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Modified Hilbert Transform Pair and Kramers-Kronig Relations for Complex Permittivities

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ABSTRACT

Hilbert transform pair and Kramers-Kronig relations given in the open literature are not applicable to permittivity representations which account for loss **only** in the plasma/dielectric model. In this paper, modified versions of these relationships are derived.

INTRODUCTION

Expressions representing the complex permittivity of a plasma/dielectric medium are derived through the equations of motion of electrons in the presence of an applied electric field [1-3]. These steady-state representations of the complex permittivities, as they appear in the open literature, yield noncausal time responses, unless the equations of motion include both loss and restoring terms [4]. Causality for the no-loss and/or no-restoring cases can be accomplished, however, provided appropriate impulsive terms are added [4,5]. These impulsive terms are derived and discussed in detail in references 4 and 5. Through inverse Fourier transformations of these modified complex permittivities, the principle of causality was verified.

For complex permittivities, which are analytic in either the lower half plane or the upper half plane of the complex-ω plane, the Hilbert transform pair or the Kramers-Kronig relations provide very useful properties; namely, if the real part of the complex permittivity is known, the imaginary part can be found and vice versa [6]. For the $e^{j\omega t}$ time convention, the complex permittivity is analytic in the lower half of the complex-ω plane.The analytical requirement is a direct consequence of the principle of causality. Therefore, the Hilbert transform pair or the Kramers-Kronig relations in their present form cannot be applied to complex permittivities which are meromorphic in the lower half plane of the complex-ω plane.

In this paper, modified versions of the Hilbert transform pair and the Kramers-Kronig relations are derived for the complex permittivity which is singular at $\omega = 0$. Such a complex permittivity exists when the plasma/dielectric model allows a loss term but no restoring term. Permittivity, in which both loss and restoring terms are included, is shown to satisfy the standard Hilbert transform pair and, thus, the Kramers-Kronig relations.For convenience, the standard transform pair and relations are derived in the Appendix.

PROOF OF HILBERT TRANSFORM PAIR FOR PERMITTIVITIES OF PLASMA MODELS WITH BOTH LOSS AND RESTORING TERMS

The complex permittivity of a plasma material in which both loss and restoring terms are assumed in the model is given as [3,4]

$$\varepsilon(\omega) = \varepsilon_o \left[1 - \frac{\omega_p^2 \left(\omega^2 - \beta^2\right)}{\left(\omega^2-\beta^2\right)^2 + \omega^2\nu^2} - j \frac{\omega_p^2\, \omega\, \nu}{\left(\omega^2-\beta^2\right)^2 + \omega^2\nu^2} \right] \tag{1}$$

where ε_o is the free space permittivity, ω_p is the plasma frequency, ν is the collision frequency of the plasma, β is a constant related to the restoring term, and ω is the frequency of the applied electric field. Equation (1) represents the permittivity of a cold plasma in the absence of a magnetic field. It is easily shown that equation (1) can be rewritten as

$$\varepsilon(\omega) = \varepsilon_o - \frac{\omega_p^2\, \varepsilon_o}{\omega^2-\beta^2-j\omega\nu} \tag{2}$$

or

$$\varepsilon(\omega) = \varepsilon_o - \frac{\omega_p^2\, \varepsilon_o}{(\omega - \omega_1)(\omega - \omega_3)} \tag{3}$$

where

$$\omega_1 = \frac{j}{2}\left[\nu + \sqrt{\nu^2 - 4\beta^2} \right] \tag{4}$$

and

$$\omega_3 = \frac{j}{2}\left[\nu - \sqrt{\nu^2 - \beta^2} \right] \tag{5}$$

For $\nu > 0$ the complex function $\varepsilon(\omega) - \varepsilon_o = - \dfrac{\omega_p^2\, \varepsilon_o}{(\omega - \omega_1)(\omega - \omega_3)}$ is meromorphic in the upper half plane of the complex-ω plane and analytic in the lower half plane of the complex-ω plane. By transposing ε_o to the left side in equation (3), the right side approaches zero as ω approaches infinity—a necessary condition for the Hilbert transform to be applicable. This, of course, implies $\varepsilon(\omega)$ approaches ε_o, as it should. Therefore, with complex permittivity $\varepsilon(\omega) = \varepsilon_R(\omega) - j\varepsilon_I(\omega)$ and $\varepsilon(\omega) - \varepsilon_o = f(\omega)$ [See Appendix], the following Hilbert transforms for $\varepsilon(\omega) - \varepsilon_o$ are defined as

$$\varepsilon_R(\omega_o) - \varepsilon_O = \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_I(\omega)}{\omega - \omega_o} d\omega \tag{6}$$

and

$$\varepsilon_I(\omega_o) = -\frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_R(\omega) - \varepsilon_o}{\omega - \omega_o} d\omega \tag{7}$$

These transforms are applicable provided $\varepsilon(\omega) - \varepsilon_o$ is analytic in the lower half plane of the complex-ω plane.

Verification of equations (6) and (7) will now be demonstrated for the relation given by equation (1). Assume the real part is known; that is,

$$\varepsilon_R(\omega) - \varepsilon_o = -\frac{\omega_p^2 \varepsilon_o (\omega^2 - \beta^2)}{(\omega^2 - \beta^2)^2 + \omega^2 \nu^2} \tag{8}$$

then equation (7) determines the imaginary part. Hence,

$$\varepsilon_I(\omega_o) = -\frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{1}{\omega - \omega_o} \left[\frac{-\omega_p^2 \varepsilon_o (\omega^2 - \beta^2)}{(\omega^2 - \beta^2)^2 + \omega^2 \beta^2} \right] d\omega \tag{9}$$

This integral can be evaluated by calculus of residues once the poles of the integrand are determined. Thus,

$$\varepsilon_I(\omega_o) = \frac{\omega_p^2 \varepsilon_o}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{1}{\omega - \omega_o} \frac{\omega^2 - \beta^2}{(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)} d\omega \tag{10}$$

where

$$\left.\begin{aligned} \omega_1 &= j \frac{1}{2} \left[\sqrt{\nu^2 - 4\beta^2} + \nu \right] \\ \omega_2 &= j \frac{1}{2} \left[\sqrt{\nu^2 - 4\beta^2} - \nu \right] \\ \omega_3 &= j \frac{1}{2} \left[-\sqrt{\nu^2 - 4\beta^2} + \nu \right] \\ \omega_4 &= j \frac{1}{2} \left[-\sqrt{\nu^2 - 4\beta^2} - \nu \right] \end{aligned}\right\} \tag{11}$$

with $\beta > 0$ and $\nu > 0$. Evaluating equation (10) by contour integration over the contour shown in figure 1, one obtains

$$\varepsilon_I(\omega_o) = \frac{\omega_p^2\varepsilon_o}{\pi}\left\{\pi j \frac{\omega_o^2 - \beta^2}{\left(\omega_o^2 - \beta^2\right)^2 + \omega_o^2\nu^2} + 2\pi j\left[\frac{\omega_1^2 - \beta^2}{(\omega_1-\omega_o)(\omega_1-\omega_2)(\omega_1-\omega_3)(\omega_1-\omega_4)}\right.\right.$$

$$\left.\left. + \frac{\omega_3^2 - \beta^2}{(\omega_3-\omega_o)(\omega_3-\omega_1)(\omega_3-\omega_2)(\omega_3-\omega_4)}\right]\right\} \qquad (12)$$

To simplify the calculations let

$$\left.\begin{aligned} \omega_1 &= a \\ \omega_2 &= b \\ \omega_3 &= -b \\ \omega_4 &= -a \end{aligned}\right\} \qquad (13)$$

With equation (13) substituted, equation (12), after some manipulations, becomes

$$\varepsilon_I(\omega_o) = j\omega_p^2\varepsilon_o\left\{\frac{\omega_o^2 - \beta^2}{\left(\omega_o^2 - \beta^2\right)^2 + \omega_o^2\nu^2} + \frac{1}{ab(a-b)}\left[\frac{ab\omega_o + \beta^2(a-b) - \beta^2\omega_o}{ab - \omega_o^2 + \omega_o(a-b)}\right]\right\} \qquad (14)$$

Noting

$$\left.\begin{aligned} ab &= \beta^2 \\ a - b &= j\nu \end{aligned}\right\} \qquad (15)$$

equation (14) reduces to

$$\varepsilon_I(\omega_o) = j\omega_p^2\varepsilon_o\left\{\frac{\omega_o^2 - \beta^2}{\left[\omega_o^2 - \beta^2\right]^2 + \omega_o^2\beta^2} + \frac{1}{\beta^2 - \omega_o^2 + j\omega_o\nu}\right\} \qquad (16)$$

And finally,

$$\varepsilon_I(\omega_o) = \frac{\omega_p^2\varepsilon_o\omega_o\nu}{\left(\omega_o^2 - \beta^2\right)^2 + \omega_o^2\nu^2} \qquad (17)$$

which is the imaginary part as given in equation (1).

The validity of equation (6) is proved next. Substitute equation (17) with $\omega_o = \omega$ into equation (6),

$$\varepsilon_R(\omega_o) - \varepsilon_o = \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{1}{\omega - \omega_o} \left[\frac{\omega_p^2 \varepsilon_o \omega \nu}{(\omega^2 - \beta^2)^2 + \omega^2 \nu^2} \right] d\omega \tag{18}$$

or

$$\varepsilon_R(\omega_o) - \varepsilon_o = \frac{1}{\pi} \omega_p^2 \varepsilon_o \nu \; \mathcal{PV} \int_{-\infty}^{\infty} \frac{\omega}{(\omega - \omega_o)(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)} d\omega \tag{19}$$

where the ω_n's are given by equation (11). Using the contour shown in figure 1, contour integration gives

$$\varepsilon_R(\omega_o) - \varepsilon_o = \frac{1}{\pi} \omega_p^2 \varepsilon_o \nu \left\{ \pi j \frac{\omega_o}{\left[\omega_o^2 - \beta^2\right]^2 + \omega_o^2 \nu^2} + 2\pi j \left[\frac{\omega_1}{(\omega_1 - \omega_o)(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_1 - \omega_4)} + \frac{\omega_3}{(\omega_3 - \omega_o)(\omega_3 - \omega_1)(\omega_3 - \omega_2)(\omega_3 - \omega_4)} \right] \right\} \tag{20}$$

In terms of the parameters defined in equation (13), equation (20) can written as

$$\varepsilon_R(\omega_o) - \varepsilon_o = j \omega_p^2 \varepsilon_o \nu \left\{ \frac{\omega_o}{\left[\omega_o^2 - \beta^2\right]^2 + \omega_o^2 \nu^2} + \frac{1}{(a-b)\left[ab - \omega_o^2 + \omega_o(a-b)\right]} \right\} \tag{21}$$

With the definitions of a and b substituted,

$$\varepsilon_R(\omega_o) - \varepsilon_o = - \omega_p^2 \varepsilon_o \frac{\omega_o^2 - \beta^2}{\left[\omega_o^2 - \beta^2\right]^2 + \omega_o^2 \nu^2} \tag{22}$$

which is the real part as given in equation (1). Clearly, equation (1) satisfies the standard Hilbert transform pair and, therefore, the standard Kramers-Kronig relations.

DERIVATIONS OF MODIFIED HILBERT TRANSFORM PAIR AND KRAMERS-KRONIG RELATIONS

The expression representing the complex permittivity with only a loss term assumed in the model is given by [1,2]

$$\varepsilon(\omega) = \varepsilon_o - \frac{\omega_p^2 \varepsilon_o}{\omega(\omega - j\nu)} \tag{23}$$

or

$$\varepsilon(\omega) = \varepsilon_o - \frac{\omega_p^2 \varepsilon_o}{\omega^2 + \nu^2} - j\,\frac{\omega_p^2 \varepsilon_o \left(\frac{\nu}{\omega}\right)}{\omega^2 + \nu^2} \tag{24}$$

which can be obtained from equation (1) with $\beta = 0$. It was shown via the inverse Fourier transform in reference 5 that the time response for this permittivity is noncausal. However, it was also shown that a causal response can be realized by adding an impulsive term to equation (23) or (24). The permittivity with this addition becomes

$$\varepsilon(\omega) = \varepsilon_o - \frac{\omega_p^2 \varepsilon_o}{\omega(\omega - j\nu)} + \boxed{\pi\,\frac{\omega_p^2 \varepsilon_o}{\nu}\,\delta(\omega)} \tag{25}$$

In this section, modified Hilbert transform pair and Kramers-Kronig relations are derived for the permittivity representation given by equation (25). To emphasize the need for the impulsive term, a dashed box around it will be maintained throughout the derivation. For a complex function to possess a Hilbert transform, it must be analytic in one half plane or the other of complex-ω plane. Obviously, equation (25) as written does not possess this property. However, rewriting as

$$\varepsilon(\omega) - \varepsilon_o - \boxed{\pi\,\frac{\omega_p^2 \varepsilon_o}{\nu}\,\delta(\omega)} + j\,\frac{\omega_p^2 \varepsilon_o}{\omega\nu} = j\,\frac{\left(\frac{\omega_p^2 \varepsilon_o}{\nu}\right)}{\omega - j\nu} \tag{26}$$

one can readily observe that the right side of equation (26) is analytic in the lower half plane of the complex-ω plane and approaches zero as ω approaches infinity. Therefore, define the right side of equation (26) as $g(\omega) = g_R(\omega) - jg_I(\omega)$. Thus, $g(\omega)$ satisfies the standard Hilbert transform pair

$$\left.\begin{aligned} g_R(\omega_o) &= \frac{1}{\pi}\,\mathcal{PV}\int_{-\infty}^{\infty} \frac{g_I(\omega)}{\omega - \omega_o}\,d\omega \\ g_I(\omega_o) &= -\frac{1}{\pi}\,\mathcal{PV}\int_{-\infty}^{\infty} \frac{g_R(\omega)}{\omega - \omega_o}\,d\omega \end{aligned}\right\} \tag{27}$$

Since $g(\omega)$ must also equal the left side of equation (26), one has

$$\varepsilon_R(\omega) - j\varepsilon_I(\omega) - \varepsilon_o - \boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega)} + j \frac{\omega_p^2 \varepsilon_o}{\omega\nu} = g_R(\omega) - jg_I(\omega) \quad (28)$$

Equating real and imaginary parts

$$\left.\begin{aligned} g_R(\omega) &= \varepsilon_R(\omega) - \varepsilon_o - \boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega)} \\ g_I(\omega) &= \varepsilon_I(\omega) - \frac{\omega_p^2 \varepsilon_o}{\omega\nu} \end{aligned}\right\} \quad (29)$$

With equation (29) substituted into equation (27),

$$\left.\begin{aligned} \varepsilon_R(\omega_o) - \varepsilon_o - \boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega_o)} &= \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_I(\omega) - \frac{\omega_p^2 \varepsilon_o}{\omega\nu}}{\omega - \omega_o} d\omega \\ \varepsilon_I(\omega_o) - \frac{\omega_p^2 \varepsilon_o}{\omega_o \nu} &= -\frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_R(\omega) - \varepsilon_o - \boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega)}}{\omega - \omega_o} d\omega \end{aligned}\right\} \quad (30)$$

Rewriting as

$$\left.\begin{aligned} \varepsilon_R(\omega_o) - \varepsilon_o &= \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_I(\omega)}{\omega - \omega_o} d\omega - \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\omega_p^2 \varepsilon_o}{\omega - \omega_o} d\omega + \boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega_o)} \\ \varepsilon_I(\omega_o) &= -\frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_R(\omega) - \varepsilon_o}{\omega - \omega_o} d\omega + \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{\boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega)}}{\omega - \omega_o} d\omega + \frac{\omega_p^2 \varepsilon_o}{\omega_o \nu} \end{aligned}\right\} \quad (31)$$

The second integral in each of these equations can be evaluated rather easily yielding

$$\boxed{\begin{aligned} \varepsilon_R(\omega_o) - \varepsilon_o &= \frac{1}{\pi}\, \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_I(\omega)}{\omega - \omega_o}\, d\omega + \pi \frac{\omega_p^2 \varepsilon_o}{\nu}\, \delta(\omega_o) \\ \varepsilon_I(\omega_o) &= -\frac{1}{\pi}\, \mathcal{PV} \int_{-\infty}^{\infty} \frac{\varepsilon_R(\omega) - \varepsilon_o}{\omega - \omega_o}\, d\omega \end{aligned}} \tag{32}$$

which are the modified versions of the Hilbert transform pair. It is readily observed that the only change from the standard form is the addition of the impulsive term. The corresponding Kramers-Kronig relations are

$$\boxed{\begin{aligned} \varepsilon_R(\omega_o) - \varepsilon_o &= \frac{2}{\pi}\, \mathcal{PV} \int_{0}^{\infty} \frac{\omega\, \varepsilon_I(\omega)}{\omega^2 - \omega_o^2}\, d\omega + \pi \frac{\omega_p^2 \varepsilon_o}{\nu}\, \delta(\omega_o) \\ \varepsilon_I(\omega_o) &= -\frac{2\omega_o}{\pi}\, \mathcal{PV} \int_{0}^{\infty} \frac{\varepsilon_R(\omega) - \varepsilon_o}{\omega^2 - \omega_o^2}\, d\omega \end{aligned}} \tag{33}$$

The significance of the impulsive term in equations (32) and (33) will now be demonstrated. Rewriting equation (26) as

$$\varepsilon_R(\omega) - j\, \varepsilon_I(\omega) - \varepsilon_o - \pi \frac{\omega_p^2 \varepsilon_o}{\nu}\, \delta(\omega) + j \frac{\omega_p^2 \varepsilon_o}{\omega \nu} = \frac{\omega_p^2 \varepsilon_o}{\nu} \left[\frac{-\nu + j\omega}{\omega^2 + \nu^2} \right] \tag{34}$$

which upon equating real and imaginary parts yields

$$\left.\begin{aligned} \varepsilon_R(\omega) - \varepsilon_o &= \pi \frac{\omega_p^2 \varepsilon_o}{\nu}\, \delta(\omega) - \frac{\omega_p^2 \varepsilon_o}{\omega^2 + \nu^2} \\ \varepsilon_I(\omega) &= \frac{\omega_p^2 \varepsilon_o}{\omega \nu} - \frac{\omega_p^2 \varepsilon_o}{\nu \left(\omega^2 + \nu^2 \right)} \end{aligned}\right\} \tag{35}$$

Verification that these equations satisfy the Hilbert transform pair given by equation (32) will now be shown. First, assume

$\varepsilon_R(\omega)-\varepsilon_o$ is given by the first equation of equation (35), $\varepsilon_I(\omega)$ is determined by the second equation of equation (32); that is,

$$\varepsilon_I(\omega_o) = -\frac{1}{\pi}\,\mathcal{PV}\int_{-\infty}^{\infty}\frac{\boxed{\pi\,\frac{\omega_p^2\varepsilon_o}{\nu}\delta(\omega)} - \left(\frac{\omega_p^2\varepsilon_o}{\omega^2+\nu^2}\right)}{\omega-\omega_o}\,d\omega \qquad (36)$$

which becomes

$$\varepsilon_I(\omega_o) = \frac{1}{\pi}\left[\boxed{\frac{\pi\frac{\omega_p^2\varepsilon_o}{\nu}}{-\,\omega_o}} - \omega_p^2\varepsilon_o\left(\pi j\,\frac{1}{\omega_o^2+\nu^2} + 2\pi j\,\frac{1}{j\nu-\omega_o}\,\frac{1}{j\nu}\right)\right] \qquad (37)$$

or

$$\varepsilon_I(\omega_o) = -\frac{1}{\pi}\left[\boxed{-\frac{\pi\omega_p^2\varepsilon_o}{\omega_o\nu}} + \frac{\pi\omega_p^2\varepsilon_o\omega_o}{\nu\left(\omega_o^2+\nu^2\right)}\right] \qquad (38)$$

Equation (38) clearly shows the necessity of the impulsive term in order to recover the second equation in equation (35).

The validity of the companion transform given in equation (32) is now proved. Knowing $\varepsilon_I(\omega)$ determine $\varepsilon_R(\omega)-\varepsilon_o$, thus,

$$\varepsilon_R(\omega_o) - \varepsilon_o = \frac{1}{\pi}\,\mathcal{PV}\int_{-\infty}^{\infty}\frac{\frac{\omega_p^2\varepsilon_o}{\omega\nu} - \frac{\omega_p^2\varepsilon_o\omega}{\nu(\omega^2+\nu^2)}}{\omega-\omega_o}\,d\omega + \boxed{\pi\,\frac{\omega_p^2\varepsilon_o}{\nu}\,\delta(\omega_o)} \qquad (39)$$

or

$$\varepsilon_R(\omega_o) - \varepsilon_o = \frac{\omega_p^2\varepsilon_o}{\pi\nu}\left[\pi j\left(-\frac{1}{\omega_o}\right)+\pi j\left(\frac{1}{\omega_o}\right) - \left(\pi j\frac{\omega_o}{\omega_o^2+\nu^2} + 2\pi j\frac{j\nu}{j\nu-\omega_o}\frac{1}{j2\nu}\right)\right] + \boxed{\pi\frac{\omega_p^2\varepsilon_o}{\nu}\delta(\omega_o)} \qquad (40)$$

And finally,

$$\varepsilon_R(\omega_o) - \varepsilon_o = - \frac{\omega_p^2 \varepsilon_o}{\omega_o^2 + \nu^2} + \boxed{\pi \frac{\omega_p^2 \varepsilon_o}{\nu} \delta(\omega_o)} \tag{41}$$

which agrees with first equation of equation (35).

APPENDIX

DERIVATIONS OF STANDARD HILBERT TRANSFORM PAIR AND KRAMERS-KRONIG RELATIONS

For a complex function f(z) of a complex variable z which is analytic in the lower half plane of the complex-ω plane, application of Cauchy's integral formula for a complex variable z_o located in the lower half plane is given as

$$\oint \frac{f(z)}{z - z_o} dz = - 2\pi j f(z_o) \qquad \text{(A-1)}$$

where the closed contour is taken along the real axis and infinite semicircle in the lower half plane (see Fig. 2a.). If f(z) approaches zero uniformly as z approaches infinity, then the contribution from the semicircle contour is zero [7]. Therefore, equation (A-1) becomes

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - z_o} dx = - 2\pi j f(z_o) \qquad \text{(A-2)}$$

As the imaginary part of z_o vanishes, the integral in equation (A-2) must be deformed in such a manner so that the contour remains on the same side of x_o as the original contour (see Fig. 2b.), thus

$$\lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x - x_o} dx + \int_{\pi}^{0} f(x_o + \varepsilon e^{j\phi}) j d\phi + \int_{\varepsilon}^{\infty} \frac{f(x)}{x - x_o} dx \right] = - 2\pi j f(x_o) \qquad \text{(A-3)}$$

By definition,

$$\lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x - x_o} dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x - x_o} dx \right] \equiv \mathcal{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_o} dx \qquad \text{(A-4)}$$

where $\mathcal{PV}$ denotes Cauchy's principle value. Upon evaluating the second term in equation (A-3) and substituting equation (A-4),

$$\mathcal{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_o} dx - j\pi f(x_o) = - 2\pi j f(x_o) \qquad \text{(A-5)}$$

and thus,

$$f(x_o) = \frac{1}{-j\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_o} dx \qquad \text{(A-6)}$$

In general, f(z) can be written as

$$f(z) = u(x,y) - j\, v(x,y) \qquad \text{(A-7)}$$

where u(x,y) and v(x,y) are real functions. Substitution of equation (A-7) with y = 0 into equation (A-6) and then equating real and imaginary parts yields

$$\left.\begin{aligned} u(x_o) &= \frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_o} dx \\ v(x_o) &= -\frac{1}{\pi} \mathcal{PV} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_o} dx \end{aligned}\right\} \qquad \text{(A-8)}$$

Equation (A-8) is defined as the Hilbert transform pair for the function f(z) = u(x,y) - jv(x,y) with y = 0 which is analytic in the lower half plane of the complex-ω plane.

According to Cauchy-Goursat integral formula, the closed integral of an analytic function g(z) is zero [8]; that is,

$$\oint g(z)\, dz = 0 \qquad \text{(A-9)}$$

Choosing an infinite semicircle in the lower half plane of the complex z-plane and the real axis as the closed contour with $g(z) = f(z)e^{jzt}$ where f(z) is analytic in lower half plane, equation (A-9) becomes

$$\int_{-\infty}^{\infty} f(x)\, e^{jxt} dx + \int_{C_r} f(z)\, e^{jzt} dz = 0 \qquad \text{(A-10)}$$

For t < 0, the contribution from contour C_r is easily shown to be zero. Therefore,

$$\int_{-\infty}^{\infty} f(x)\, e^{jxt} dx = 0 \qquad \text{(A-11)}$$

From Fourier analysis, the following transform pair is defined

$$\left.\begin{aligned} \mathcal{f}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{jxt}dx \\ f(x) &= \int_{-\infty}^{\infty} \mathcal{f}(t)\, e^{-jxt}dt \end{aligned}\right\} \qquad \text{(A-12)}$$

If $\mathcal{f}(t)$ represents a real time function then, according to equation (11), $\mathcal{f}(t)$ is zero for $t<0$, and therefore, the lower limit in the second equation of equation (12) can be replaced with zero. With these conditions f(x) is written as

$$f(x) = \int_{0}^{\infty} \mathcal{f}(t)\cos(xt)\, dt - j\int_{0}^{\infty} \mathcal{f}(t)\sin(xt)\, dt \qquad \text{(A-13)}$$

Comparing this equation with equation (7) for y = 0, one can write

$$\left.\begin{aligned} u(x) &= \int_{0}^{\infty} \mathcal{f}(t)\cos(xt)\, dt \\ v(x) &= \int_{0}^{\infty} \mathcal{f}(t)\sin(xt)\, dt \end{aligned}\right\} \qquad \text{(A-14)}$$

Clearly, u(x) is an even function and v(x) is an odd function. These properties permit the conversion of the Hilbert transform pair (equation (A-8)) into the Kramers-Kronig relations. The Hilbert transform pair, equation (A-8), is expanded as

$$\left.\begin{aligned} u(x_o) &= \frac{1}{\pi}\,\mathcal{PV}\left[\int_{0}^{\infty}\frac{v(x)}{x-x_o}dx + \int_{-\infty}^{0}\frac{v(x)}{x-x_o}dx\right] \\ v(x_o) &= -\frac{1}{\pi}\,\mathcal{PV}\left[\int_{0}^{\infty}\frac{u(x)}{x-x_o}dx + \int_{-\infty}^{0}\frac{u(x)}{x-x_o}dx\right] \end{aligned}\right\} \qquad \text{(A-15)}$$

With $u(x) = u(-x)$ and $v(x) = -v(-x)$,

$$\left.\begin{aligned} u(x_o) &= \frac{1}{\pi}\, \mathcal{PV} \left[\int_0^\infty \frac{v(x)}{x - x_o}\, dx + \int_0^\infty \frac{v(x)}{x + x_o}\, dx \right] \\ v(x_o) &= -\frac{1}{\pi}\, \mathcal{PV} \left[\int_0^\infty \frac{u(x)}{x - x_o}\, dx - \int_0^\infty \frac{u(x)}{x + x_o}\, dx \right] \end{aligned}\right\} \qquad \text{(A-16)}$$

And finally,

$$\left.\begin{aligned} u(x_o) &= \frac{2}{\pi}\, \mathcal{PV} \int_0^\infty \frac{x\, v(x)}{x^2 - x_o^2}\, dx \\ v(x_o) &= -\frac{2}{\pi}\, x_o\, \mathcal{PV} \int_0^\infty \frac{u(x)}{x^2 - x_o^2}\, dx \end{aligned}\right\} \qquad \text{(A-17)}$$

Equation (A-17) is defined as the Kramers-Kronig relations for a complex function $f(z) = u(x,y) - jv(x,y)$ with $y = 0$, which is analytic in the lower half plane of the complex-z plane.

CONCLUDING REMARKS

The Hilbert transform pair and the Kramers-Kronig relations given in the open literature are not applicable to permittivity representations which account for loss **only** in the plasma/dielectric model. In this paper, modified versions of these relationships are derived for these permittivity representations. The necessity of the additional term (impulsive term) to the standard relationships is demonstrated. These modified versions thus provide the necessary relationships that guarantee satisfaction of the principle of causality—a requirement demanded of time-domain solutions of Maxwell's equations.

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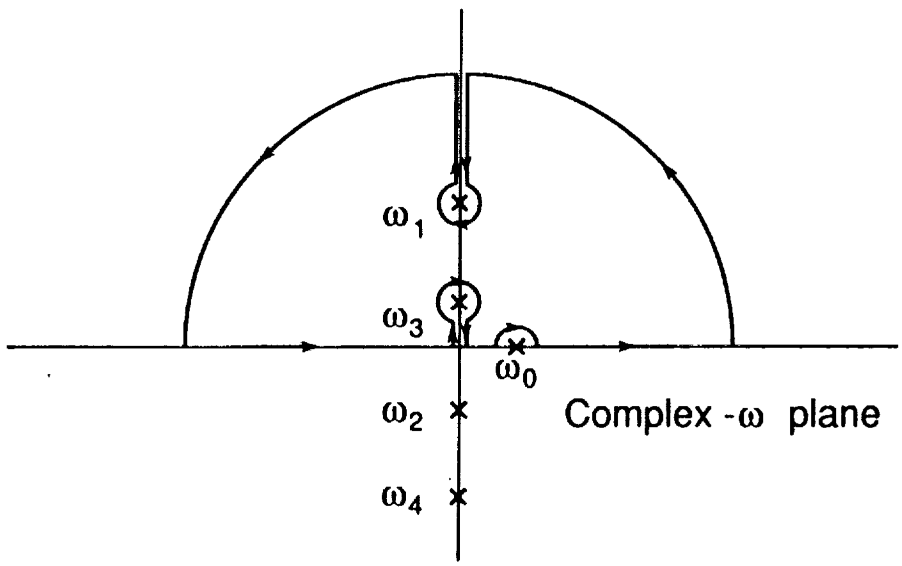
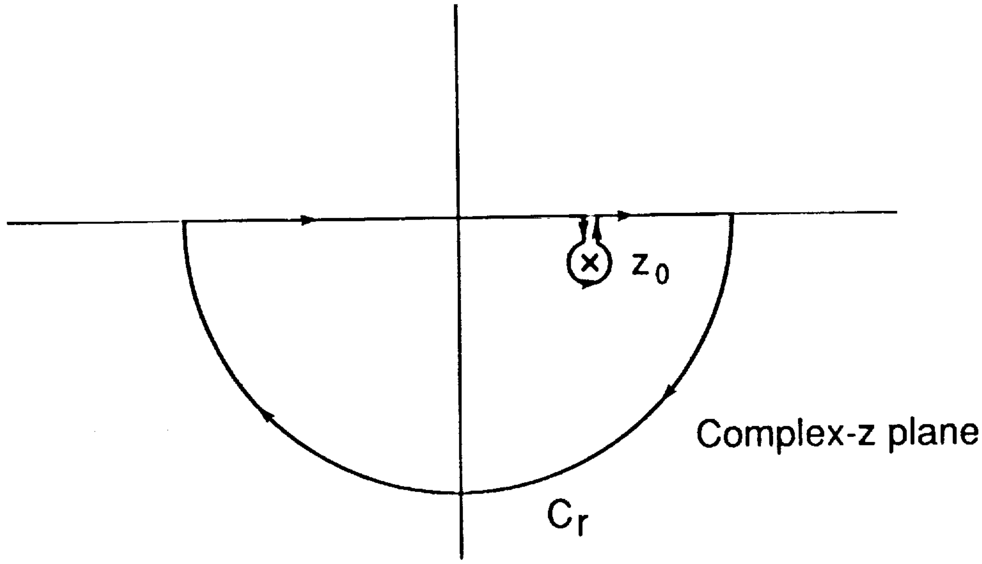
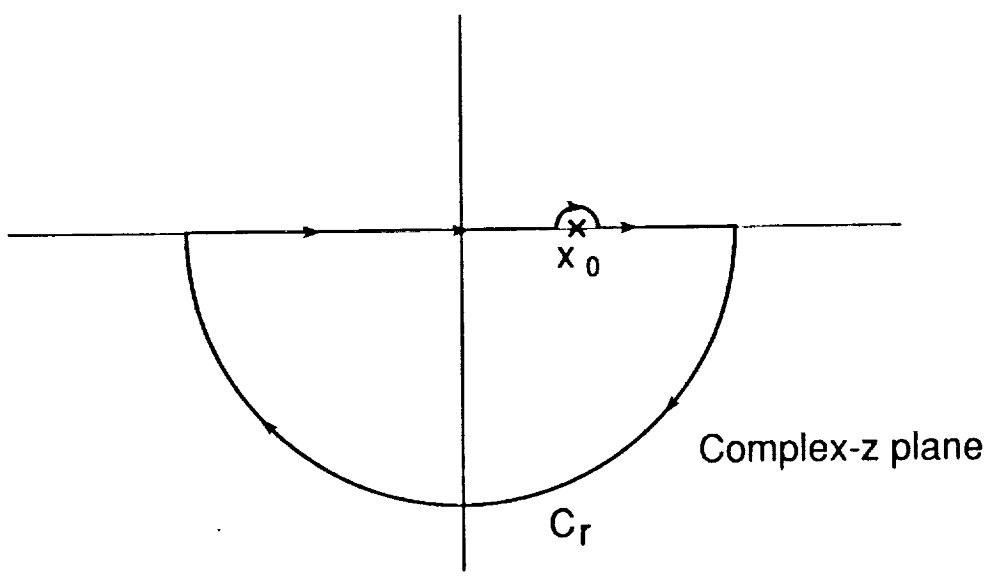


Figure 1.-Contour for equations (10) and (19).



a. Imaginary part of $z_0 < 0$



b. Imaginary part of $z_0 = 0$

Figure 2.-Contours for equations (A-1) and (A-2).

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15. Supplementary Notes

16. Abstract

Hilbert transform pair and Kramers-Kronig relations given in the open literature are not applicable to permittivity representations which account for loss only in the plasma/dielectric model. In this paper, modified versions of these relationships are derived.

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