^{\lambda_n(s_0-s_1)} \sum_{p,q} A_{npq}(\alpha(\tau)) d_p^{(0)}(s_1) d_q^{(0)}(s_1).d_m^{(0)}(s_0) \right\} \\ &+ \ldots + \int_0^{t_+} ds_0 \int_0^{s_0} ds_1 \int_0^{s_1} ds_2 \ldots \end{aligned} \tag{26}$$

The convergence of the series is insured by imposing, for instance, a Lipschitz condition on the integrands of the four integrals in the right-hand side member of equation (24). Calculation of the series lends itself to symbolic computation, as shown in reference 6.

To complete the solution for the $d_j(t_+)(j = 1, \ldots)$ at any level of approximation, we are required to specify their initial values at $t_+ = 0$. The values of $d_j(t_+ = 0)$ must be determined from a match of the velocity fields on either side of $\xi = \tau$:

$$\vec{u}(\vec{x}, \xi = \tau_-) = \vec{u}(\vec{x}, \xi = \tau_+) \tag{27}$$

On the negative side of $\xi = \tau$, in general the velocity field is the resultant of the entire history of the motion $\alpha(\xi)$ up to the "present" time $\xi = \tau$. Through specification of the d_j at $t_+ = 0$ in terms of this velocity field, the velocity response for $t_+ \geq 0$ acknowledges its dependence on the past motion. The representation of this past motion in a suitable way is all that remains to be done to realize a complete characterization of the velocity response for $t_+ \geq 0$. We shall defer a discussion of this important step until we have derived the velocity response to the motion $\alpha_\tau^\star(\xi)$ and formed the indicial response. For the present, we simply designate the dependence of $d_j(t_+)$ on the past motion as a functional

$$d_j(t_+) = d_j\left[\alpha(\xi : \xi \in [0, \tau]), t_+\right] \tag{28}$$

and write the velocity response to the motion $\alpha_\tau(\xi)$ for $t_+ \geq 0$ in the form

$$\vec{u}[\vec{x}, \alpha_\tau(\xi); t, \tau] = \vec{u}_{equil}(\vec{x}, \alpha(\tau)) + \sum_n d_n[\alpha(\xi \in [0, \tau]); t_+] \vec{\gamma}_n(\vec{x}, \alpha(\tau)) + c.c. \tag{29}$$

CONSTRUCTION OF VELOCITY RESPONSE TO MOTION HISTORY: $\alpha_\tau^\star(\xi)$

In addition to constructing the velocity response to the motion $\alpha_\tau^\star(\xi)$ containing a step change in α at $\xi = \tau$, we need to be assured that the derived response will satisfy Postulate 2. To that end, consider the interval $\tau < \xi < \tau + \Delta\tau$ in which the velocity response changes continuously from $\vec{u}[\alpha_\tau(\xi)]$ to $\vec{u}[\alpha_{\tau+\Delta\tau}(\xi)]$ as a result of the continuous change in α over the interval. We intend to show that if the continuous change in α is replaced by a step change, applied anywhere between τ and $\tau + \Delta\tau$, the resulting velocity response at any observation time $t \geq \tau + \Delta\tau$ will differ from the response to the continuous change in α, $\vec{u}[\alpha_{\tau+\Delta\tau}(\xi)]$, by terms of $\vartheta(\Delta\alpha)^2$.

In the equation of motion (11), for brevity let

$$\mathcal{L}[\vec{u}] = (\vec{u}.\vec{\nabla})\vec{u} + \vec{\nabla}p - \nu\nabla^2\vec{u} \tag{30}$$

and let $\vec{u}_1$ be the velocity response to the continuous change in α for $\tau < \xi < \tau + \Delta\tau$. With $\vec{u} = \vec{u}_1$ and with the time variable $t_+ = t - \tau$, integrate equation (11) from $t_+ = 0$ to $t_+ = \Delta\tau$

$$\int_0^{\Delta\tau} \frac{\partial}{\partial t_+}\vec{u}_1 dt_+ + \int_0^{\Delta\tau} \mathcal{L}[\vec{u}_1]dt_+ = -\int_0^{\Delta\tau} \dot{\vec{v}}_c dt_+ \tag{31}$$

which gives

$$\vec{u}_1(\Delta\tau) - \vec{u}_1(0) + \int_0^{\Delta\tau} \mathcal{L}[\vec{u}_1]dt_+ = -(\vec{v}_c(\Delta\tau) - \vec{v}_c(0)) = -\Delta\vec{v}_c \tag{32}$$

Now let the continuous change in α over the interval $\tau < \xi < \tau + \Delta\tau$ be replaced by a step change of the same net amplitude $\Delta\alpha = \alpha(\tau + \Delta\tau) - \alpha(\tau)$, applied at an arbitrary time $\tau^\star$ anywhere between τ and $\tau + \Delta\tau$. Let $\vec{u}_2$ be the corresponding velocity response over the interval $0 < t_+ < \Delta\tau$, and integrate equation (11) again (note that the initial state at $t_+ = 0$ remains unaltered so that $\vec{u}_2(0) = \vec{u}_1(0)$):

$$\int_0^{\Delta\tau} \frac{\partial}{\partial t_+}\vec{u}_2 dt_+ + \int_0^{\Delta\tau} \mathcal{L}[\vec{u}_2]dt_+ = -\Delta\vec{v}_c \int_0^{\Delta\tau} \delta(t_+ - (\tau^\star - \tau))dt_+ \tag{33}$$

which gives

$$\vec{u}_2(\Delta\tau) - \vec{u}_2(0) + \int_0^{\Delta\tau} \mathcal{L}[\vec{u}_2]dt_+ = -\Delta\vec{v}_c \tag{34}$$

Taking the difference between equations (32) and (34) yields

$$\vec{u}_2(\Delta\tau) - \vec{u}_1(\Delta\tau) + \int_0^{\Delta\tau} (\mathcal{L}[\vec{u}_2] - \mathcal{L}[\vec{u}_1])\, dt_+ = 0 \tag{35}$$

But

$$\left.\begin{array}{rcl} \vec{u}_1 & = & \vec{u}[\alpha_\tau(\xi)] + terms \quad of \quad \vartheta(\Delta\alpha) \\ \vec{u}_2 & = & \vec{u}[\alpha_\tau(\xi)] + terms \quad of \quad \vartheta(\Delta\alpha) \end{array}\right\} \tag{36}$$

Hence, in equation (35)

$$\int_0^{\Delta\tau} (\mathcal{L}[\vec{u}_2] - \mathcal{L}[\vec{u}_1])\, dt_+ = \frac{\Delta\alpha}{\dot{\alpha}}\left(\underline{\mathcal{L}[\vec{u}_2]} - \underline{\mathcal{L}[\vec{u}_1]}\right) = \vartheta(\Delta\alpha)^2 \tag{37}$$

where the underline denotes an average over time.

Therefore, $\vec{u}_2$ and $\vec{u}_1$ differ by terms of $\vartheta(\Delta\alpha)^2$ at $t_+ = \Delta\tau$. For $t_+ \geq \Delta\tau$, note that $\vec{u}_1 \equiv \vec{u}[\alpha_{\tau+\Delta\tau}(\xi)]$, whereas $\vec{u}_2 \equiv \vec{u}[\alpha^\star_{\tau^\star}(\xi)]$. Since for all $t_+ \geq \Delta\tau$, boundary conditions and equations of motion are identical for the two velocity responses, their solutions will differ only by terms of the order of the difference in initial conditions, which, as we have just seen, is of $\vartheta(\Delta\alpha)^2$. Thus, we are assured that the velocity response $\vec{u}[\alpha^\star_{\tau^\star}(\xi)]$ will satisfy Postulate 2. For simplicity, we put, henceforth, $\tau^\star = \tau$. It remains to develop a form for $\vec{u}[\alpha^\star_\tau(\xi)]$ analogous to that for $\vec{u}[\alpha_\tau(\xi)]$.

The derivation for $\vec{u}[\alpha^\star_\tau(\xi)]$ is required to account for the effect of a step change in α at $\xi = \tau$. This is reflected in the equation of motion by the presence of an impulsive forcing term arising from the time-derivative of the step change in $\vec{v}_c$,

$$\frac{\partial\vec{u}}{\partial t_+} + (\vec{u}.\vec{\nabla})\vec{u} + \vec{\nabla}p - \nu\nabla^2\vec{u} = -\dot{\vec{v}}_c = -\Delta\vec{v}_c\delta(t_+) \tag{38}$$

where, to first order in $\Delta\,\alpha$, the magnitude of the step change in $\vec{v}_c$, denoted $\Delta\,v_c$, is

$$\Delta\,v_c = U_0\,sec^2(\alpha(\tau))\Delta\,\alpha \tag{39}$$

The derivation parallels that for $\vec{u}[\,\alpha_\tau(\xi)\,]$. Again, the velocity response is decomposed into a time-invariant equilibrium state and a transient component,

$$\vec{u}(\vec{x},t_+) = \vec{u}_{equil}(\vec{x},\alpha(\tau) + \Delta\,\alpha) + \vec{u}_{trans}(\vec{x},t_+,\alpha(\tau) + \Delta\,\alpha) \tag{40}$$

where now both components are to be evaluated at the new level of angle of attack, $\alpha(\tau) + \Delta\,\alpha$. The equation for $\vec{u}_{trans}$ takes the same form as that of equation (14) with the exception that the right-hand side now contains the impulsive forcing term $-\Delta\,\vec{v}_c\delta(t_+)$.

As before, we try to form a solution by a suitable projection onto the functional space of the set of eigensolutions of the linearized homogenous equation

$$\left.\begin{array}{rcl} \vec{u}_{trans}(\vec{x},t_+) & = & \sum_n \left(h_n(t_+)\vec{\gamma}_n(\vec{x}) + \overline{h}_n(t_+)\overline{\vec{\gamma}}_n(\vec{x})\right) \\ p_{trans}(\vec{x},t_+) & = & \sum_n \left(h_n(t_+)p_n'(\vec{x}) + \overline{h}_n(t_+)\overline{p}_n'(\vec{x})\right) + \tilde{p}(\vec{x},t_+) \end{array}\right\} \tag{41}$$

where the eigensolutions $(\vec{\gamma}_n, p_n')$ are now those appropriate to the new level of $\alpha, \alpha(\tau) + \Delta\,\alpha$. Multiplying the resulting equation (the counterpart of eqs. (21)) by the adjoint vector $\overline{\vec{\gamma}}_j^\star(\vec{x},\alpha(\tau) + \Delta\,\alpha)$, integrating over space and simplifying, we get the counterpart of equations (23):

$$\dot{h}_j(t_+) - \lambda_j h_j - \sum_{n,m}\left(A_{jnm}h_n h_m + \ldots + D_{jnm}\overline{h}_n\overline{h}_m\right) = -\Delta\,\alpha\; c_j\delta(t_+) \qquad j = 1,\ldots \tag{42}$$

where

$$\Delta\,\alpha\; c_j(\alpha(\tau) + \Delta\,\alpha) = \int_V \overline{\vec{\gamma}}_j^\star(\vec{x},\alpha(\tau) + \Delta\,\alpha).\Delta\,\vec{v}_c dV \tag{43}$$

and the eigenvalues λ_j as well as the coefficients $A_{jnm},\ldots,D_{jnm}$ have the same definitions as before, albeit, now referred to $\alpha(\tau) + \Delta\,\alpha$ rather than to $\alpha(\tau)$.

Writing equation (42) as integral equations yields

$$\begin{array}{rcl} h_j(t_+) & = & C_j e^{\lambda_j t_+} - \Delta\,\alpha\; c_j H(t_+)e^{\lambda_j t_+} + \sum_{n,m} A_{jnm}\int_0^{t_+} ds\; e^{\lambda_j(t_+-s)}h_n(s)h_m(s) + \ldots \\ & & + \sum_{n,m} D_{jnm}\int_0^{t_+} ds\; e^{\lambda_j(t_+-s)}\overline{h}_n(s)\overline{h}_m(s) \qquad j = 1,\ldots \end{array} \tag{44}$$

which again can be solved by successive approximations, once the open constants C_j have been determined. To determine them, a condition on the velocity fields on either side of $\xi = \tau$ must be imposed. Here, it is no longer appropriate to require the continuity of the velocity fields across $\xi = \tau$. Rather, we must admit the existence of a discontinuity in the velocity fields across $\xi = \tau$, reflecting the step change in α that occurs there. Accordingly, we shall require that the velocity fields on either side of $\xi = \tau$ differ by an amount proportional to the step change in α at $\xi = \tau$. The condition is

$$\vec{u}[\,\alpha_\tau^\star(\xi),\xi = \tau_+\,] - \vec{u}[\,\alpha_\tau(\xi),\xi = \tau_-\,] = \Delta\,\alpha H(0)\vec{\Phi}(\vec{x},\tau) \tag{45}$$

Using equations (12) and (19) for $\vec{u}[\alpha_\tau(\xi), \xi = \tau_-]$, and equations (40) and (41) for $\vec{u}[\alpha_\tau^\star(\xi), \xi = \tau_+]$, we have in equation (45),

$$\left(\vec{u}_{equil}(\vec{x}, \alpha(\tau) + \Delta\alpha) - \vec{u}_{equil}(\vec{x}, \alpha(\tau))\right) + \sum_n \left(h_n(t_+ = 0)\vec{\gamma}_n(\vec{x}, \alpha(\tau) + \Delta\alpha) + \overline{h}_n\overline{\vec{\gamma}}_n\right)$$
$$- \sum_n \left(d_n(t_+ = 0)\vec{\gamma}_n(\vec{x}, \alpha(\tau)) + \overline{d}_n\overline{\vec{\gamma}}_n\right) = \Delta\alpha\; H(t_+ = 0)\; \vec{\Phi}(\vec{x}, \tau) \tag{46}$$

Multiplying equation (46) by the adjoint vector $\overline{\vec{\gamma}}_j^\star(\vec{x}, \alpha(\tau) + \Delta\alpha)$ and integrating over space,we get

$$\begin{aligned} h_j(t_+ = 0) \;=\; & \langle\, \vec{\gamma}_j^\star(\alpha + \Delta\alpha), -\left(\vec{u}_{equil}(\alpha + \Delta\alpha) - \vec{u}_{equil}(\alpha)\right) \\ & + \sum_n \left(d_n(t_+ = 0)\vec{\gamma}_n(\alpha) + \overline{d}_n\overline{\vec{\gamma}}_n\right) + \Delta\alpha\; H(t_+ = 0)\; \vec{\Phi}(\vec{x}, \tau)\, \rangle \end{aligned} \tag{47}$$

Comparing equation (47) with (44) evaluated at $t_+ = 0$, we have

$$C_j = \langle\, \vec{\gamma}_j^\star(\alpha + \Delta\alpha), -\left(\vec{u}_{equil}(\alpha + \Delta\alpha) - \vec{u}_{equil}(\alpha)\right) + \sum_n \left(d_n(t_+ = 0)\vec{\gamma}_n(\alpha) + \overline{d}_n\overline{\vec{\gamma}}_n\right) \rangle \tag{48}$$

and

$$\langle\, \vec{\gamma}_j^\star(\vec{x}, \alpha(\tau) + \Delta\alpha), \vec{\Phi}(\vec{x}, \tau)\, \rangle = -c_j \tag{49}$$

Equation (48) determines the form of C_j, and a comparison of the form of c_j (cf. eq. (43)) with equation (49) reveals that $\Delta\alpha\,\vec{\Phi}$ is simply $\Delta\,\vec{v}_c$. The latter result implies that the velocity step imposed to the body, through the step change in angle of attack, is transferred instantaneously to the velocity field. The form of C_j can be simplified by retaining only terms to the first order in $\Delta\alpha$, since that is all we shall need to form the indicial response. After making extensive use of the properties of scalar products of adjoint vectors, we get

$$C_j = d_j(t_+ = 0) + \Delta\alpha\left(\frac{\partial}{\partial\alpha}d_j(t_+ = 0) + \mathcal{D}_j\right) + \vartheta(\Delta\alpha)^2 \tag{50}$$

with

$$\mathcal{D}_j = -\langle \vec{\gamma}_j^\star(\vec{x}, \alpha(\tau)), \frac{\partial}{\partial\alpha}\vec{u}\,[\alpha_\tau(\xi), \xi = \tau]\rangle \tag{51}$$

Returning to the integral equation (44) for the $h_j(t_+)$ with the values of C_j from equation (48) inserted, we get the following equation where only terms to the first order in $\Delta\alpha$ are retained:

$$\begin{aligned} h_j(t_+) \;=\; & e^{\lambda_j(\alpha(\tau))t_+} d_j(t_+ = 0) + \Delta\alpha.\left\{\frac{\partial}{\partial\alpha}\left(e^{\lambda_j(\alpha(\tau))t_+} d_j(t_+ = 0)\right) + e^{\lambda_j(\alpha(\tau))t_+}\left(\mathcal{D}_j - H(t_+)c_j\right)\right\} \\ & + \sum_{n,m}\int_0^{t_+} ds\left\{A_{jnm}e^{\lambda_j(t_+ - s)}h_n(s)h_m(s) + \ldots + D_{jnm}e^{\lambda_j(t_+ - s)}\overline{h}_n(s)\overline{h}_m(s)\right\} + \Delta\alpha.\sum_{n,m}\int_0^{t_+} ds \\ & \times\left\{\frac{\partial}{\partial\alpha}\left(A_{jnm}e^{\lambda_j(t_+ - s)}\right)h_n(s)h_m(s) + \ldots + \frac{\partial}{\partial\alpha}\left(D_{jnm}e^{\lambda_j(t_+ - s)}\right)\overline{h}_n(s)\overline{h}_m(s)\right\} \\ & + \vartheta(\Delta\alpha)^2 \end{aligned} \tag{52}$$

Anticipating the form of the result, we let

$$h_j(t_+) = d_j(t_+) + \Delta\alpha.\left(\frac{\partial}{\partial\alpha}d_j(t_+) + f_j(t_+)\right) + \vartheta(\Delta\alpha)^2 \tag{53}$$

Inserting equation (53) in (52), we get linear integral equations for the $f_j(t_+)(j = 1, \ldots)$ of the form

$$
\begin{aligned}
f_j(t_+) &= (\mathcal{D}_j - c_j H(t_+))\, e^{\lambda_j t_+} + \sum_{n,m} A_{jnm} \int_0^{t_+} e^{\lambda_j(t_+ - s)} \left(d_n(s) f_m(s) + d_m(s) f_n(s) \right) ds \\
&\quad + \ldots + \sum_{n,m} D_{jnm} \int_0^{t_+} e^{\lambda_j(t_+ - s)} \left(\overline{d}_n(s) \overline{f}_m(s) + \overline{d}_m(s) \overline{f}_n(s) \right) ds
\end{aligned} \tag{54}
$$

where the eigenvalues λ_j and the coefficients $A_{jnm}, \ldots, D_{jnm}$ are to be evaluated at $\alpha(\tau)$. Equation (54) can be solved by successive approximations. Given the form (53) and assuming the f_j determinable from (54), we are now able to write the form of $\vec{u}[\alpha^\star_\tau(\xi)]$ (cf. eq. (40)) as

$$
\vec{u}[\alpha^\star_\tau(\xi)] = \vec{u}_{equil}(\vec{x}, \alpha(\tau) + \Delta\alpha) + \sum_n \left(d_n(t_+) + \Delta\alpha \frac{\partial}{\partial\alpha} d_n(t_+) + \Delta\alpha f_n(t_+) \right) \vec{\gamma}_n(\vec{x}, \alpha(\tau) + \Delta\alpha) + \text{c.c.} \tag{55}
$$

Again systematically retaining only terms to the first order in $\Delta\alpha$ in (55), we have finally,

$$
\vec{u}[\alpha^\star_\tau(\xi)] = \vec{u}[\alpha_\tau(\xi)] + \Delta\alpha . \left\{ \frac{\partial}{\partial\alpha} \vec{u}[\alpha_\tau(\xi)] + \sum_n \left(f_n(t_+) \vec{\gamma}_n(\vec{x}, \alpha(\tau)) + \overline{f}_n \overline{\vec{\gamma}}_n \right) \right\} \tag{56}
$$

INDICIAL AND TOTAL RESPONSES OF THE VELOCITY FIELD

The forms we have derived for $\vec{u}[\alpha_\tau(\xi)]$ in equation (29) and $\vec{u}[\alpha^\star_\tau(\xi)]$ in equation (55) satisfy Postulates 1 and 2. Postulate 3 presumes the existence of a limit as $\Delta\alpha \to 0$ such that the indicial response can be defined as

$$
\vec{u}_\alpha[\vec{x}, \alpha(\xi); t, \tau] = \lim_{\Delta\alpha \to 0} \left(\frac{\vec{u}[\vec{x}, \alpha^\star_\tau(\xi)] - \vec{u}[\vec{x}, \alpha_\tau(\xi)]}{\Delta\alpha} \right) \tag{57}
$$

We see from the way we have derived equation (56) that the limit (defined by the right-hand side of eq. (57)) will in fact exist, so that Postulate 3 can be satisfied. From equation (56), we have for the indicial response of the velocity field the following:

$$
\vec{u}_\alpha = \frac{\partial}{\partial\alpha} \vec{u}(\vec{x}, \alpha(\tau); t, \tau) + \sum_n f_n(t_+) \vec{\gamma}_n(\vec{x}, \alpha(\tau)) + \text{c.c.} \tag{58}
$$

The total response of the velocity field is obtained by summing the indicial response along the motion history $\alpha_t(\xi)$ according to equation (10)

$$
\vec{u}(\vec{x}, t) = \vec{u}(\vec{x}, \tau_0) + \int_{\tau_0}^{t} d\tau \frac{d\alpha}{d\tau} \left\{ \frac{\partial}{\partial\alpha} (\vec{u}_{equil} + \vec{u}_{trans}) + \sum_n f_n(t_+) \vec{\gamma}_n(\vec{x}, \alpha(\tau)) + \text{c.c.} \right\} \tag{59}
$$

that is,

$$
\begin{aligned}
\vec{u}(\vec{x}, t) &= \vec{u}(\vec{x}, \tau_0) + \vec{u}_{equil}(\vec{x}, \alpha(t)) - \vec{u}_{equil}(\vec{x}, \alpha(\tau_0)) \\
&\quad + \int_{\tau_0}^{t} d\tau \frac{d\alpha}{d\tau} \left\{ \frac{\partial}{\partial\alpha} \left(\sum_n d_n(t_+) \vec{\gamma}_n(\vec{x}, \alpha(\tau) \right) + \sum_n f_n(t_+) \vec{\gamma}_n(\vec{x}, \alpha(\tau)) + \text{c.c.} \right\}
\end{aligned} \tag{60}
$$

In experiments, one usually maintains the angle of attack of the system constant for some sufficiently long time before letting it oscillate. Under these circumstances, the velocity field at $\xi = \tau_0$ is practically equal to its equilibrium value. Therefore, equation (60) is reduced to

$$\vec{u}(\vec{x},t) = \vec{u}_{equil}(\vec{x},\alpha(t)) + \int_{\tau_0}^{t} d\tau \frac{d\alpha}{d\tau} \left\{ \frac{\partial}{\partial \alpha} \left(\sum_n d_n(t_+) \vec{\gamma}_n(\vec{x},\alpha(\tau) \right) + \sum_n f_n(t_+) \vec{\gamma}_n(\vec{x},\alpha(\tau)) + \text{c.c.} \right\} \tag{61}$$

Equation (61) has the following form:

$$\vec{u}(\vec{x},t) = \vec{u}_{equil}(\vec{x},\alpha(t)) + \int_{\tau_0}^{t} d\tau \frac{d\alpha}{d\tau} \vec{\mathcal{F}} \left[d_j \left[\alpha_\tau(\xi), t_+ = 0 \right]; t_+ \right] \tag{62}$$

with the functional dependence of $\vec{\mathcal{F}}$ determined by the initial condition $d_j(t_+ = 0)(j = 1, \ldots)$, according to equations (25), (26), and (54).

DETERMINATION OF THE INITIAL CONDITIONS $d_j(t_+ = 0)(j = 1, \ldots)$

According to equations (25), (26), (54), and (61), the indicial response of the velocity field is completely known if the initial values $d_j(t_+ = 0)(j = 1, \ldots)$ are determined. It is possible to derive an equation governing the behavior of $d_j(t_+ = 0)$. Indeed, as stated previously, the value of the velocity field at $\xi = \tau$ is the resultant of the entire motion history up to the present time $\xi = \tau$, $\alpha_\tau(\xi : \xi \in]-\infty, \tau[)$. According to relation (10), it is equal to

$$\vec{u}(\vec{x},\xi = \tau) = \vec{u}(\vec{x},\xi = -\infty) + \int_{-\infty}^{\tau} ds\ \dot{\alpha}(s) \vec{u}_\alpha \tag{63}$$

By using equations (29) and (61), equation (63) can be transformed as

$$\begin{aligned} &\vec{u}_{equil}(\vec{x},\alpha(\tau)) + \sum_n d_n(t_+ = 0) \vec{\gamma}_n(\vec{x},\alpha(\tau)) + \text{c.c.} \\ &= \vec{u}_{equil}(\vec{x},\alpha(\tau)) + \int_{-\infty}^{\tau} ds\ \dot{\alpha}(s) \left\{ \frac{\partial}{\partial \alpha(s)} \left(\sum_n d_n(\tau - s) \vec{\gamma}_n(\vec{x},\alpha(s)) \right) \right. \\ &\left. + \sum_n f_n(\tau - s) \vec{\gamma}_n(\vec{x},\alpha(s)) + \text{c.c.} \right\} \end{aligned} \tag{64}$$

It is shown in the appendix that the $d_j(t_+ = 0)$ are governed by the following equation, derived from equation (64):

$$d_j(t_+ = 0) = \int_{-\infty}^{\tau} ds\ \dot{\alpha}(s) \langle\ \vec{\gamma}_j^{\dagger}(\vec{x},\alpha(\tau)), \sum_n d_n(\tau - s) \frac{\partial}{\partial \alpha(s)} \vec{\gamma}_n(\vec{x},\alpha(s)) + \sum_n g_n(\tau - s) \vec{\gamma}_n(\vec{x},\alpha(s))\ \rangle \tag{65}$$

where the quantities $g_n(\tau - s)$ obey linear integral equations involving the unknown quantities $d_j(s = 0)(j = 1, \ldots)$ and other quantities that are known such as $\vec{\gamma}_n$, $\vec{u}_{equil}$, A_{jnm}, $\ldots$, and D_{jnm}. Accordingly, equation (65) corresponds to a Volterra integral equation of the second kind.

To proceed further, we need to consider two cases, first a body undergoing an externally forced motion, and second, a body undergoing an unforced motion. In the former case, the motion history is externally imposed and therefore known; in the latter case, it is unknown.

Externally Driven Motion

In this case, the angle of attack $\alpha(s)$ is known. The Volterra integral equation (65) governing $d_j(t_+ = 0)$ can be solved in principle, once the values of the equilibrium state of the velocity field $\vec{u}_{equil}$, of the eigenvalues $\lambda_j(j = 1,\ldots)$ and of the eigenvectors $\vec{\eta}_j(j = 1,\ldots)$ are determined from the known values of the angle of attack.

Experimentalists commonly take the forcing as a periodic function of time, athough one can imagine different types of forcing. We shall show in the following that in the case of periodic external excitation, the coefficients $d_j(t_+ = 0)$ can be decomposed into a Fourier series of frequency ω, the forcing frequency. Indeed, according to equation (65), the initial values $d_j(t = \tau)(j = 1,\ldots)$ have the following analytical form:

$$d_j(t = \tau) = \int_{-\infty}^{\tau} ds\, \dot{\alpha}(s)\Phi_j\left(d_n(t = s)(n = 1,\ldots), \alpha(s), \alpha(\tau), \tau - s\right) \qquad j = 1,\ldots \quad (66)$$

We show first that the integral constituting the right-hand side of equation (66) is periodic in time with same period T as the external forcing. To that end, let us make the following change of variables:

$$\left.\begin{array}{rcl} \tau & = & \tau' + T \\ s & = & s' + T \end{array}\right\} \quad (67)$$

The integral becomes, by using the periodic property of $\alpha(s)$,

$$\int_{-\infty}^{\tau} ds\, \dot{\alpha}(s)\Phi_j\left(d_n(t = s)(n = 1,\ldots), \alpha(s), \alpha(\tau), \tau - s\right)$$
$$= \int_{-\infty}^{\tau'} ds\, \dot{\alpha}(s')\Phi_j\left(d_n(s' + T)(n = 1,\ldots), \alpha(s'), \alpha(\tau'), \tau' - s'\right) \quad (68)$$

Therefore, according to (68)

$$d_n(s + T) = d_n(s) \quad (69)$$

which is the property announced previously. Because the $d_j(t_+ = 0)(j = 1,\ldots)$ are periodic in time with period equal to T, they can be decomposed into a Fourier series of frequency $\omega = 2\pi/T$:

$$d_j(t_+ = 0) = \sum_n d_{jn}\, e^{in\omega\tau} \quad (70)$$

Since the value of $d_j(t_+)$ is governed by equation (26), it has the same property as $d_j(t_+ = 0)$ of being periodic in time with the same frequency ω. On returning to the value of the velocity field given by equation (29), it is evident that $\vec{u}(\vec{x}, \alpha, \varsigma)$ is also periodic in time with the same frequency as the excitation. Accordingly, the velocity field can be decomposed into a Fourier series of frequency ω. This result is in agreement with theoretical predictions of Joseph (ref. 7) related to a body excited by a periodic forcing, in a Reynolds number range such that the equilibrium state at fixed α is time-invariant and stable.

One can approximate equation (70) in the case of a slowly varying motion as

$$d_j(t_+ = 0) \simeq d_{j0} + d_{j1}e^{i\omega\tau} + d_{j-1}e^{-i\omega\tau} \quad (71)$$

that is,

$$d_j(t_+ = 0) = d_j\left(\alpha(\tau), \dot{\alpha}(\tau)\right) \quad (72)$$

Elastically Mounted Body in Unforced Motion

In the case of an elastically mounted body in unforced motion, the angle-of-attack motion history $\alpha(s)$ is unknown. One is faced with solving equation (65) coupled with the equation of motion governing the body, according to a scheme of advancing step by step in angle of attack. In addition to being complicated, the above procedure does not give any insight into the form of $d_j(t_+ = 0)$. It is possible, however, to get an approximate analytical form of $d_j(t_+ = 0)$ by making use of the following reasoning. Let us first note that according to equations (25), (26), (54), and (65), the value of the velocity field contains terms such as

$$\exp\{\lambda_j(\alpha(\tau))(t-\tau)\} \qquad j = 1,\ldots$$

under an integration over time. Hence, because all the real parts of λ_j are negative, the effect of a distant-past motion history of the angle of attack on the present value of the velocity field is fading away, so that it is principally the recent motion history of α that determines the present value of the velocity field. In as much as one is concerned only with a recent-past motion history, one can consider fitting it over the recent past to a known motion history of the system. The latter can be provided, for instance, by experiments made under external periodic forcing. Under this approximation, the analytical dependence of $d_j(t_+ = 0)$ for an elastically mounted body will be the same as for a periodically driven body.

For a slowly varying unforced motion, $d_j(t_+ = 0)$ obeys equation (72). As $\dot{\alpha}(\tau) \ll \alpha(\tau)$, one can expand d_j:

$$d_j(t_+ = 0) = d_j(\alpha(\tau), \dot{\alpha}(\tau)) = d_j(\alpha(\tau), 0) + \dot{\alpha}(\tau)\frac{\partial d_j}{\partial \alpha(\tau)} + \vartheta(\dot{\alpha}^2(\tau)) \tag{73}$$

As a consequence of the results of this section, one can rewrite relation (62) as the following, by using relation (72):

$$\vec{u}(\vec{x}, t) \simeq \vec{u}_{equil}(\vec{x}, \alpha(t)) + \int_0^t d\tau \frac{d\alpha}{d\tau}\vec{\mathcal{F}}(\alpha(\tau), \dot{\alpha}(\tau), t_+) \tag{74}$$

DETERMINATION OF NORMAL FORCE

The pressure can be derived from the velocity field by using the Navier-Stokes equations. According to equations (12), (20), and (21), the pressure depends on $d_j(t_+)(j = 1,\ldots)$ as

$$p(\vec{x}, t_+) = p_{equil}(\vec{x}) + p_{trans}(d_j(t_+), j = 1, \ldots) \tag{75}$$

Since the $d_j(t_+)(j = 1,\ldots)$ depend on their initial value according to equations (25) and (26), one has

$$p(\vec{x}, t_+) = p_{equil}(\vec{x}, \alpha(t)) + p_{trans}(d_j(t_+ = 0), j = 1, \ldots) \tag{76}$$

The value of the normal force N (normalized with respect to density ρ) is composed of a contribution from the (normalized) surface pressure and a viscous contribution arising from the skin friction at the surface. We derive a form for the normal force containing the two components that is applicable in the case of a two-dimensional flow. The restriction is imposed for simplicity only; generalization to accommodate three-dimensional flow is straightforward. Let S denote the body cross section, $\vec{j}$ a unit vector directed normal to the axial velocity U_0, $\vec{n}$ and $\vec{t}$ unit vectors, respectively, normal and tangential to the surface element dS. Let ζ be the magnitude of the vorticity vector, determinable from the velocity field through $\zeta = \mid \vec{\nabla} \times \vec{u} \mid$. Then,

$$N = \int_S p(\vec{x},t)|_S\, \vec{j}.\vec{n}dS + \int_S \nu\, \zeta(\vec{x},t)|_S\, \vec{j}.\vec{t}dS \tag{77}$$

As shown previously, p and $\vec{u}$ (and hence ζ) can be separated, respectively, into an equilibrium component, which depends only on the instantaneous value of α, and a transient component, which depends on the initial values of $d_j(j = 1, \ldots)$. It follows that C_N, the dimensionless form of the normal force, may be given the form

$$C_N(t) = C_N^{equil}(\alpha(t)) - \int_0^{t_+} d\tau \frac{d\alpha}{d\tau} \mathcal{F}\left[d_j\left[\alpha_\tau(\xi), t_+ = 0\right], t_+, \tau\right] \tag{78}$$

where the value of the function $\mathcal{F}$ approaches zero as $t - \tau \to \infty$, a result of the properties of $d_n(t_+)$. Relation (78) was derived in previous work (ref. 8) on the basis of a functional analysis approach; in reference 8, the function $\mathcal{F}$ was called the deficiency function. The value of the normal force can be approximated as the following, by using equations (72):

$$C_N(t) \simeq C_N^{equil}(\alpha(t)) - \int_0^t d\tau \frac{d\alpha}{d\tau} \mathcal{F}(\alpha(\tau), \dot{\alpha}(\tau), t_+) \tag{79}$$

and by using equation (73):

$$\begin{aligned} C_N(t) &= C_N^{equil}(\alpha(t)) - \int_0^t d\tau \frac{d\alpha}{d\tau} \mathcal{F}(\alpha(\tau), t_+) + \vartheta(\dot{\alpha}^2(t)) && (80) \\ &\simeq C_N^{equil}(\alpha(t)) + \dot{\alpha}(t) C_{N_{\dot{\alpha}}}(\alpha(t)) + \vartheta(\dot{\alpha}^2(t)) && (81) \end{aligned}$$

According to relation (81), the value of the normal force depends only on the value of angle of attack α and of its derivative $\dot{\alpha}$ at the time of observation t. In other words, only a recent-past motion history of the angle of attack at $\xi < t$ is taken into account through the dependence on $\dot{\alpha}(t)$. The logic underlying relation (81) forms the basis for most mathematical models in flight dynamics. In spite of its simplicity, it has been capable of reproducing a great variety of the characteristics of aircraft motions (see, e.g., ref. 9).

CONCLUSIONS

We conclude that in the case of a time-invariant equilibrium state, derivation of the indicial response of the velocity field directly from the Navier-Stokes equations leads to an analytical form of the normal force which confirms the form of existing mathematical models. Applied to the next physical situation of a periodic time-varying equilibrium state, our approach will show that the same level of mathematical modeling will no longer hold, but rather will require considerable amendment, owing to the presence of the phase characteristic of the periodic equilibrium state.

APPENDIX
DERIVATION OF EQUATIONS GOVERNING INITIAL CONDITIONS

In the following, we propose to derive the equations governing the initial conditions $d_j(t_+ = 0)(j = 1, \ldots)$.

Let us start with the following equation derived directly from equation (64) of the main text:

$$\sum_n d_n(t_+ = 0)\vec{\gamma}_n(\vec{x}, \alpha(\tau)) + c.c. = \int_{-\infty}^{\tau} ds\, \dot{\alpha}(s) \left\{ \frac{\partial}{\partial \alpha(s)} \left(\sum_n d_n(\tau - s)\vec{\gamma}_n(\vec{x}, \alpha(s)) \right) + \sum_n f_n(\tau - s)\vec{\gamma}_n(\vec{x}, \alpha(s)) + c.c. \right\} \tag{A1}$$

Multiplying equation (A1) by the adjoint vector $\vec{\gamma}_j^{\star}(\vec{x}, \alpha(\tau))$ and integrating over space, one gets

$$\begin{aligned} d_j(t_+ = 0) &= \int_{-\infty}^{\tau} ds\, \dot{\alpha}(s) \langle\, \vec{\gamma}_j^{\star}(\vec{x}, \alpha(\tau)), \sum_n d_n(\tau - s) \frac{\partial}{\partial \alpha(s)} \vec{\gamma}_n(\vec{x}, \alpha(s)) \\ &+ \sum_n \left(\frac{\partial}{\partial \alpha(s)} d_n(\tau - s) + f_n(\tau - s) \right) \vec{\gamma}_n(\vec{x}, \alpha(s)) \,\rangle \end{aligned} \tag{A2}$$

Equation (A2) would constitute a Volterra integral equation of the second kind if the quantities $\frac{\partial}{\partial \alpha} d_j(\tau - s)(j = 1, \ldots)$ were not present. We shall show in the following that, indeed, the quantities $g_j(j = 1, \ldots)$ defined as

$$g_j = \frac{\partial}{\partial \alpha} d_j(\tau - s) + f_j(\tau - s) \tag{A3}$$

do not contain any element such as $\frac{\partial}{\partial \alpha} d_j(\tau - s)(j = 1, \ldots)$.

Let us first write the following equations governing $f_j(\tau - s)$, derived from equations (54):

$$\begin{aligned} f_j(\tau - s) &= -\frac{\partial}{\partial \alpha} d_j . e^{\lambda_j(\tau - s)} + \left(\mathcal{D}_j + \frac{\partial}{\partial \alpha} d_j - c_j H(\tau - s) \right) e^{\lambda_j(\tau - s)} \\ &+ \sum_{n,m} A_{jnm} \int_0^{\tau - s} ds_1 e^{\lambda_j(\tau - s - s_1)} \left(d_n(s_1) f_m(s_1) + d_m(s_1) f_n(s_1) \right) + \ldots \\ &+ \sum_{n,m} D_{jnm} \int_0^{\tau - s} ds_1 e^{\lambda_j(\tau - s - s_1)} \left(\bar{d}_n(s_1) \bar{f}_m(s_1) + \bar{d}_m(s_1) \bar{f}_n(s_1) \right) \end{aligned} \tag{A4}$$

where the quantities $\left(\mathcal{D}_j + \frac{\partial}{\partial \alpha} d_j \right)$ do not contain any element $\frac{\partial}{\partial \alpha} d_j(\tau - s)(j = 1, \ldots)$. Indeed, one can show by using equation (51) that

$$\begin{aligned} \mathcal{D}_j + \frac{\partial}{\partial \alpha} d_j &= -\langle\, \vec{\gamma}_j^{\star}(\vec{x}, \alpha(s)), \frac{\partial}{\partial \alpha(s)} \vec{u}_{equil}(\vec{x}, \alpha(s)) \\ &+ \sum_n \left(d_n(t_+ = 0) \frac{\partial}{\partial \alpha(s)} \vec{\gamma}_n(\vec{x}, \alpha(s)) + \bar{d}_n \frac{\partial}{\partial \alpha(s)} \bar{\vec{\gamma}}_n \right) \,\rangle \end{aligned} \tag{A5}$$

The quantities $\frac{\partial}{\partial\alpha}d_j(j=1,\ldots)$ can be obtained from equation (24) as

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}d_j &= e^{\lambda_j(\tau-s)}.\frac{\partial}{\partial\alpha}d_j(s) + \sum_{n,m} A_{jnm}\int_0^{\tau-s} ds_1\ e^{\lambda_j(\tau-s-s_1)}\left(\frac{\partial}{\partial\alpha}d_n(s_1).d_m(s_1)\right.\\
&\left. + d_n(s_1).\frac{\partial}{\partial\alpha}d_m(s_1)\right) + \ldots + \sum_{n,m} D_{jnm}\int_0^{\tau-s} ds_1\ e^{\lambda_j(\tau-s-s_1)}\left(\frac{\partial}{\partial\alpha}\overline{d}_n(s_1).\overline{d}_m(s_1)\right.\\
&\left. + \overline{d}_n(s_1).\frac{\partial}{\partial\alpha}\overline{d}_m(s_1)\right) + Q_j \qquad j=1,\ldots
\end{aligned}
\tag{A6}
$$

with $Q_j(j=1,\ldots)$ defined as

$$
\begin{aligned}
Q_j &= d_j(s).\frac{\partial}{\partial\alpha}e^{\lambda_j(\tau-s)} + \sum_{n,m}\int_0^{\tau-s} ds_1\ \frac{\partial}{\partial\alpha}\left(A_{jnm}e^{\lambda_j(\tau-s-s_1)}\right) d_n(s_1)d_m(s_1)\\
&+ \ldots + \sum_{n,m}\int_0^{\tau-s} ds_1\ \frac{\partial}{\partial\alpha}\left(D_{jnm}e^{\lambda_j(\tau-s-s_1)}\right).\overline{d}_n(s_1).\overline{d}_m(s_1) \qquad j=1,\ldots
\end{aligned}
\tag{A7}
$$

that is, the quantities Q_j are independent of $\frac{\partial}{\partial\alpha}d_j(j=1,\ldots)$.

Adding equation (A4) to (A6), one gets

$$
\begin{aligned}
g_j &= \left(\mathcal{D}_j + \frac{\partial}{\partial\alpha}d_j - c_j H(t_+)\right)e^{\lambda_j(\tau-s)} + Q_j\\
&+ \sum_{n,m} A_{jnm}\int_0^{\tau-s} ds_1\ e^{\lambda_j(\tau-s-s_1)}\left(d_n(s_1)g_m(s_1) + d_m(s_1)g_n(s_1)\right) + \ldots\\
&+ \sum_{n,m} D_{jnm}\int_0^{\tau-s} ds_1\ e^{\lambda_j(\tau-s-s_1)}\left(\overline{d}_n(s_1)\overline{g}_m(s_1) + \overline{d}_m(s_1)\overline{g}_n(s_1)\right) \qquad j=1,\ldots
\end{aligned}
\tag{A8}
$$

The equations governing $g_j(j=1,\ldots)$ do not contain any element $\frac{\partial}{\partial\alpha}d_j(j=1,\ldots)$, as announced previously. Thus, equation (A2) does in fact constitute a Volterra integral equation of the second kind, as stated in the text.

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is RECAST is DEMONS.

16. Abstract

~~The aim of this research is to recast~~ the indicial response approach in a form appropriate to the study of vortex-induced oscillations phenomena. ~~We demonstrate that~~ an appropriate form for the indicial response of the velocity field may be derived directly from the Navier-Stokes equations. ~~For convenience, the study is divided into three parts. In Part I, we demonstrate~~, on the basis of the Navier-Stokes equations, it is DEMONS. how a form of the velocity response to an arbitrary motion may be determined. To establish its connection with THE ~~our~~ previous work, the new approach is applied first to the simple situation wherein the indicial response has a time-invariant equilibrium state. Results for the aerodynamic response to an arbitrary motion are shown to confirm the form ~~that we have~~ obtained previously.

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